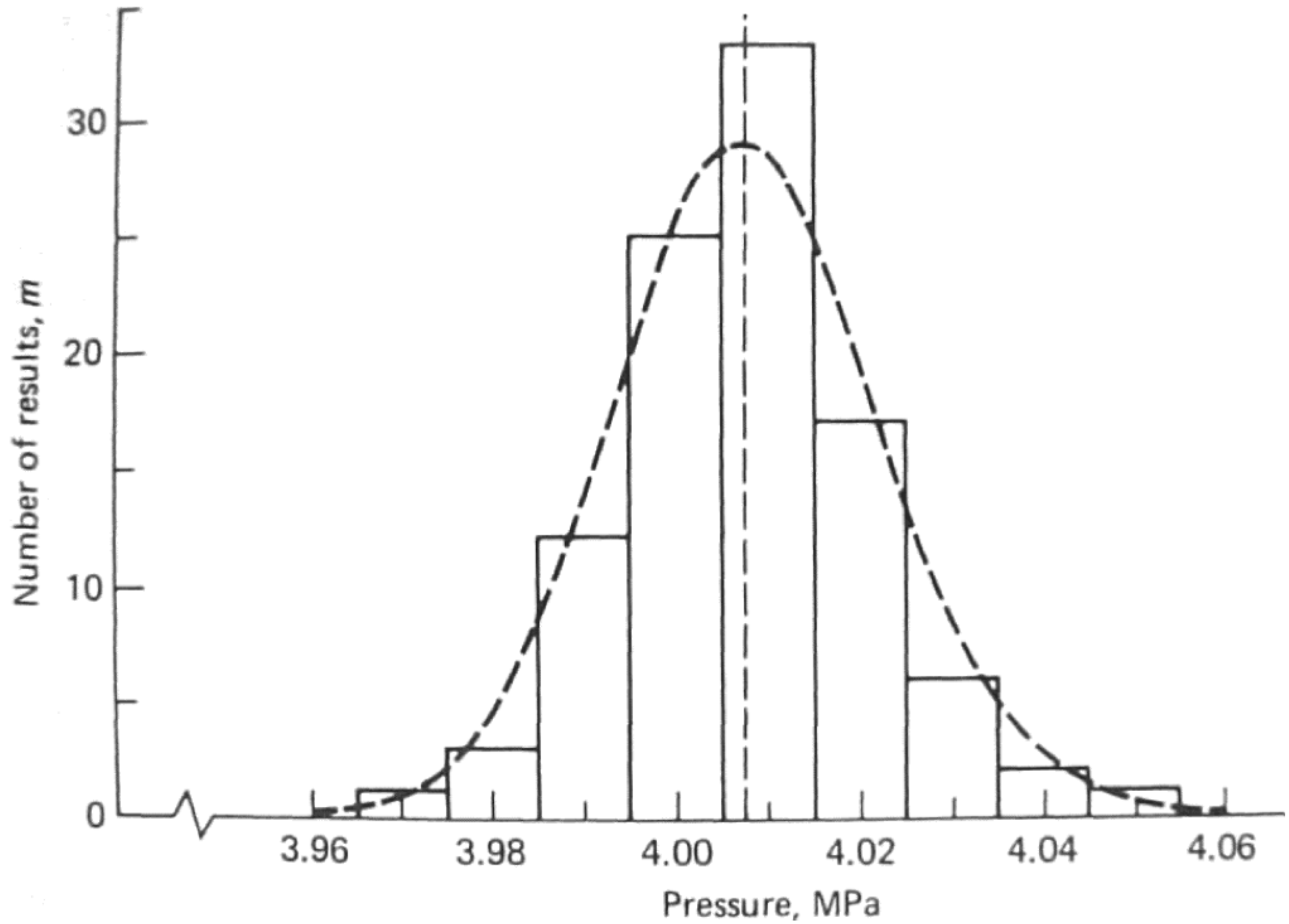
Section 3.1 Introduction to Probability Theory

(B.M.L. Chapter 3, pp. 43-73)



Sample Mean and Standard Deviation(1)

μ is the true mean of the distribution, or the actual value without any error. If we take a sample and average the results, we obtain the most probable value of the mean: ***sample mean***

$$\mu \approx \overline{x} = \frac{x_1 + x_2 + x_3 + \ldots x_n}{n} = \sum_{i=1}^{n} \frac{x_i}{n}$$

Define the deviation to be the the sample mean and any value

$$d_i = x_i - \mu$$

The mean squared deviation can be approximated by averaging the squared deviation of the sample: ***(sample standard deviation)***

$$\sigma \approx S_x = \sqrt{\frac{\left(x_1 - \bar{x}\right)^2 + \left(x_2 - \bar{x}\right)^2 + \ldots \left(x_n - \bar{x}\right)^2}{n-1}} = \sqrt{\sum_{i=1}^{n} \frac{\left(x_i - \bar{x}\right)^2}{n-1}}$$

Sample Mean and Standard Deviation(2)

For the Sample standard deviation … ***n-1*** is the degrees of freedom *(number of samples minus what we calculate from them)* …. Since the sample mean is already computed from the samples, the degrees of freedom are reduced by 1

$$\sigma \approx S_x = \sqrt{\frac{\left(x_1 - \bar{x}\right)^2 + \left(x_2 - \bar{x}\right)^2 + \ldots \left(x_n - \bar{x}\right)^2}{n-1}} = \sqrt{\sum_{i=1}^{n} \frac{\left(x_i - \bar{x}\right)^2}{n-1}}$$

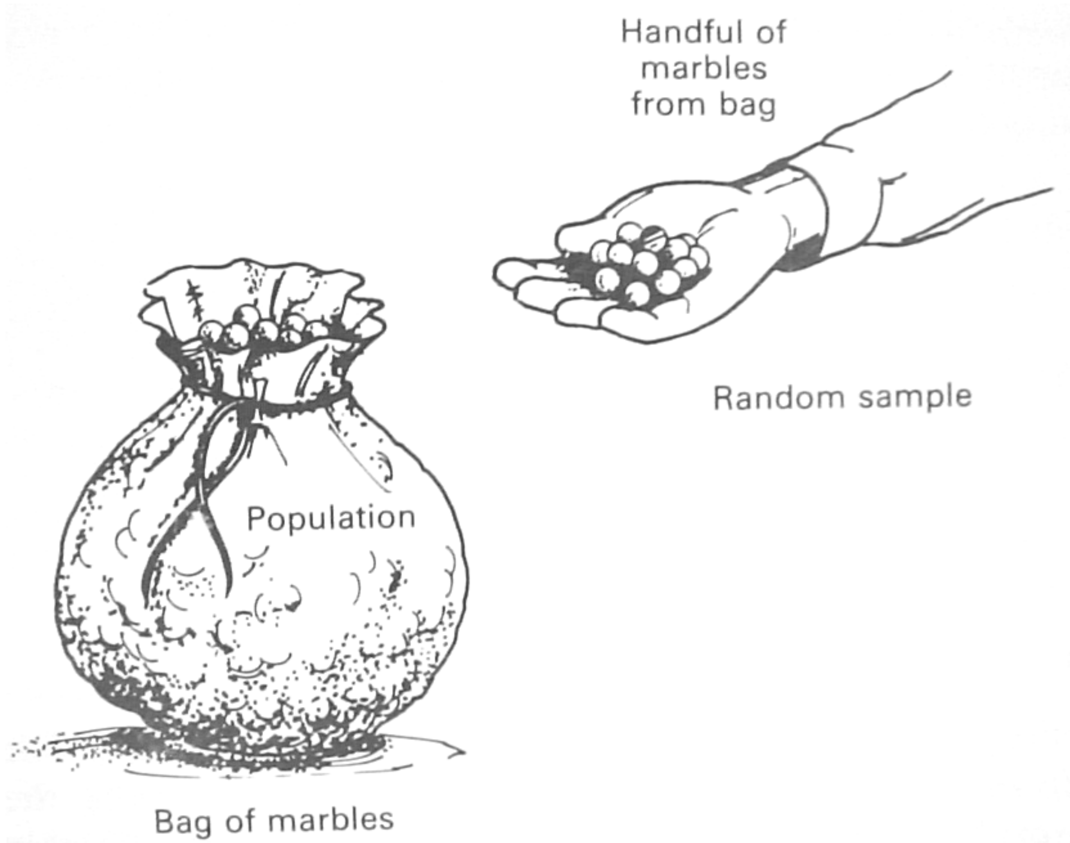
- If the samples within the population are independent of each other (as in Gaussian population) … then

$$\sigma^2 \approx S_x{}^2 = \sum_{i=1}^{n}\left[\frac{x_i{}^2}{n-1}\right] + \frac{n}{n-1}\left(\sum_{i=1}^{x} x_i\right)^2 \rightarrow \Psi_x{}^2 \equiv \sum_{i=1}^{n}\left[\frac{x_i{}^2}{n-1}\right] \text{ "mean square"}$$

$$S_x{}^2 = \Psi_x{}^2 + \frac{n}{n-1}\left(\bar{x}\right)^2$$

Estimation of Uncertainty (1):

Sample Statistics



- Based on Measurements of a hand full of Marbles what can We conclude About the Diameters Of the marbles in the bag?

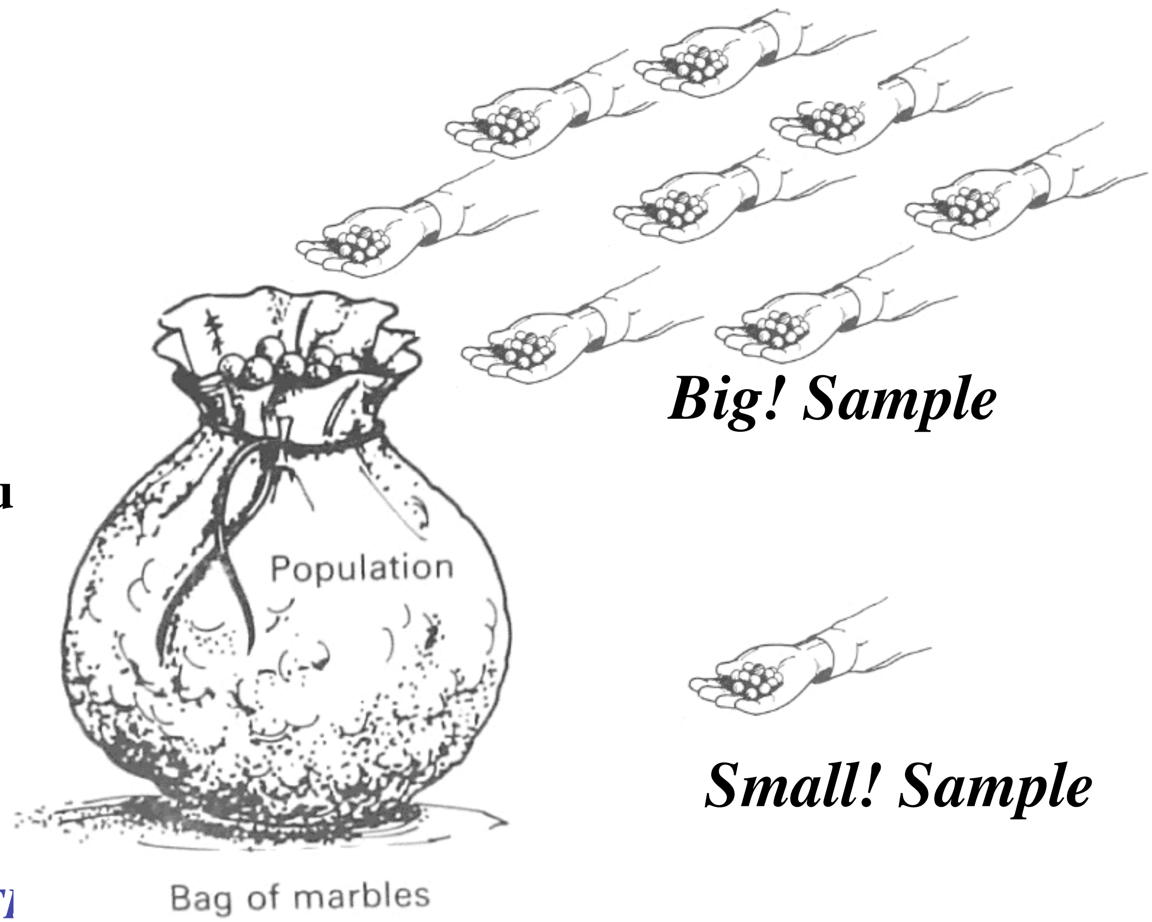
Estimation of Uncertainty (2)

Sample Statistics

• In real life we deal with samples of a population and NOT the entire population itself … thus we must use averages From the sample to infer the properties of the population

• As the sample population gets very large … not a problem … But for smaller samples … its a bit trickier

• **Which sample would you Expect provides the best information about the population of marbles in the bag?**



Introduction to Uncertainty Analysis

• The overall uncertainty of a measurement will be a combination of the bias uncertainty and the precision

• If we can account for the bias we take it out … otherwise bias is modeled as an uncertainly

• The overall uncertainty is the Root-sum-square (RSS) of the Bias and random uncertainty + other classifiable errors like hysterysis, calibration, etc.

$$U_x = (B_x^2 + R_x^2 + \mathrm{Oe}^2)^{1/2}$$

Probabilistic Description of Error (1)

• If an error is purely random … then it will tend to give a different Value each time … and the occurrence of a given value is Just as likely as the occurrence of another value

• Flipping a coin is a good example … 50% probability of Heads, 50% probability of tails

Probabilistic Description of Error (2)

• What is the probability that a coin 4 times in a row and having Them all be heads? … look at sample space …

(H,H,H,H), (H,H,H,T), (H,H,T,H), (H,H,T,T), (H,T,H,H), (H,T,H,T), (H,T,T,H), (H,T,T,T), (T,H,H,H), (T,H,H,T), (T,H,T,H), (T,H,T,T), (T,T,H,H), (T,T,H,T), (T,T,T,H), (T,T,T,T)

$$P(H,H,H,H)=N(H,H,H,H)/N_{possible}=1/16$$

• As a shortcut, we could say that the probability of getting heads on any one throw is 1/2. The probability of getting four heads in a row therefore is $(1/2)(1/2)(1/2)(1/2) =$ or $(1/2)^4 = 1/16$.

Probabilistic Description of Error (3)

- Example of Non-uniform probability distribution

- How many ways to get Seven?

{1,6},{2,5},{3,4}
{6,1},{5,2},{4,3}

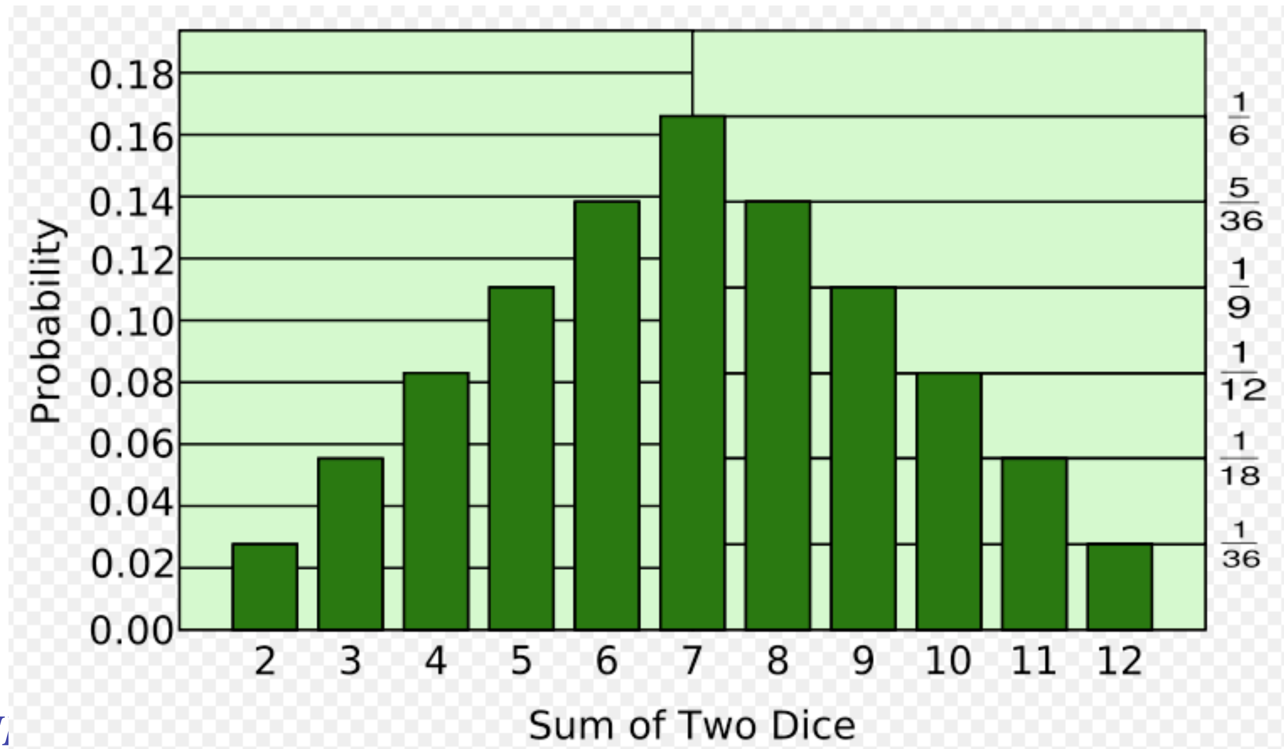
How about four?
{1,3},{2,2},{3,1}

… so seven is twice
As likely as 4!

Probabilistic Description of Error (4)

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
Probability (simplified)	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36

- **“7” is Most likely**
- **“2” is least likely**



Probabilistic Description of Error (5)

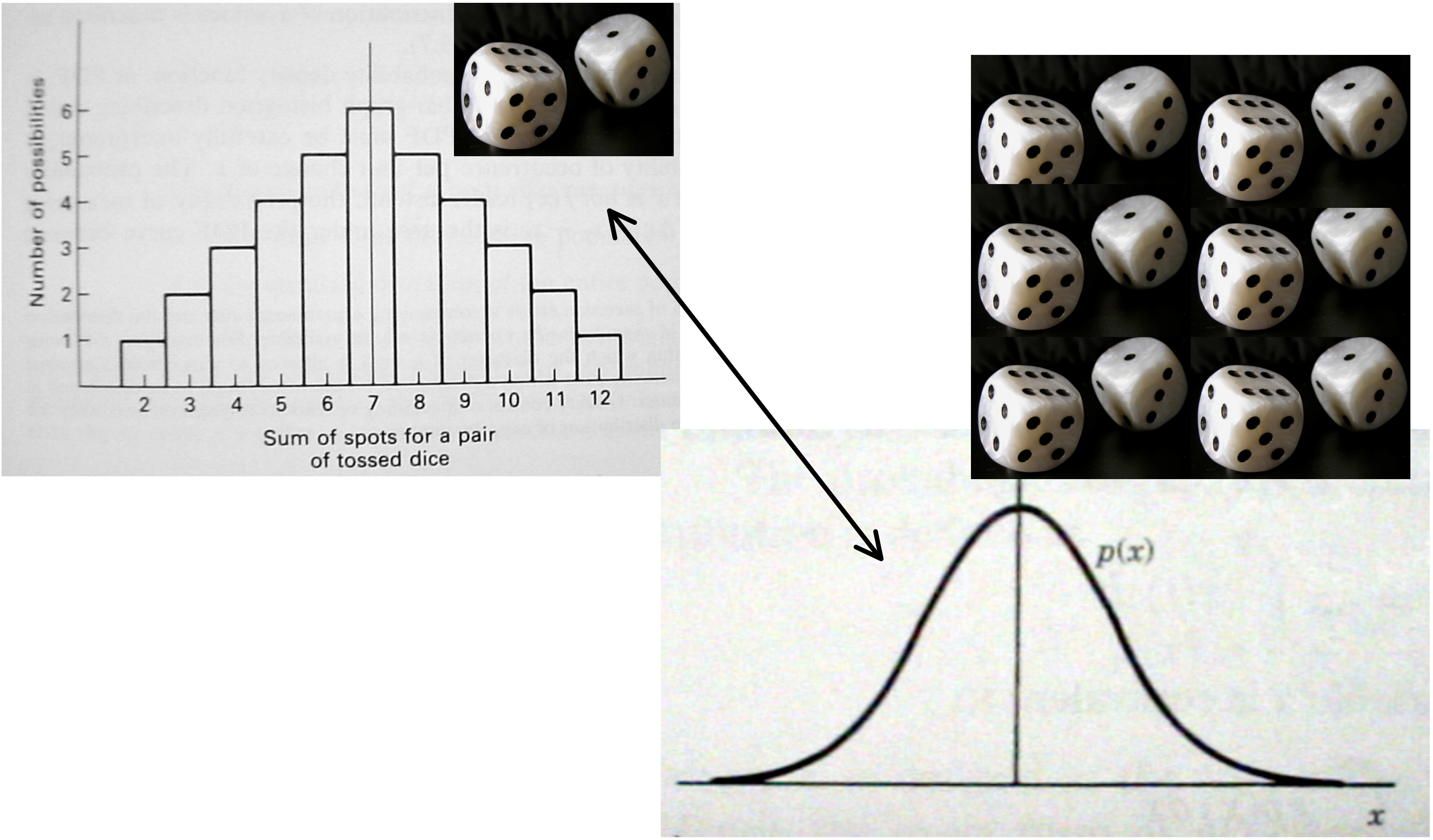
- For three or more die rolls, the curve becomes more bell-shaped with each additional die added to system … central limit theorem

- The “Bell-shaped” curve is referred to as the Normal or Gaussian distribution

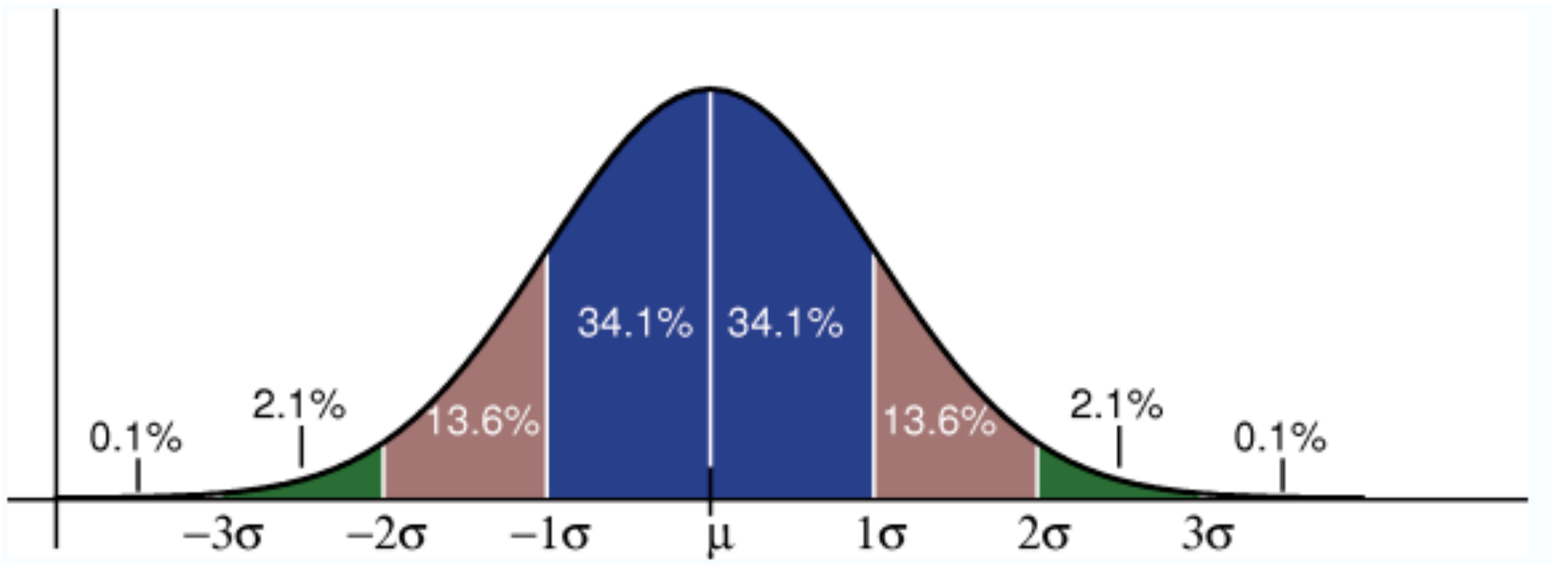
- The Gaussian distribution describes the population of possible Outcomes when a large number of independent sources contribute To the final outcome

- It is typically used for a probabilities description of uncorrelated errors … ***empirical result based on observation***

Central Limit Theorem



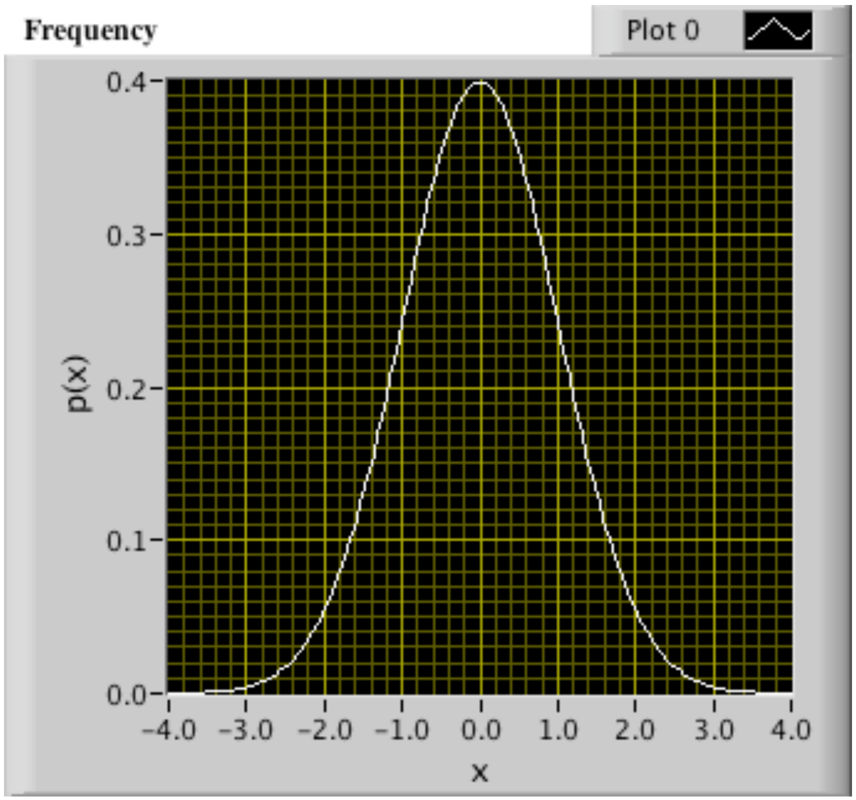
Gaussian Probability Density Function



$$p(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

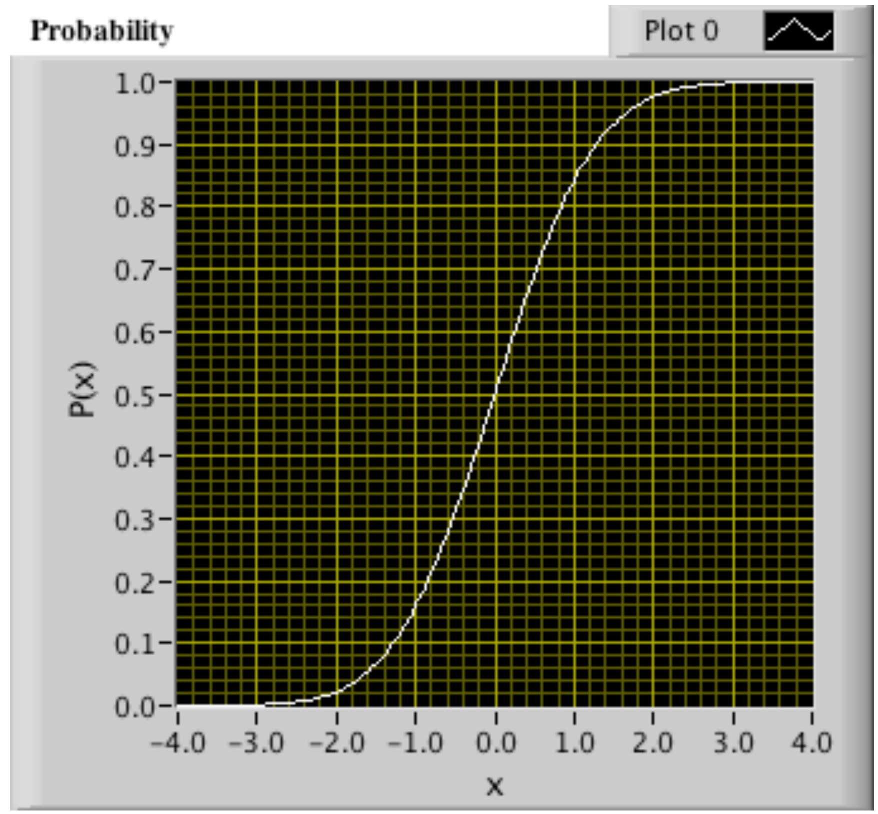
- μ --> "*mean*" most likely value
- σ --> "*standard deviation*" … Describes likelihood of deviation from the mean --> σ^2 = ***"variance"***

Probability Density versus Distribution (1)



Density

$$p(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



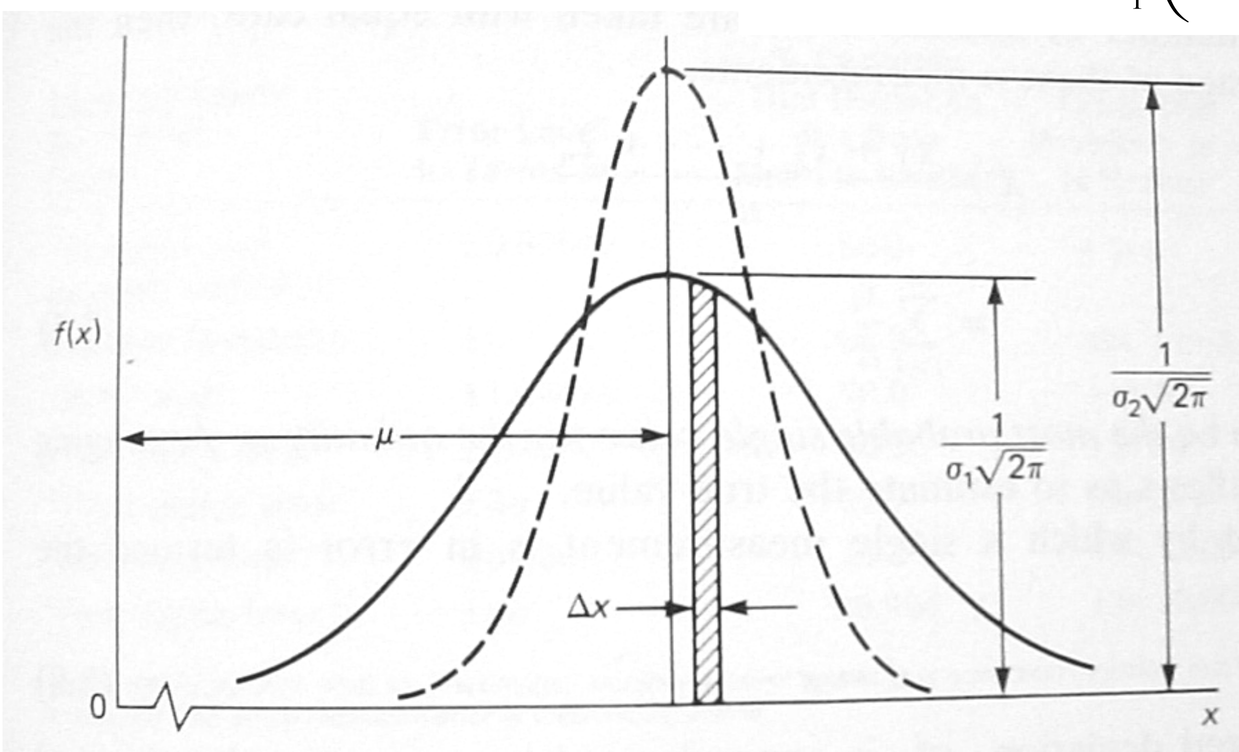
Distribution

$$P_{(x)} = \int_{-\infty}^{x} \left(\frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) dx$$

Probability Density versus Distribution (2)

• Probability of an occurrence with in a given range is the integral
Of the density function over that range

$$P_{(x \geq x_1 \,\&\, x \leq x_2)} = \int_{x_1}^{x_2} \left(\frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) dx$$



• Integral cannot
Be analytically
evaluated

• Numerical
Calculation
Is used

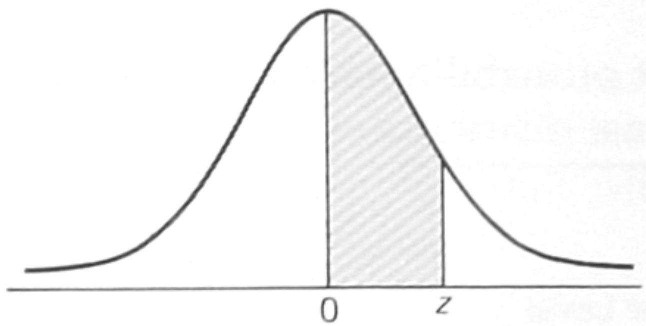
Tabulation of Normal Data

$z = (x - \mu)/\sigma$

$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2}$$

"not very convenient"

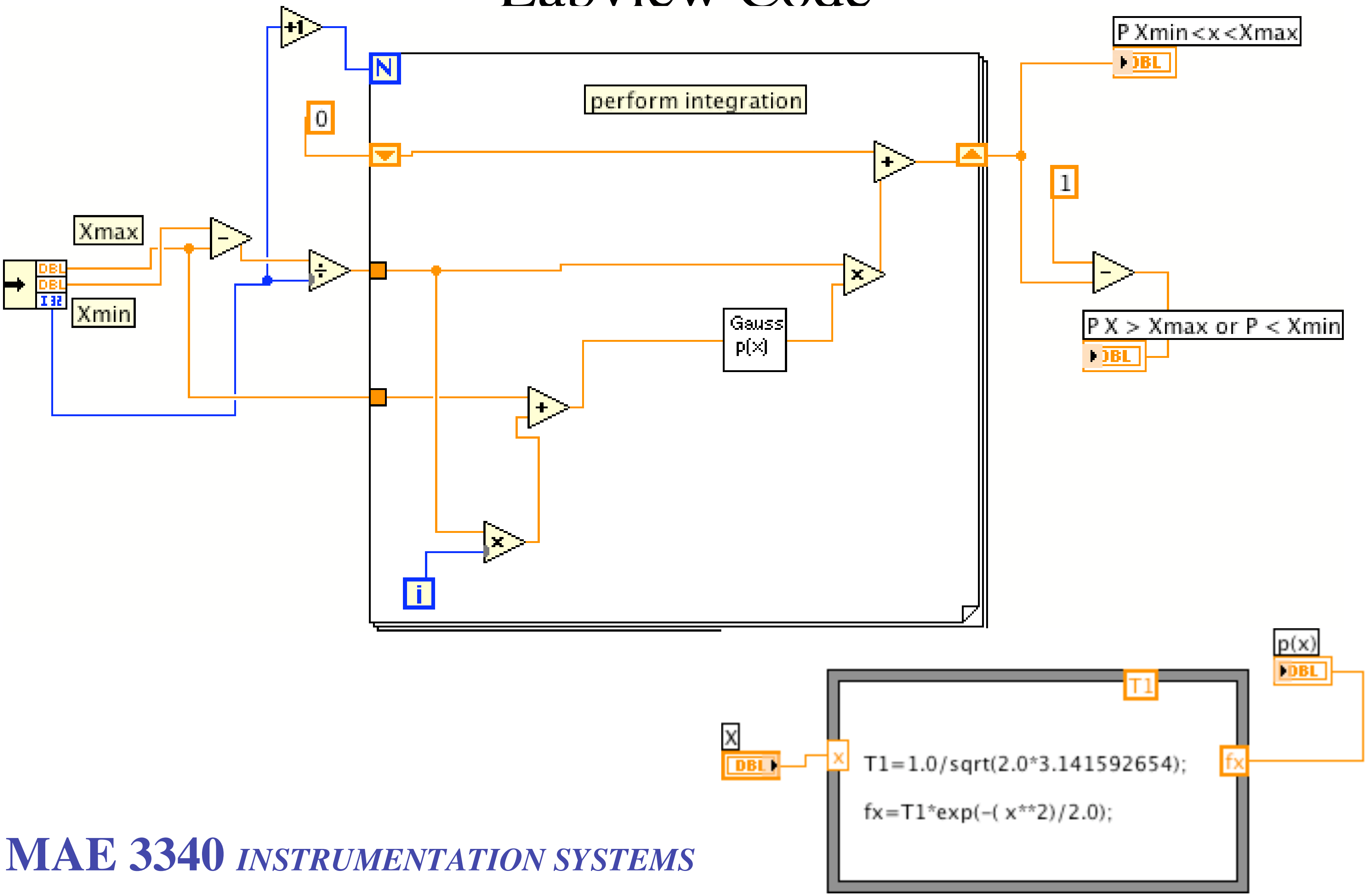
Table 3.2 Areas under the standard normal curve



Second Decimal Place in z

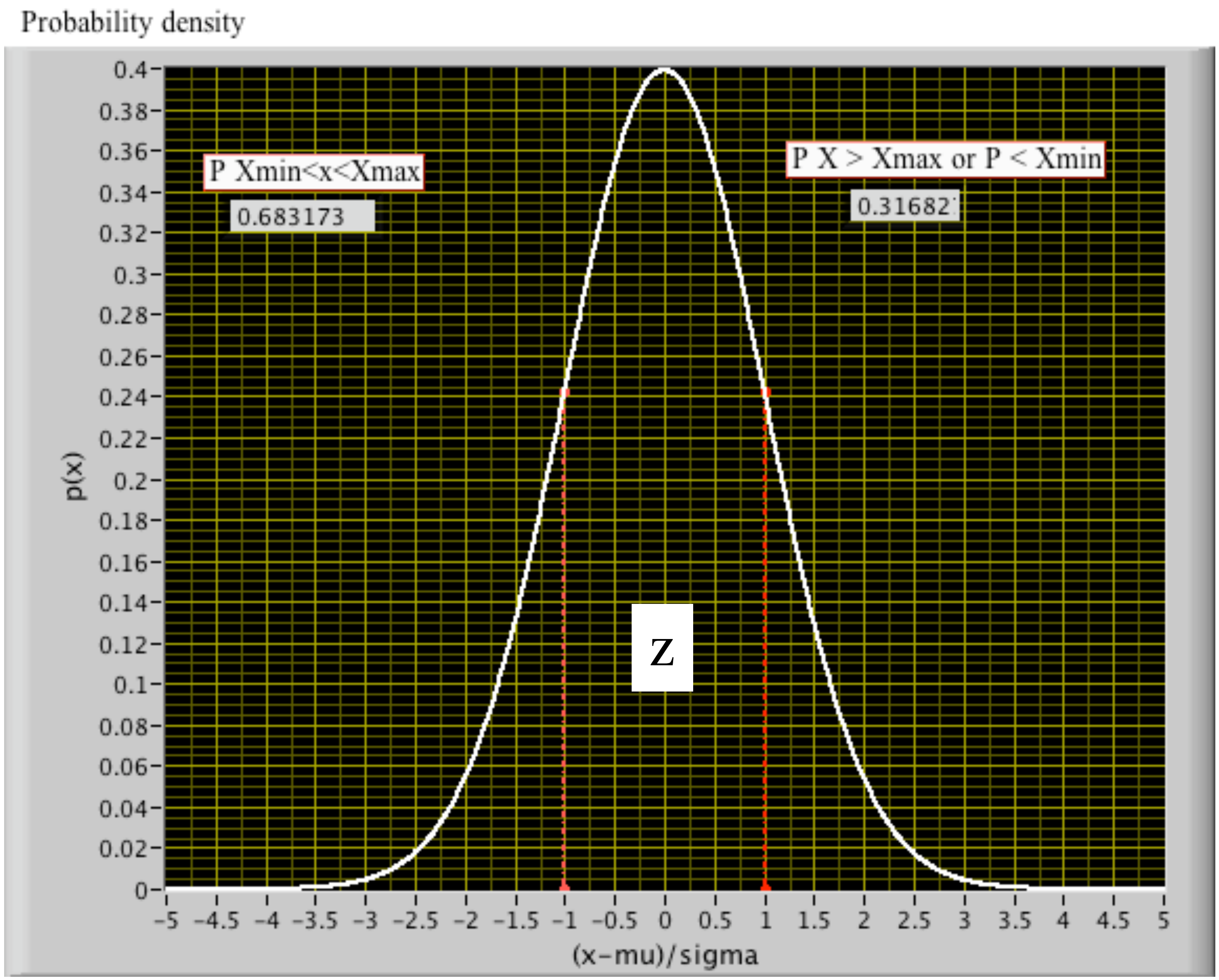
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879

Labview Code



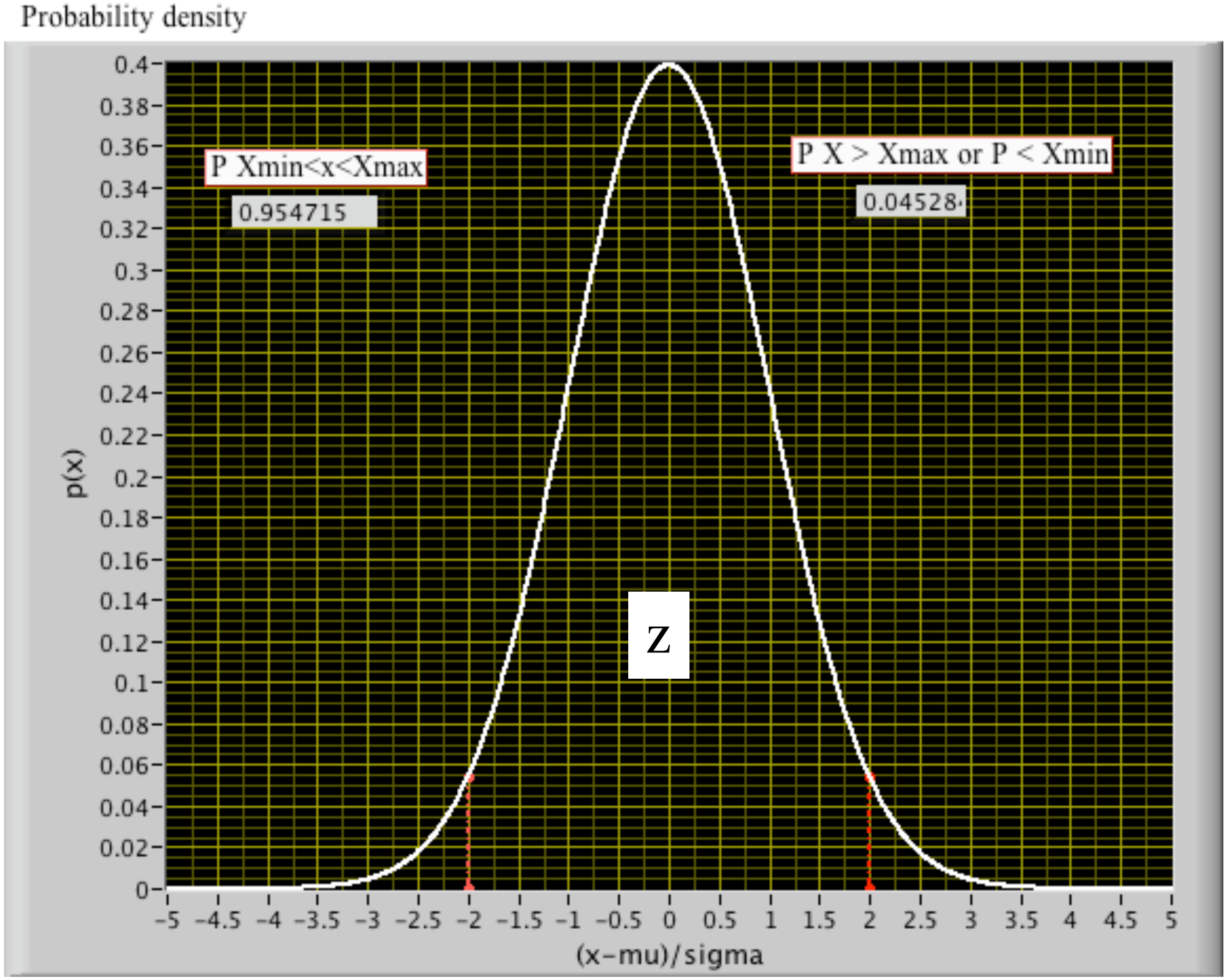
Probabilities of Deviation (1)

1-σ "one-sigma"



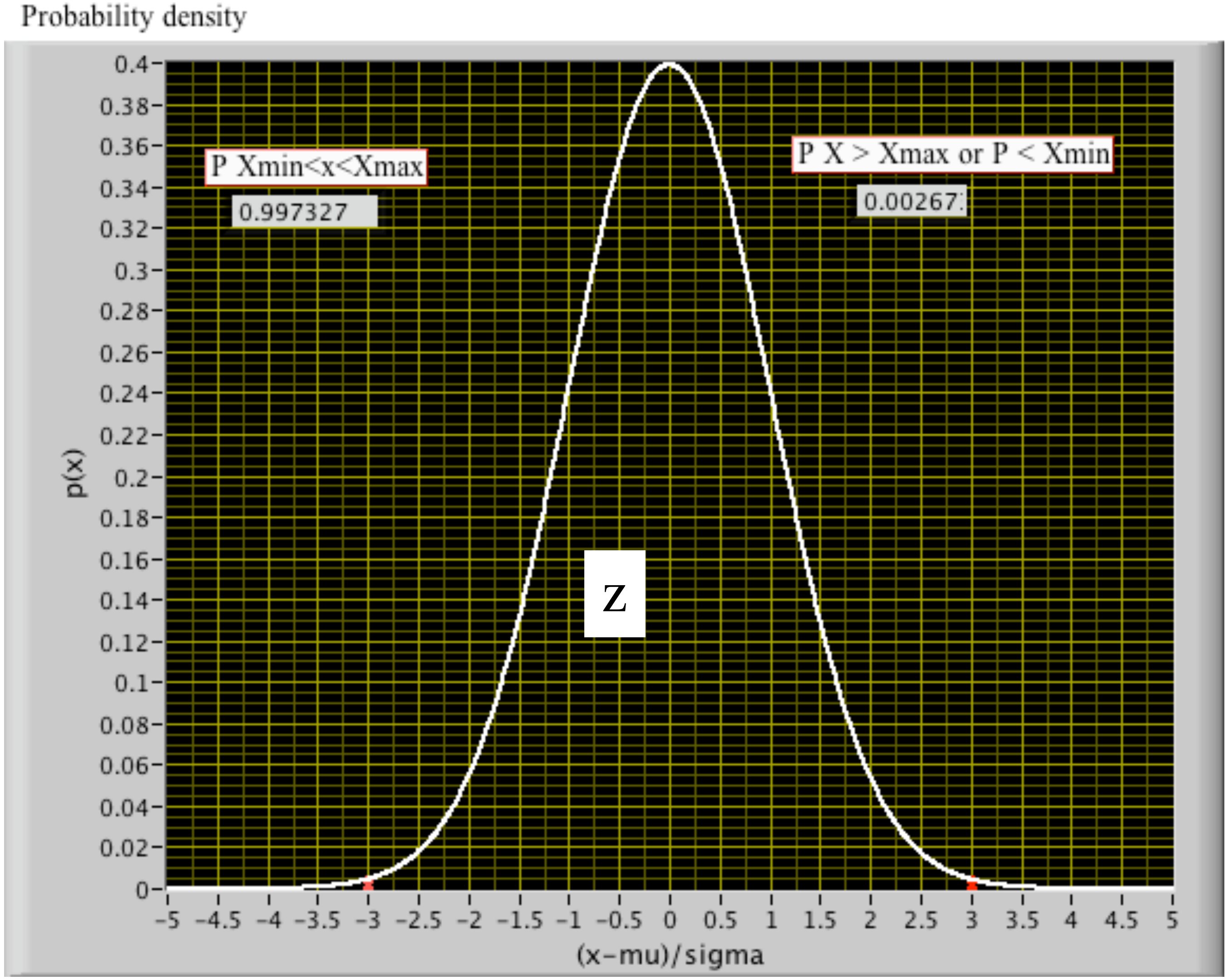
Probabilities of Deviation (2)

2-σ ”two-sigma"



Probabilities of Deviation (3)

3-σ ”three-sigma



Probabilities of Deviation (4)

Summary of Probability Estimates Based on the Normal Distribution

Common Name for "Error" Level	Error Level in Terms of σ	% Confidence That Deviation of x from Mean is Smaller	Odds That Deviation of x is Greater
Standard deviation	$\pm\sigma$	68.3	abt. 1 in 3
Two-sigma error	$\pm 1.96\sigma$	95.0	1 in 20
Three-sigma error	$\pm 3\sigma$	99.7	1 in 370

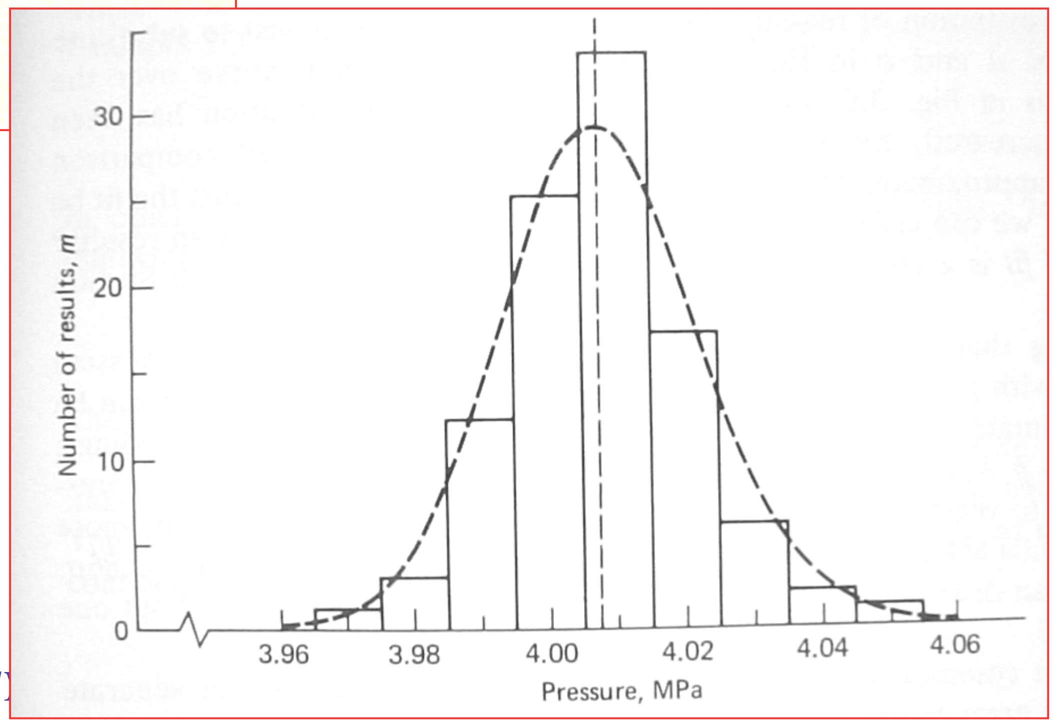
Example 3.6.1 (pp. 52-53 B.M.L)

Pressure p, in MPa	Number of Results, m
3.970	1
3.980	3
3.990	12
4.000	25
4.010	33
4.020	17
4.030	6
4.040	2
4.050	1

- # of Pressure readings Taken that line within $\pm$ 0.005 Mpa of listed value

- Histogram of Data (number of occurrences Within each bin)

- Compared with Normal Distribution based on sampled Mean and standard deviation



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Example 3.6.1 (2)

Pressure p, in MPa	Number of Results, m	Deviation, d	d^2
3.970	1	−0.038	144.4×10^{-5}
3.980	3	−0.028	78.4
3.990	12	−0.018	32.4
4.000	25	−0.008	6.4
4.010	33	0.002	0.4
4.020	17	0.012	14.4
4.030	6	0.022	48.4
4.040	2	0.032	102.4
4.050	1	0.042	176.4
$\sum p = 400.77$	$n = \sum m = 100$		$\sum d^2 = 1858 \times 10^{-5}$

$\bar{p} = 400.77/100 = 4.008$ MPa

$S_p = \sqrt{1858 \times 10^{-5}/99} = 0.014$ MPa

- ***Sample mean and standard deviation***

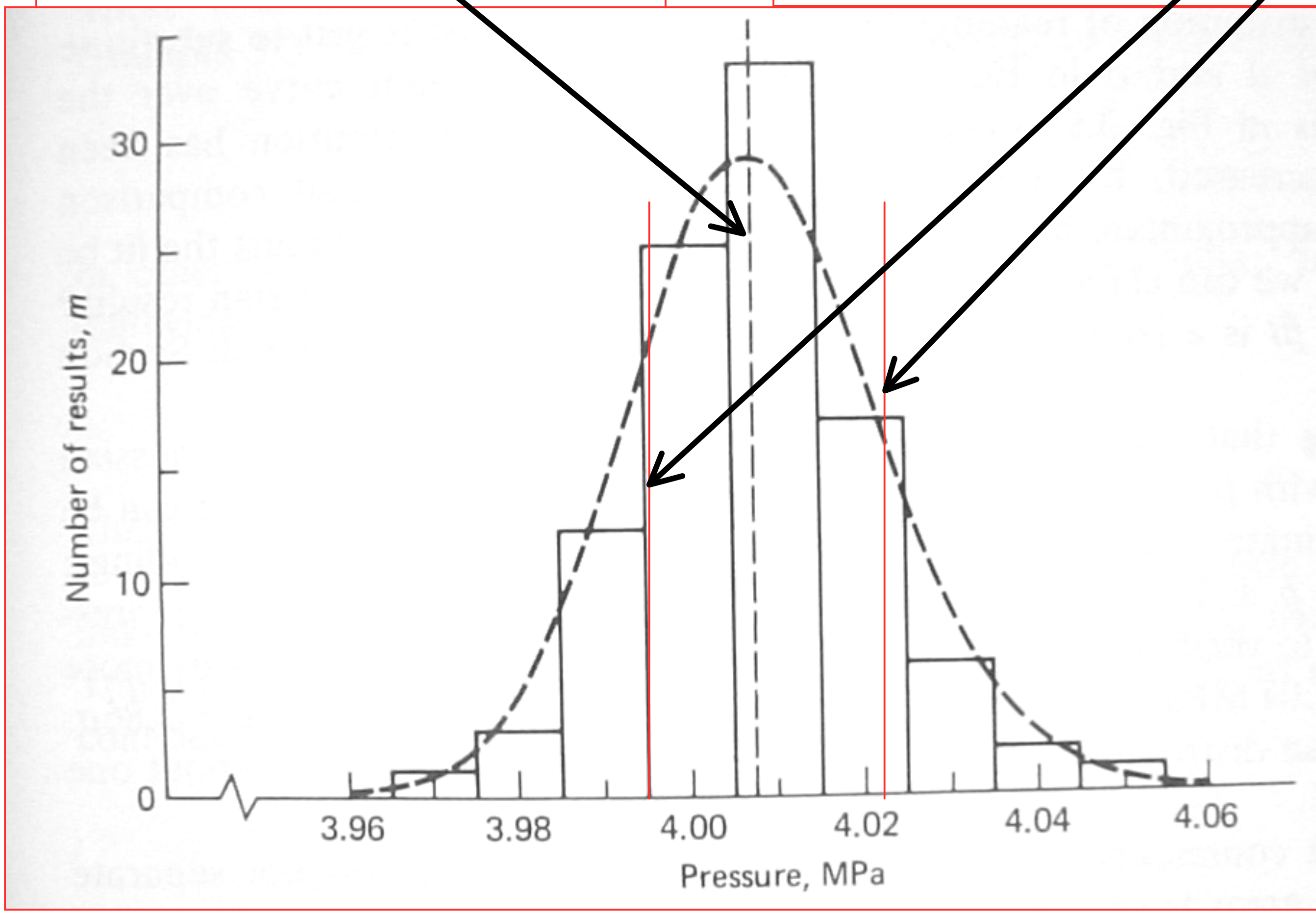
$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} = 4.008\,Mpa$$

$$S_x = \sqrt{\sum_{i=1}^{n} \frac{\left(x_i - \bar{x}\right)^2}{n-1}} = 0.014\,Mpa$$

Example 3.6.1 (3)

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} = 4.008\,Mpa$$

$$S_x = \sqrt{\sum_{i=1}^{n} \frac{\left(x_i - \bar{x}\right)^2}{n-1}} = 0.014\,Mpa$$



- Sample *"Not quite"* Gaussian
- How much *"not quite"?*

Confidence Intervals for Finite Samples

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Estimate of the mean

$$S_x = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}[x_i - \bar{x}]^2} = \sqrt{\frac{\left(\sum_{i=1}^{n} x_i^2\right) - n\bar{x}^2}{n-1}}$$

Estimate of the Standard Deviation

Based on a finite sample, we would like to:

1) *Estimate the mean and standard deviation, and their uncertainty*

2) *Infer the probability distribution of the data*

Confidence Intervals (1)

• For a Gaussian distributed population … the sum of any selected sample is also Gaussian distributed … consequently … the sample mean *(for n points)*

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \approx \mu$$

… is a Gaussian distributed variable

with a standard deviation given by $\sigma^2_{\bar{x}} = \frac{(\sigma(x))^2}{n}$

… more data you use … the better your estimate

Of course if we take another equally large, but different random sample from The population … we will get another equally valid estimate of the mean …Which estimate is "more correct"

• Our estimate of the mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \approx \mu$

is a Gaussian distributed variable with Variance … →

In terms of Normalized value

$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

Confidence Intervals (2)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \approx \mu \longrightarrow z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \rightarrow \bar{x} = \mu \pm z \cdot \frac{\sigma}{\sqrt{n}}$$

We'd like to be able to say how sure we are of this estimate. Let's look at the probability that our estimate of the mean is within some bound. We can say that there is a *c%* chance that our estimate of the mean lies within

$$\mu \pm z_{c/2} \frac{\sigma}{\sqrt{n}} \rightarrow \left[\mu - z_{c/2} \cdot \frac{\sigma}{\sqrt{n}} < x < \mu + z_{c/2} \cdot \frac{\sigma}{\sqrt{n}} \right]$$

- Or Alternatively

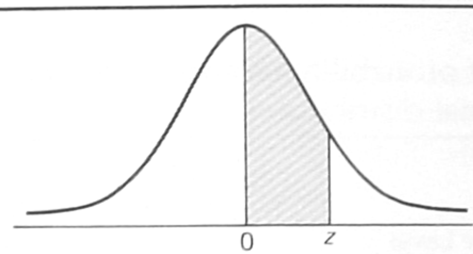
$$\bar{x} - z_{c/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{c/2} \frac{\sigma}{\sqrt{n}}$$

Confidence Intervals (3)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \approx \mu \longrightarrow \quad \bar{x} - z_{c/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{c/2}\frac{\sigma}{\sqrt{n}}$$

The larger we make the confidence interval c …. the larger $z_{c/2}$ becomes … and the larger the range for the mean estimate

Table 3.2 Areas under the standard normal curve



$z_{c/2}$ c/2

Second Decimal Place in z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879

Confidence Intervals (4)

This means that we are $c\%$ confident that the true mean μ lies within the interval about our measurement:

$$\bar{x} - z_{c/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{c/2}\frac{\sigma}{\sqrt{n}}$$

The only trouble is that we don't know the value of σ either. If n is large enough, we can use our estimate S_x, so

$$\bar{x} - z_{c/2}\frac{S_x}{\sqrt{n}} < \mu < \bar{x} + z_{c/2}\frac{S_x}{\sqrt{n}}$$

Standard Error of the Sample Mean

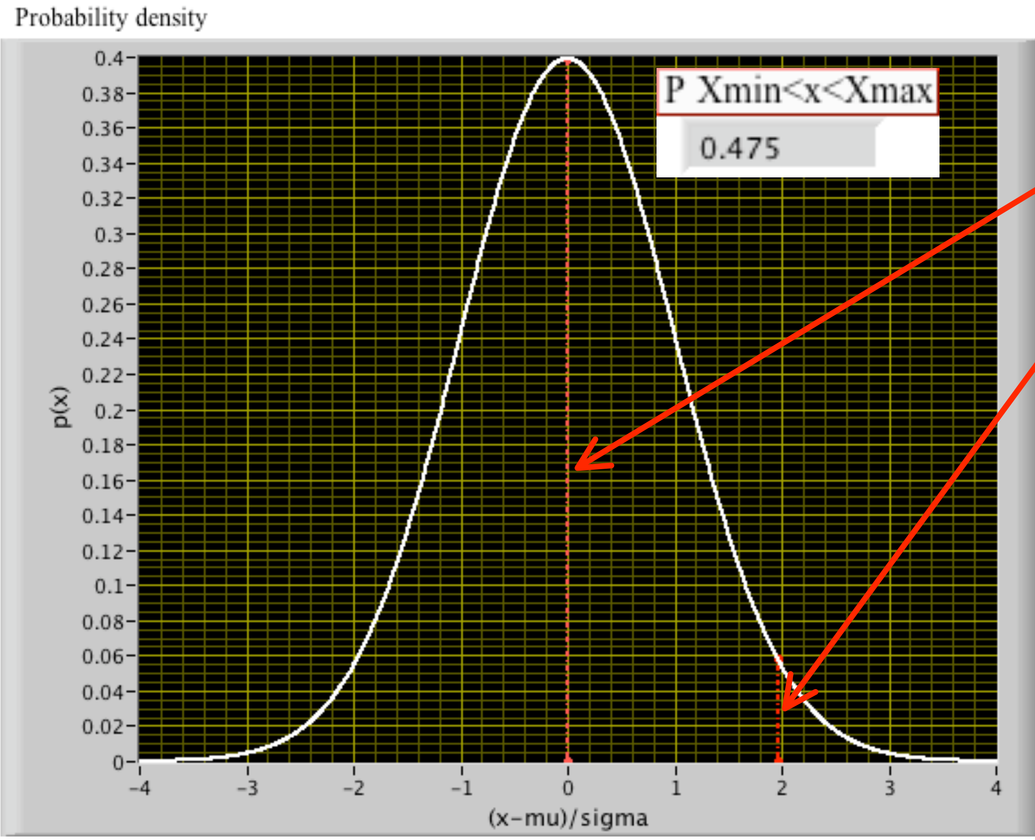
$$S_{\bar{x}} = \frac{S_x}{\sqrt{n}}$$

Confidence Intervals (5)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \approx \mu \longrightarrow$$

$$\bar{x} - z_{c/2}\frac{S_x}{\sqrt{n}} < \mu < \bar{x} + z_{c/2}\frac{S_x}{\sqrt{n}}$$

Same effect using computer code … i.e .. For *95% confidence* level … *c/2 = 0.475*



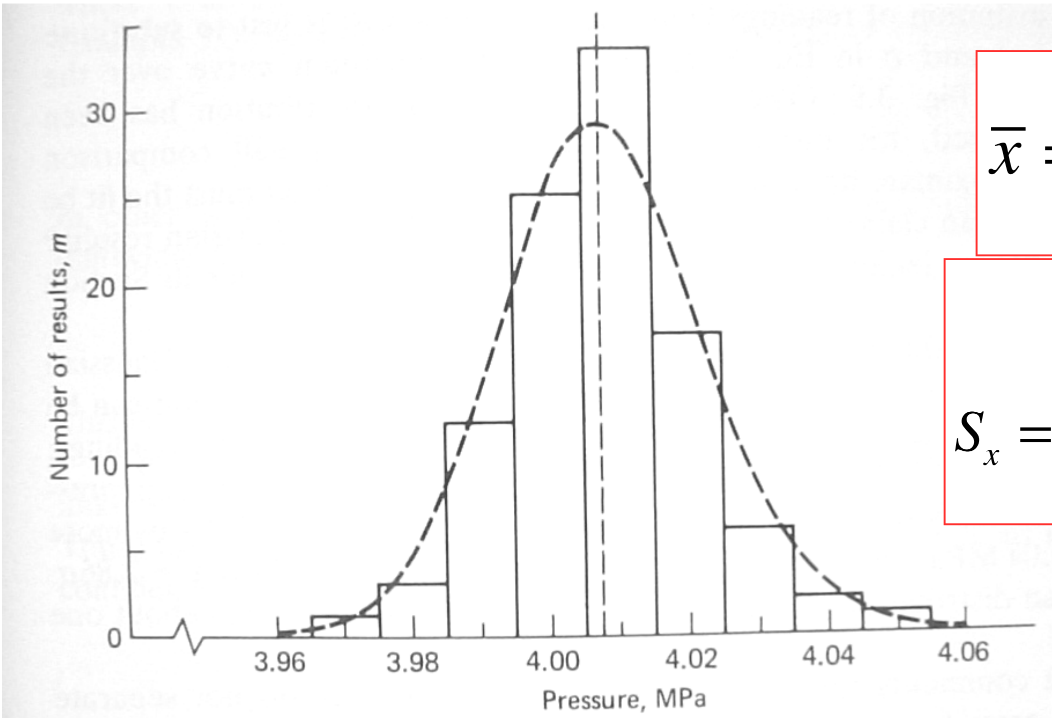
$z = 0$

$z = 1.96 = z_{c/2}$

0.4750 - area Under curve between lines

$$\bar{x} - 1.96\frac{S_x}{\sqrt{n}} < \mu < \bar{x} + 1.96\frac{S_x}{\sqrt{n}}$$

Confidence Interval for Example 3.6.1(1)



$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} = 4.008\,Mpa$$

$$S_x = \sqrt{\sum_{i=1}^{n} \frac{\left(x_i - \bar{x}\right)^2}{n-1}} = 0.014\,Mpa$$

What is 99% confidence level for this sample mean?

- Can use table 3.2 with c = 49.5%

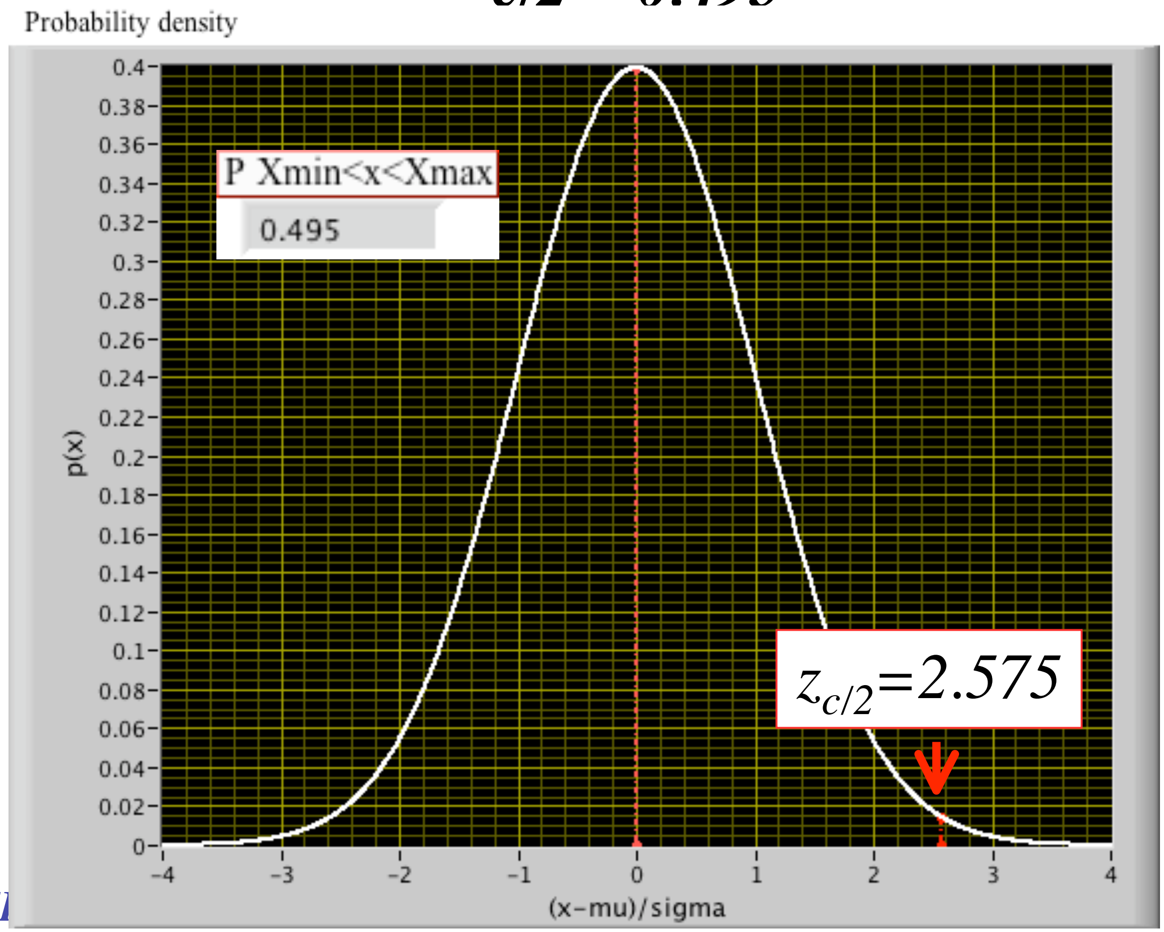
Which is kinda Kludgy

Confidence Interval for Example 3.6.1(2)

- Or use your numerical program

99% confidence level
c/2 = 0.495

What is 99% confidence level for this sample mean?

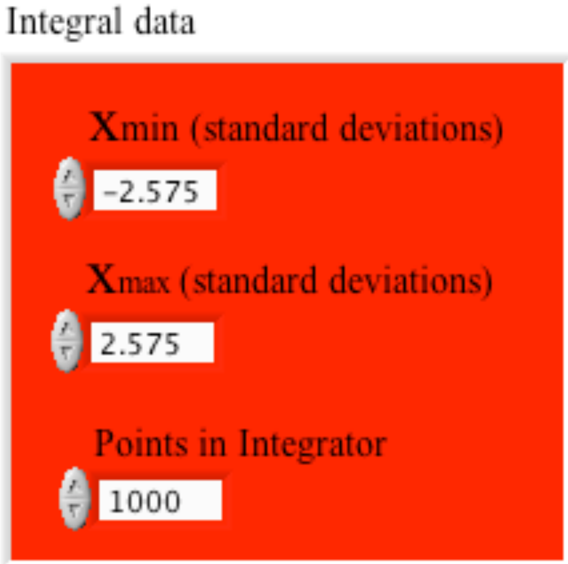


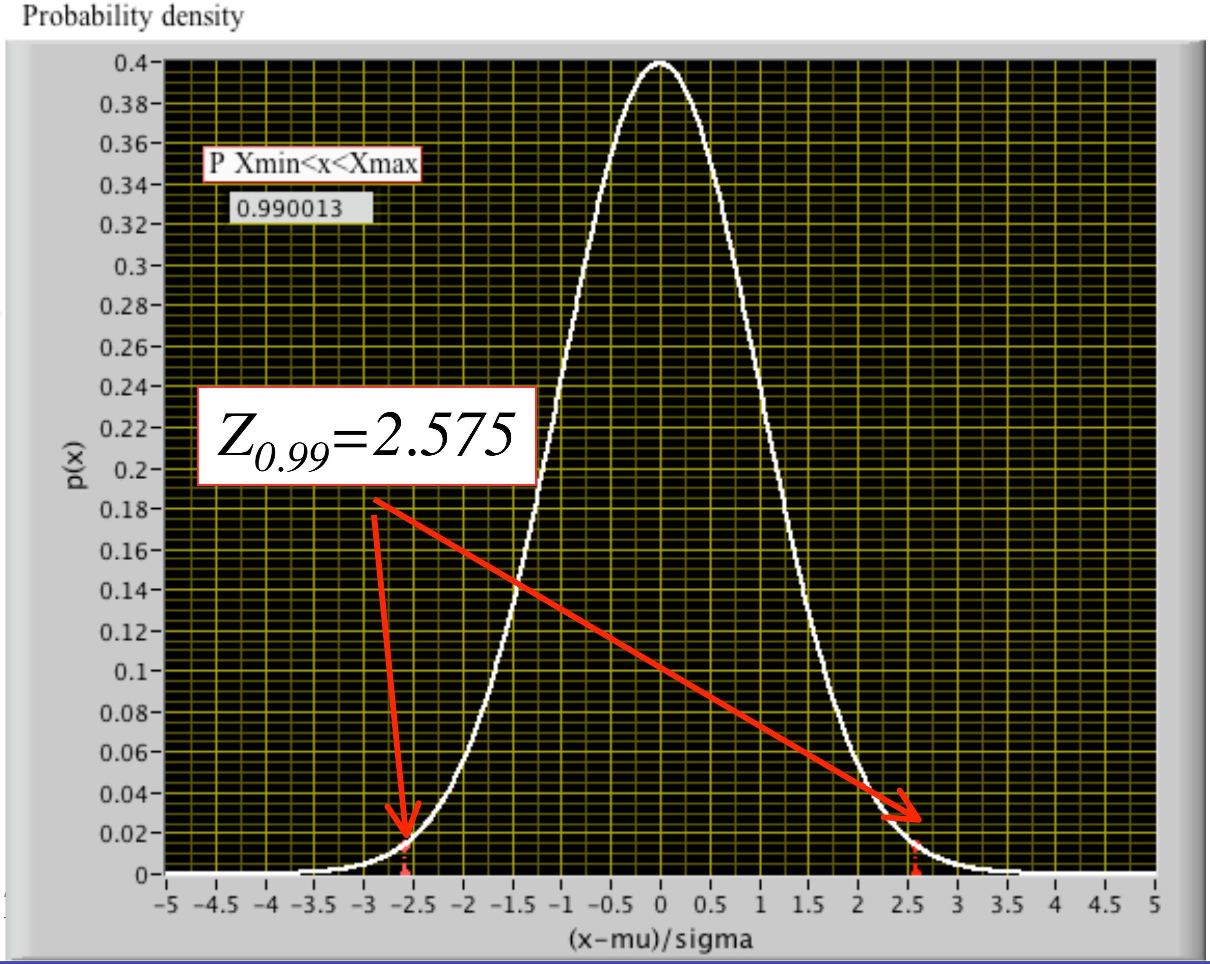
Confidence Interval for Example 3.6.1(3)

- or more directly use two sided probability

→ 99% confidence level

What is 99% confidence level for this sample mean?





Confidence Interval for Example 3.6.1(3)

→ 99% confidence level $z_{c/2}=2.575$

$$\bar{x} - z_{c/2}\frac{S_x}{\sqrt{n}} < \mu < \bar{x} + z_{c/2}\frac{S_x}{\sqrt{n}} \rightarrow$$

$$4.008 - 2.575\frac{0.014}{\sqrt{100}} < \mu < 4.008 + 2.575\frac{0.014}{\sqrt{100}} \rightarrow$$

$$4.004395 < \mu < 4.01165 \rightarrow \boxed{\mu = 4.008 \pm 0.003605}$$

Confidence Interval for Example 3.6.1(4)

Or Using the tables

$c = 0.99$, $c/2 = 0.495$ …

Easier to mechanize using Computer .. And less error

	Second Decimal Place in z									
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986

$z_{0.495} = 2.575$

$$\mu \pm z_{c/2} \frac{\sigma}{\sqrt{n}}$$

$\mu = 4.008 \pm 2.575\ (0.014)/10 = 4.008 \pm 0.003605$ (99%)

Table 4.3 Probability Values for Normal Error Function: One-Sided Integral Solutions for $p(z_1) = \frac{1}{(2\pi)^{1/2}} \int_0^{z_1} e^{-\beta^2/2} d\beta$

$z_1 = \frac{x_1 - x'}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1809	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2794	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4292	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4758	0.4761	0.4767
2.0	0.4772	0.4778	0.4803	0.4788	0.4793	0.4799	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.49865	0.4987	0.4987	0.4988	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990

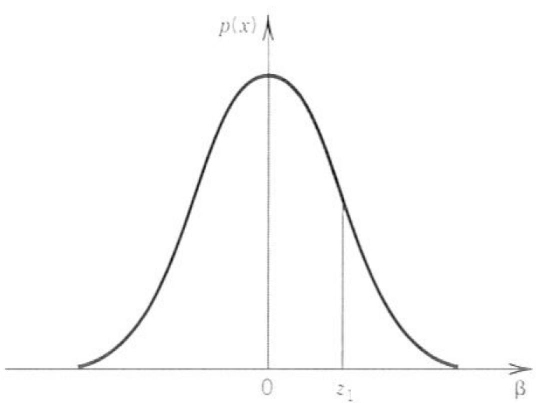


Figure 4.3 Integration terminology for the normal error function and Table 4.3.

Confidence Intervals for Small Samples

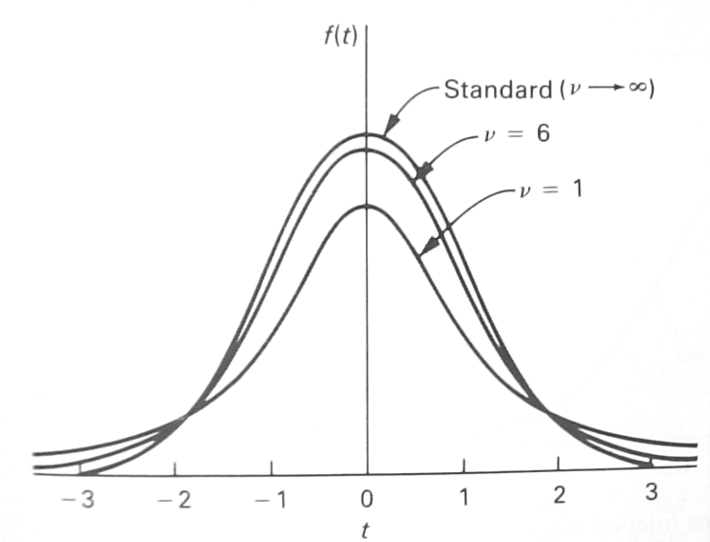
We do not always have the luxury of taking large samples ($n > 30$). For smaller sample sizes, we cannot assume that $\sigma \sim S_x$. If we derive the distribution of the quantity

$$t = \frac{\bar{x} - \mu}{S_x / \sqrt{n}}$$

• Dependent upon the number of Degrees of freedom, ***ν=n-1***

assuming that the population is gaussian,we get the Student t-distribution

The derivation of the *t*-distribution was first published in 1908 by William Sealy Gosset, while he worked at a Guinness brewery in Dublin. He was not allowed to publish under his own name, so the paper was written under the pseudonym ***Student***.



Student's-t distribution (1)

• The **Student's *t*-distribution** is a probability distribution that arises in the problem of estimating the mean of a normally distributed population when the sample size is small. It is the basis of the popular Student's *t*-tests for the statistical significance of the difference between two sample means, and for confidence intervals for the difference between two population means.

• Given a sample set …

$$\{x_1, x_2, x_3, \ldots x_n\} \rightarrow \begin{bmatrix} mean : \mu \\ variance : \sigma^2 \end{bmatrix} \rightarrow sample\ mean : \bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$$

• The variable $z = \dfrac{\bar{x} - \mu}{\sigma_{\bar{x}}} = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$

is normally distributed with mean 0 and variance 1

Student's-t distribution (2)

- Gosset studied a related quantity, *for small samples*

$$t = \frac{\bar{x} - \mu}{S_x / \sqrt{n}}$$

"t" distribution

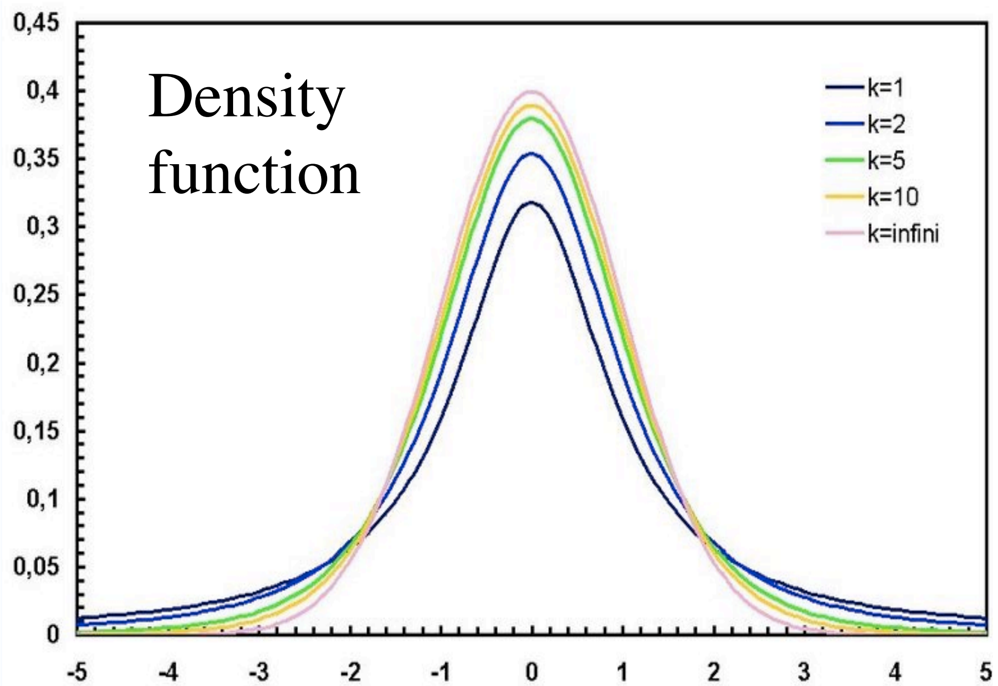
And showed that it had the probability density function

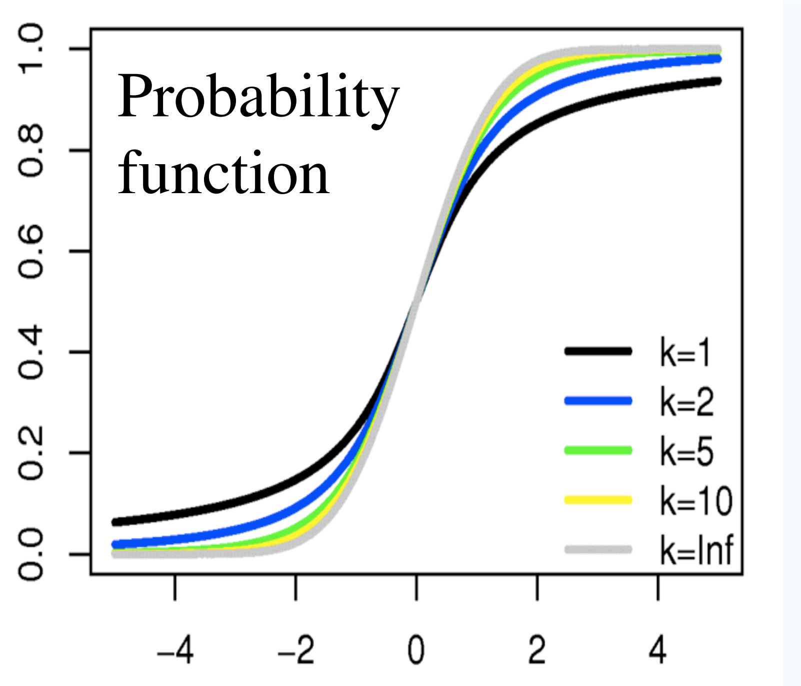
$$p(t) = \frac{\Gamma\left[\frac{\nu+1}{2}\right]}{\sqrt{\nu\pi}\,\Gamma\left[\frac{\nu}{2}\right]\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}} \rightarrow \left[\begin{array}{c} \nu = n-1 \\ \Gamma = \text{"gamma function"} \end{array}\right]$$

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du \approx \frac{e^{-0.5772156649\,x}}{x} \prod_{i=1}^{\infty} \left(\frac{e^{x/i}}{1+\frac{x}{i}} \right)$$

Student's-t distribution (3)

• The **Student's *t*-distribution** is a probability distribution that arises in the problem of estimating the mean of a normally distributed population when the sample size is small. It is the basis of the popular Student's *t*-tests for the statistical significance of the difference between two sample means, and for confidence intervals for the difference between two population means.





Small-Sample Confidence Interval(1)

- Done exactly in the same was as for large samples … only Now you use the "*t-distribution*" for $\nu = n\text{-}1$ *degrees of freedom* and ***not*** the Gaussian distribution

- Want to evaluate …. To evaluate precision of estimate At some *c* --> *confidence* level

$$\bar{x} - t_{c/2,\nu} \cdot \frac{S_x}{\sqrt{n}} \le \mu_x \le \bar{x} + t_{c/2,\nu} \cdot \frac{S_x}{\sqrt{n}}$$

- S_x --> sample standard deviation

Small-Sample Confidence Interval(2)

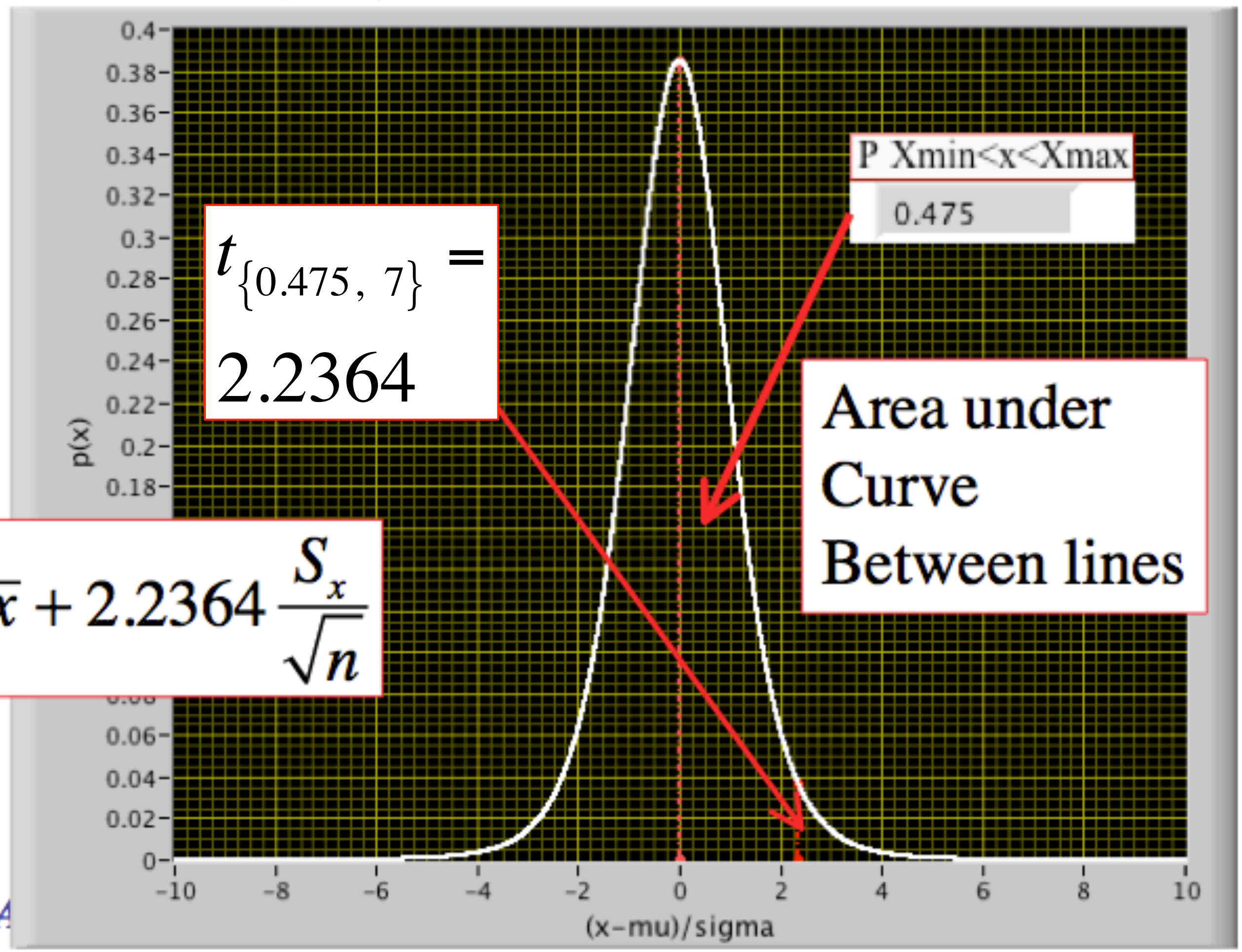
- *example* … for a sample with 8 data points estimate precision Bounds for 95% level of certainty

$\nu = n-1 = 7$ -->

$c/2, \nu=7 = 0.475$

$$t_{\{0.475,\ 7\}} = 2.2364$$

Student's-t Probability density



$t_{\{0.475,\ 7\}}$ = 2.2364

Area under Curve Between lines

$$\bar{x} - 2.2364\frac{S_x}{\sqrt{n}} < \mu < \bar{x} + 2.2364\frac{S_x}{\sqrt{n}}$$

Small-Sample Confidence Interval(3)

- *compare to "large population" Gaussian ν =infinity*

For 95% confidence level

$$\bar{x} - 2.2364\frac{S_x}{\sqrt{n}} < \mu < \bar{x} + 2.2364\frac{S_x}{\sqrt{n}}$$

8 of data points
(ν --> 7)

$$\bar{x} - 1.96\frac{S_x}{\sqrt{n}} < \mu < \bar{x} + 1.96\frac{S_x}{\sqrt{n}}$$

"lots" of data points
(ν --> infinity)

"uncertainty is obviously larger for small sample"

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Small-Sample Confidence Interval(4)

- *Example ~~3.6 in B.M.L.~~* **2**

Postal scale calibration …. 14 one-ounce weights chosen & weighed

Value	Σx	δx	δx²	Σδx²
1.080	1.080	0.071	0.005	0.005
1.030	2.110	0.021	0.000	0.005
0.960	3.070	-0.049	0.002	0.008
0.950	4.020	-0.059	0.003	0.011
1.040	5.060	0.031	0.001	0.012
1.010	6.070	0.001	0.000	0.012
0.980	7.050	-0.029	0.001	0.013
0.990	8.040	-0.019	0.000	0.014
1.050	9.090	0.041	0.002	0.015
1.080	10.170	0.071	0.005	0.020
0.970	11.140	-0.039	0.002	0.022
1.000	12.140	-0.009	0.000	0.022
0.980	13.120	-0.029	0.001	0.023
1.010	14.130	0.001	0.000	0.023

Sample Statistics

$\bar{x} = 1.00929$

$S_x = 0.04178$

- ***Compute***

95% confidence interval (precision) for population mean

Small-Sample Confidence Interval(5)

- *Example* 2 .

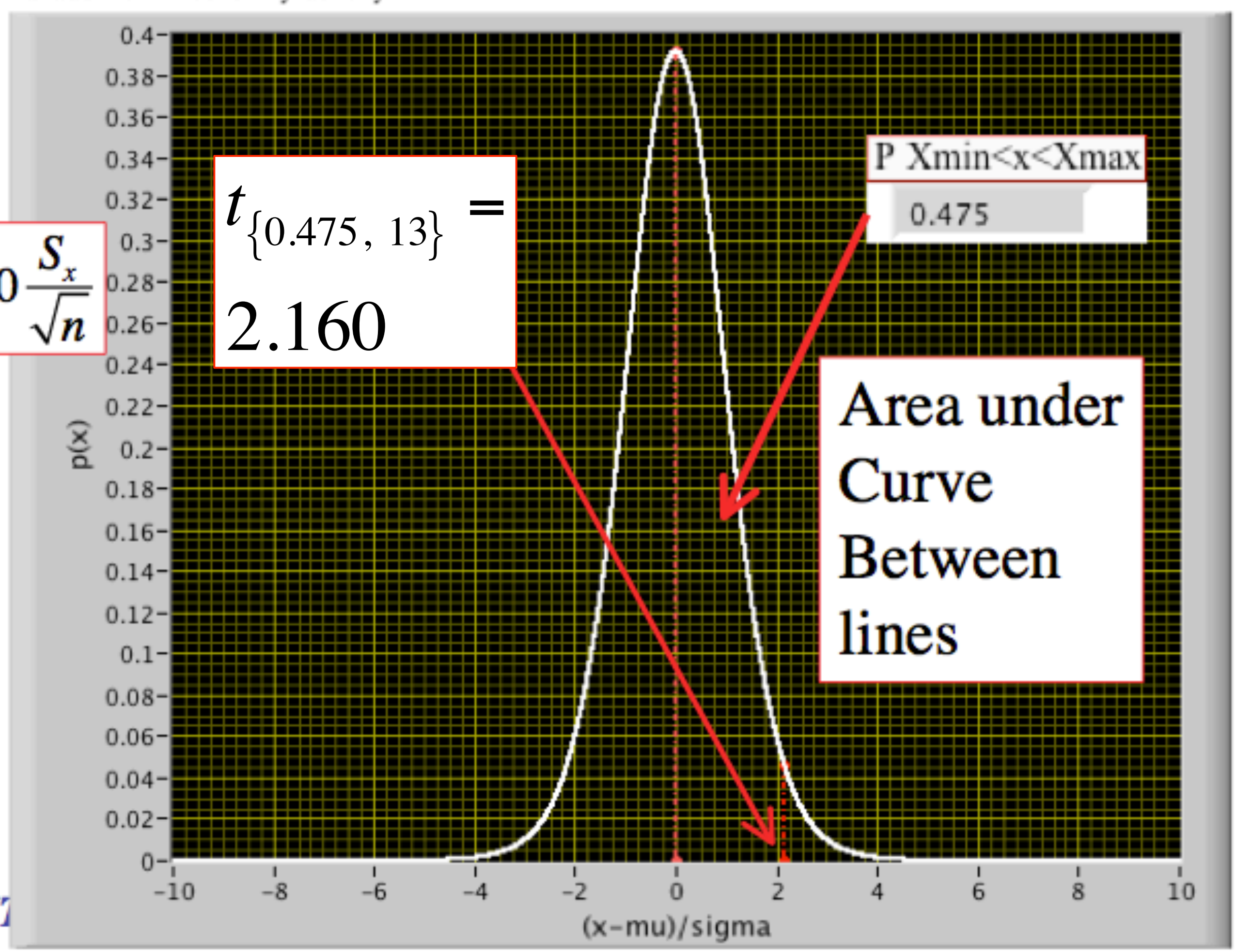
*95 % confidence level --> **c/2,ν=0.475 --> ν= n-1 = 13***

$\bar{x}$ =1.00929

S_x =0.04178

$$\bar{x} - 2.160\frac{S_x}{\sqrt{n}} < \mu < \bar{x} + 2.160\frac{S_x}{\sqrt{n}}$$

Student's-t Probability density



Small-Sample Confidence Interval(6)

- *Example* 2 .

*95 % confidence level --> **c/2=0.475 --> ν= n-1 = 13***

$\bar{x}$ **=1.00929**
S_x **=0.04178**

$$\bar{x} - 2.160\frac{S_x}{\sqrt{n}} < \mu < \bar{x} + 2.160\frac{S_x}{\sqrt{n}} \rightarrow$$

$$1.00929 - 2.160 \cdot \frac{0.04178}{\sqrt{14}} < \mu < 1.00929 + 2.160 \cdot \frac{0.04178}{\sqrt{14}}$$

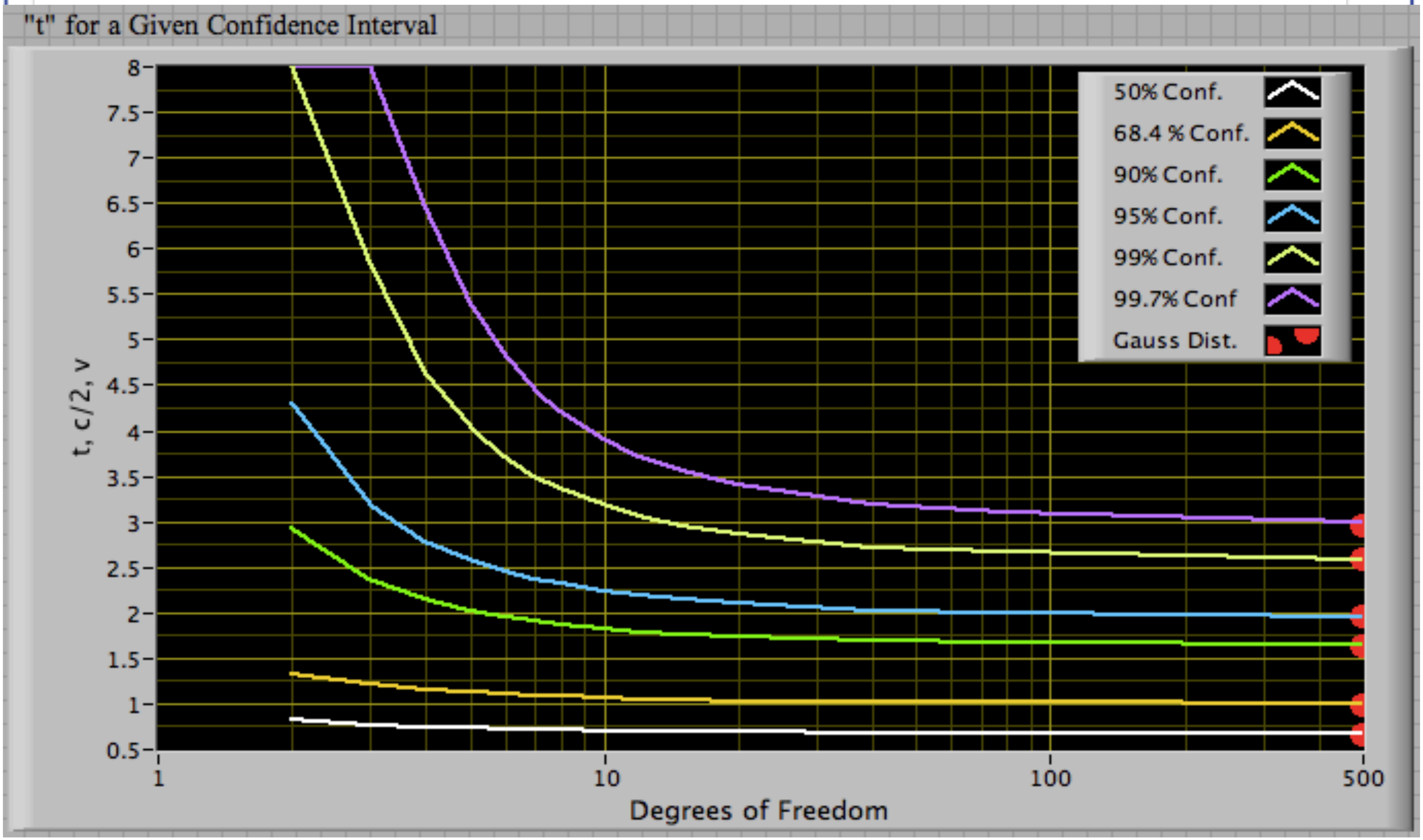
$$0.098517 < \mu < 1.03341$$

---> Measurement precision .... At 95% confidence level

$$P_{recision} \approx t_{\{0.475,\ 13\}} \cdot \frac{S_x}{\sqrt{n}} = 2.160 \cdot \frac{0.04178}{\sqrt{14}} = 0.02412$$

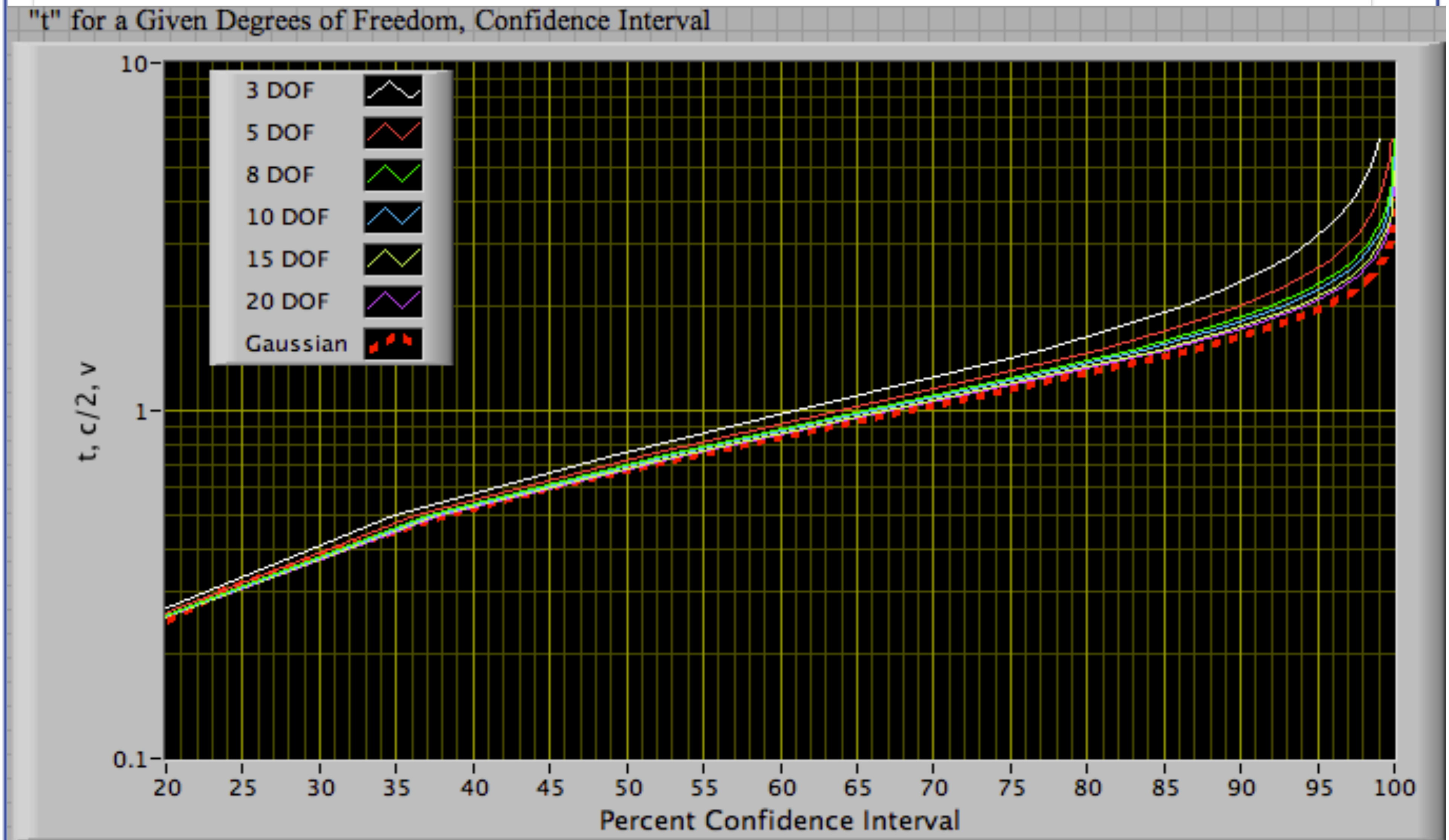
Small Sample Confidence Intervals (7)

Plotted Data from Table 4.4 for *Student's t* Distribution (Figliola and Beasley)



Small Sample Confidence Intervals (8)

Re-Plotted Data from Table 4.4 for *Student's t* Distribution (Figliola and Beasley)



The t-Test Comparison (1)

If we take two small samples, and we wish to determine whether or not the resultant means are statistically identical, we use this test.

$$t = \frac{\bar{x} - \mu}{S_x / \sqrt{n}} \longrightarrow t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(S_1^2 / n_1\right) + \left(S_2^2 / n_2\right)}}$$

We find t by choosing a confidence interval. In order to do that, we need to know the number of degrees of freedom. In general, the number of samples in 1 and 2 may be different. *The effective degrees of freedom can be approximated by:*

$$\nu = \frac{\left[\left(S_1^2 / n_1\right) + \left(S_2^2 / n_2\right)\right]^2}{\dfrac{\left(S_1^2 / n_1\right)^2}{n_1 - 1} + \dfrac{\left(S_2^2 / n_2\right)^2}{n_2 - 1}}$$

to the nearest integer. If the computed value of t lies inside of the interval $\pm t_{\alpha/2,\nu}$, then the two means are statistically identical within the confidence assumed.

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The t-Test Comparison (2)

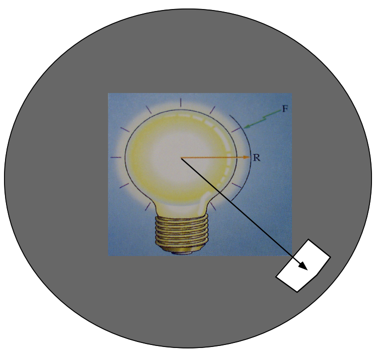
- Want to determine lifetimes of two different Brands of light bulbs

  - *At 95% confidence level*

    Is there any statistical difference?

Lifetime, months

Brand A	*Brand B*
7.2	**7.5**
7.6	**8.7**
6.9	**7.7**
8.2	**7.5**
7.3	**6.7**
7.8	**11.2**
6.6	**7.0**
6.9	**10.7**
5.5	**7.0**
7.4	**8.6**
5.7	**6.1**
6.2	**6.3**
	7.8
	8.7
	6.1



Lifetime Statistics , months

Brand A	*Brand B*
$x=6.94$	$\bar{x}=7.84$
$S_x=0.82$	$S_x=1.53$
$n=12$	$n=15$

$$\nu = \frac{\left[\left(S_1^2/n_1\right)+\left(S_2^2/n_2\right)\right]^2}{\dfrac{\left(S_1^2/n_1\right)^2}{n_1-1}+\dfrac{\left(S_2^2/n_2\right)^2}{n_2-1}} = \frac{\left(\dfrac{0.82^2}{12}+\dfrac{1.53^2}{15}\right)^2}{\left(\dfrac{\left(\dfrac{0.82^2}{12}\right)^2}{12-1}+\dfrac{\left(\dfrac{1.53^2}{15}\right)^2}{15-1}\right)}$$

$= 22.213$ .. ***Round to 22***

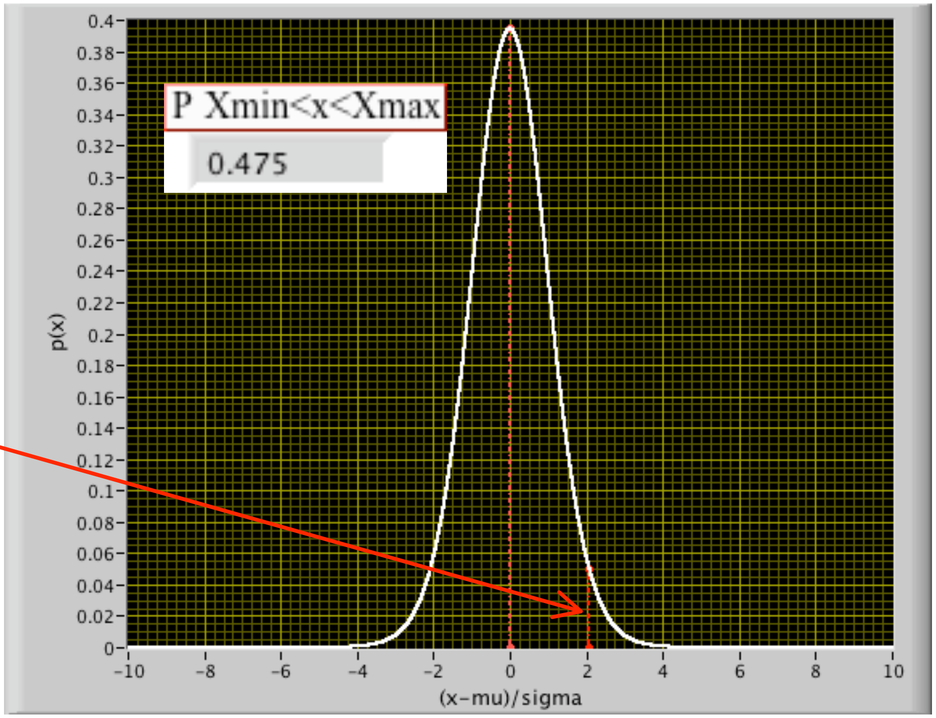
The t-Test Comparison (3)

Lifetime Statistics , months	
Brand A	*Brand B*
$\bar{x}=6.94$	$\bar{x}=7.84$
$S_x=0.82$	$S_x=1.53$
$n=12$	$n=15$

$\nu_{effective} \sim 22$

$z_{c/2=0.475,\nu=22} = 2.074$

For 95% --> $c/2=0.475$



- Look at test statistic

$$t = \left| \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(S_1^2/n_1\right)+\left(S_2^2/n_2\right)}} \right| = \left| \left(\frac{6.94-7.84}{\left(\frac{0.82^2}{12}+\frac{1.53^2}{15}\right)^{0.5}} \right) \right| = 1.954 < 2.074$$

At 95% level no Statistical significance

Bias and Single Sample Uncertainty

What can you do about estimating the your precision uncertainty if you only take 1 or 2 samples?

You can use the instruments specs (non repeatability) to estimate the uncertainty and treat it like it is a bias error.

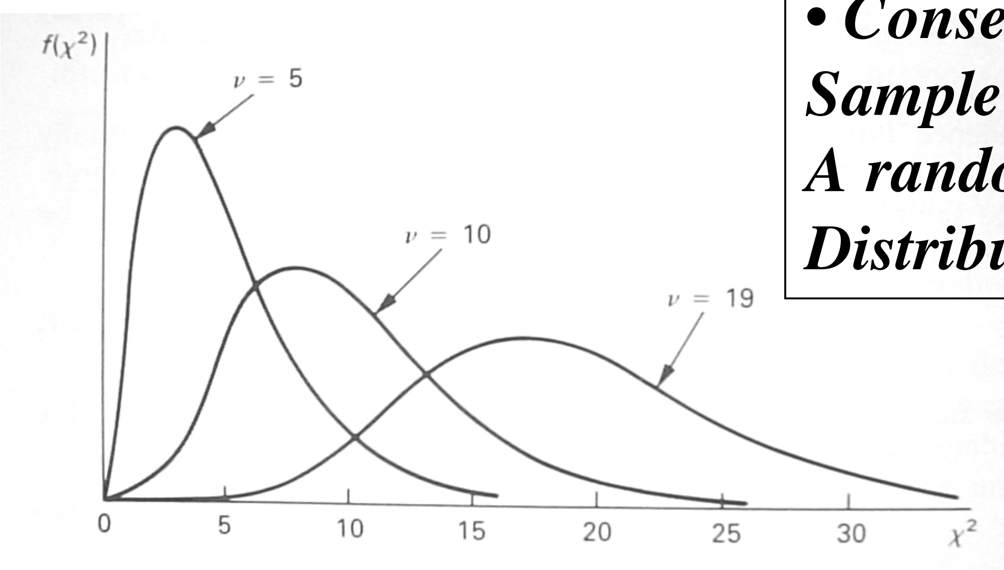
Better approach is to *"take more samples"*

χ^2 Distribution (1)

• As we saw earlier … For a Gaussian distributed population
… the sum of any selected sample is also Gaussian distributed
… consequently … **the sample mean *(for n points)* … is a Gaussian … distributed variable**

• However the sum of the squares of any set of points is NOT Gaussian distributed .. The distribution is instead described By a *χ2 distribution for ν = n-1 degrees of freedom.*

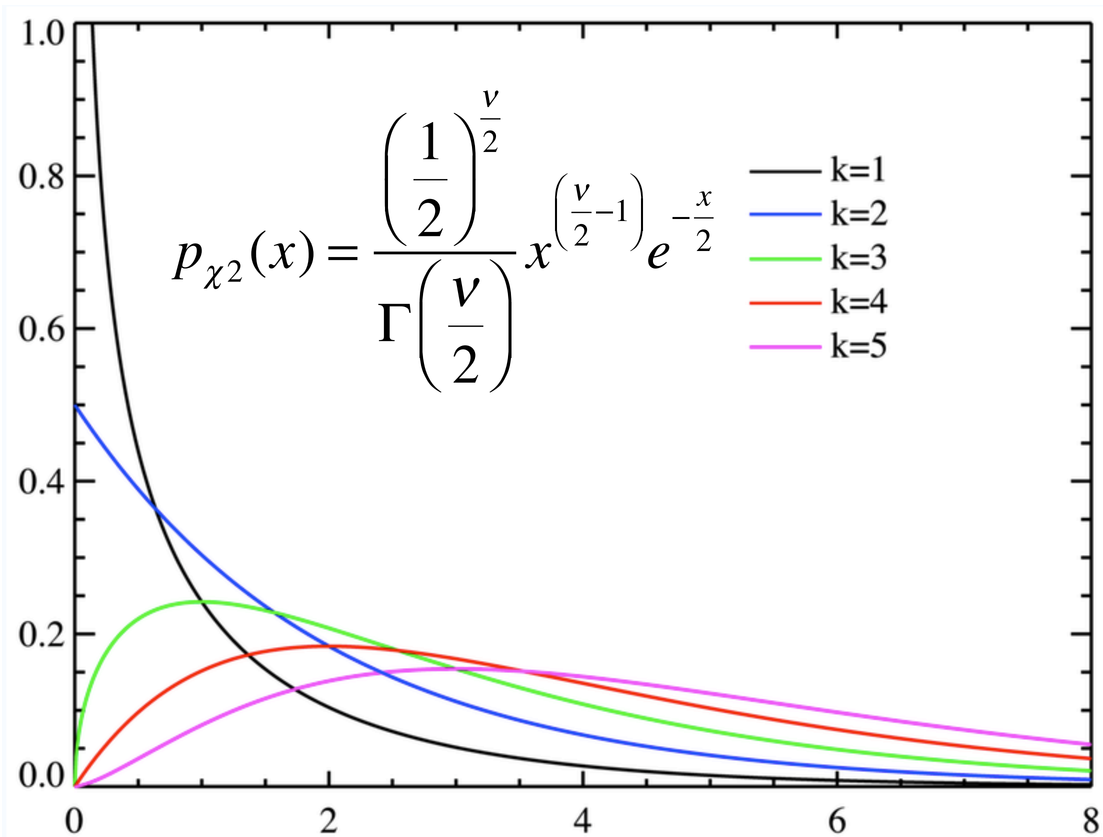
$$S_x^{\ 2} = \sum_{i=1}^{n} \frac{\left(x_i - \bar{x}\right)^2}{n-1}$$



• Consequently, the Sample variance is A random variable Distributed as χ2.

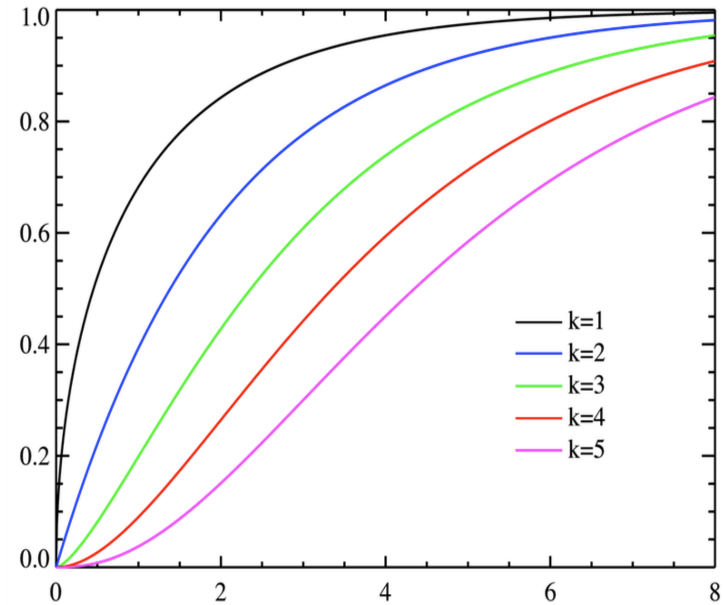
$$p_{\chi 2}(x) = \frac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} x^{\left(\frac{\nu}{2}-1\right)} e^{-\frac{x}{2}}$$

χ^2 Distribution (2)



$$p_{\chi^2}(x) = \frac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} x^{\left(\frac{\nu}{2}-1\right)} e^{-\frac{x}{2}}$$

Cumulative Distribution function



- For a Gaussian Distributed population with $\mu=0$, $\sigma^2=1$

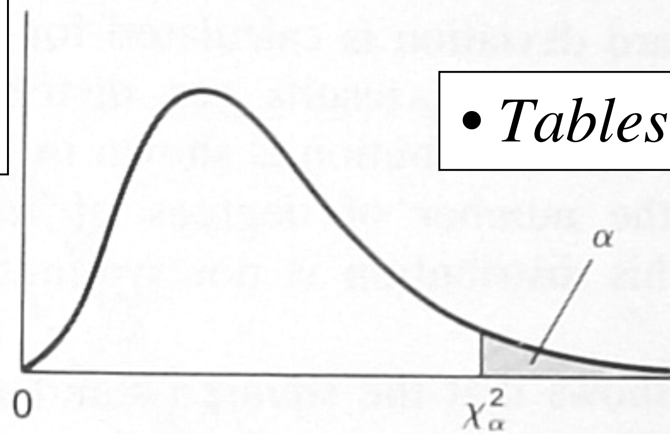
$$S_x^{\ 2} = \sum_{i=1}^{n} \frac{\left(x_i - \bar{x}\right)^2}{n-1}$$

- One-sided density function … because of "squared" components

χ^2 Distribution (3)

$$P_{\chi 2}(\alpha,\nu) = \int_{\alpha}^{\infty} \frac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} a^{\left(\frac{\nu}{2}-1\right)} e^{-\frac{a}{2}} da$$

- *Tables of $\chi 2$ probability*



ν	$\chi^2_{0.995}$	$\chi^2_{0.99}$	$\chi^2_{0.975}$	$\chi^2_{0.95}$	$\chi^2_{0.05}$	$\chi^2_{0.025}$	$\chi^2_{0.01}$	$\chi^2_{0.005}$	ν
1	0.000	0.000	0.001	0.004	3.841	5.024	6.635	7.879	1
2	0.010	0.020	0.051	0.103	5.991	7.378	9.210	10.597	2
3	0.072	0.115	0.216	0.352	7.815	9.348	11.345	12.838	3
4	0.207	0.297	0.484	0.711	9.488	11.143	13.277	14.860	4
5	0.412	0.554	0.831	1.145	11.070	12.832	15.086	16.750	5

χ^2 Significance Testing (1)

- Recall that the *Gauss/Student's-t* distributions allow us to Assess the precision of an estimate of the population Mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \longrightarrow \sigma^2_{\bar{x}} = \frac{(\sigma(x))^2}{n}$$

1) large sample .. *Gaussian distribution*
2) small sample … *Student's t distribution*

- The $\chi 2$ distribution allows up to perform the same evaluation For the sample variance *(square of the standard deviation)*

$$S_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left[x_i - \bar{x}\right]^2 = \frac{\left(\sum_{i=1}^{n} x_i^2\right) - n\bar{x}^2}{n-1}$$

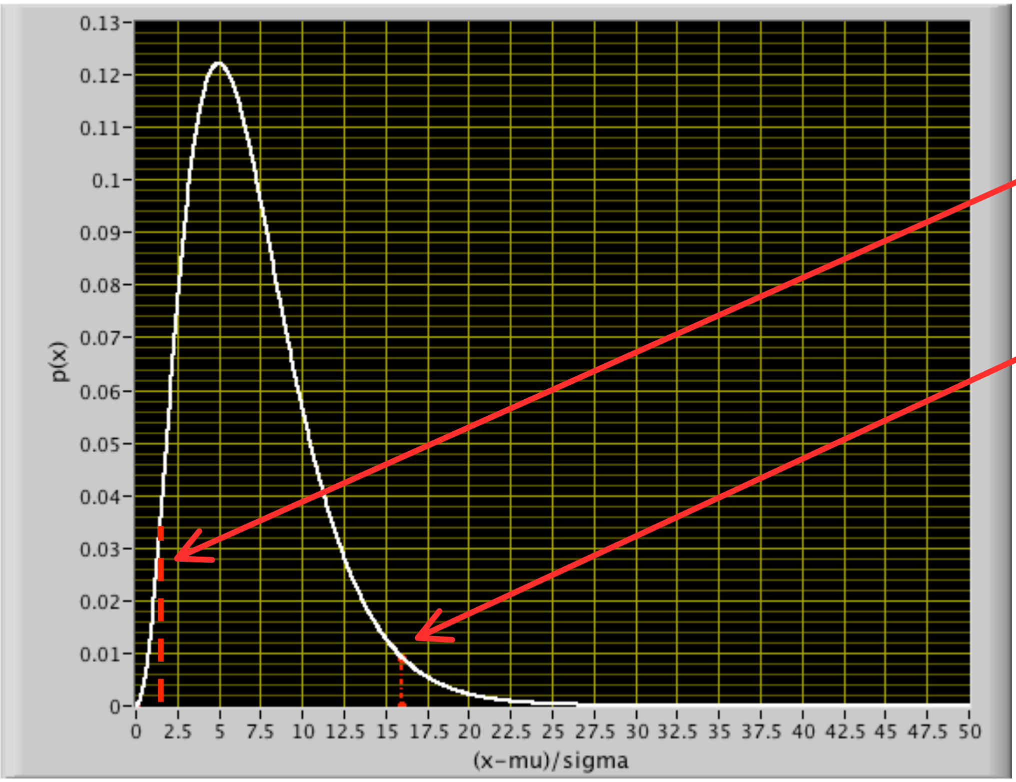
χ^2 Significance testing (2)

$$\frac{(n-1)S_x^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2}} .....(c\%) \qquad c\% = 1 - \alpha$$

- Example 1 … 8 data points … *ν=7, 95% confidence level*
- $\alpha = 1 - 0.95 = 0.05$-->

$\alpha/2 = 0.025$
$1 - \alpha/2 = 0.975$

χ2 Probability density



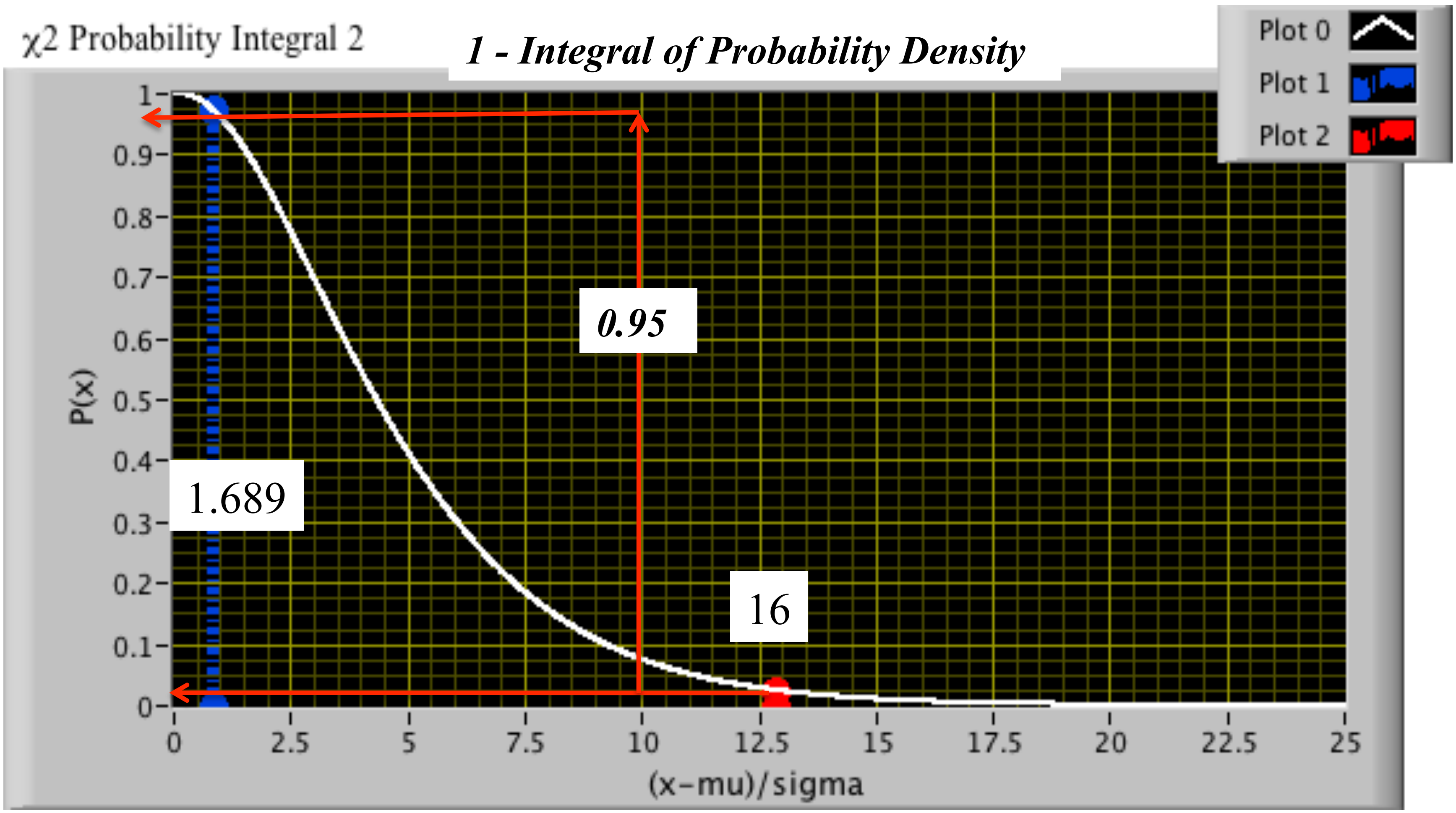
$\chi 2\,(1-\alpha/2) = 1.689$

$\chi 2\,(\alpha/2) = 16$

$$\frac{7S_x^2}{16} < \sigma^2 < \frac{7S_x^2}{1.689} .....(95\%)$$

χ^2 Significance testing (4)

$$95\%\ldots conf. \rightarrow \frac{(n-1)S_x^2}{\chi^2_{0.025}} \le \sigma_x^2 \le \frac{(n-1)S_x^2}{\chi^2_{0.975}}$$



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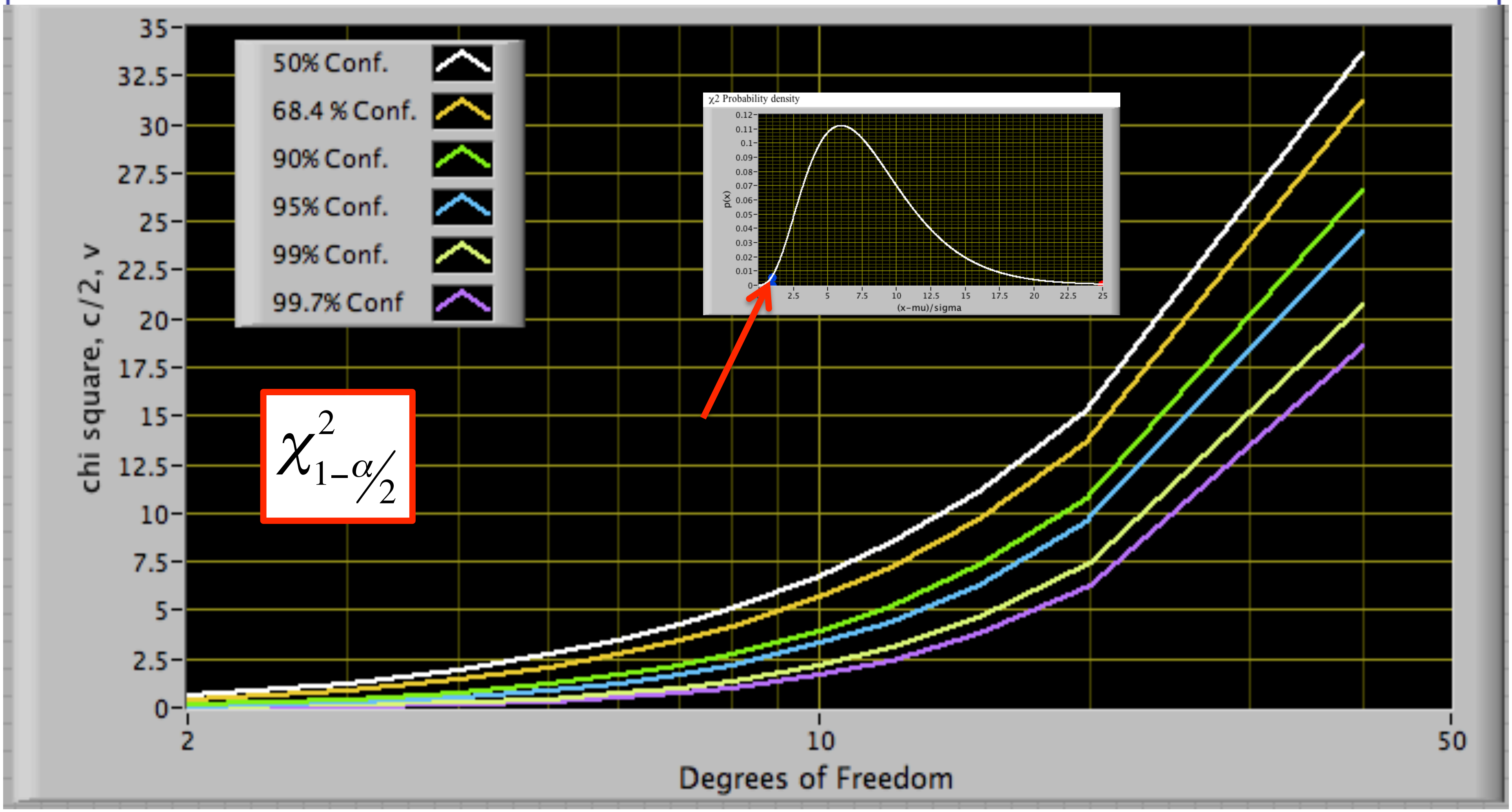
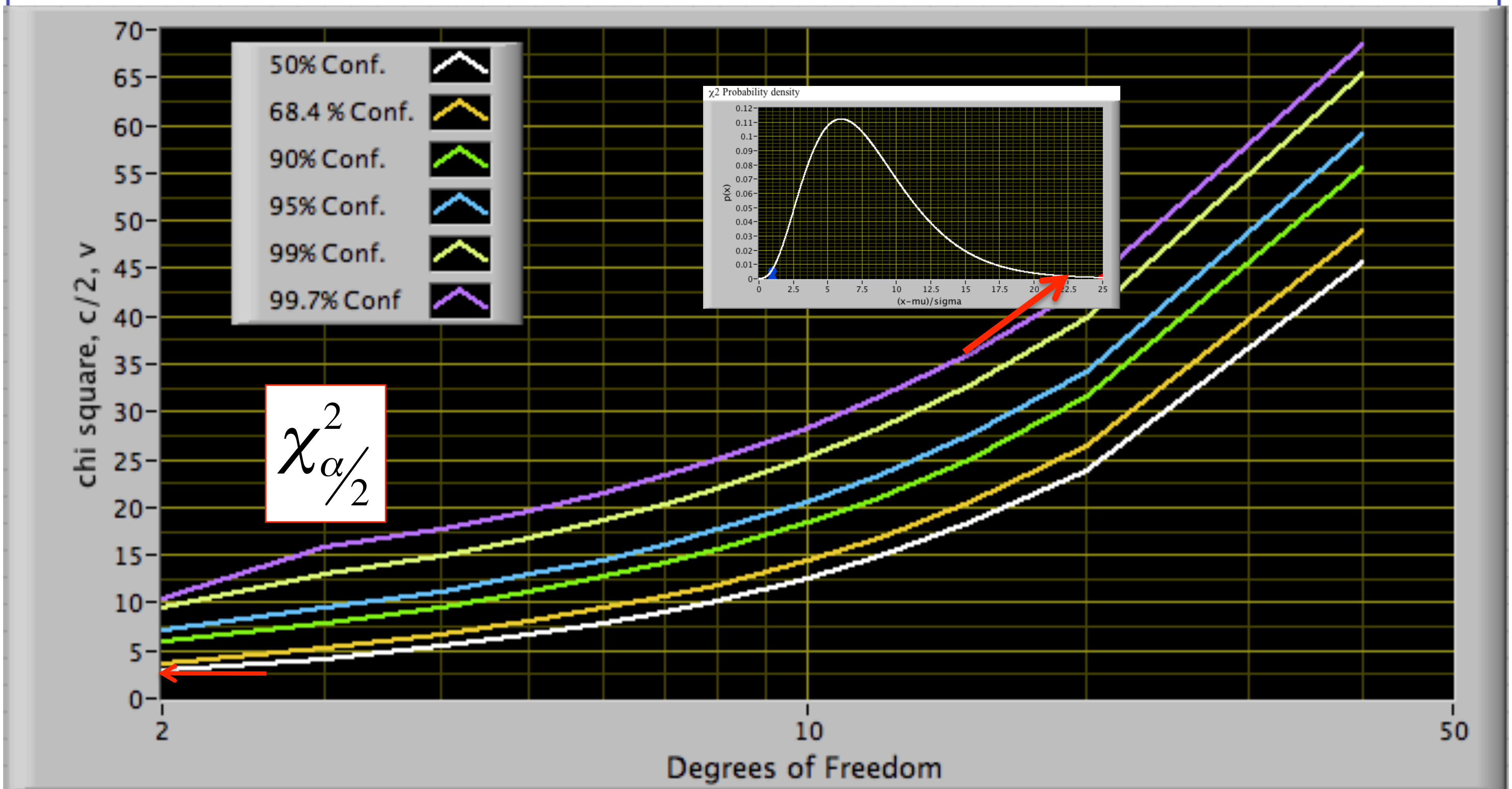
χ^2 Significance testing (5)

$$\frac{(n-1)S_x^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2}} \text{.....}(c\%) \quad c\% = 1 - \alpha$$



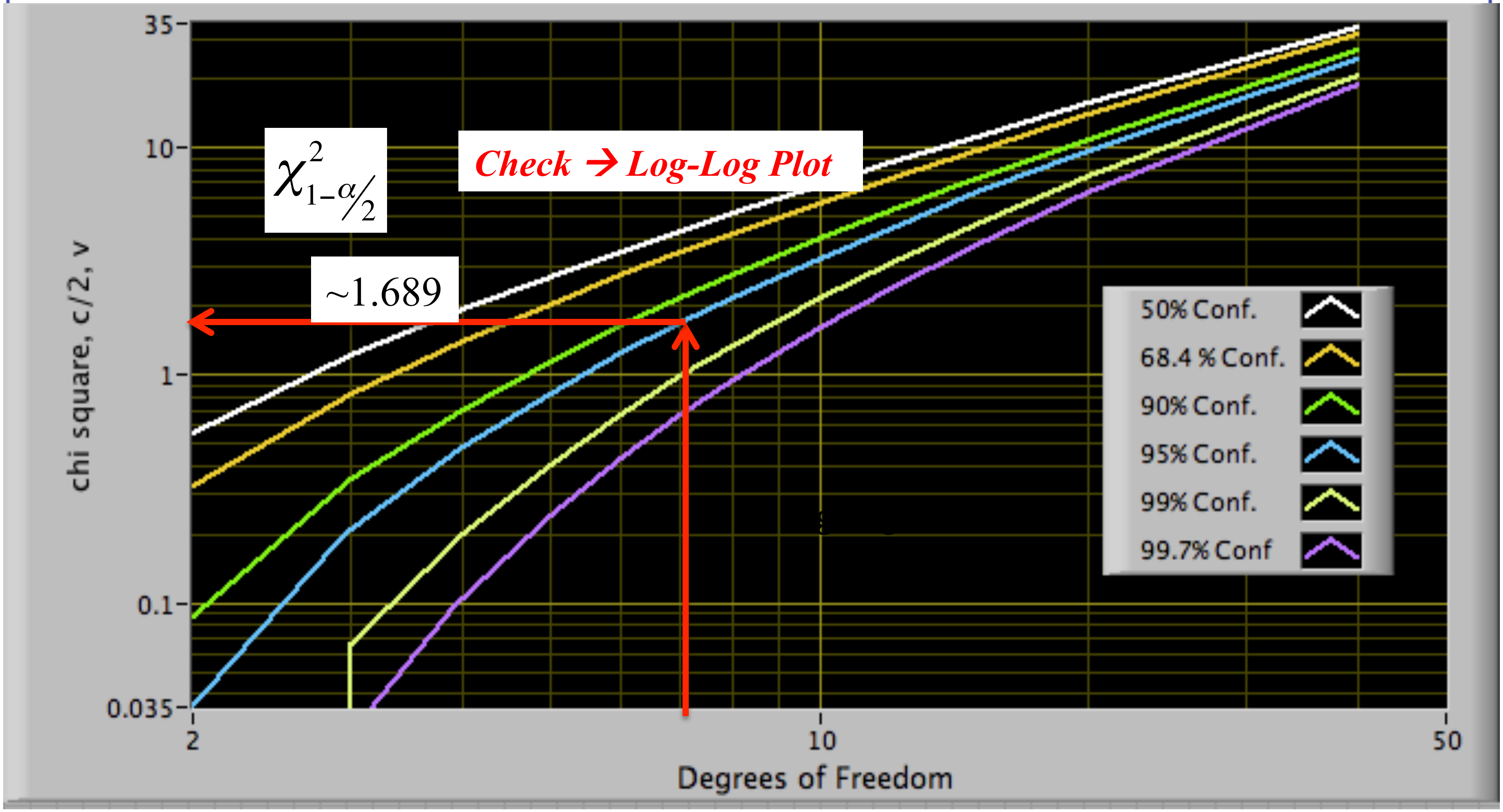
χ^2 Significance testing (6)

$$\frac{(n-1)S_x^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2}} \ldots\ldots(c\%) \qquad c\% = 1 - \alpha$$



χ^2 Significance testing (7)

$$\frac{(n-1)S_x^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2}} \ldots\ldots(c\%) \qquad c\% = 1 - \alpha$$



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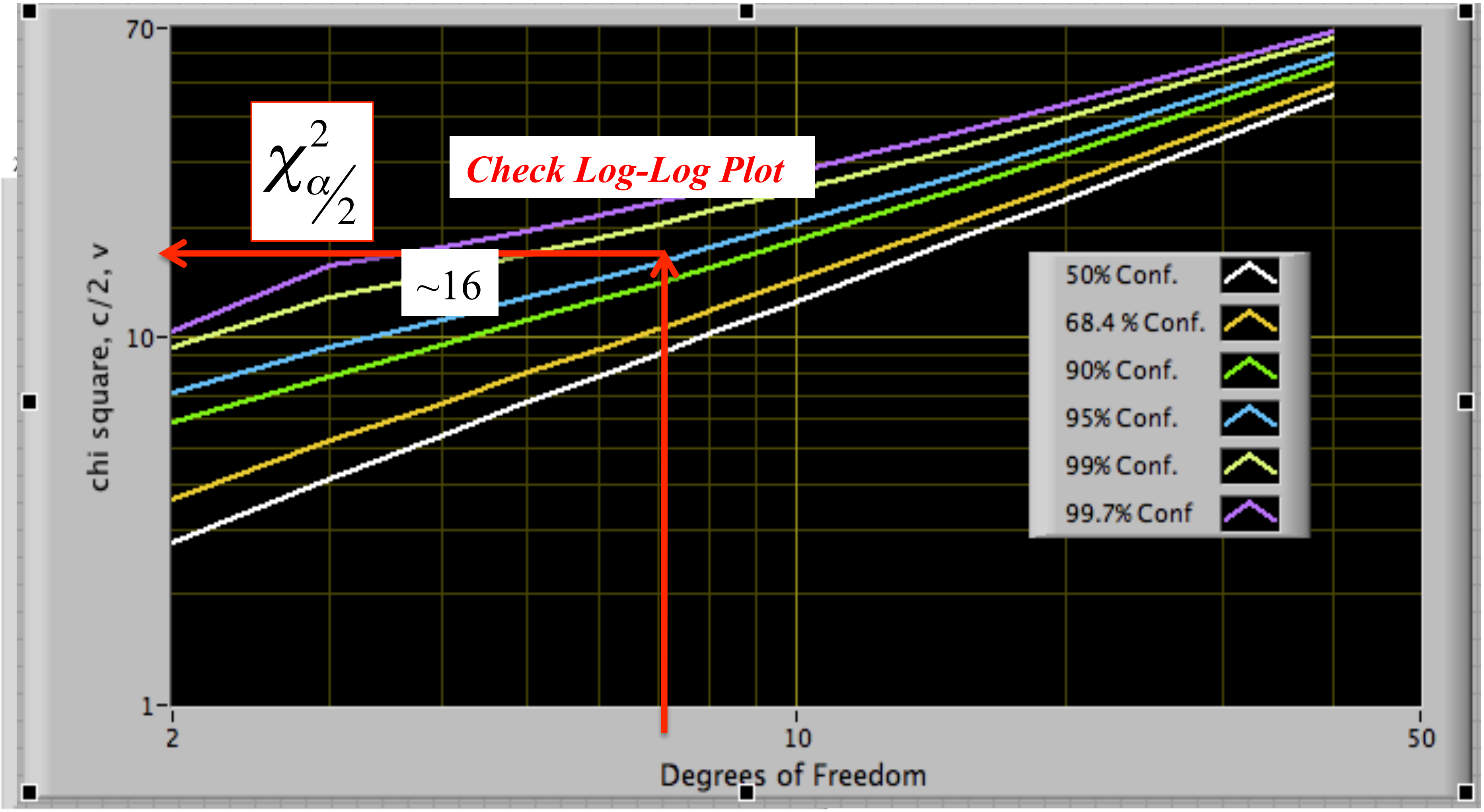
χ^2 Significance testing (8)

$$\frac{(n-1)S_x^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2}} \;.....(c\%) \qquad c\% = 1 - \alpha$$



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χ^2 Significance testing ~~(3)~~

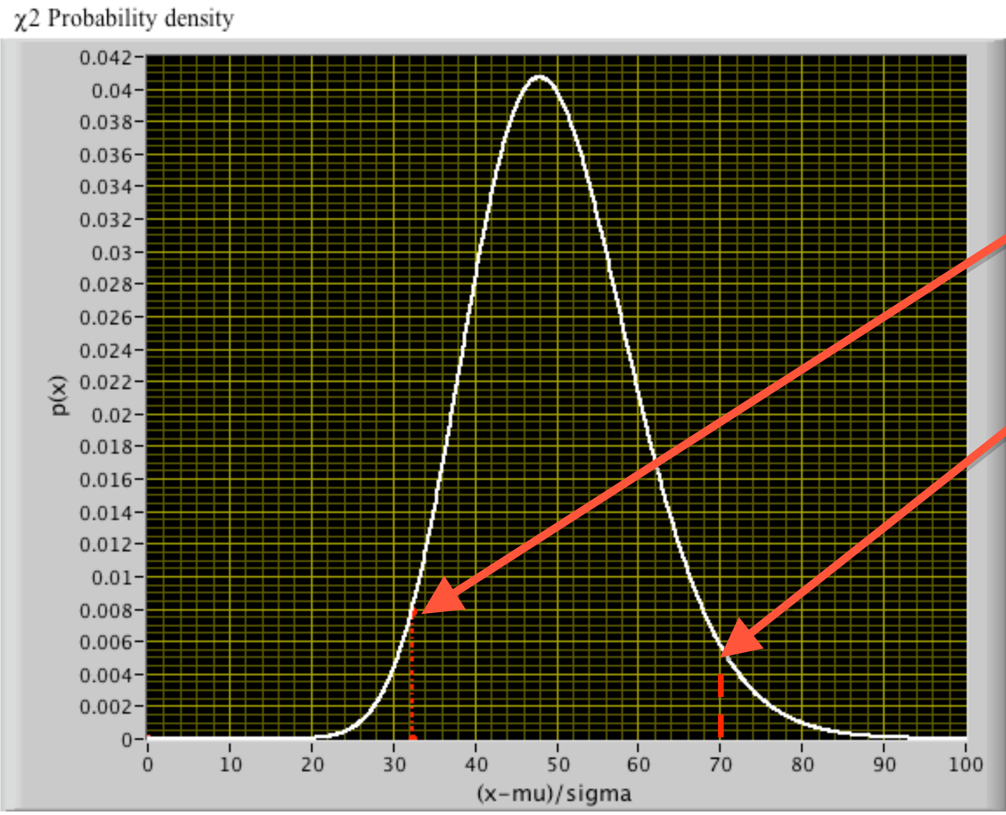
$$\frac{(n-1)S_x^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2}} \text{.....}(c\%) \qquad c\% = 1 - \alpha$$

- Example 2 … 51 data points … *ν=50, 95% confidence level*
- $\alpha = 1\text{-}0.95 = 0.05\text{-->}$

$\alpha/2 = 0.025$
$1\text{-}\alpha/2 = 0.975$

$\chi 2\ (1\text{-}\alpha/2) = 32.33$

$\chi 2\ (\alpha/2) = 70.75$



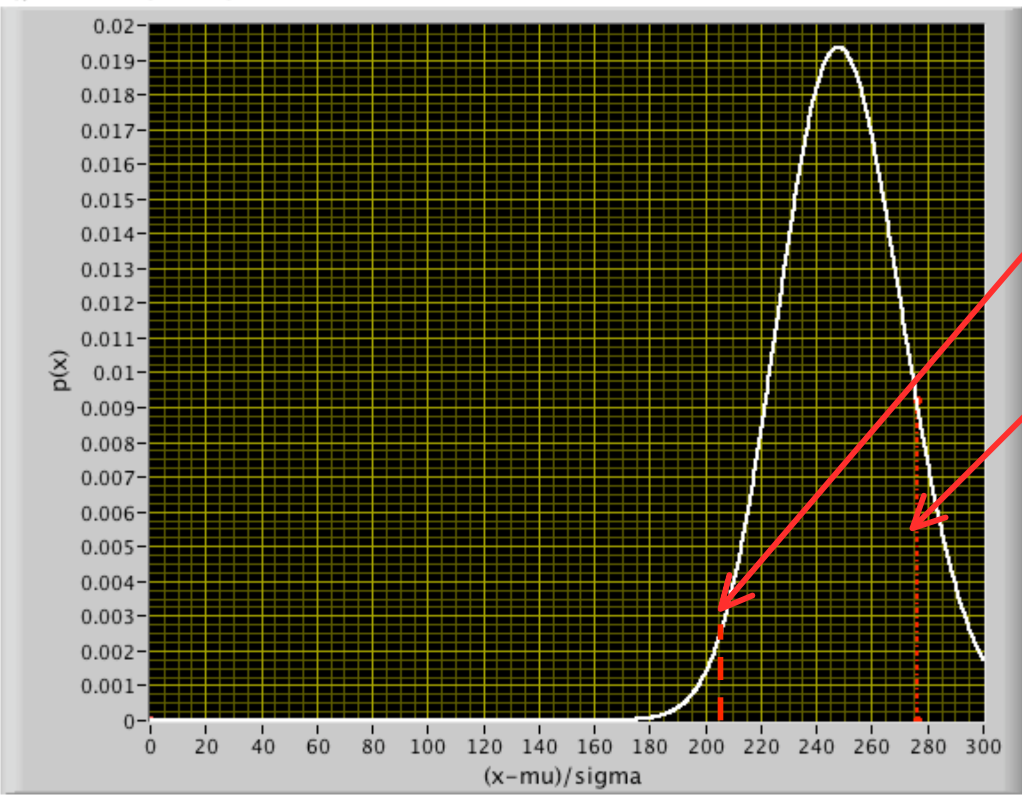
$$\frac{50S_x^2}{70.75} < \sigma^2 < \frac{50S_x^2}{32.33} \text{.....}(95\%)$$

χ^2 Significance testing ~~(4)~~ 9

$$\frac{(n-1)S_x^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S_x^2}{\chi^2_{1-\alpha/2}} \ldots\ldots(c\%) \qquad c\% = 1 - \alpha$$

- Example 2.. 251 data points … *ν=250, 95% confidence level*
- $\alpha = 1$ 0.95 = 0.05 →

χ2 Probability density



$\alpha/2 = 0.025$
$1-\alpha/2 = 0.975$

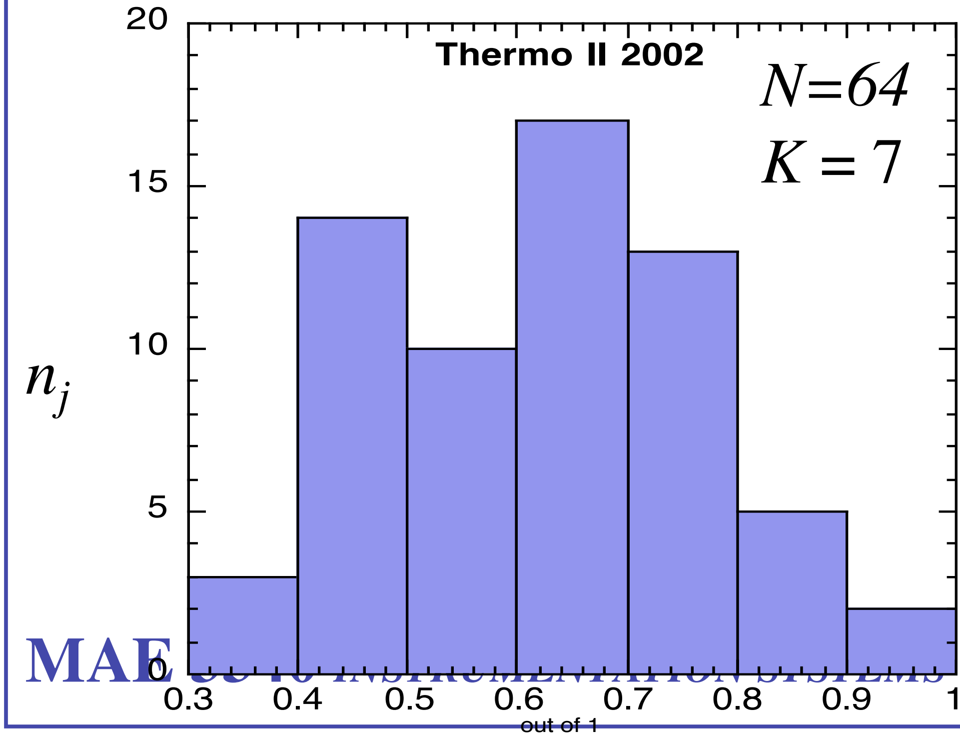
$\chi 2\ (1-\alpha/2) = 207.35$

$\chi 2\ (\alpha/2) = 276.2$

$$\frac{250 S_x^2}{276.2} < \sigma^2 < \frac{250 S_x^2}{207.25} \ldots\ldots(95\%)$$

Other uses for χ2 distribution

We can use Chi-squared to estimate our confidence in our estimate of the standard deviation S_x. However, there is seldom much call for this. A more useful application of Chi-squared is to check our assumption that the data we are dealing with fits a certain distribution. We are going to assuming in this class that our data fits a normal (gaussian) distribution. If we have a set of data and we want to make sure this is a good fit, we use this test.

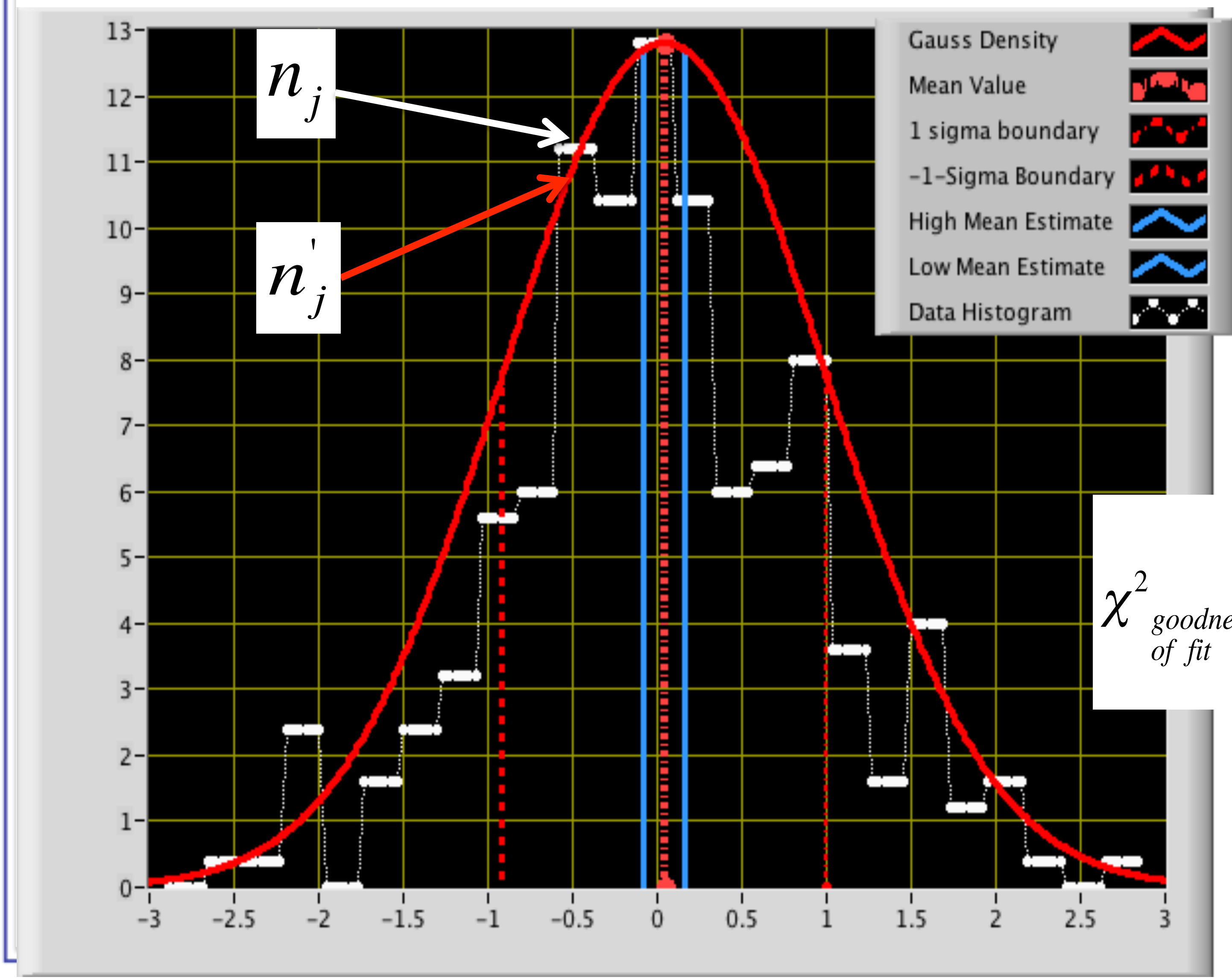


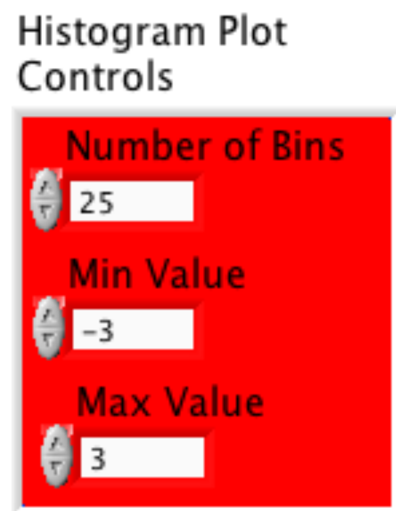
$$\chi^2 = \sum_j \frac{\left(n_j - n'_j\right)^2}{n'_j}$$

$j = 1, 2, \ldots K$

See example 4.7, Pages 138-139 in your text book

χ2 Goodness of Fit Test on Experimental Set





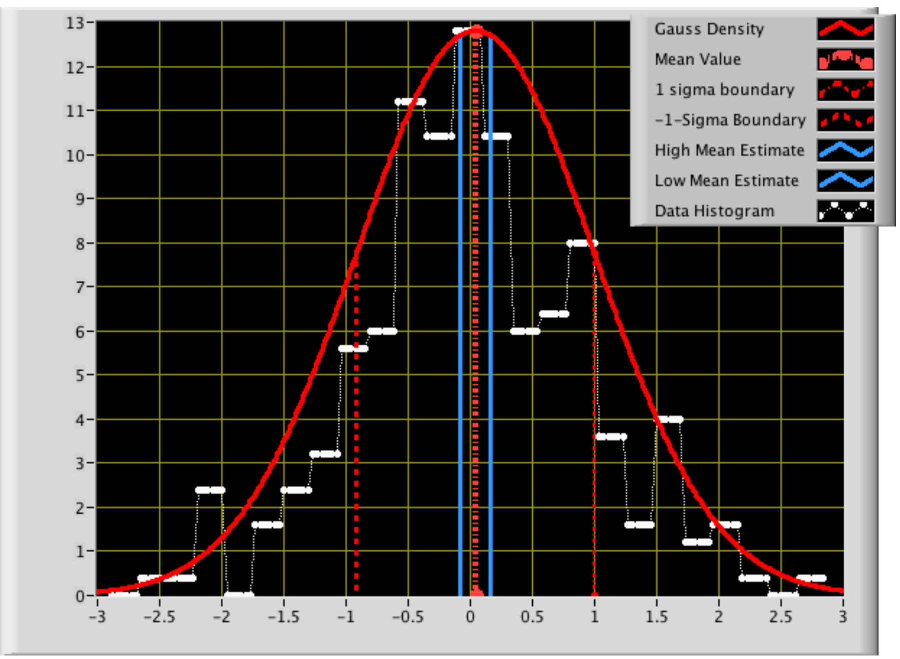
$$\chi^2_{\substack{goodness\\ of\ fit}} = \sum_{j=1}^{N_{bins}} \left[\frac{\left(n_j - n'_j\right)^2}{n'_j} \right]$$

$\chi2$ Goodness of Fit Test on Experimental Set (2)

$$\chi^2_{\substack{goodness\\ of\ fit}} = \sum_{j=1}^{N_{bins}} \left[\frac{\left(n_j - n'_j\right)^2}{n'_j} \right]$$

$$n_j \rightarrow histogram\ results\ at\ center\ of\ bins\ ...\{1, ...N_{bins}\}$$

$$n'_j \rightarrow gauss\ p(x)\ calculated\ at\ center\ bins\ value$$



Histogram Plot Controls

Number of Bins: 25

Min Value: -3

Max Value: 3

Bin Center Values

0

-2.88
-2.64
-2.4
-2.16
-1.92
-1.68
-1.44
-1.2
-0.96
-0.72
-0.48
-0.24
-1.1102
0.24
0.48
0.72
0.96
1.2
1.44
1.68
1.92
2.16
2.4
2.64

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Bin Center Values	Histogram % Bin Value { nj}	Gauss % Bin Value {n'j}
0	0	0
-2.88	0	0.11937
-2.64	0.4	0.24934
-2.4	0.4	0.48894
-2.16	2.4	0.90016
-1.92	0	1.5559
-1.68	1.6	2.52486
-1.44	2.4	3.84674
-1.2	3.2	5.50237
-0.96	5.6	7.38933
-0.72	6	9.31656
-0.48	11.2	11.0282
-0.24	10.4	12.256
-1.1102	12.8	12.7878
0.24	10.4	12.5267
0.48	6	11.5207
0.72	6.4	9.94756
0.96	8	8.064
1.2	3.6	6.13739
1.44	1.6	4.38546
1.68	4	2.94202
1.92	1.2	1.85299
2.16	1.3	1.09571
2.4	0.2	0.60828
2.64	0.3	0.31704

Histogram Plot Controls

Number of Bins: 25

Min Value: -3

Max Value: 3

$$\chi^2_{\substack{goodness\\of\ fit}} = \sum_{j=1}^{N_{bins}} \left[\frac{\left(n_j - n'_j\right)^2}{n'_j} \right]$$

$n_j \rightarrow$ *histogram results at center of bins ...*$\{1, ... N_{bins}\}$

$n'_j \rightarrow$ *gauss* $p(x)$ *calculated at center bins value*

out of 1

Bin Center Values	Histogram % Bin Value { nj}	Gauss % Bin Value {n'j}
0	0	0
-2.88	0	0.11937
-2.64	0.4	0.24934
-2.4	0.4	0.48894
-2.16	2.4	0.90016
-1.92	0	1.5559
-1.68	1.6	2.52486
-1.44	2.4	3.84674
-1.2	3.2	5.50237
-0.96	5.6	7.38933
-0.72	6	9.31656
-0.48	11.2	11.0282
-0.24	10.4	12.256
-1.1102	12.8	12.7878
0.24	10.4	12.5267
0.48	6	11.5207
0.72	6.4	9.94756
0.96	8	8.064
1.2	3.6	6.13739
1.44	1.6	4.38546
1.68	4	2.94202
1.92	1.2	1.85299
2.16	1.3	1.09571
2.4	0.2	0.60828
2.64	0.3	0.31704

out of 1

$$\chi^2_{goodness\ of\ fit} = \sum_{j=1}^{N_{bins}} \left[\frac{\left(n_j - n'_j\right)^2}{n'_j} \right]$$

$$= 16.1947$$

$$\text{DOF} = N_{bins} - 2 = 23$$

Histogram bins are tied together by estimates of

$$\bar{x}_{(\mu)} \ldots and \ldots S_{\bar{x}\left(\frac{\sigma_x}{\sqrt{n}}\right)}$$

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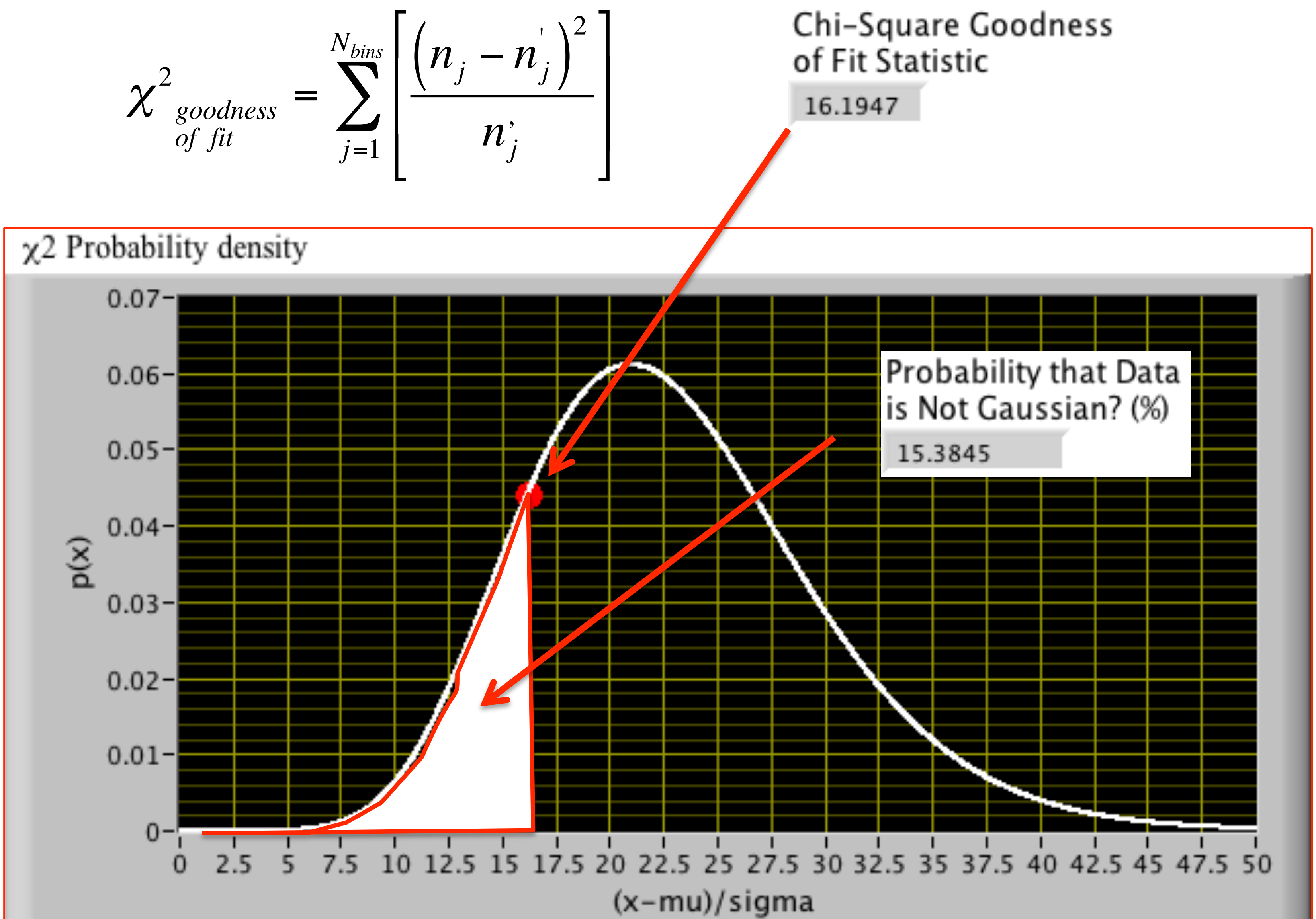
χ2 Goodness of Fit Test on Experimental Set (3)

$$\chi^2_{\substack{goodness \\ of\ fit}} = \sum_{j=1}^{N_{bins}} \left[\frac{\left(n_j - n'_j\right)^2}{n'_j} \right]$$

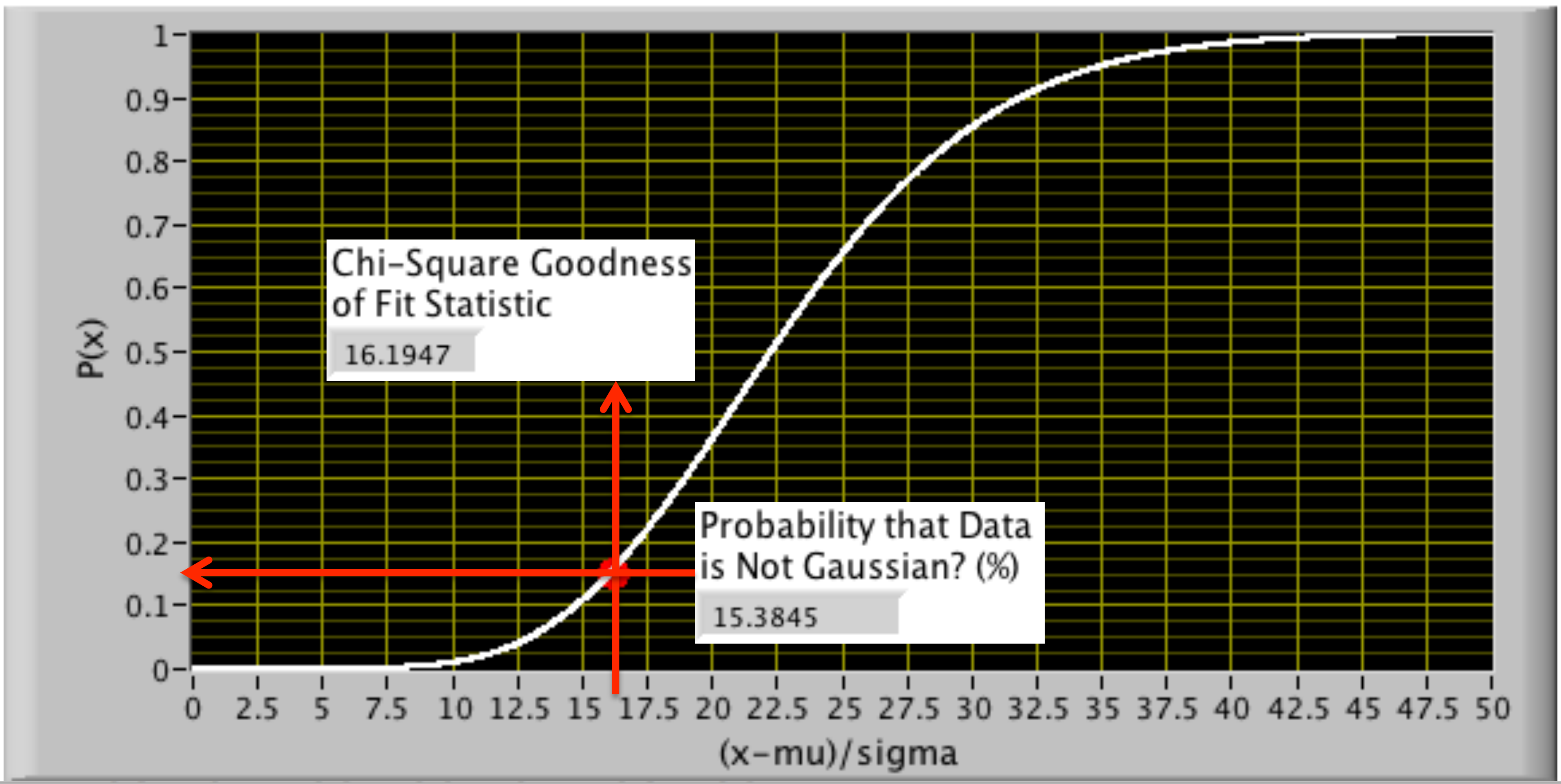


out of 1

χ2 Goodness of Fit Test on Experimental Set (4)

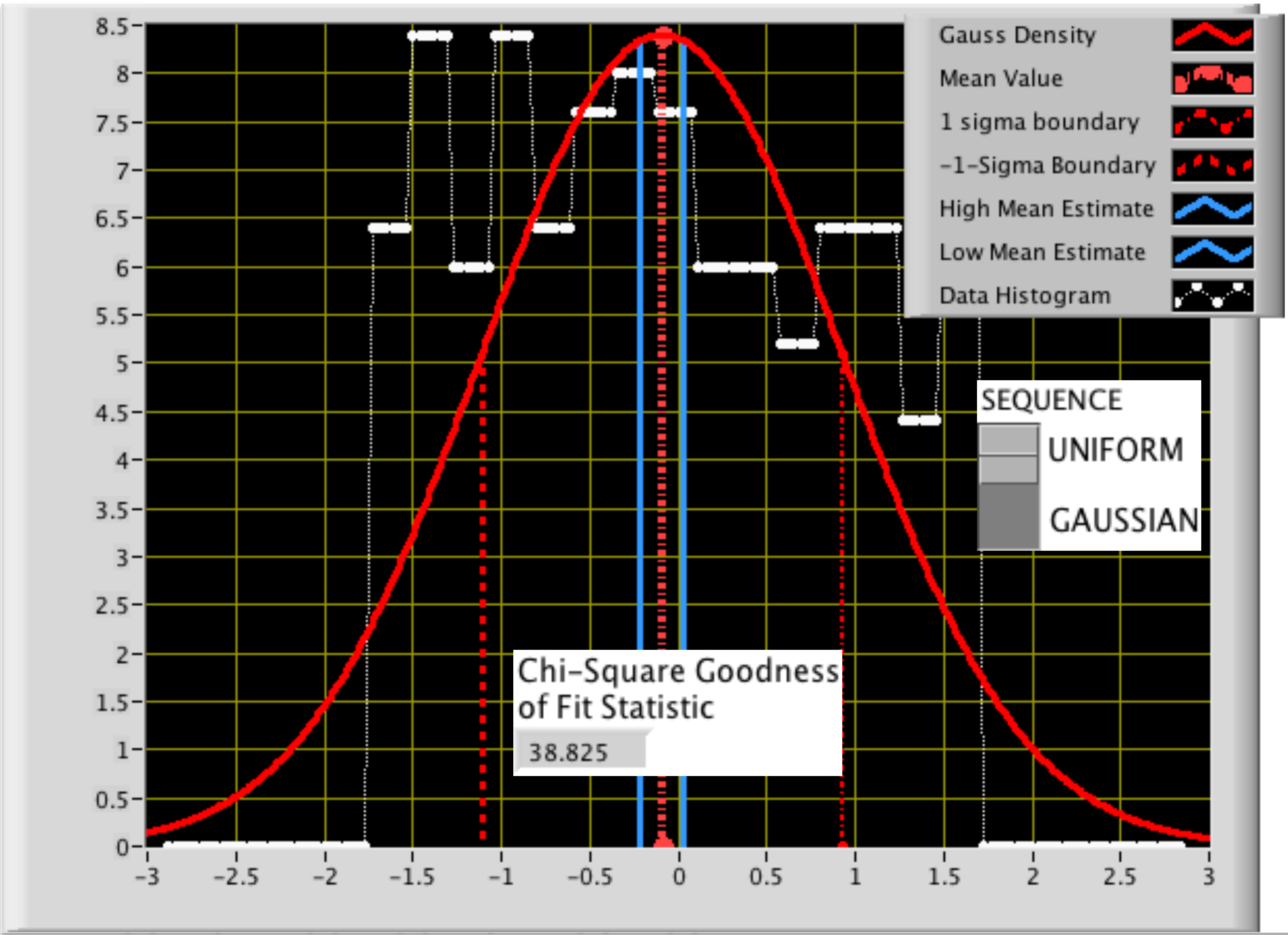
$$\chi^2_{\substack{goodness\\ of\ fit}} = \sum_{j=1}^{N_{bins}} \left[\frac{\left(n_j - n'_j\right)^2}{n'_j} \right]$$

χ2 Probability Distribution



out of 1

χ2 Goodness of Fit Test on Experimental Set (5)

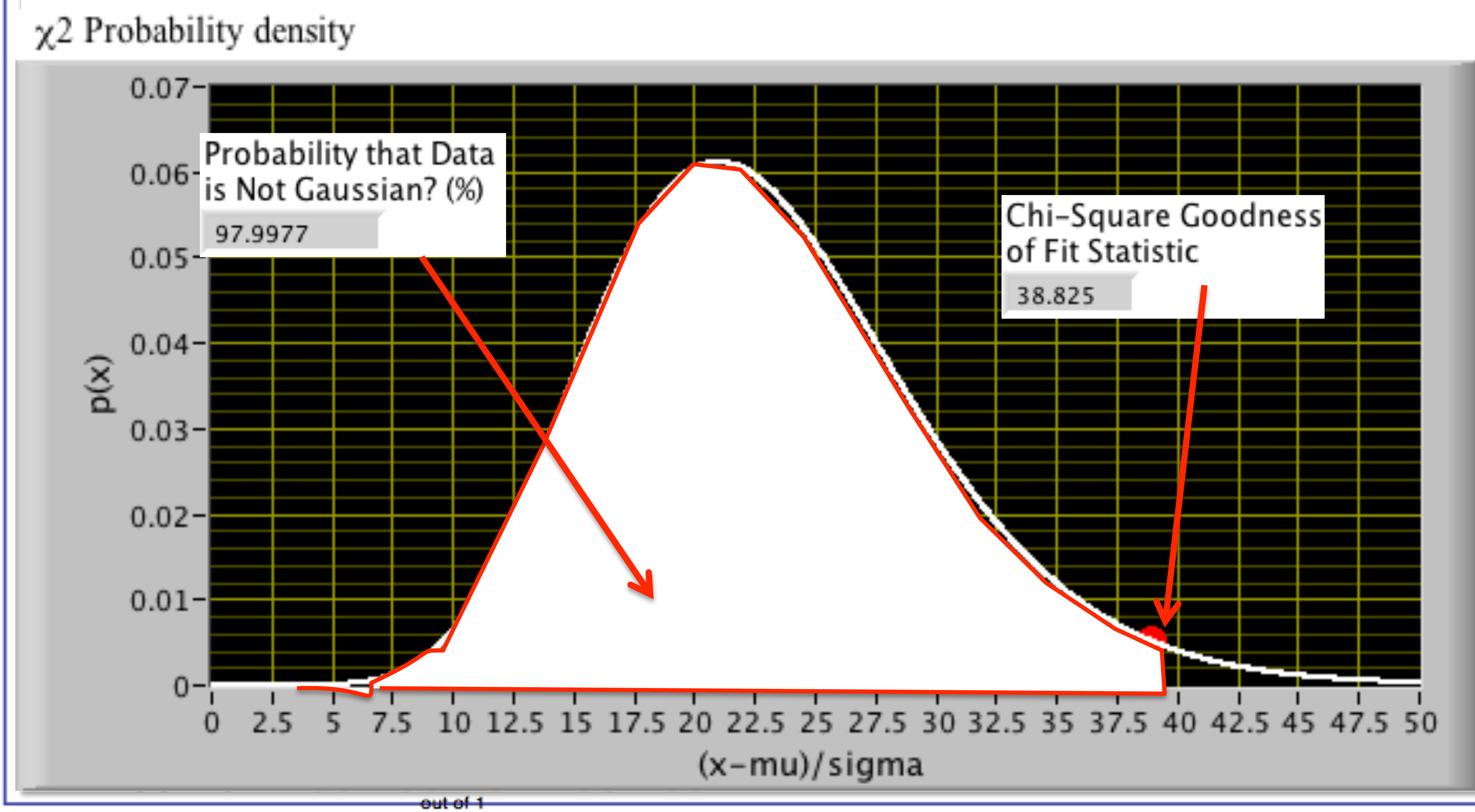


Uniformly Distributed Sequence Example

out of 1

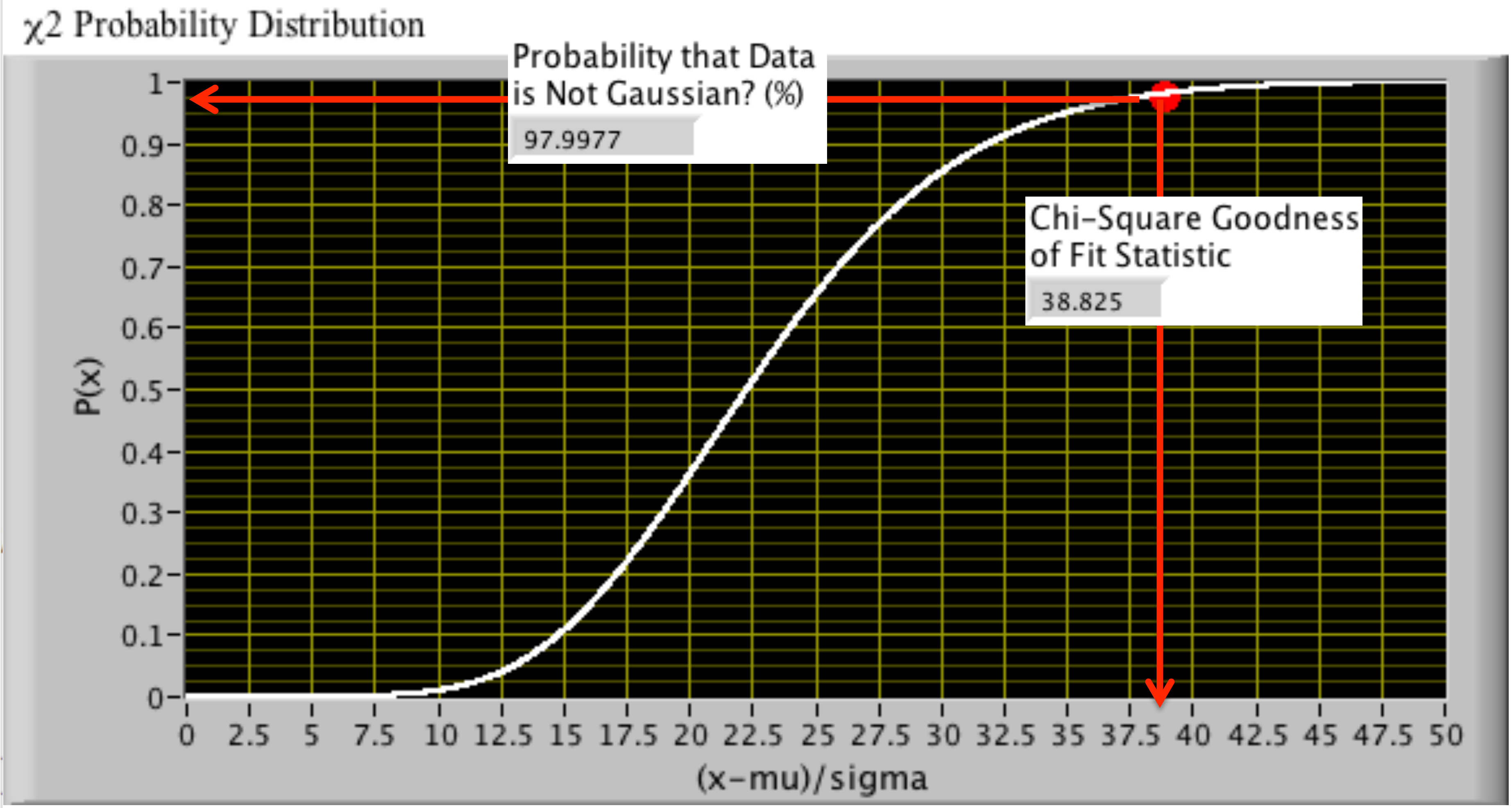
χ2 Goodness of Fit Test on Experimental Set (6)

Uniformly Distributed Sequence



χ2 Goodness of Fit Test on Experimental Set (6)

Uniformly Distributed Sequence



out of 1