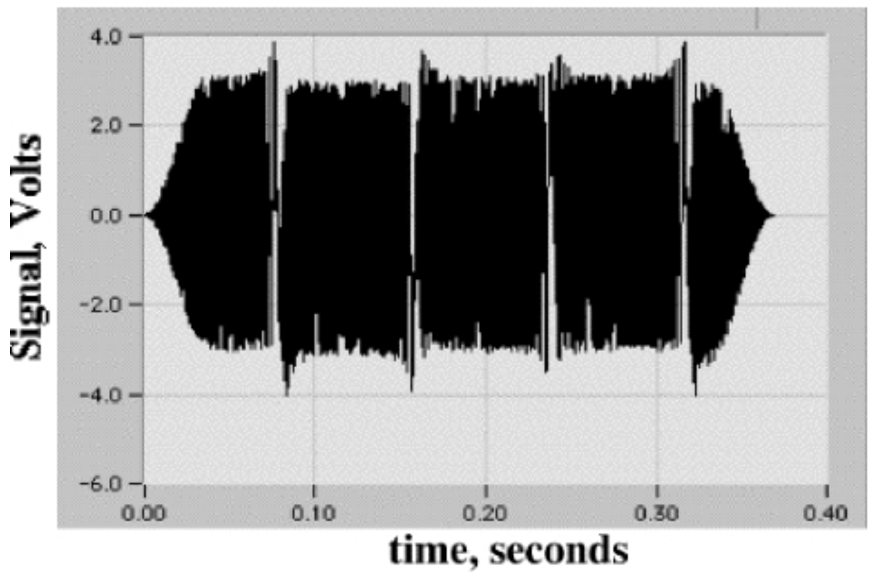
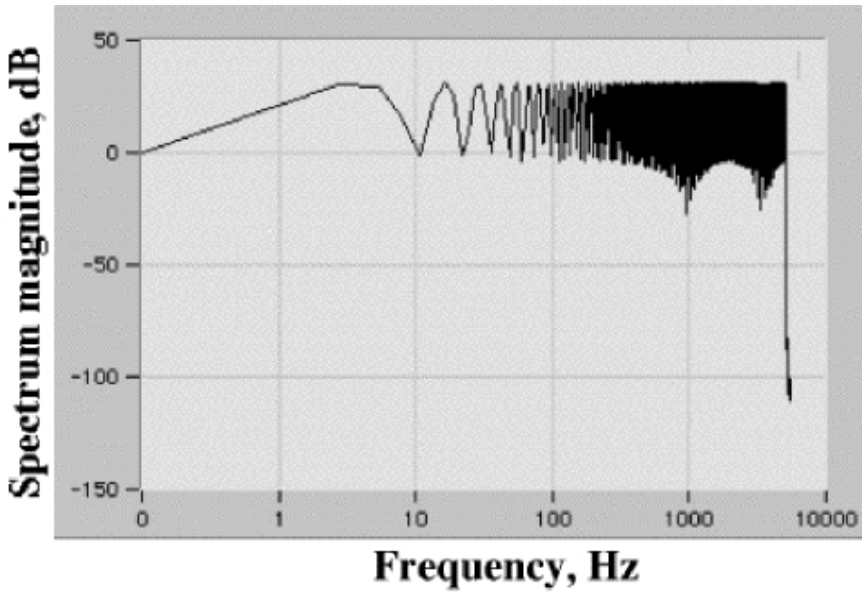
Section 2.2: The Analog Measurand: Time-Dependent Characteristics Beckwith Chapter 4.



a) Excitation Wave Form



b) Wave Form Spectrum

Analog Signals (1)

• An analog signal is the time history of any Physical quantity as it varies with respect to time.

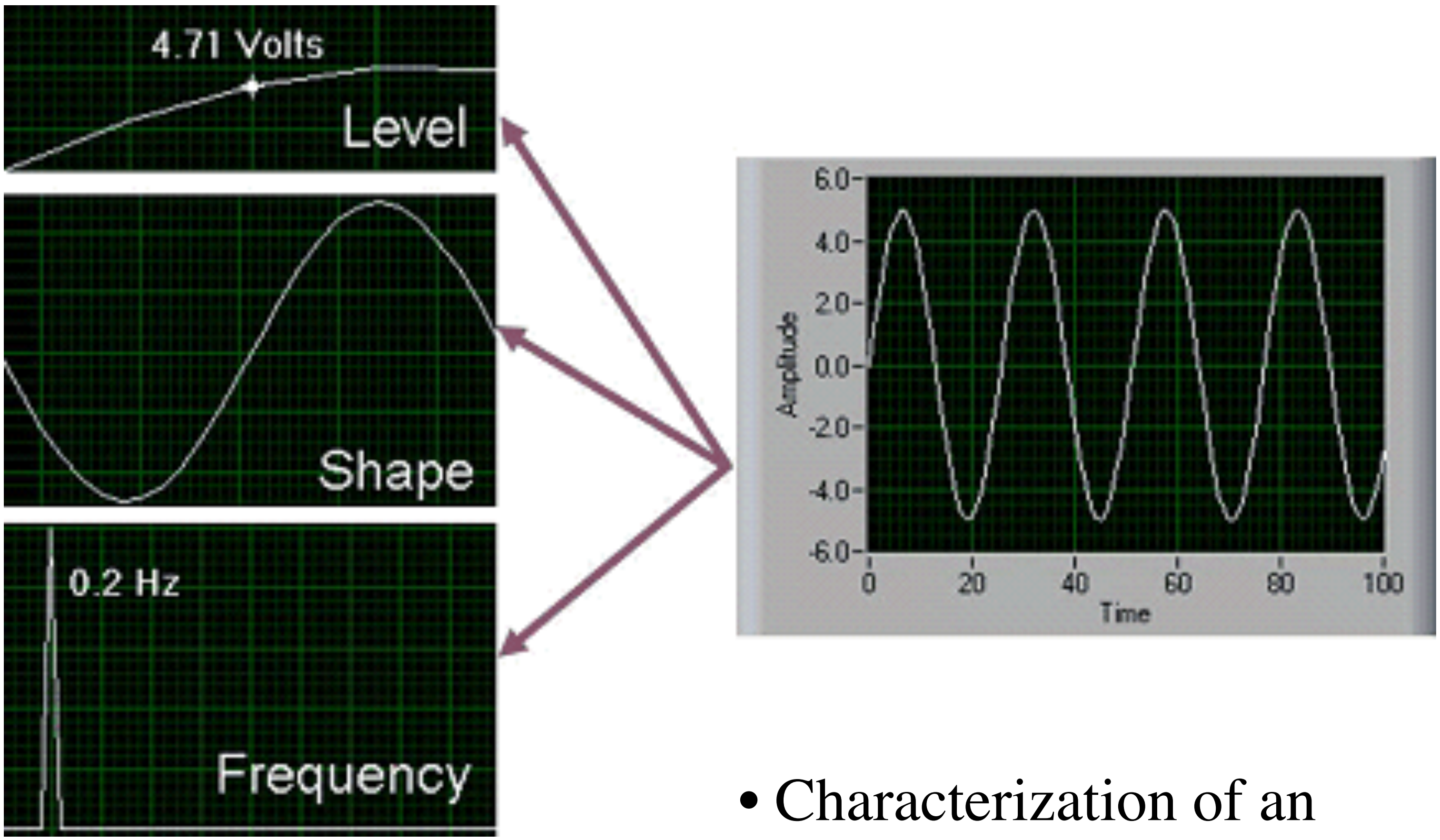
• Examples of analog signals include voltage, temperature, pressure, sound, and load.

• The three primary characteristics of an analog signal are

1) level
2) shape
3) frequency

• Analog signals are examples of a broader class of mathematical entities called *"Waveforms"*

Analog Signals (2)



- Characterization of an analog signal

Analog Signals (3)

• ***Level***

Because analog signals can have arbitrary values, level gives vital information about a measured analog signal. Intensity of a light source, temperature in a room, and pressure in a chamber are examples demonstrating importance of signal level.

• ***Shape***

Some signals are named after their shape - sine, square, sawtooth, and triangle. Shape of an analog signal is as important as level -- because by measuring shape of an analog signal one can further analyze signal properties, including peak values, DC values, and slope. Analysis of heartbeats, video signals, sounds, vibrations, and circuit responses are applications involving shape measurements.

•***Frequency***

All analog signals are categorized by their frequency. Unlike level or shape, frequency cannot be directly measured. Signal must be analyzed using software to determine the frequency information. This analysis is usually done using an algorithm known as the Fourier transform. ***(more on this later)***

Classification of Waveforms

The shape of the signal on an x-y plot is called a Waveform. These are categorized as such:

Static Does not change with time. $y(t) = A_o$

Dynamic Does vary with time

Deterministic Varies in time in predictable way

Periodic Repeats at regular intervals

$$y(t) = A_o \sin \omega t$$

Complex Periodic More than a single frequency

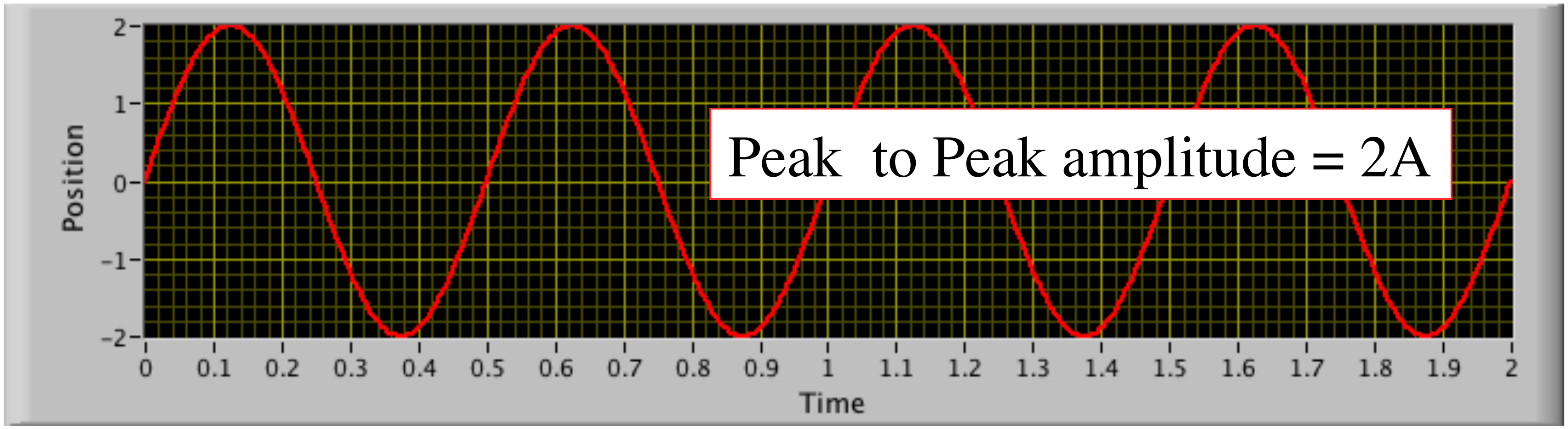
$$y(t) = A_o + \sum_{n=1}^{\infty} C_n \sin\left(n\omega t + \phi_n\right)$$

Nonperiodic Does not repeat

$$y(t) = A_o \text{ for } t > 0$$

Waveform Amplitude

Cyclic Frequency Plot



$$y_{RMS} \equiv \sqrt{\frac{1}{T}\int_0^T y(t)^2\,dt} = \sqrt{\frac{1}{T}\int_0^T \left\{A\sin[\omega t]\right\}^2 dt} = A\sqrt{\frac{1}{T}\int_0^T \sin^2[\omega t]\,dt} =$$

$$A\sqrt{\frac{1}{T}\int_0^T \frac{1}{2}\left(1-\cos[2\omega t]\right)dt} = A\sqrt{\frac{1}{2T}\cdot T - \frac{1}{2}\int_0^T \cos[2\omega t]\,dt} = A\sqrt{\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2\omega}\sin 2\omega T} =$$

$$A\sqrt{\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2\omega}\sin 2\cdot\frac{2\pi}{T}T} = \boxed{\frac{\sqrt{2}}{2}A}$$

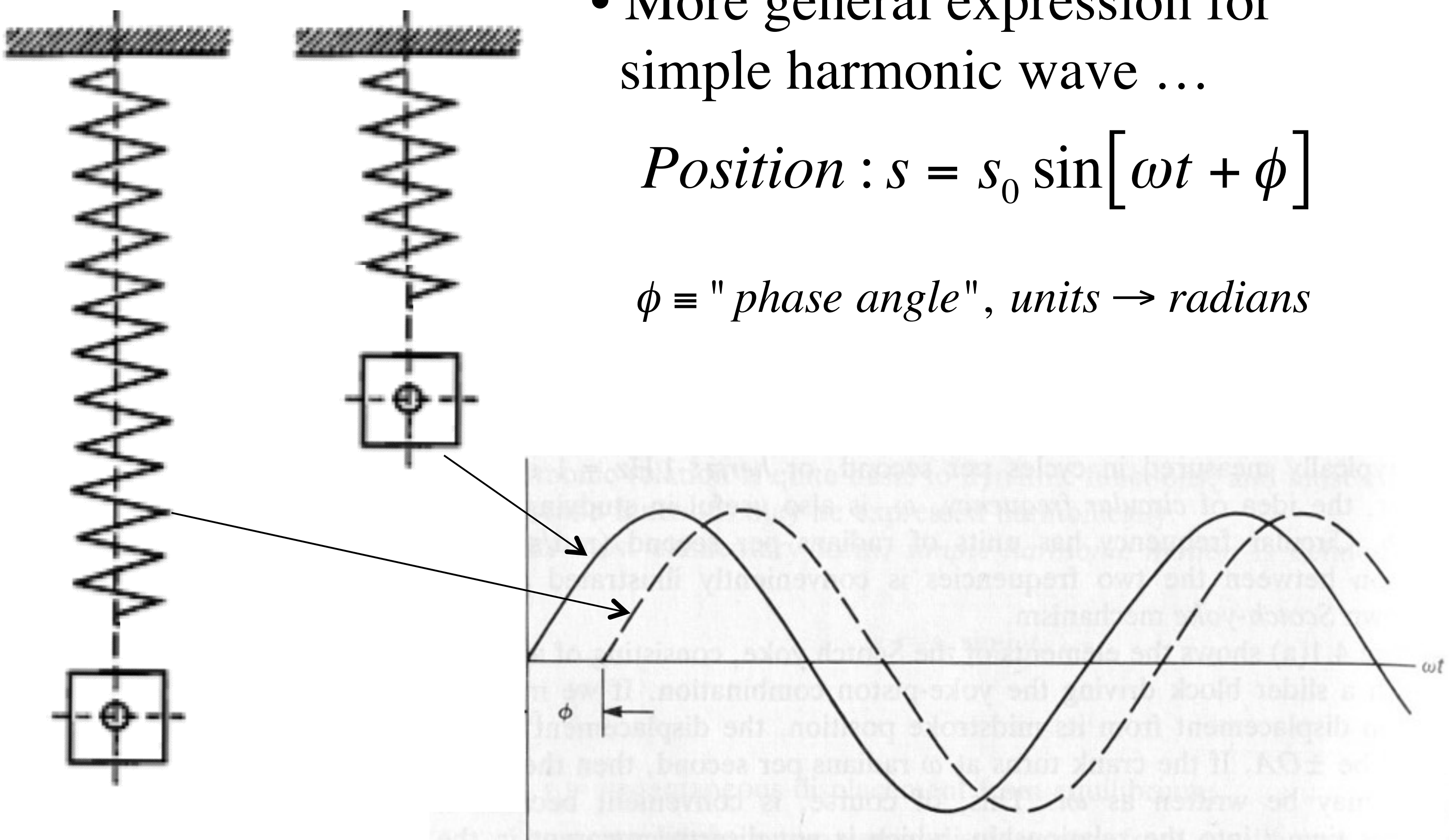
"Root Mean Squared" amplitude

Phase Angle (1)

- More general expression for simple harmonic wave …

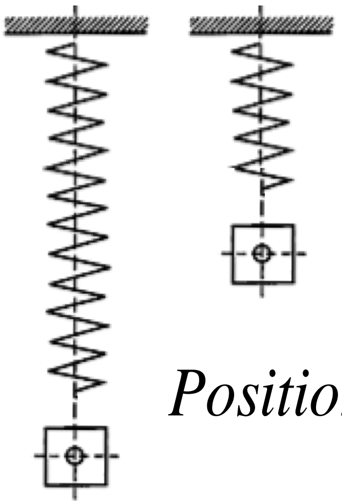
$$Position : s = s_0 \sin\left[\omega t + \phi\right]$$

$$\phi \equiv \text{"} phase\ angle \text{"},\ units \rightarrow radians$$



- Has the effect of *shifting the signal in time*

Phase Angle (2)



$$Position: s = s_0 \sin\left[\omega t + \phi\right]$$

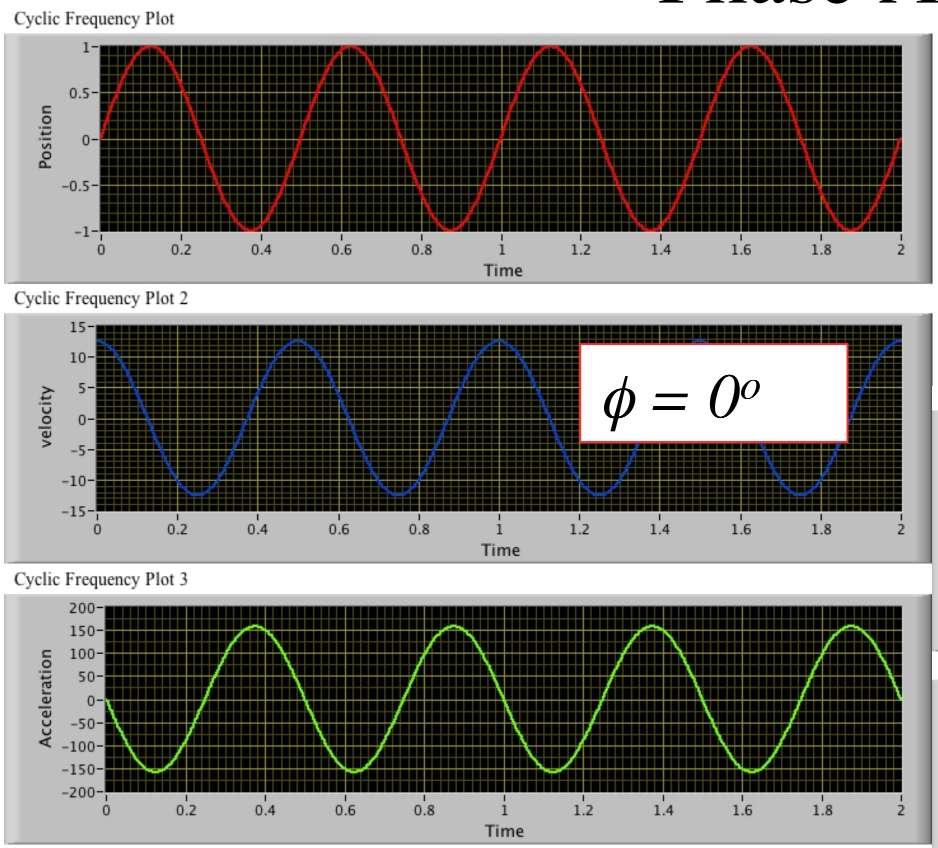
$$Velocity: v = \frac{d}{dt}\left[\sin\left[\omega t + \phi\right]\ \right] \rightarrow$$

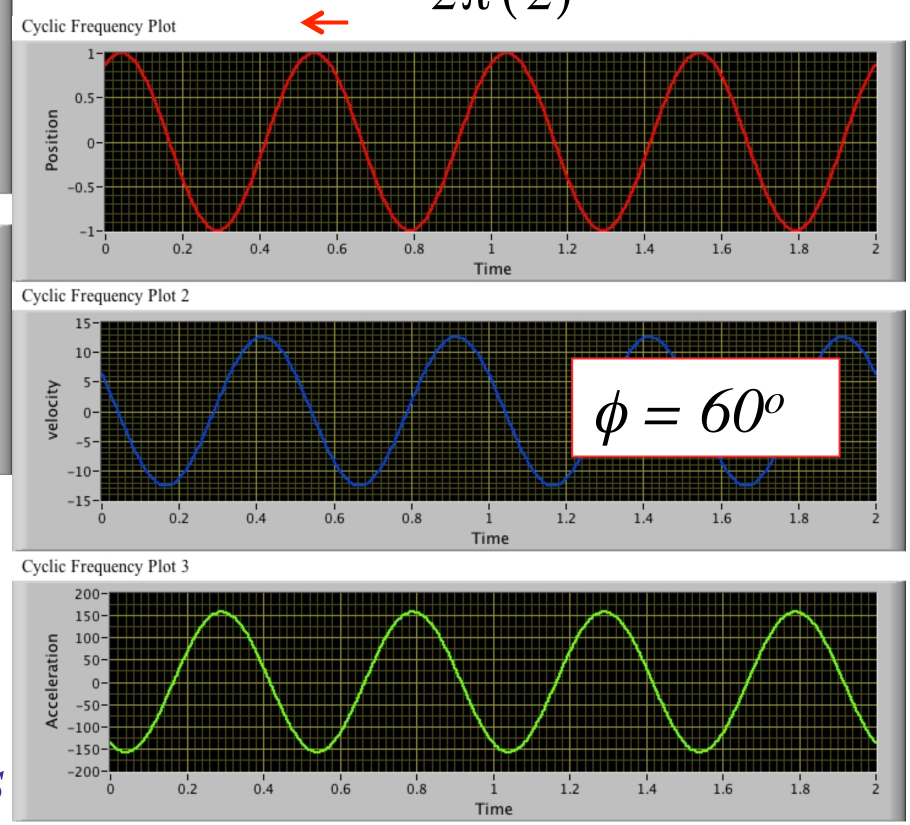
$$s_0, \omega, \phi = const \rightarrow v = s_0 \cos\left[\omega t + \phi\right] \cdot \frac{d}{dt}\left[\omega t + \phi\right] = s_0 \omega \cos\left[\omega t + \phi\right]$$

$$Acceleration: a = \frac{dv}{dt} = \frac{d}{dt}\left[s_0 \omega \cos\left[\omega t + \phi\right]\right] = -s_0 \omega^2 \sin\left[\omega t + \phi\right]$$

Phase Angle (3)

$$\Delta t = \frac{\phi^\circ \cdot \frac{180}{\pi} \frac{deg}{radian}}{2\pi f} = \frac{60\,\frac{\pi}{180}}{2\pi(2)} = 0.0833 sec$$





Simple Harmonic Relations (1)

s is the departure from the equilibrium

s_0 is the amplitude

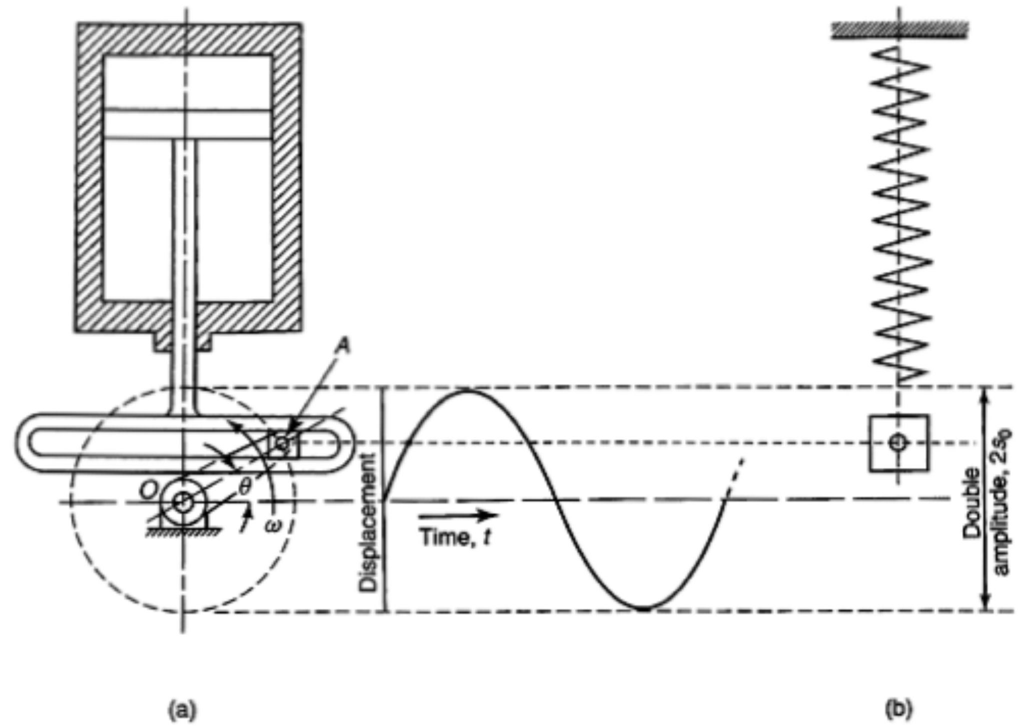
ω is the ***circular frequency*** *(rad/s)*

t is time.

- look at the scotch yoke below. If the flywheel turns at θ radians/sec, then the motion of the piston is described by

$$s = s_0 \sin \omega t$$

- Can write $\theta = \omega t$ to introduce time. The cycle repeats every 2π radians, so $\omega = 2\pi f$.

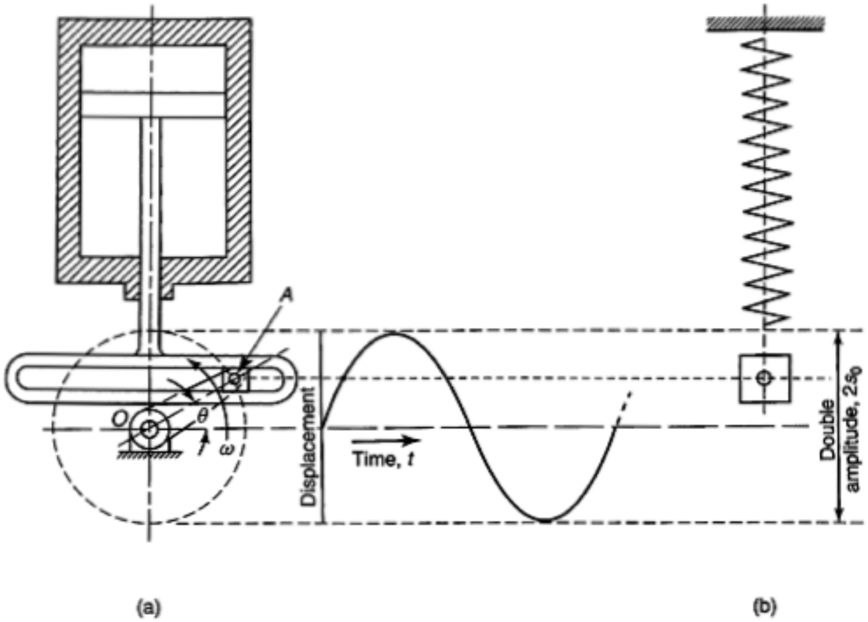


(a) The Scotch-yoke mechanism provides a simple harmonic motion to the piston; (b) a spring-mass system that moves with simple harmonic motion.

- … event occurs every T seconds, … ***Cyclic frequency*** *$f = 1/T$* Hz.

T … period

Simple Harmonic Relations (2)



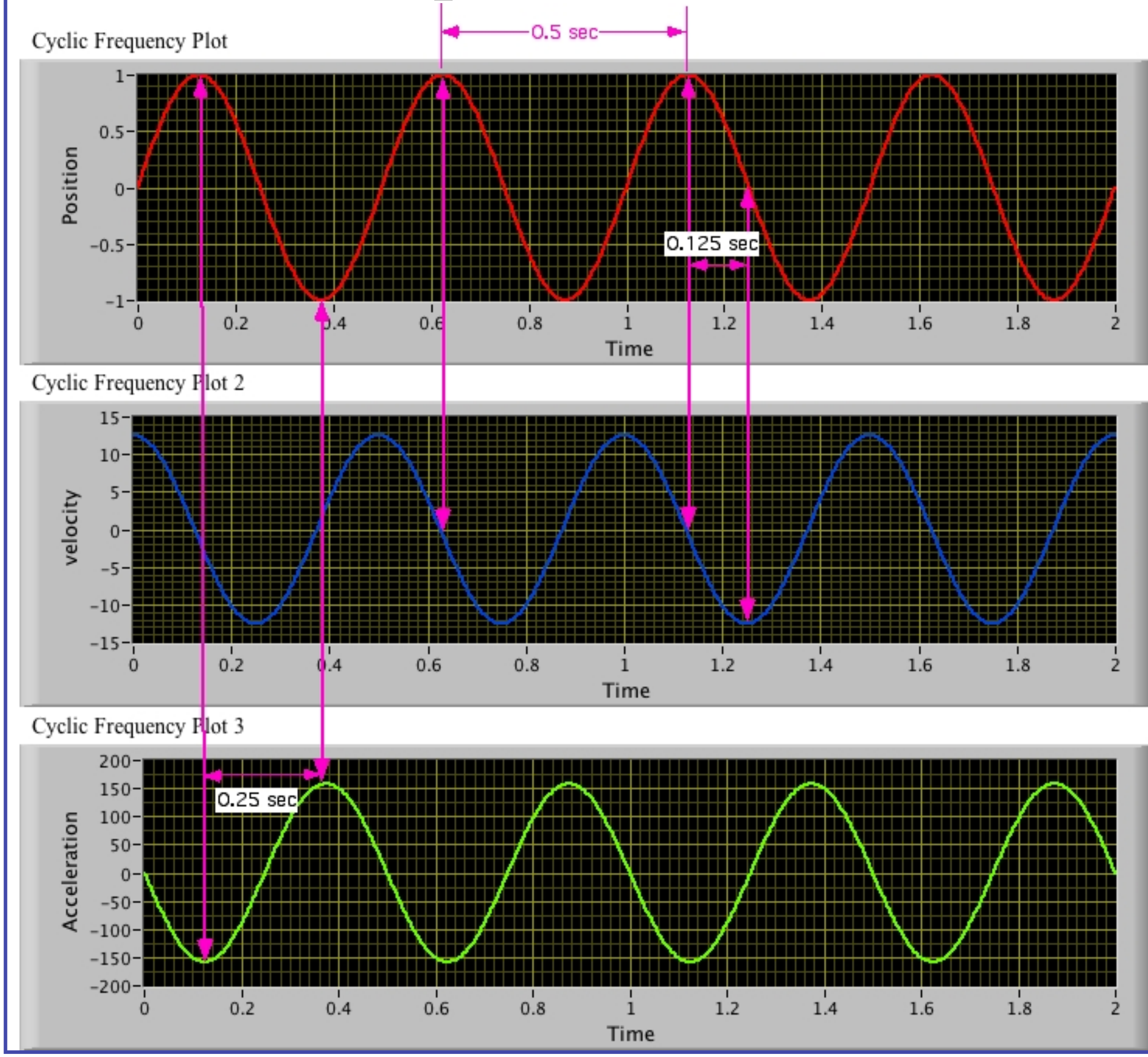
(a) The Scotch-yoke mechanism provides a simple harmonic motion to the piston; (b) a spring-mass system that moves with simple harmonic motion.

$$Position : s = s_0 \sin \omega t$$

$$Velocity : v = \frac{d}{dt}\left[s_0 \sin \omega t\right] \rightarrow s_0, \omega = const \rightarrow v = s_0 \cos \omega t \cdot \frac{d}{dt}\left[\omega t\right] = s_0 \omega \cos \omega t$$

$$Acceleration : a = \frac{dv}{dt} = \frac{d}{dt}\left[s_0 \omega \cos \omega t\right] = -s_0 \omega^2 \sin \omega t$$

Simple Harmonic Relations (3), example



- $T = 0.5$ sec
- $f = 1/T = 2$ Hz
- $\omega = 2\pi f =$ 12.5664 rad/sec
- Velocity is "out of phase" by 0.125 sec *w.r.t*. position
- Acceleration is "out Of phase" by 0.250 sec .*w.r.t*. position

Simple Harmonic Relations (4)

- $T = 0.5$ sec
- $f = 1/T = 2$ Hz
- $\omega = 2\pi f = 12.5664$ rad/sec

What is the "phasing" between Position, acceleration, and velocity?

- Velocity is "out of phase" by 0.125 sec *w.r.t.* position →

$$\omega \cdot \Delta t_{velocity} = 12.5664 \cdot 0.125 = 1.5708_{radians} = \frac{\pi}{2}_{radians} \times \frac{180}{\pi}_{\frac{deg}{radian}} = 90^\circ$$

- Acceleration is "out Of phase" by 0.250 sec *.w.r.t.* position →

$$\omega \cdot \Delta t_{acceleration} = 12.5664 \cdot 0.25 = 3.14159_{radians} = \pi_{radians} \times \frac{180}{\pi}_{\frac{deg}{radian}} = 180^\circ$$

Simple Harmonic Relations (5)

What is the "phasing" between Position and velocity?

- Look at velocity waveform

$$Position: s = s_0 \sin \omega t \rightarrow$$

$$Velocity: v = s_0 \omega \cos \omega t = s_0 \omega \cos\left[\omega t + \frac{\pi}{2} - \frac{\pi}{2}\right] =$$

$$s_0 \omega \left(\cos\left[\omega t + \frac{\pi}{2}\right] \cos\left[-\frac{\pi}{2}\right] - \sin\left[\omega t + \frac{\pi}{2}\right] \sin\left[-\frac{\pi}{2}\right] \right) = \boxed{s_0 \omega \sin\left[\omega t + \frac{\pi}{2}\right]}$$

See end of lecture for this trig relation

$$\omega \cdot \Delta t_{velocity} = 12.5664 \cdot 0.125 = 1.5708_{radians} = \frac{\pi}{2}_{radians} \times \frac{180}{\pi_{\frac{deg}{radian}}} = 90^\circ$$

$$\frac{\pi}{2} = \text{"} phase\ angle \text{"},\ \phi$$

Simple Harmonic Relations (6)

What is the "phasing" between Position and acceleration?

- Look at acceleration waveform

$$Position: s = s_0 \sin\omega t \rightarrow$$

$$Acceleration: a = -s_0\omega^2 \sin\omega t = -s_0\omega^2 \sin[\omega t + \pi - \pi] =$$

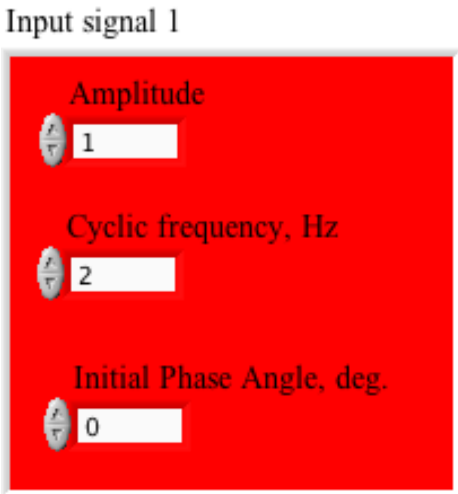
$$-s_0\omega^2 \left(\cos[\omega t + \pi]\sin[\pi] + \sin[\omega t + \pi]\cos[-\pi]\right) = \boxed{s_0\omega^2 \sin[\omega t + \pi]}$$

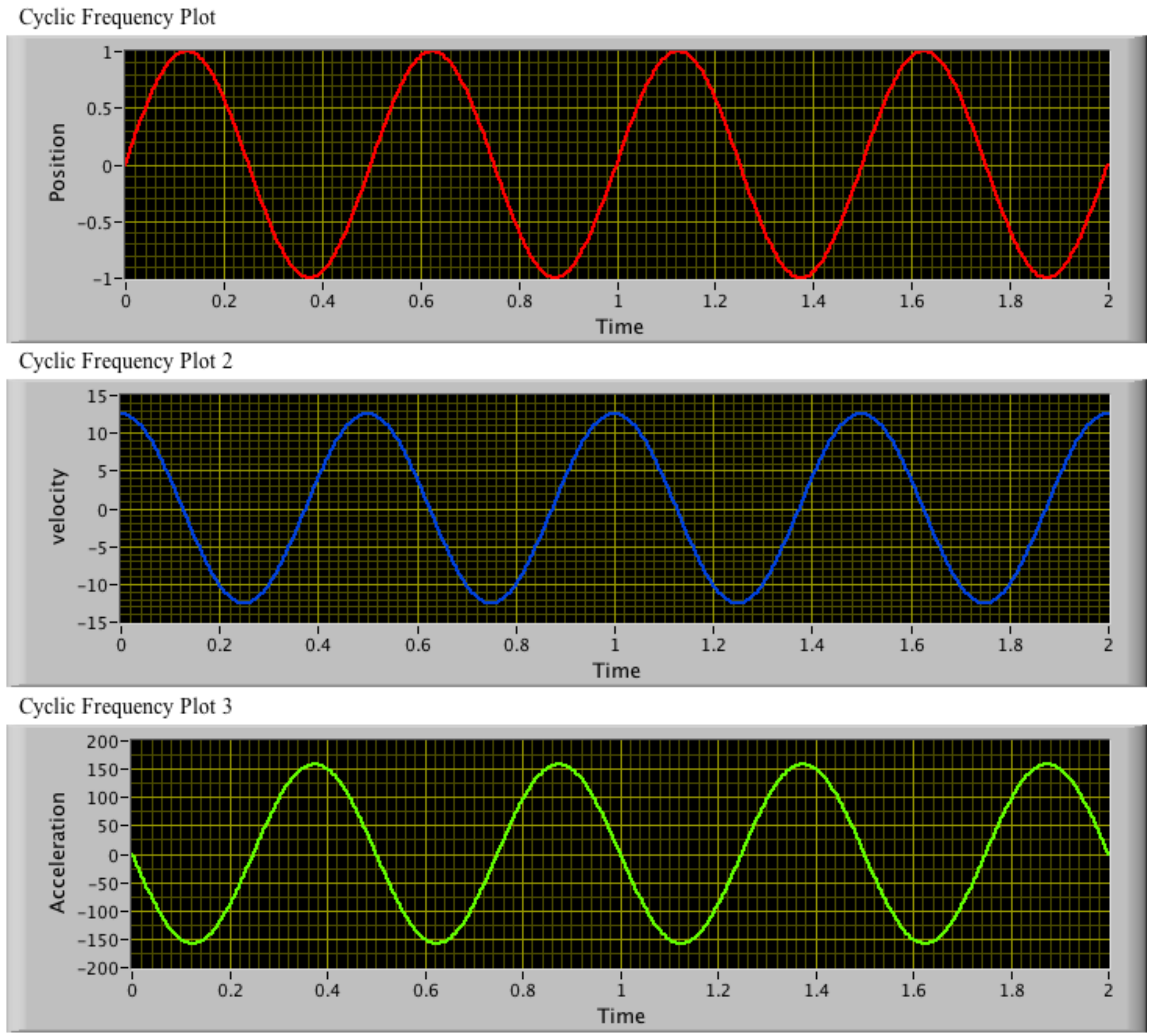
See end of lecture for this trig relation

$$\boxed{\omega \cdot \Delta t_{acceleration} = 12.5664 \cdot 0.25 = 3.14159_{radians} = \pi_{radians} \times \frac{180}{\pi}\,\frac{\deg}{radian} = 180^\circ}$$

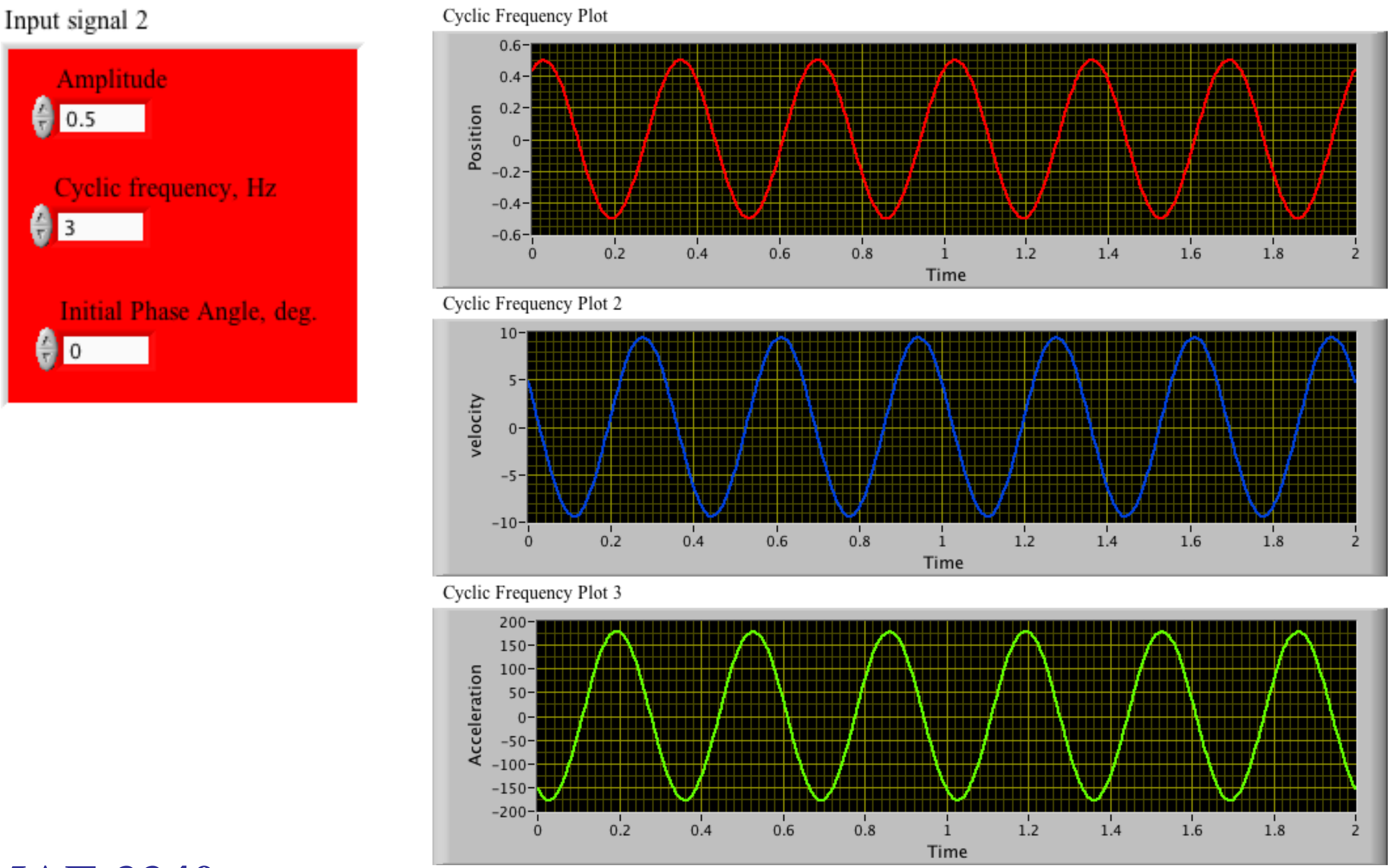
$$\pi = \text{"}phase\ angle\text{"},\ \phi$$

Two Wave Harmonic Motions (1)





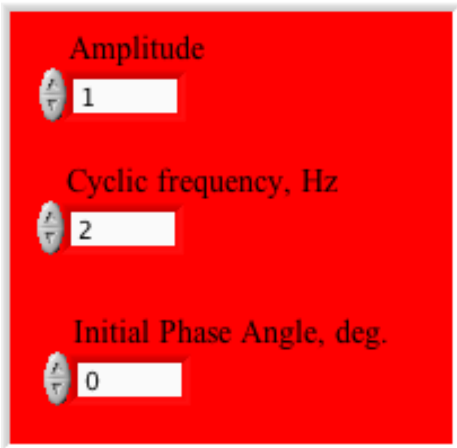
Two Wave Harmonic Motions (2)



Two Wave Harmonic Motions (3)

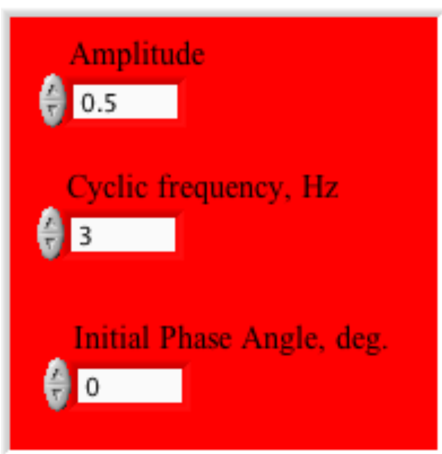
$$s_{1,2}(t) = s_1 \sin\left[\omega_1 t + \phi_1\right] + s_2 \sin\left[\omega_2 t + \phi_2\right]$$

Input signal 1

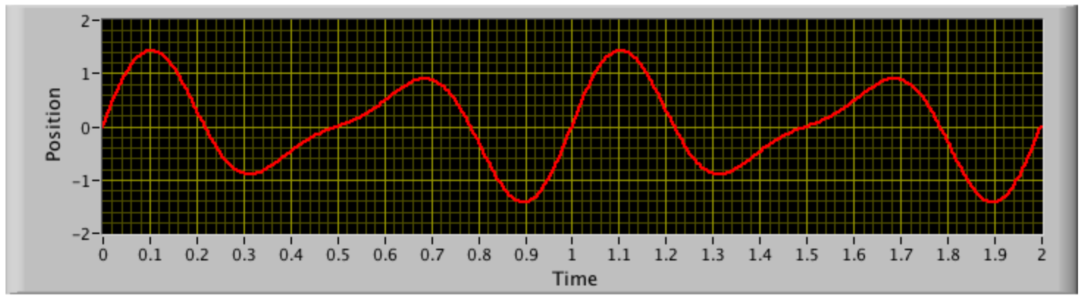


+ →

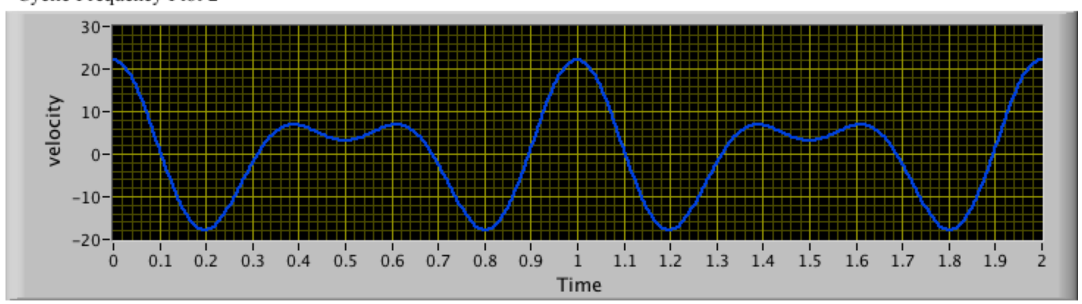
Input signal 2



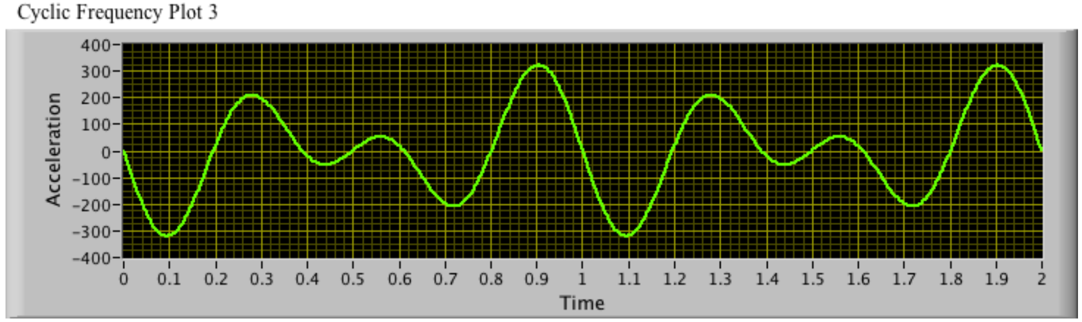
Cyclic Frequency Plot



Cyclic Frequency Plot 2



Cyclic Frequency Plot 3



"combined signal"

Two Wave Harmonic Motions (4)

- Combined signal

$$s_{1,2}(t) = s_1 \sin\left[\omega_1 t + \phi_1\right] + s_2 \sin\left[\omega_2 t + \phi_2\right] \rightarrow \textit{expand} \ \sin \ \textit{terms}$$

$$s_1\left[\sin\left[\omega_1 t\right]\cos\left[\phi_1\right] + \sin\left[\phi_1\right]\cos\left[\omega_1 t\right]\right] + s_2\left[\sin\left[\omega_2 t\right]\cos\left[\phi_2\right] + \sin\left[\phi_2\right]\cos\left[\omega_2 t\right]\right] =$$

$$\left(s_1 \cos\left[\phi_1\right]\right)\sin\left[\omega_1 t\right] + \left(s_1 \sin\left[\phi_1\right]\right)\cos\left[\omega_1 t\right] + \left(s_2 \cos\left[\phi_2\right]\right)\sin\left[\omega_2 t\right] + \left(s_2 \sin\left[\phi_2\right]\right)\cos\left[\omega_2 t\right]$$

- Define

See end of lecture for these trig relations

$$\begin{bmatrix} B_1 = s_1 \cos\left[\phi_1\right] \\ A_1 = s_1 \sin\left[\phi_1\right] \end{bmatrix} \cdots\cdots \begin{bmatrix} B_2 = s_2 \cos\left[\phi_2\right] \\ A_2 = s_2 \sin\left[\phi_2\right] \end{bmatrix} \rightarrow$$

"combined signal"

$$s_{1,2}(t) = B_1 \sin\left[\omega_1 t\right] + A_1 \cos\left[\omega_1 t\right] + B_2 \sin\left[\omega_2 t\right] + A_2 \cos\left[\omega_2 t\right]$$

Two Wave Harmonic Motions (5)

- Combined signal written as summation

$$s_{1,2}(t) = s_1 \sin\left[\omega_1 t + \phi_1\right] + s_2 \sin\left[\omega_2 t + \phi_2\right] =$$

$$B_1 \sin\left[\omega_1 t\right] + A_1 \cos\left[\omega_1 t\right] + B_2 \sin\left[\omega_2 t\right] + A_2 \cos\left[\omega_2 t\right] = \sum_{i=1}^{2} \left\{B_i \sin\left[\omega_i t\right] + A_i \cos\left[\omega_i t\right]\right\}$$

- Relation between phase angle and coefficients $\{A_i, B_i\}$

$$\begin{bmatrix} B_i = s_i \cos\left[\phi_i\right] \\ A_i = s_i \sin\left[\phi_i\right] \end{bmatrix} \ldots\ldots\ldots \frac{A_i}{B_i} = \frac{\sin\left[\phi_i\right]}{\cos\left[\phi_i\right]} = \tan\left[\phi_i\right] \rightarrow \phi_i = \tan^{-1}\left[\frac{A_i}{B_i}\right]$$

Waveform Examples (1)

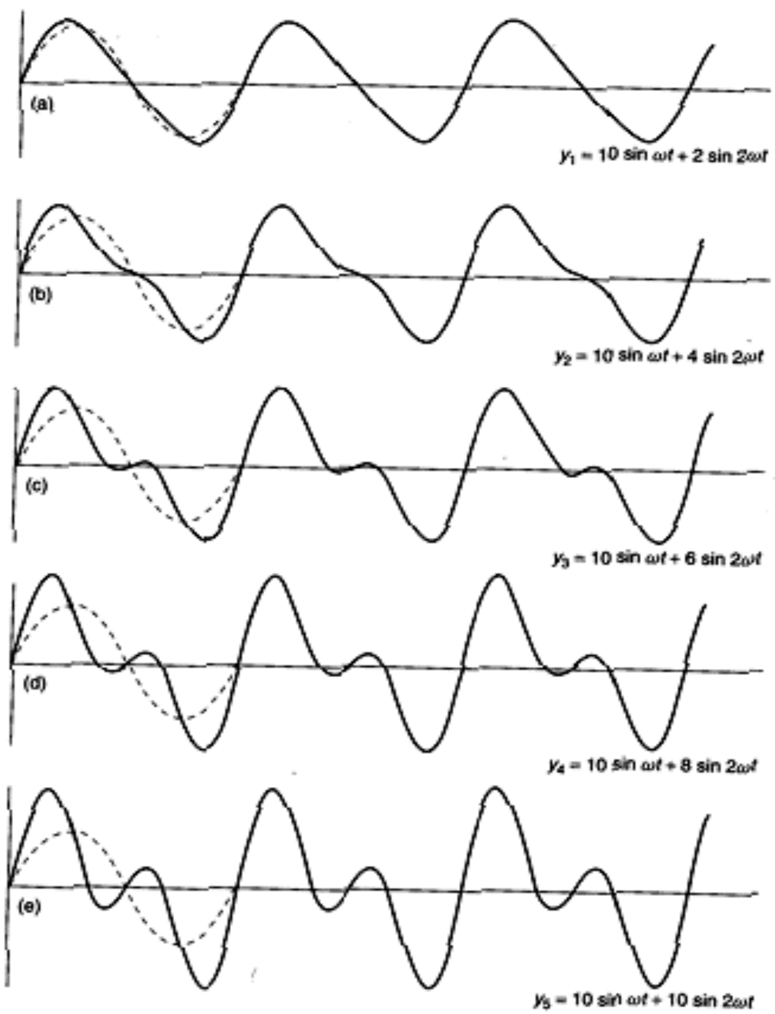


FIGURE 4.3: Examples of two-component waveforms with second-harmonic component of various relative amplitudes.

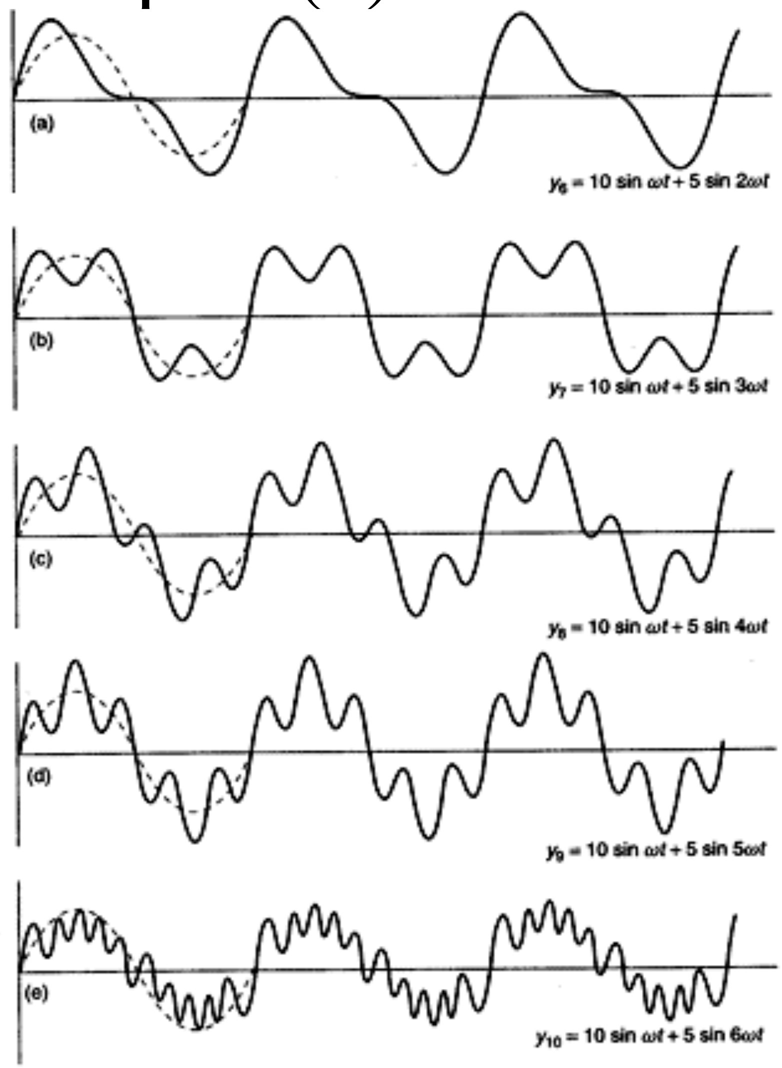


FIGURE 4.4: Examples of two-component waveforms with second term of various relative frequencies.

Waveform Examples (2)

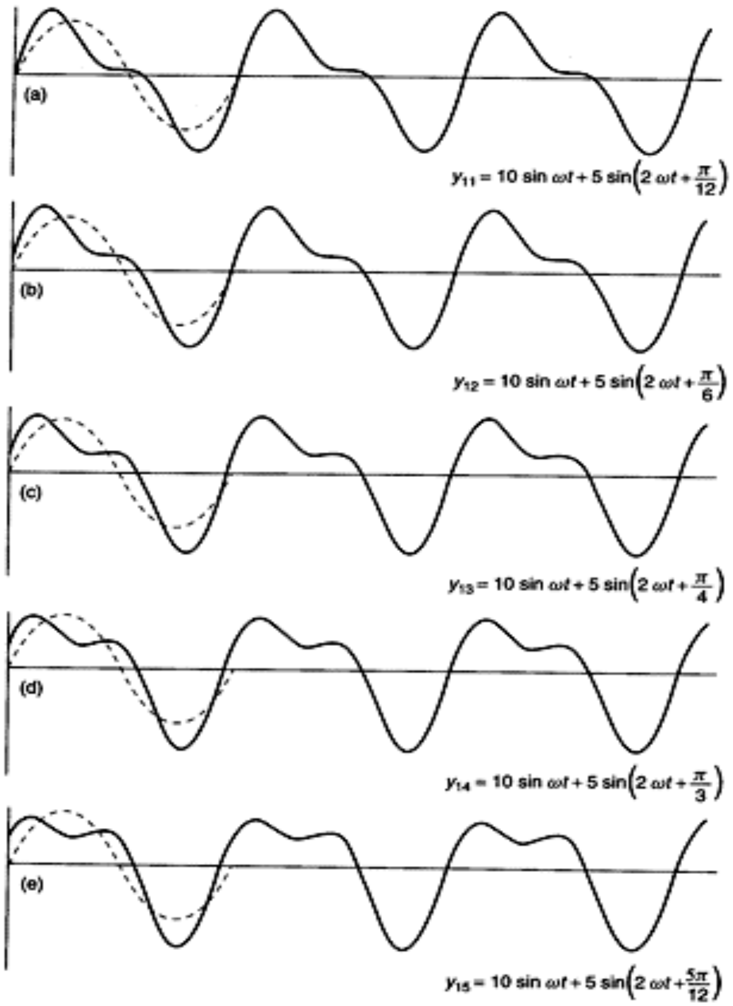


FIGURE 4.5: Examples of two-component waveforms with second harmonic having various degrees of phase shift.

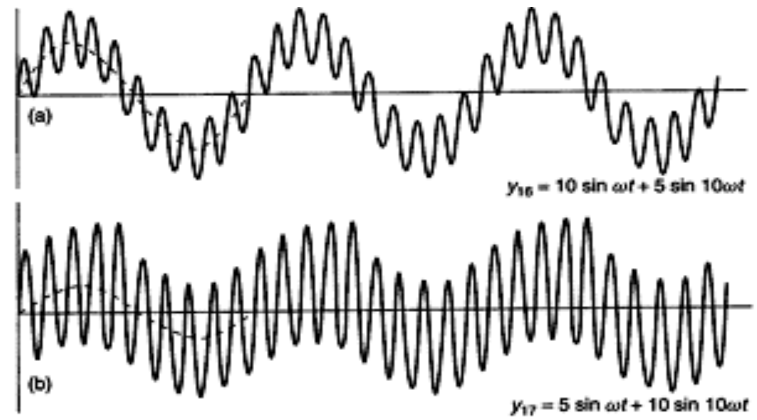


FIGURE 4.6: Examples of waveforms with the two components having considerably different amplitudes.

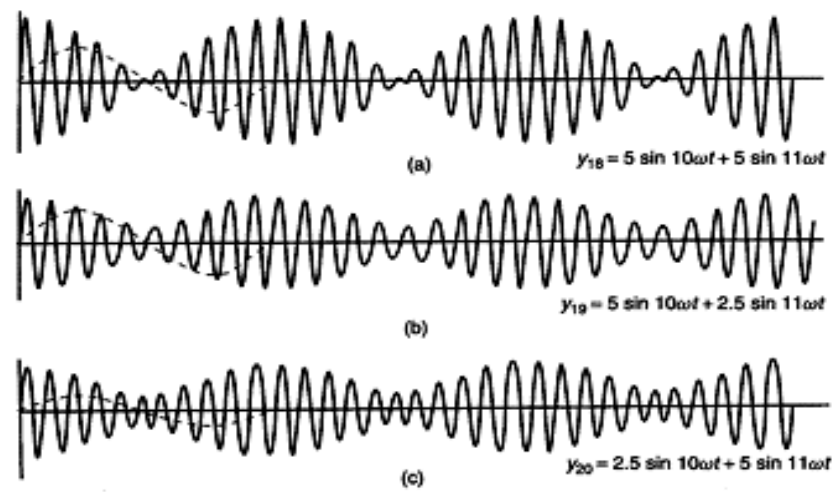


FIGURE 4.7: Examples of waveforms with the two components having frequencies that are nearly the same.

Beat Frequency and Heterodyning (1)

• In telecommunications and radio astronomy, **heterodyning** is the generation of new frequencies by mixing two or more signals in a nonlinear device such as a vacuum tube, transistor, diode mixer, Josephson junction, or bolometer. The mixing of each two frequencies results in the creation of two new frequencies, one at the sum of the two frequencies mixed, and the other at their difference. A low frequency produced in this manner is sometimes referred to as a beat frequency.

A beat frequency, or "beating," can be heard when multiple engines of an aircraft are running at close but not identical speeds, or two musical instruments are playing slightly out of tune. For example, a frequency of 3,000 hertz and another of 3,100 hertz would beat together, producing an audible beat frequency of 100 hertz. A *heterodyne* radio or infrared receiver is one which uses such a frequency shifting process.

Beat Frequency and Heterodyning (2)

- Consider two waveforms of same amplitude AND TWO NEARLY EQUAL FREQUENCIES

$$y(t) = A\sin\left[\omega_0 t\right] + A\sin\left[\omega_1 t\right]$$

Let …. $\omega_1 = \omega_0 + \Delta\omega$ --> $\Delta\omega$ ~ small

- Then

$$y(t) = A\sin\left[\omega_0 t\right] + A\sin\left[\omega_1 t\right] = A\left\{\sin\left[\omega_0 t\right] + \sin\left[\left(\omega_0 + \Delta\omega\right)t\right]\right\} =$$

$$A\left\{\sin\left[\omega_0 t\right] + \sin\left[\omega_0 t\right]\cos\left[\Delta\omega t\right] + \cos\left[\omega_0 t\right]\sin\left[\Delta\omega t\right]\right\} =$$

$$A\sin\left[\omega_0 t\right]\left\{1 + \cos\left[\Delta\omega t\right]\right\} + A\cos\left[\omega_0 t\right]\sin\left[\Delta\omega t\right]$$

Beat Frequency and Heterodyning (3)

- From Trig substitutions

$$\left\{1+\cos\left[\Delta\omega t\right]\right\} = 2\cos^2\left[\frac{\Delta\omega}{2}t\right] \ldots\ldots and \ldots\ldots \sin\left[\Delta\omega t\right] = 2\cos\left[\frac{\Delta\omega}{2}t\right]\sin\left[\frac{\Delta\omega}{2}t\right]$$

- Substituting into ….

$$y(t) = 2A\sin\left[\omega_0 t\right]\cos^2\left[\frac{\Delta\omega}{2}t\right] + A\cos\left[\omega_0 t\right]\cos\left[\frac{\Delta\omega}{2}t\right]\sin\left[\frac{\Delta\omega}{2}t\right] =$$

$$2A\cos\left[\frac{\Delta\omega}{2}t\right]\left\{\sin\left[\omega_0 t\right]\cos\left[\frac{\Delta\omega}{2}t\right] + \cos\left[\omega_0 t\right]\sin\left[\frac{\Delta\omega}{2}t\right]\right\} =$$

$$2A\cos\left[\frac{\Delta\omega}{2}t\right]\left\{\sin\left[\omega_0 t + \frac{\Delta\omega}{2}t\right]\right\} = 2A\cos\left[\left(\frac{\omega_1-\omega_0}{2}\right)t\right]\left\{\sin\left[\omega_0 t + \left(\frac{\omega_1-\omega_0}{2}\right)t\right]\right\} =$$

$$2A\cos\left[\left(\frac{\omega_1-\omega_0}{2}\right)t\right]\sin\left[\left(\frac{\omega_1+\omega_0}{2}\right)t\right]$$

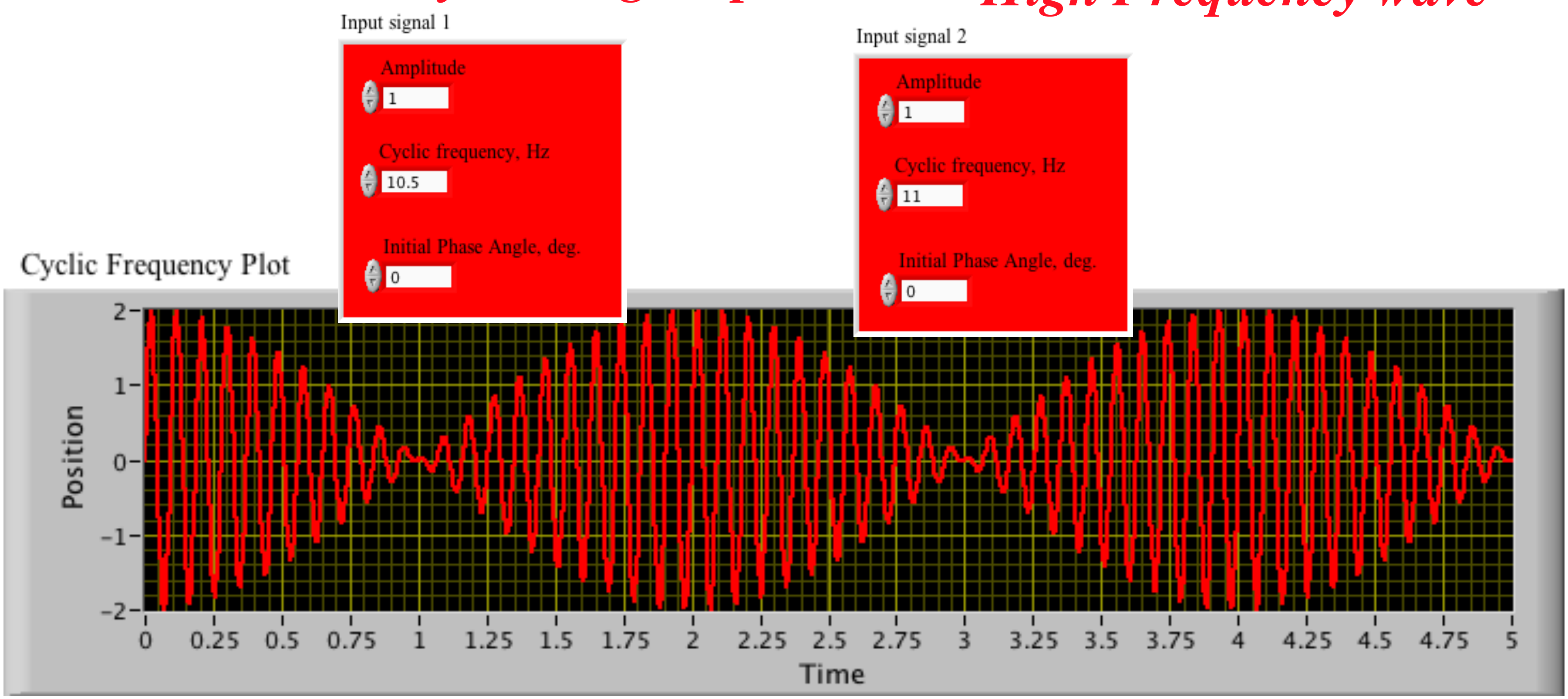
Beat Frequency and Heterodyning (4)

- Collected waveform

$$y(t) = A\sin\left[\omega_0 t\right] + A\sin\left[\omega_1 t\right] = 2A\cos\left[\left(\frac{\omega_1 - \omega_0}{2}\right)t\right]\sin\left[\left(\frac{\omega_1 + \omega_0}{2}\right)t\right]$$

Slowly beating amplitude

High Frequency wave

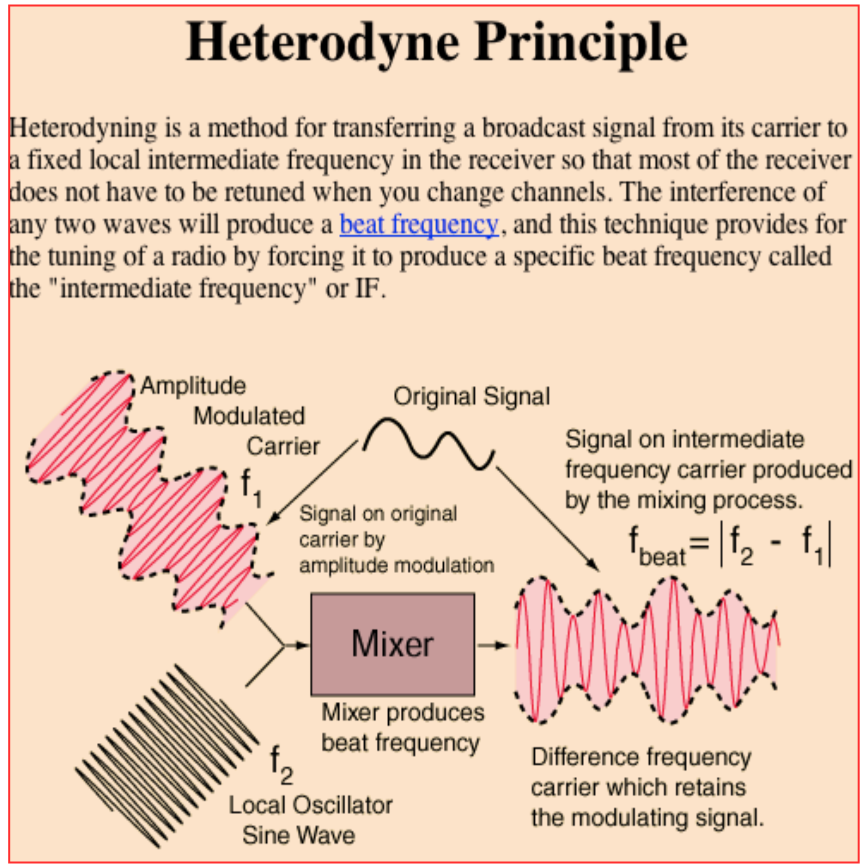


Beat Frequency and Heterodyning (5)

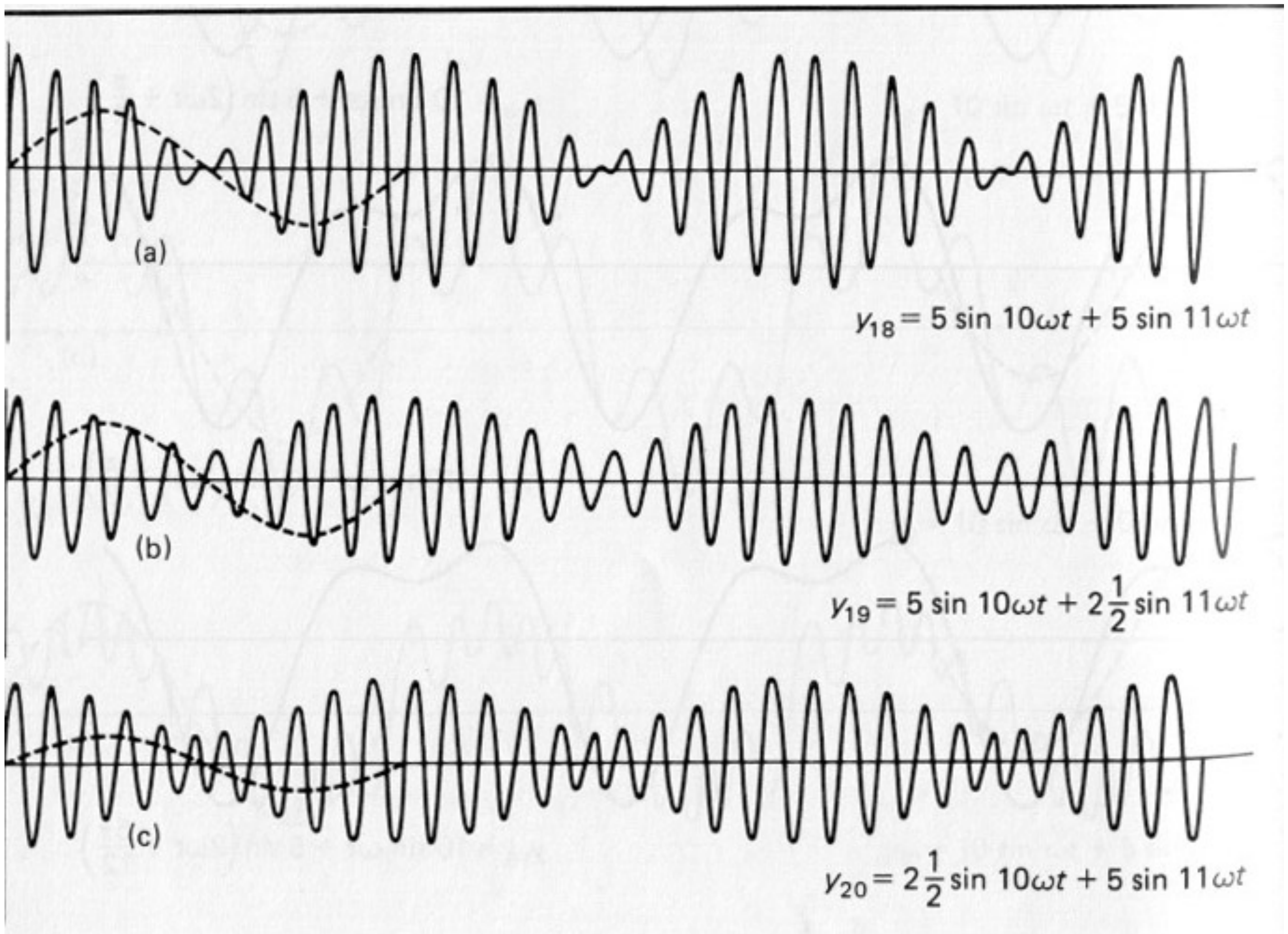
• An electromagnetic carrier wave which is carrying a signal by means of amplitude modulation or frequency modulation can transfer that signal to a carrier of different frequency by means of a process called heterodyning. This transfer is accomplished by mixing the original modulated carrier with a sine wave of another frequency. This process produces a beat frequency equal to the difference between the frequencies, and this difference frequency constitutes a third carrier which will be modulated by the original signal.

• Heterodyning is extremely important in radio transmission -- in fact, the development of heterodyning schemes was one of the major developments which led to mass communication by radio.

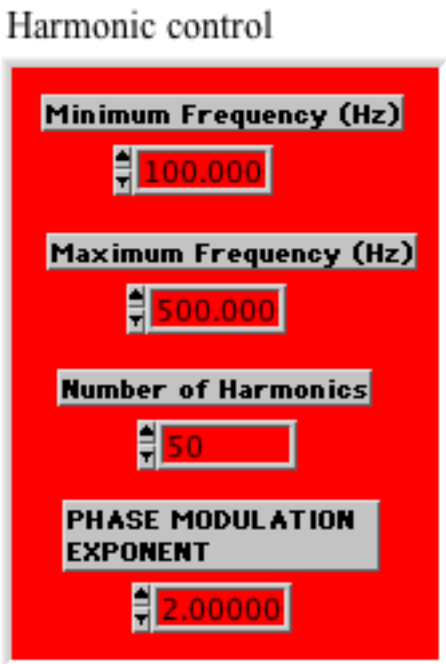
Source: http://hyperphysics.phy-astr.gsu.edu/hbase/audio/radio.html

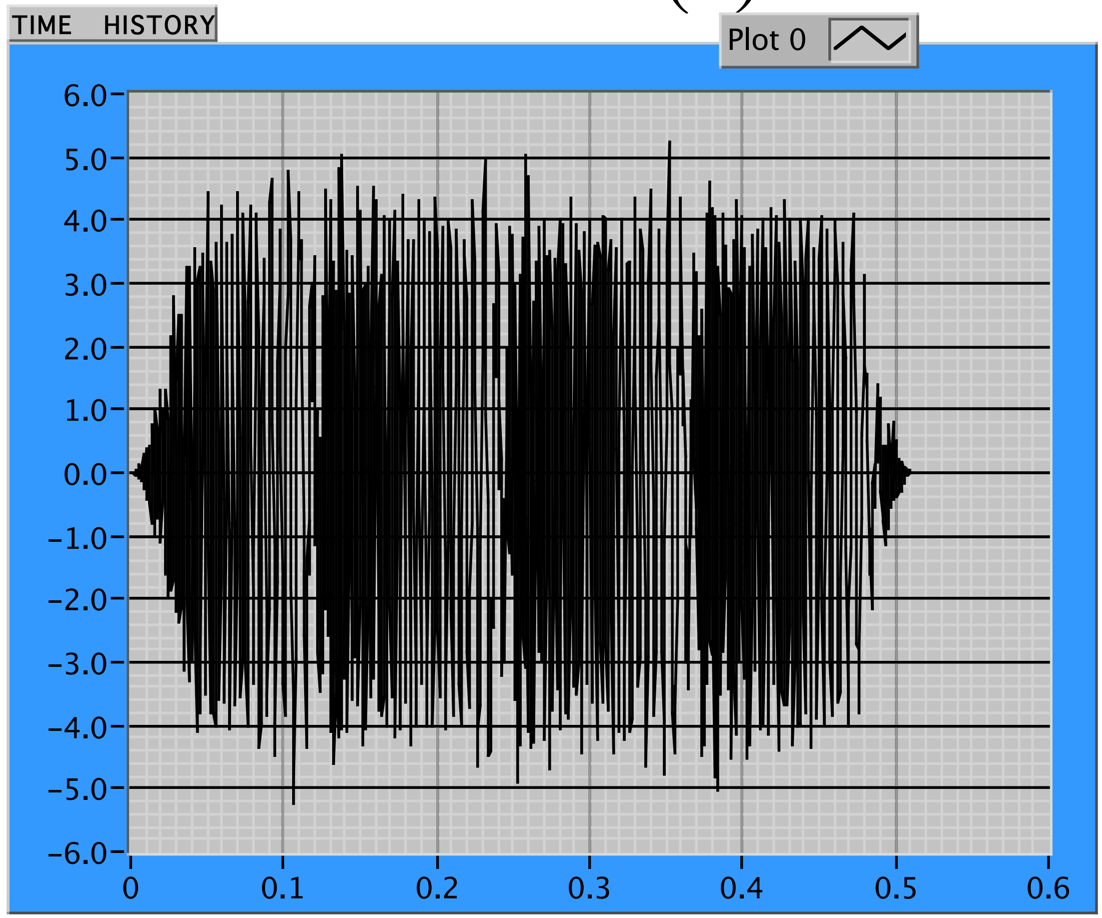


More Heterodyning Waveform Examples



Multi Harmonic Waveforms (1)



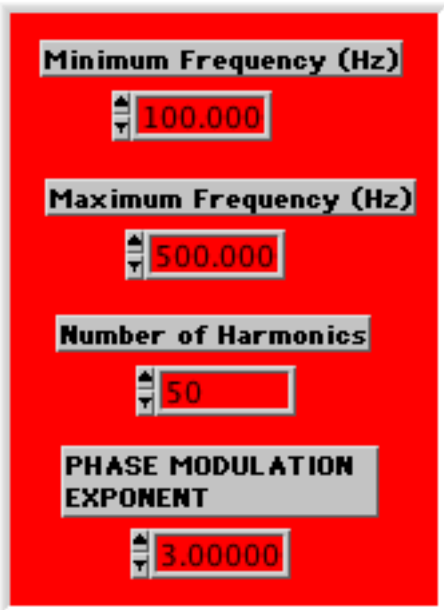


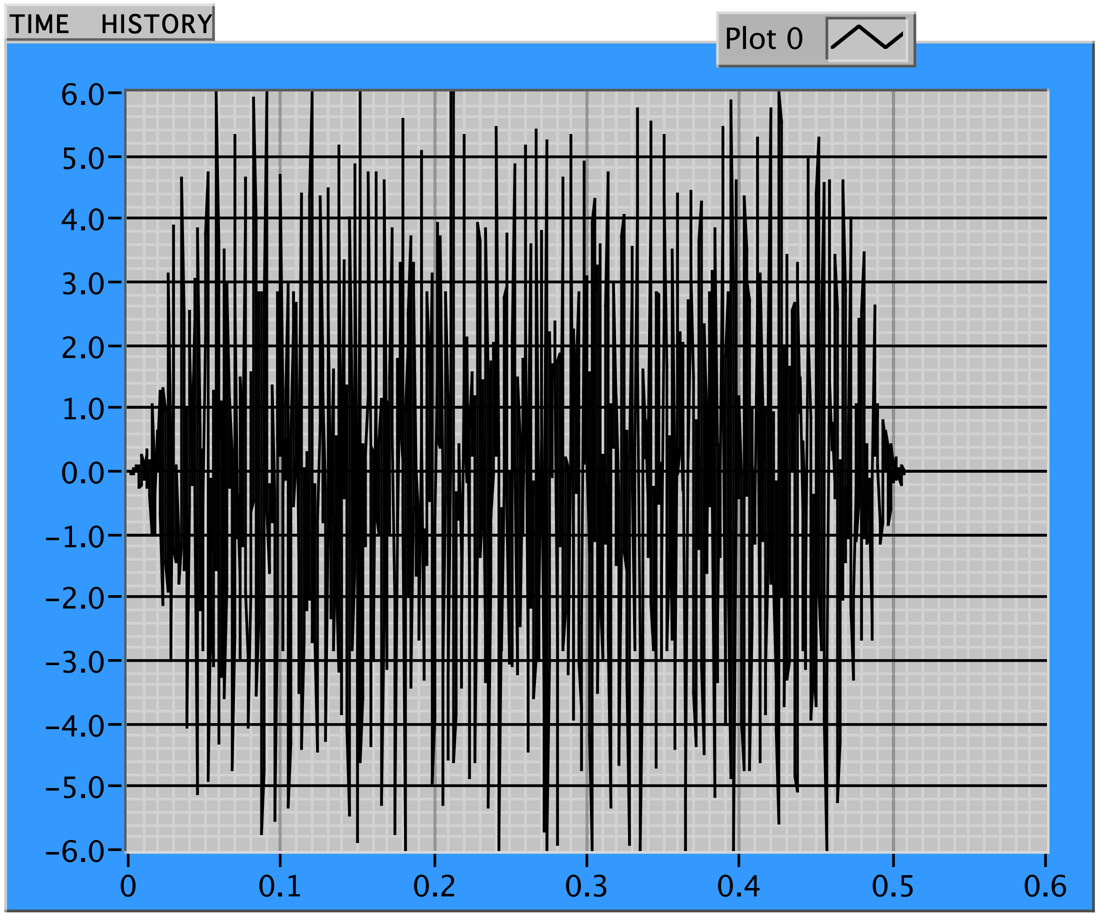
$$S(t) = \sum_{i=1}^{N_h} \sin\left[\left(2\pi f_0 + i\delta f\right)t + \frac{i^n \pi}{N_h}\right]$$

Still just a summation of simple sine wave harmonics, produces an extremely complex waveform

Multi Harmonic Waveforms (2)

Harmonic control





$$S(t) = \sum_{i=1}^{N_h} \sin\left[\left(2\pi f_0 + i\delta f\right)t + \frac{i^n \pi}{N_h}\right]$$

Same Harmonics, different phase angle modulation

Fourier Series (1)

- Any periodic function can be broken up into an infinite Series of sines and cosines … *using Fourier series*

"Harmonic Coefficients"

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos n\omega_0 t + B_n \sin n\omega_0 t\right) \quad \textbf{(Fourier Series)}$$

"Harmonic Order" **"Fundamental frequency"**

$$\begin{bmatrix} A_n = \left[\frac{\omega}{\pi}\right]\int_0^{\frac{2\pi}{\omega}} y(\tau)\cos(n\omega_0\tau)d\tau \\ B_n = \left[\frac{\omega}{\pi}\right]\int_0^{\frac{2\pi}{\omega}} y(\tau)\sin(n\omega_0\tau)d\tau \end{bmatrix} = \begin{bmatrix} A_n = \left[\frac{2}{T}\right]\int_0^{T} y(\tau)\cos(\frac{2\pi}{T}n\tau)d\tau \\ B_n = \left[\frac{2}{T}\right]\int_0^{T} y(\tau)\sin(\frac{2\pi}{T}n\tau)d\tau \end{bmatrix}$$

- T --> period

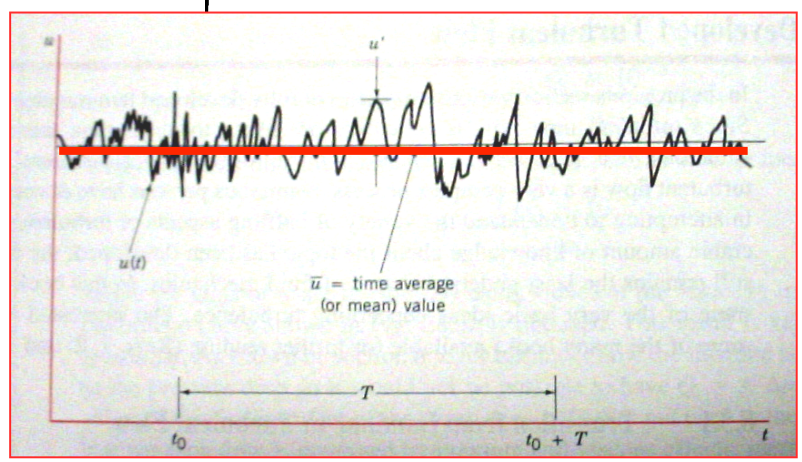
$\omega_0 = 2\pi f_0 = 2\pi/T$

- ***$A_0/2$ is the "mean" or DC level of the signal***

$$\frac{A_0}{2} = \frac{1}{2}\left[\frac{2}{T}\right]\int_0^{t} y(\tau)\cos(0\cdot\omega_0\tau)d\tau \rightarrow$$

$$\frac{A_0}{2} = \left[\frac{1}{T}\right]\int_0^{t} y(\tau)d\tau = y_{mean}$$

$y(0)$----->



Generalized Waveforms: Fourier Series (2)

- Any periodic wave form may be represented by its Fourier Series

"Harmonic Coefficients"

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos n\omega_0 t + B_n \sin n\omega_0 t\right)$$ ***(Fourier Series)***

"Harmonic Order" "Fundamental frequency"

"Harmonic Coefficients"

$$\left[\begin{array}{l} A_n = \left[\dfrac{\omega_0}{\pi}\right] \displaystyle\int_0^{\frac{2\pi}{\omega_0}} y(\tau)\cos(n\omega_0\tau)d\tau \\ B_n = \left[\dfrac{\omega_0}{\pi}\right] \displaystyle\int_0^{\frac{2\pi}{\omega_0}} y(\tau)\sin(n\omega_0\tau)d\tau \end{array}\right]$$

- Can be used to represent Some pretty interesting waveforms

$$f_0 = \text{"fundamental frequency"}$$

$$f_0 = \frac{1}{T} \rightarrow \omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

$$\boxed{\rightarrow \frac{2\pi}{\omega_0} = T} \rightarrow \boxed{\frac{\omega_0}{\pi} = \frac{T}{2}}$$

Fourier Series (3)

- Periodic Waveform, $T = Waveform\ Period$

$$f_0 = \text{"}fundamental\ frequency\text{"}$$

$$f_0 = \frac{1}{T} \rightarrow \omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

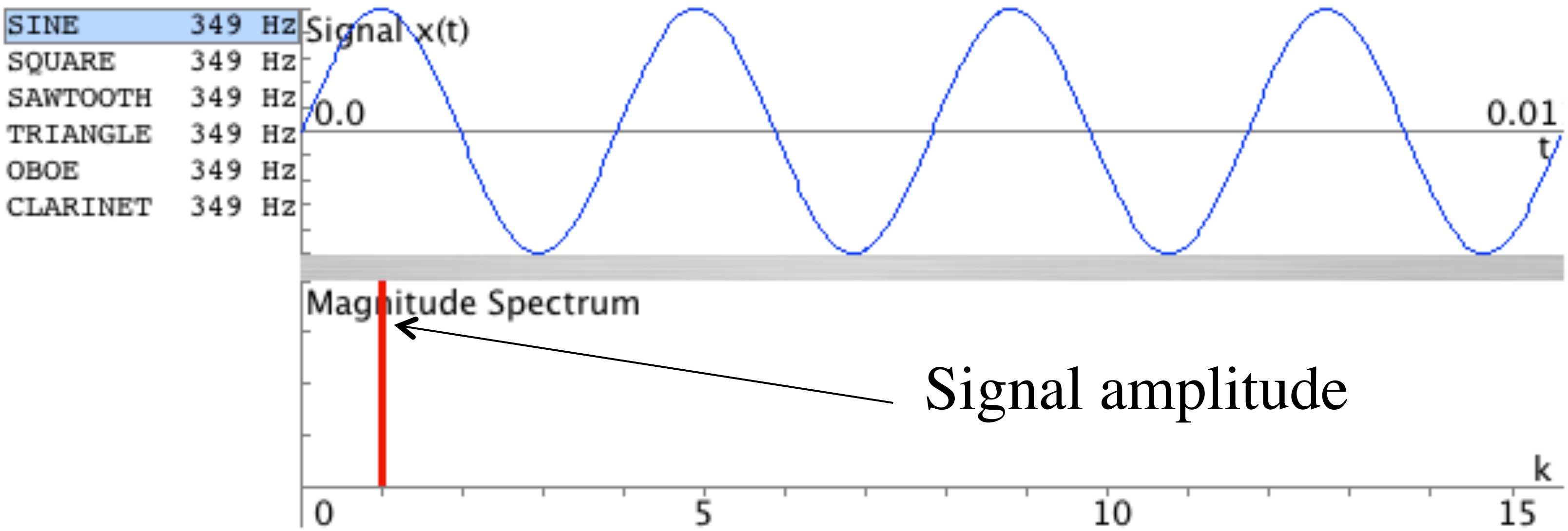
$$\boxed{\rightarrow \frac{2\pi}{\omega_0} = T} \rightarrow \boxed{\frac{\omega_0}{\pi} = \frac{2}{T}}$$

"equivalent forms"

$$\left[\begin{array}{l} A_n = \left[\frac{\omega_0}{\pi}\right] \int_0^{\frac{2\pi}{\omega_0}} y(\tau)\cos(n\omega_0\tau)d\tau = \left(\frac{2}{T}\right)\int_0^T y(\tau)\cos\left(\frac{2\pi}{T}\cdot n\cdot\tau\right)d\tau \\ B_n = \left[\frac{\omega_0}{\pi}\right] \int_0^{\frac{2\pi}{\omega_0}} y(\tau)\sin(n\omega_0\tau)d\tau = \left(\frac{2}{T}\right)\int_0^T y(\tau)\sin\left(\frac{2\pi}{T}\cdot n\cdot\tau\right)d\tau \end{array}\right]$$

Fourier Series Examples (1)

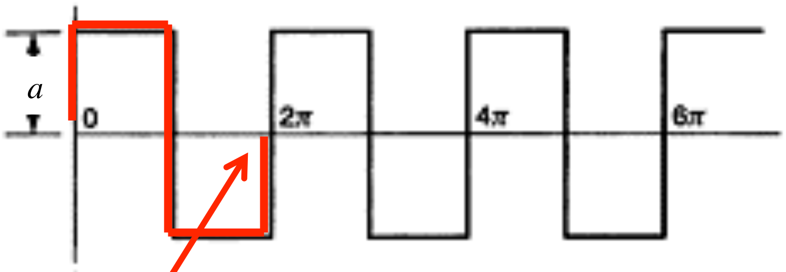
- Consider the Sinusoidal Waveform:



- Fourier Series represented by a single waveform (obviously)

Fourier Series Examples (2)

- Now Consider the Square (boxcar) waveform with period 2π:



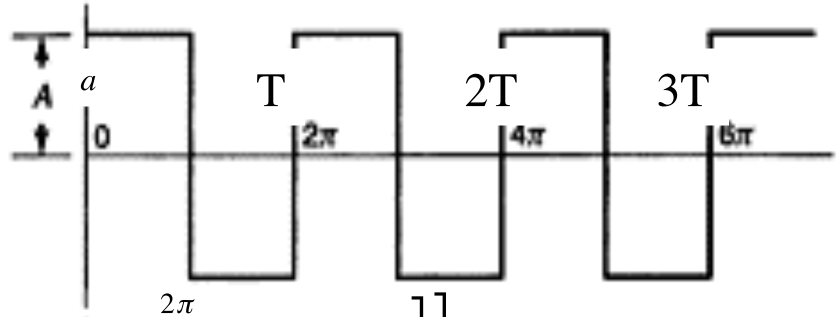
$$\begin{bmatrix} A_n = \left[\frac{\omega_0}{\pi}\right] \int_0^{\frac{2\pi}{\omega_0}} y(\tau)\cos(n\omega_0\tau)d\tau \\ B_n = \left[\frac{\omega_0}{\pi}\right] \int_0^{\frac{2\pi}{\omega_0}} y(\tau)\sin(n\omega_0\tau)d\tau \end{bmatrix}$$

$$T = 2\pi \rightarrow f = \frac{1}{2\pi} \rightarrow \boxed{\omega_0 = 2\pi f = 1}$$

$$\begin{bmatrix} A_n = \left[\frac{1}{\pi}\right] \int_0^{2\pi} y(\tau)\cos(n\tau)d\tau \\ B_n = \left[\frac{1}{\pi}\right] \int_0^{2\pi} y(\tau)\sin(n\tau)d\tau \end{bmatrix} = \begin{bmatrix} \left[\frac{1}{\pi}\right] a \left[\int_0^{\pi}\cos(n\tau)d\tau - \int_{\pi}^{2\pi}\cos(n\tau)d\tau\right] \\ \left[\frac{1}{\pi}\right] a \left[\int_0^{\pi}\sin(n\tau)d\tau - \int_{\pi}^{2\pi}\sin(n\tau)d\tau\right] \end{bmatrix}$$

Fourier Series Examples (3)

- Now Consider the Square (boxcar) waveform with period $2\,\pi$:



$$\begin{bmatrix} A_n = \left[\frac{1}{\pi}\right] \int_0^{2\pi} y(\tau)\cos(n\tau)d\tau \\ B_n = \left[\frac{1}{\pi}\right] \int_0^{2\pi} y(\tau)\sin(n\tau)d\tau \end{bmatrix} = \begin{bmatrix} \left[\frac{1}{\pi}\right] a \left[\int_0^{\pi}\cos(n\tau)d\tau - \int_{\pi}^{2\pi}\cos(n\tau)d\tau\right] \\ \left[\frac{1}{\pi}\right] a \left[\int_0^{\pi}\sin(n\tau)d\tau - \int_{\pi}^{2\pi}\sin(n\tau)d\tau\right] \end{bmatrix}$$

$$\int_0^{\pi} \text{Cos}[n*\tau]\,d\tau - \int_{\pi}^{2\pi} \text{Cos}[n*\tau]\,d\tau$$

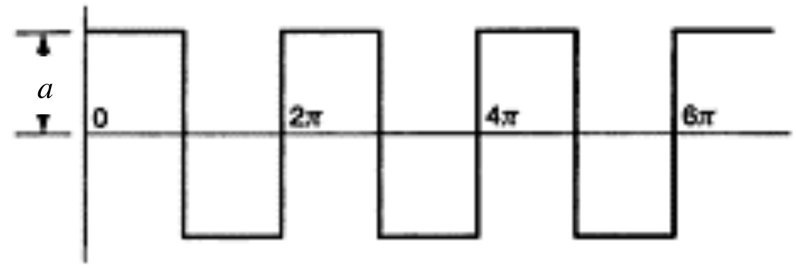
$$\frac{2\,\text{Sin}[n\,\pi]}{n} - \frac{\text{Sin}[2\,n\,\pi]}{n}$$

$$\int_0^{\pi} \text{Sin}[n*\tau]\,d\tau - \int_{\pi}^{2\pi} \text{Sin}[n*\tau]\,d\tau$$

$$\frac{1}{n} - \frac{2\,\text{Cos}[n\,\pi]}{n} + \frac{\text{Cos}[2\,n\,\pi]}{n}$$

Fourier Series (4)

- Example: Boxcar Function



Calculate A_i coefficients

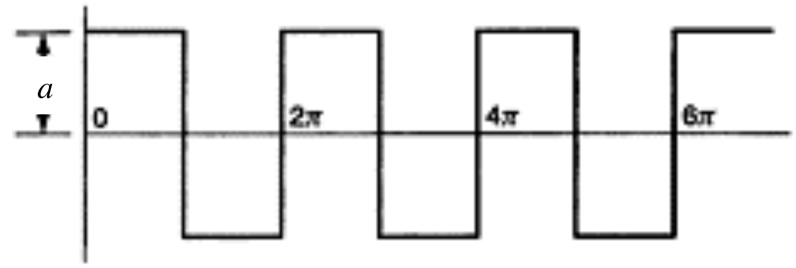
$$A_n = \left[\frac{a}{\pi}\right]\left(\frac{2\sin[n\pi]}{n} - \frac{\sin[2n\pi]}{n}\right)$$

$$A_0 = \left[\frac{a}{\pi}\right]\frac{\lim}{n\to 0}\left(\frac{2\sin[n\pi]}{n} - \frac{\sin[2n\pi]}{n}\right) \to L'Hopital \to \frac{2\pi\cdot 0\cdot\cos[0\cdot\pi]}{1} - \frac{2\pi\cdot 0\cdot\cos[2\cdot 0\cdot\pi]}{1} = 0$$

$$A_1 = \left[\frac{a}{\pi}\right]\left(\frac{2\sin[1\cdot\pi]}{1} - \frac{\sin[2\cdot 1\cdot\pi]}{1}\right) = 0,\ A_2 = \left[\frac{a}{\pi}\right]\left(\frac{2\sin[2\cdot\pi]}{2} - \frac{\sin[2\cdot 2\cdot\pi]}{1}\right) = 0......$$

Fourier Series (5)

- Example: Boxcar Function



Calculate B_i coefficients

$$B_n = \left[\frac{a}{\pi}\right]\left(\frac{1}{n} - \frac{2\cos[n\pi]}{n} + \frac{\cos[2n\pi]}{n}\right)$$

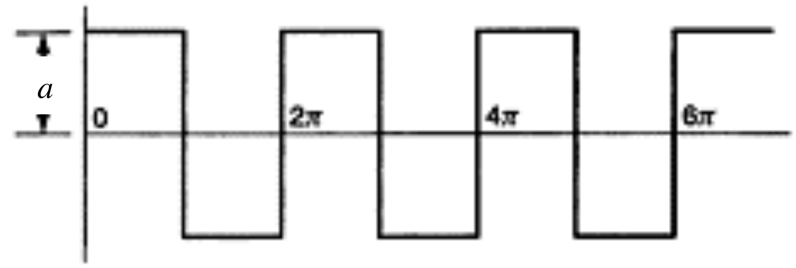
$$B_0 = \left[\frac{a}{\pi}\right]\frac{\lim}{n\to 0}\left(\frac{1}{n} - \frac{2\cos[n\pi]}{n} + \frac{\cos[2n\pi]}{n}\right) \to L'Hopital \to 0 + \frac{2\pi\cdot 0\cdot\sin[0\cdot\pi]}{1} - \frac{2\pi\cdot 0\cdot\sin[2\cdot 0\cdot\pi]}{1} = 0$$

$$B_1 = \left[\frac{a}{\pi}\right]\left(\frac{1}{1} - \frac{2\cos[1\cdot\pi]}{1} + \frac{\cos[2\cdot 1\cdot\pi]}{1}\right) = \frac{4a}{\pi},\ B_2 = \left[\frac{a}{\pi}\right]\left(\frac{1}{2} - \frac{2\cos[2\cdot\pi]}{2} + \frac{\cos[2\cdot 2\cdot\pi]}{2}\right) = 0,$$

$$B_3 = \left[\frac{a}{\pi}\right]\left(\frac{1}{3} - \frac{2\cos[3\cdot\pi]}{3} + \frac{\cos[2\cdot 3\cdot\pi]}{3}\right) = \frac{4a}{3\pi},\ldots\ldots B_4 = 0, B_5 = \frac{4a}{5\pi}$$

Fourier Series (6)

- Example: Boxcar Function



- Collect terms

$$A_0 = 0, A_1 = 0,\ A_2 = 0, A_3 = 0, A_4 = 0, A_5 = 0, A_6 = 0, A_7 = 0....$$

$$B_0 = 0, B_1 = \frac{4a}{\pi},\ B_2 = 0, B_3 = \frac{4a}{3\pi}, B_4 = 0, B_5 = \frac{4a}{5\pi}, B_6 = 0, B_7 = \frac{4a}{5\pi}....$$

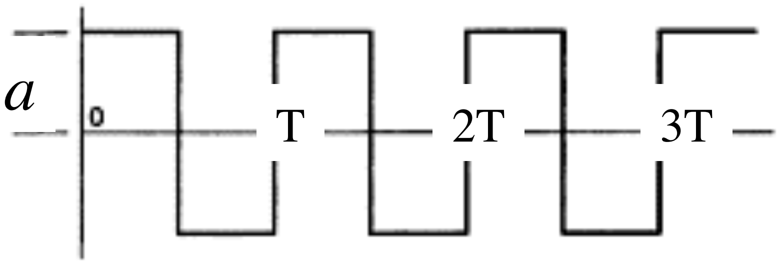
$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos n\omega_0 t + B_n \sin n\omega_0 t\right)$$

$$\omega_0 = \frac{2\pi}{T} = 1$$

$$y(t) = \frac{4a}{\pi}\left[\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + ...\right] = \frac{4a}{\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left[(2n-1)\cdot t\right]\right\}$$

Fourier Series Examples (7)

- Now Consider the More General Boxcar waveform with period T:



$$\begin{bmatrix} A_n = \left[\frac{2}{T}\right] \int_0^T y(\tau)\cos(\frac{2\pi}{T} n\tau)d\tau \\ B_n = \left[\frac{2}{T}\right] \int_0^T y(\tau)\sin(\frac{2\pi}{T} n\tau)d\tau \end{bmatrix}$$

$$A_0 = 0, A_1 = 0,\ A_2 == 0, A_3 = 0, A_4 = 0, A_5 = 0, A_6 = 0, A_7 = 0....$$

$$B_0 = 0, B_1 = \frac{4a}{\pi},\ B_2 == 0, B_3 = \frac{4a}{3\pi}, B_4 = 0, B_5 = \frac{4a}{5\pi}, B_6 = 0, B_7 = \frac{4a}{5\pi}....$$

$$y(t) = \frac{4a}{\pi}\left[\sin(\frac{2\pi}{T}t) + \frac{1}{3}\sin(3\frac{2\pi}{T}t) + \frac{1}{5}\sin(5\frac{2\pi}{T}t) + ...\right] = \frac{4a}{\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left[(2n-1)\frac{2\pi}{T}t\right]\right\}$$

$$y_{mean} = 0$$

- T --> period

$\omega_0 = 2\pi f_0 = 2\pi/T$

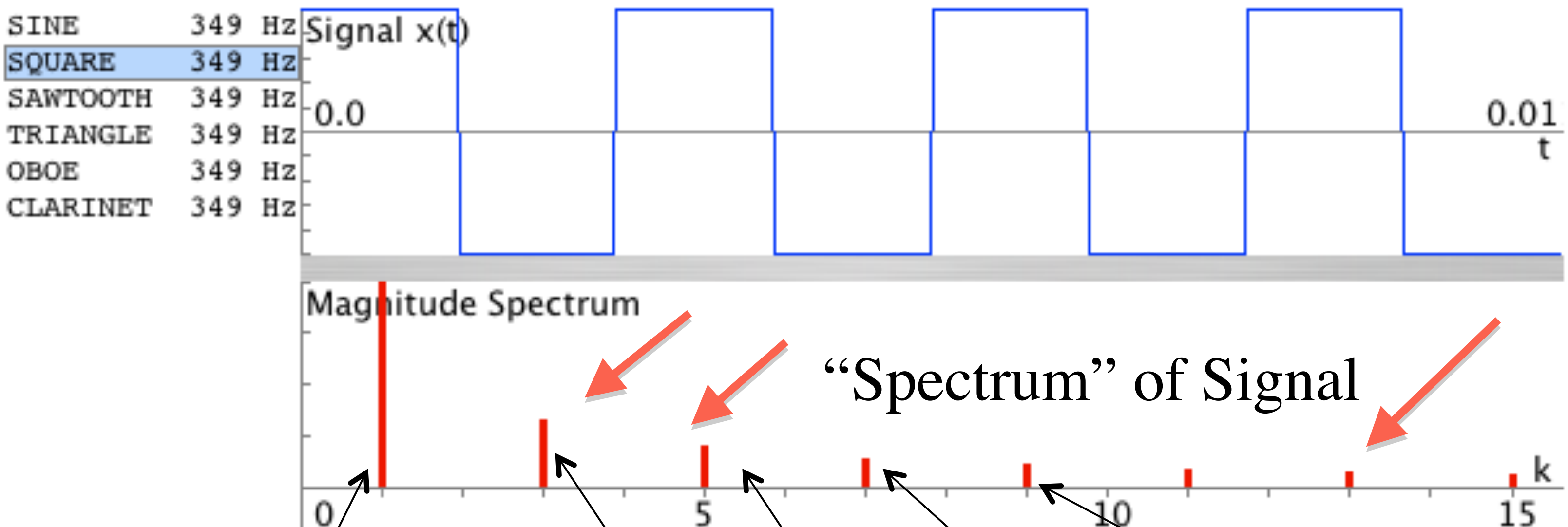
Fourier Series Examples (9)

- Square Waveform:

Multiple frequencies required to describe the wave

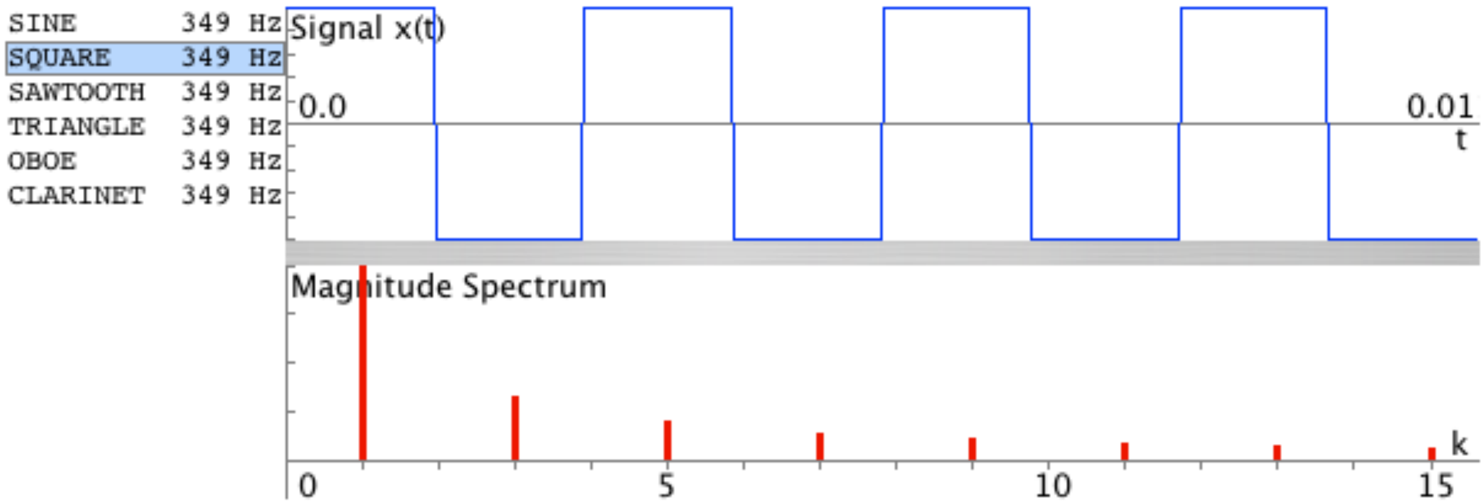
- T --> period

$\omega_0 = 2\pi f_0 = 2\pi/T$



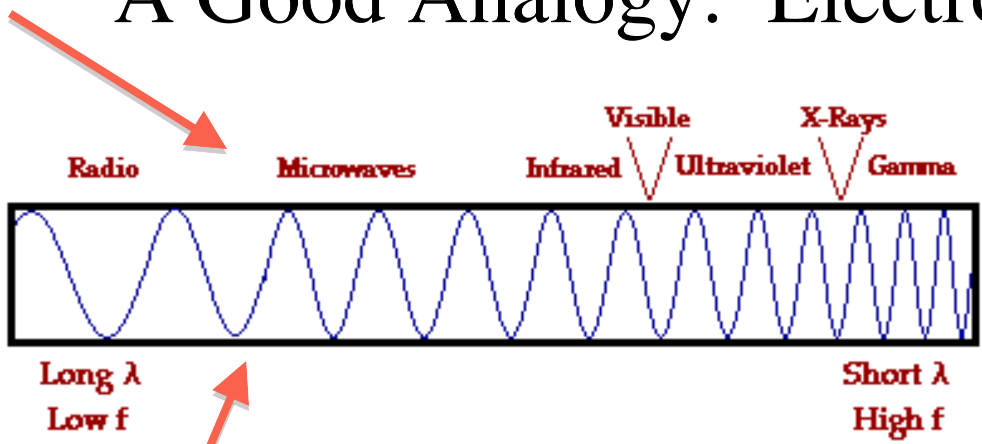
$$y(t) = \frac{4a}{\pi}\left[\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t) + \frac{1}{7}\sin(7\omega_0 t) + \frac{1}{9}\sin(9\omega_0 t) + \frac{1}{11}\sin(11\omega_0 t) + \frac{1}{13}\sin(13\omega_0 t) + \frac{1}{15}\sin(15\omega_0 t)\right]$$

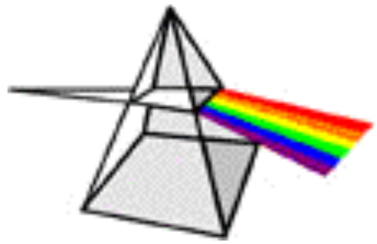
Fourier Series Examples (10)

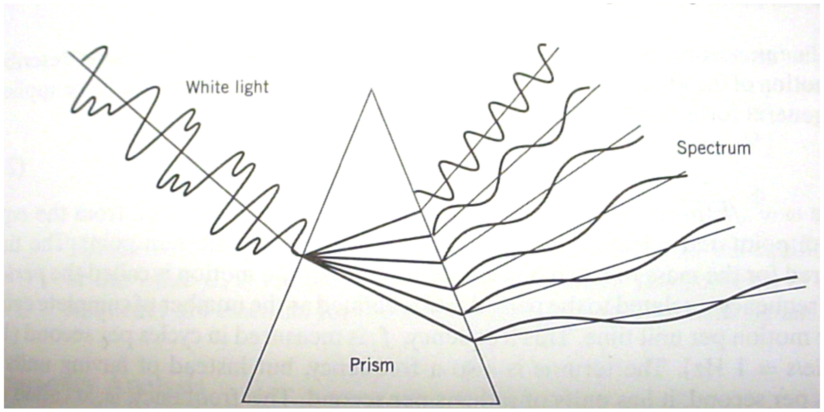


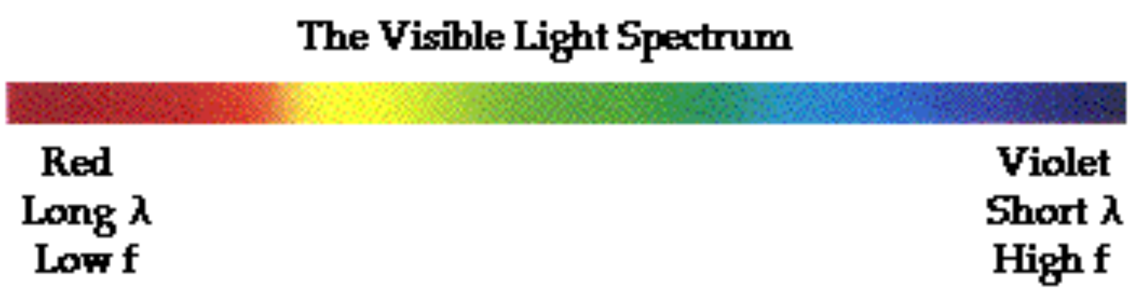
- ***Fourier Transform acts as a mathematical "prism" for a time dependent signal .. Decomposes it into spectrum components***

A Good Analogy: Electromagnetic Spectrum



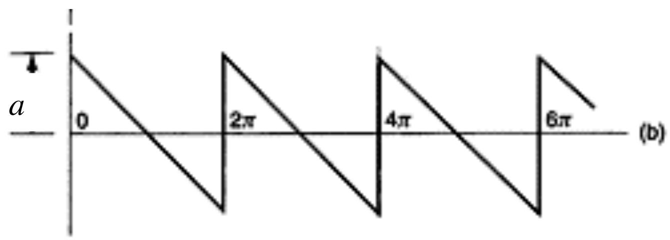




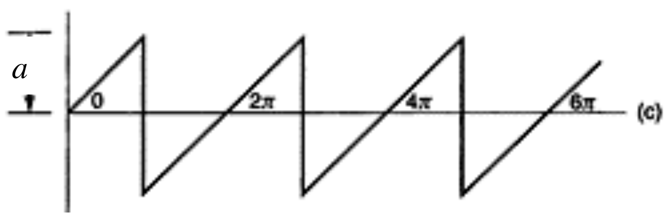


More Fourier Series Examples (1)

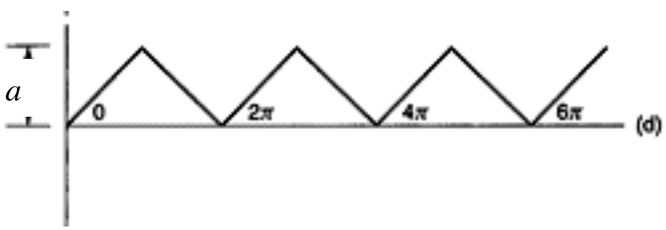
Similarly … via Fourier Series …. we get other distinctly “non sinusoidal” ***but periodic! waveforms*** *(TABLE 4.1 B.M.L.)*



$$y(t) = \frac{2a}{\pi}\left[\sin(t) + \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) + ...\right] = \frac{2a}{\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[n \cdot t\right]\right\}$$



$$y(t) = \frac{2a}{\pi}\left[\sin(t) - \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) + ...\right] = \frac{2a}{\pi}\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2}\sin\left[n \cdot t\right]\right\}$$

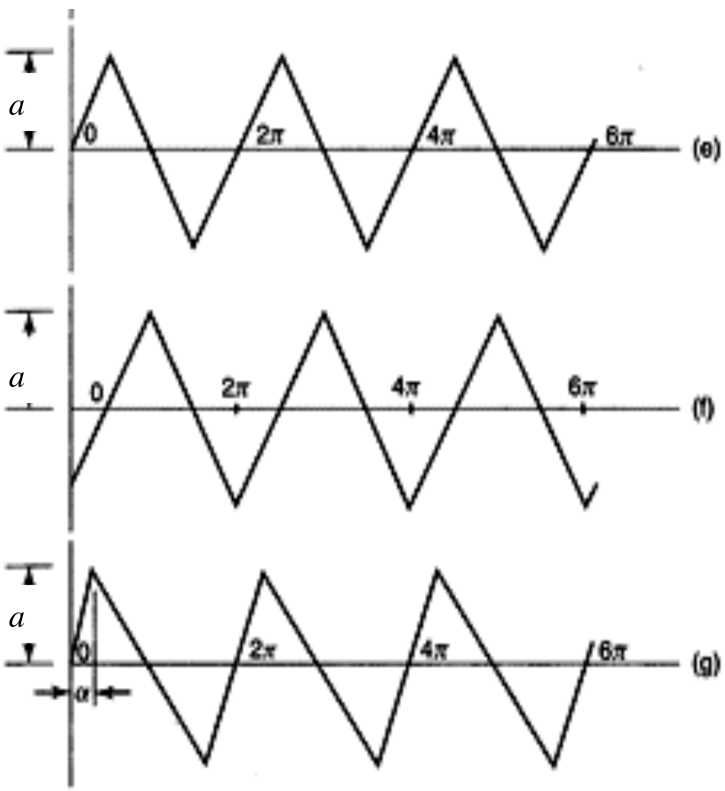


$$y(t) = \frac{a}{2} - \frac{4a}{\pi^2}\left[\cos(t) - \frac{1}{3^2}\cos(3t) + \frac{1}{5^2}\sin(5t) + ...\right] =$$

$$\frac{a}{2} - \frac{4a}{\pi^2}\left\{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left[(2n-1)\cdot t\right]\right\}$$

More Fourier Series Examples (2)

Similarly … via Fourier Series …. we get other distinctly “non sinusoidal” but ***periodic waveforms*** *(TABLE 4.1 Beckwith)*



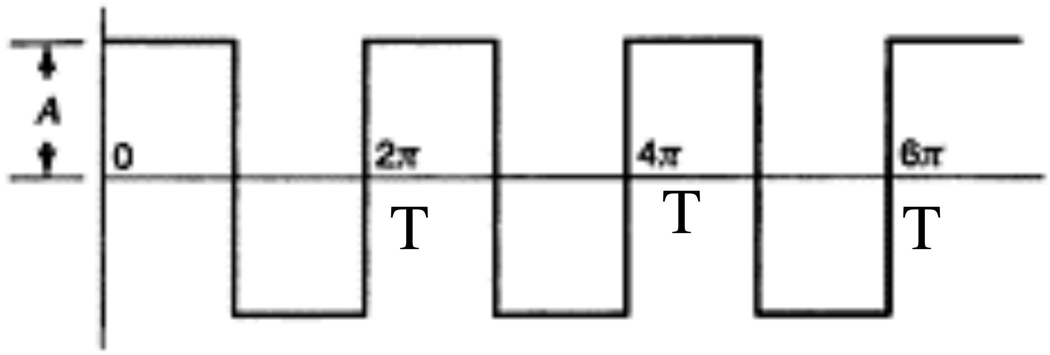
$$y(t) = \frac{8a}{\pi^2}\left[\sin(t) - \frac{1}{3^2}\sin(3t) + \frac{1}{5^2}\sin(5t) + ...\right] =$$
$$\frac{8a}{\pi^2}\left\{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\left[(2n-1)\cdot t\right]\right\}$$

$$y(t) = \frac{8a}{\pi^2}\left[\cos(t) - \frac{1}{3^2}\cos(3t) + \frac{1}{5^2}\cos(5t) + ...\right] =$$
$$\frac{8a}{\pi^2}\left\{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left[(2n-1)\cdot t\right]\right\}$$

$$y(t) = \frac{2a}{\alpha(\pi-\alpha)}\left[\sin\alpha\sin(t) - \frac{1}{2^2}\sin 2\alpha\sin(2t) + \frac{1}{3^2}\sin 3\alpha\sin(3t) + ...\right] =$$
$$\frac{2a}{\alpha(\pi-\alpha)}\left\{\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left[n\cdot\alpha\right]\sin\left[n\cdot t\right]\right\}$$

What happens if we truncate the series? (1) at less than infinity?

a



$$y(t)=\frac{4a}{\pi}\left[\sin(\omega_0 t)+\frac{1}{3}\sin(3\omega_0 t)+\frac{1}{5}\sin(5\omega_0 t)+....\right]=\frac{4a}{\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left[(2n-1)\omega_0 t\right]\right\}$$

What happens if we truncate the series? (2)

First three terms…

$$y(t)=\frac{4a}{\pi}\left[\sin(\omega_0 t)+\frac{1}{3}\sin(3\omega_0 t)+\frac{1}{5}\sin(5\omega_0 t)\right]$$

First five terms…

$$y(t)=\frac{4a}{\pi}\left[\sin(\omega_0 t)+\frac{1}{3}\sin(3\omega_0 t)+\frac{1}{5}\sin(5\omega_0 t)+\frac{1}{7}\sin(7\omega_0 t)+\frac{1}{9}\sin(9\omega_0 t)\right]$$

First eight terms…

$$y(t)=\frac{4a}{\pi}\left[\begin{array}{l}\sin(\omega_0 t)+\frac{1}{3}\sin(3\omega_0 t)+\frac{1}{5}\sin(5\omega_0 t)+\frac{1}{7}\sin(7\omega_0 t)+\frac{1}{9}\sin(9\omega_0 t)+\\ \frac{1}{11}\sin(11\omega_0 t)+\frac{1}{13}\sin(13\omega_0 t)+\frac{1}{15}\sin(15\omega_0 t)\end{array}\right]$$

Plot of square-wave function: (a) plot of first three terms only (includes the fifth harmonic); (b) plot of the first five terms (includes the ninth harmonic); (c) plot of the first eight terms (includes the fifteenth harmonic).

- Higher Frequency model is better representative of waveform

Appendix I:

Useful Trig Relationships

Useful Trigonometric Relationships (1)

$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$
$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$
$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

Useful Trigonometric Relationships (2)

$\cos(2x) = \cos^2(x) - \sin^2(x)$
$\sin(2x) = 2\sin(x)\cos(x)$
$2\cos^2(x) = 1 + \cos(2x)$
$2\sin^2(x) = 1 - \cos(2x)$

Useful Trigonometric Relationships (3)

$\cos^2(x) + \sin^2(x) = 1$
$2\cos(x)\cos(y) = \cos(x-y) + \cos(x+y)$
$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$
$2\sin(x)\cos(y) = \sin(x-y) + \sin(x+y)$