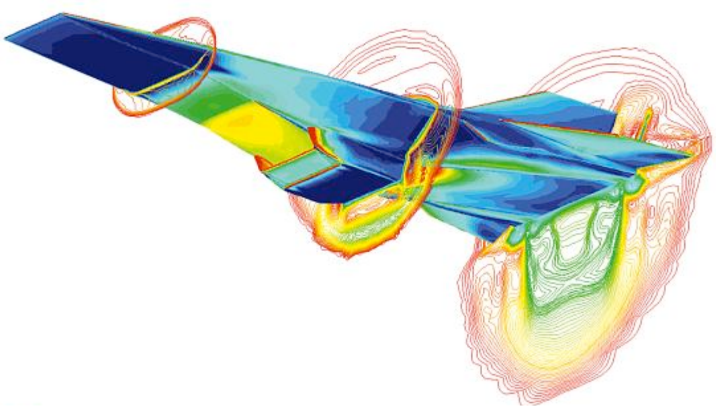
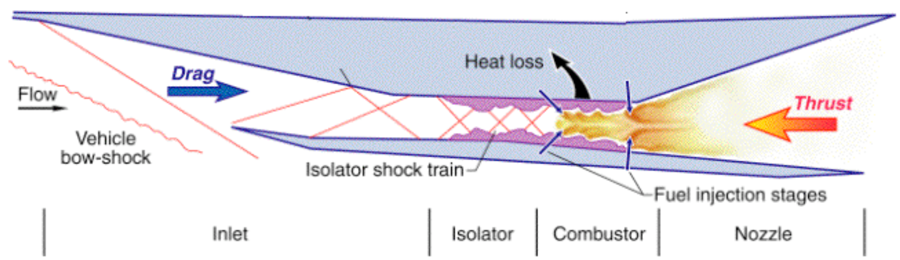
Section 12 Lecture 2: Reformulation of Normal Shock Equations for Hypersonic Flow





Anderson: Chapter 16 pp. 610-613, Chapter 3 pp. 102-111
Chapter 17 pp. 648-658

Flow Across a Hypersonic Shock Wave

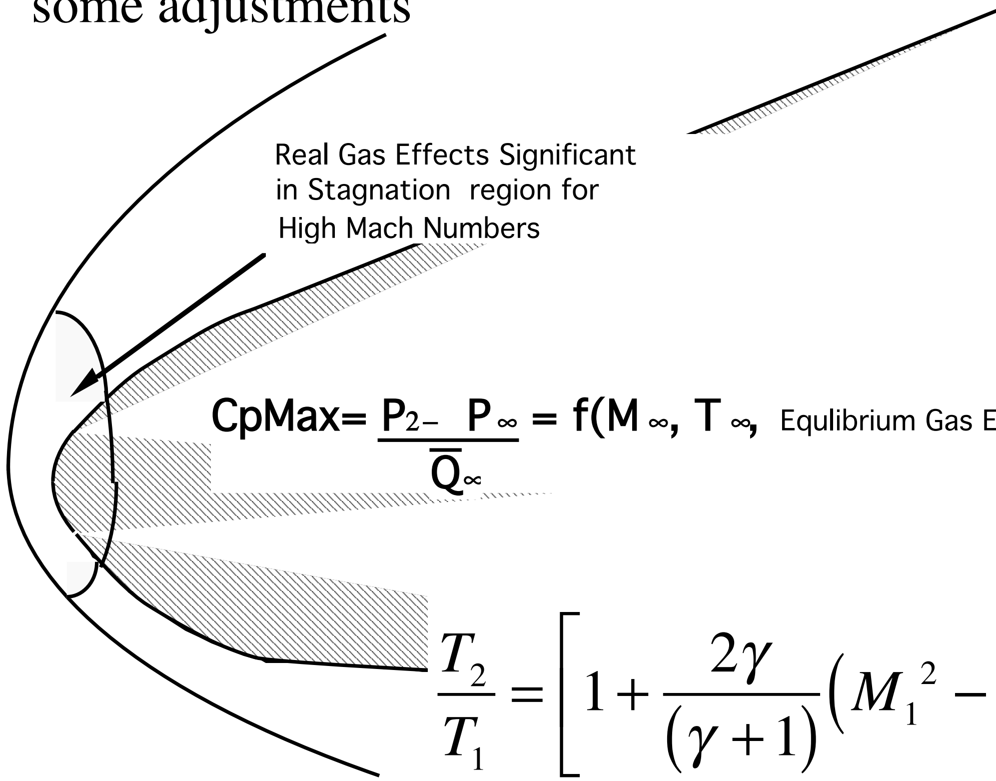
- Across a Hypersonic Shock Wave, Temperature Rises Dramatically

- Stagnation Point Properties can be modeled by Normal Shock Wave equations … with "some adjustments"

P_∞, T_∞

Q_∞, Pt_∞,

$M_\infty >> 1$



$$Cp_{Max} = \frac{P_2 - P_\infty}{\overline{Q}_\infty} = f(M_\infty, T_\infty, \text{Equlibrium Gas Energy, State Equations})$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\,2} - 1\right)\right]\left[\frac{\left(2 + (\gamma-1)M_1^{\,2}\right)}{(\gamma+1)M_1^{\,2}}\right]$$

Flow Across a Hypersonic Shock Wave

- Across a Hypersonic Shock Wave, Temperature Rises Dramatically

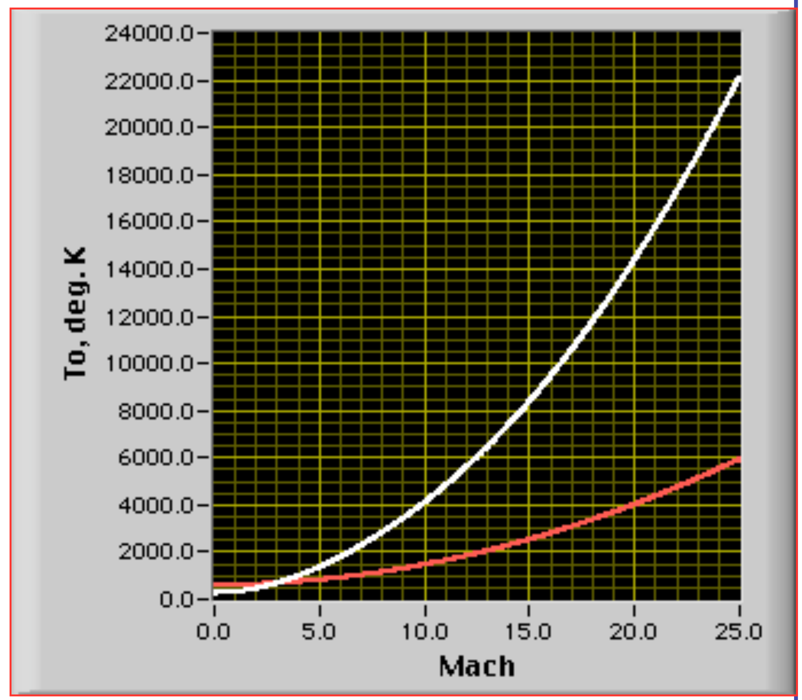
- Thermal Properties (c_p, c_v, γ) of Gas Change

- T_0 not constant across shock

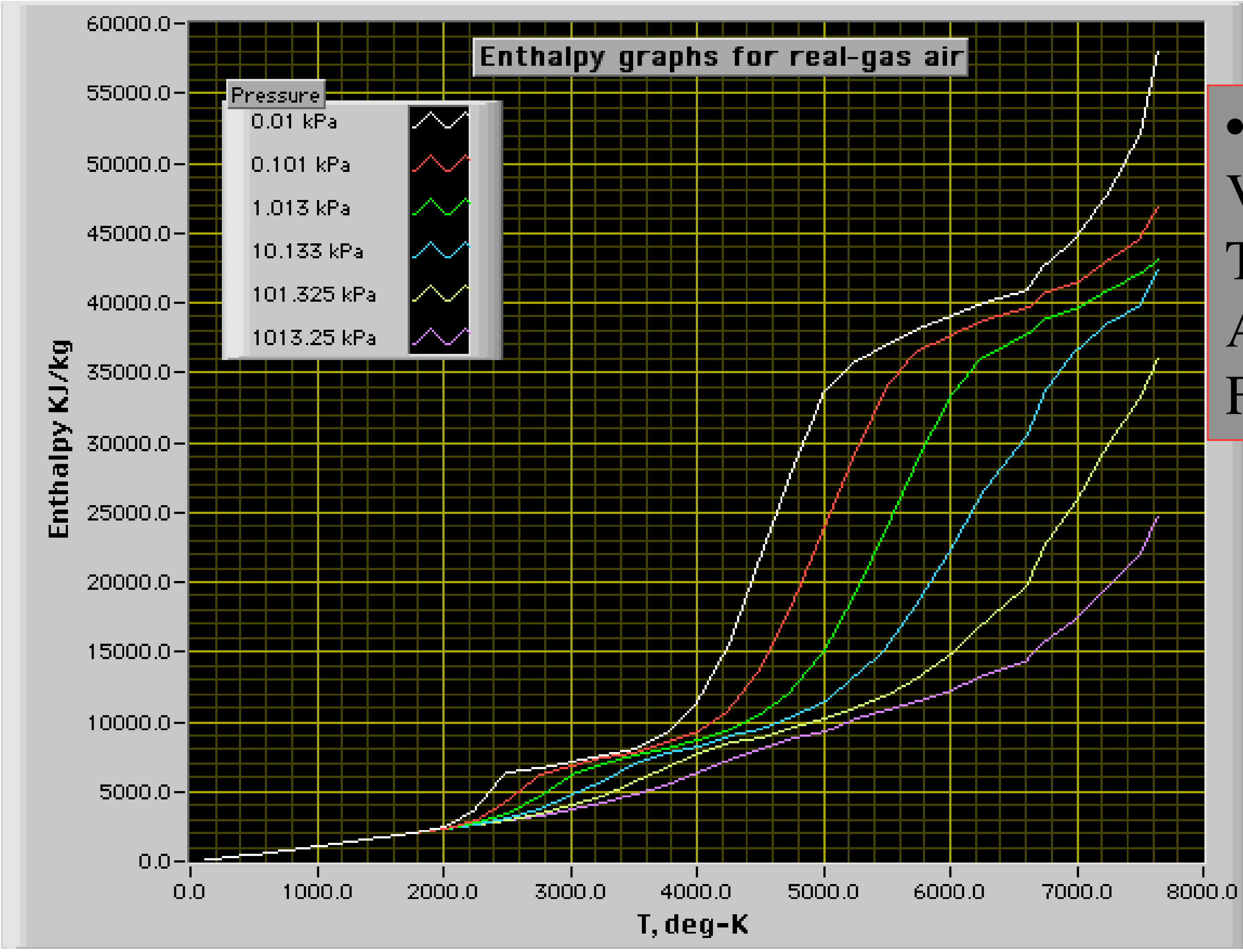
- Gas Dissociation, chemical reaction, molecular Vibration Significantly lower the Stagnation temperature Behind the shock wave when compared To "calorically perfect" gas

- In general Enthalpy is implicit function of Pressure and temperature

$$h = \eta(T, P) \Rightarrow \text{"}non-analytical-function\text{"}$$



Flow Across a Hypersonic Shock Wave



- Enthalpy Versus Temperature And pressure For air

Flow Across a Hypersonic Shock Wave (cont'd)

- For Non-ideal gas

$$\frac{\partial h}{\partial T} = c_p \Rightarrow c_p \neq const$$

$$\frac{\partial e}{\partial T} = c_v \Rightarrow c_v \neq const$$

1-D energy Equation

$$\dot{q} = 0$$

$$h_0 = h_1 + \frac{V_1^{\,2}}{2} = h_2 + \frac{V_2^{\,2}}{2}$$

- Cannot directly integrate to get temperature,

However enthalpy *Is constant across shock … so let's start there*

Flow Across a Hypersonic Shock Wave (cont'd)

- Enthalpy is constant across shock wave

$$\dot{q} = 0 \Rightarrow h_0 = h + \frac{V^2}{2} = const$$

- Take Differential of Enthalpy Equation

$$\partial h + \partial\left[\frac{V^2}{2}\right] = 0 = \partial h + V\partial V$$

Flow Across a Hypersonic Shock Wave (cont'd)

- Plug in definition of specific heat

$$0 = \frac{\partial h}{\partial T} = c_p(T) \Rightarrow \partial h = c_p(T)\partial T \Downarrow$$

$$c_p(T)\partial T + V\partial V = 0$$

- *integrate from state 1 to state 2*

$$\int_{T_1}^{T_2} c_p(T)\partial T + \int_{V_1}^{V_2} V\partial V = 0 \Downarrow$$

$$\int_{T_1}^{T_2} c_p(T)\partial T + \frac{V_2^{\ 2} - V_1^{\ 2}}{2} = 0$$

Flow Across a Hypersonic Shock Wave (cont'd)

- Look at integral

$$\int_{T_1}^{T_2} c_p(T)\partial T$$

- c_p is a complex function of temperature .. involves complex Chemistry, with reacting ionized gases …..beyond scope of this Class … but we can approximate solution for "single species" Non-reacting flows

How Does Specific heat vary with Temperature?

- Based on Theories of Statistical Thermodynamics for for Single Species gas (non-dissociated, non-reacting)

Rotational energy

Ionization/ Dissociation energy

$$e = \frac{3}{2} R_g T + R_g T + \frac{\frac{h\nu}{\kappa T}}{\left\{ e^{\frac{h\nu}{\kappa T}} - 1 \right\}} R_g T + e_{electronic}$$

Translational energy

Vibrational energy

$h \to Plank.Const == 6.6262 \times 10^{-34}{}_{j-\sec}$

$k \to Boltzmann.Const = 1.3807 \times 10^{-23} \frac{j}{{}^oK}$

$\nu \to \mathrm{Pr}\,incipal.Vibration.Mode = 8.335 \times 10^{13}\,hertz$

$\frac{h\nu}{\kappa} \equiv T_r \Rightarrow "quantum.reference.temperature"$

$\simeq$ 4000 °K for air

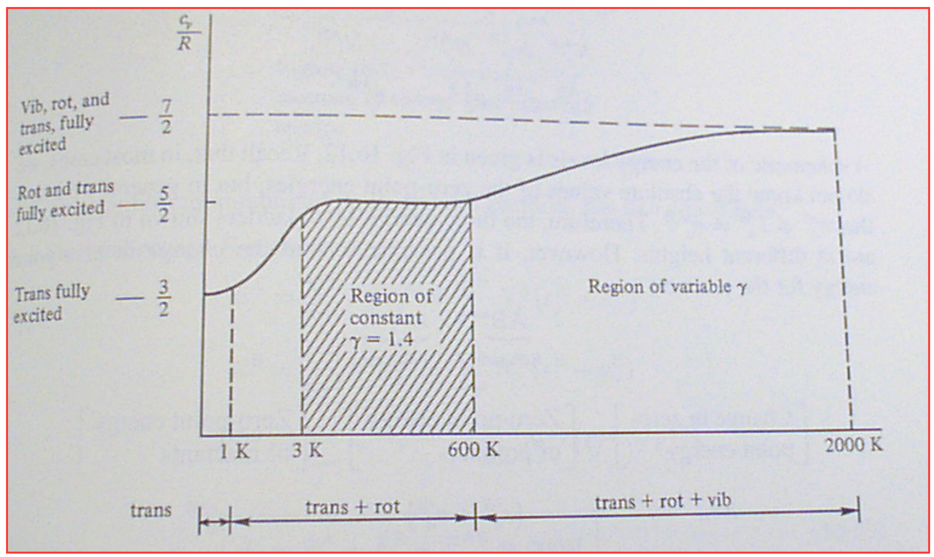
How Does Specific heat vary with Temperature?

(cont'd)

$$\frac{\partial e}{\partial T} = c_v$$

$$\frac{\partial e}{\partial T} = \frac{\partial}{\partial T}\left[\frac{3}{2}R_g T + R_g T + \frac{\frac{T_r}{T}}{\left\{e^{\frac{T_r}{T}} - 1\right\}} R_g T + e_{electronic}\right] = \frac{3}{2}R_g + R_g + \frac{T_r e^{\frac{T_q}{T}}\left(-\frac{T_r}{T^2}\right)}{-\left\{e^{\frac{T_r}{T}} - 1\right\}^2} R_g =$$

$$\frac{3}{2}R_g + R_g + \frac{e^{\frac{T_r}{T}}\left(\frac{T_r}{T}\right)^2}{\left\{e^{\frac{T_r}{T}} - 1\right\}^2} R_g = c_v$$



Write Model in terms of c_p

$$c_p = R_g + c_v = R_g \left[1 + \frac{5}{2} + \left(\frac{T_r}{T} \right)^2 \frac{e^{\frac{T_r}{T}}}{\left\{ e^{\frac{T_r}{T}} - 1 \right\}^2} \right] =$$

$$R_g \left[\frac{7}{2} + \left(\frac{T_r}{T} \right)^2 \frac{e^{\frac{T_r}{T}}}{\left\{ e^{\frac{T_r}{T}} - 1 \right\}^2} \right]$$

Evaluate Integral

$$\int_{T_1}^{T_2} c_p(T)\partial T = \int_{T_1}^{T_2} R_g \left[\frac{7}{2} + \left(\frac{T_r}{T} \right)^2 \frac{e^{\frac{T_r}{T}}}{\left\{ e^{\frac{T_r}{T}} - 1 \right\}^2} \right] dT$$

$$c_p = R_g \left[\frac{7}{2} + \left(\frac{T_r}{T} \right)^2 \frac{e^{\frac{T_r}{T}}}{\left\{ e^{\frac{T_r}{T}} - 1 \right\}^2} \right]$$

Evaluate Integral

(cont'd)

- Substituting in for c_p and integrating the "easy parts"

$$\int_{T_1}^{T_2} R_g \left[\frac{7}{2} + \left(\frac{T_r}{T}\right)^2 \frac{e^{\frac{T_r}{T}}}{\left\{ e^{\frac{T_r}{T}} - 1 \right\}^2} \right] dT =$$

$$\left(\frac{7}{2} R_g T_2 - \frac{7}{2} R_g T_1 \right) + R_g \int_{T_1}^{T_2} \left[\left(\frac{T_r}{T}\right)^2 \frac{e^{\frac{T_r}{T}}}{\left\{ e^{\frac{T_r}{T_o}} - 1 \right\}^2} \right] dT$$

Evaluate Integral

(cont'd)

- Now integrate the "hard part"

$$R_g \int_{T_1}^{T_2} \left[\left(\frac{T_r}{T} \right)^2 \frac{e^{\frac{T_r}{T}}}{\left\{ e^{\frac{T_r}{T}} - 1 \right\}^2} \right] dT \Rightarrow u = \frac{T_r}{T} \rightarrow du = -\frac{T_r}{T^2} dT$$

$$\Rightarrow dT = -du \frac{T^2}{T_r} = -du \left(\frac{T}{T_r} \right)^2 T_r = -\frac{du}{u^2} T_r$$

Evaluate Integral

(cont'd)

- Substitute in for "u" and "dT" and Integrating

$$R_g \int_{T_1}^{T_2} \left[\left(\frac{T_r}{T} \right)^2 \frac{e^{\frac{T_r}{T}}}{\left\{ e^{\frac{T_r}{T}} - 1 \right\}^2} \right] dT_0 = R_g \int_{\frac{T_r}{T_1}}^{\frac{T_r}{T_2}} \left[u^2 \frac{e^u}{\left\{ e^u - 1 \right\}^2} \right] \left(-\frac{du}{u^2} T_r \right) =$$

$$-R_g T_r \int_{\frac{T_r}{T_1}}^{\frac{T_r}{T_2}} \left[\frac{e^u}{\left\{ e^u - 1 \right\}^2} \right] du = R_g T_r \left[\frac{1}{\left\{ e^{\frac{T_r}{T_{02}}} - 1 \right\}} - \frac{1}{\left\{ e^{\frac{T_r}{T_{01}}} - 1 \right\}} \right]$$

Evaluate Integral

(cont'd)

- Substituting in to earlier result

$$\int_{T_1}^{T_2} R_g \left[\frac{7}{2} + \left(\frac{T_r}{T}\right)^2 \frac{e^{\frac{T_r}{T}}}{\left\{ e^{\frac{T_r}{T}} - 1 \right\}^2} \right] dT = \left(\frac{7}{2} R_g T_2 + \frac{R_g T_r}{\left\{ e^{\frac{T_r}{T_2}} - 1 \right\}} \right) - \left(\frac{7}{2} R_g T_1 + \frac{R_g T_r}{\left\{ e^{\frac{T_r}{T_1}} - 1 \right\}} \right)$$

Evaluate Integral

(cont'd)

- And ….

$$T_2 + \frac{V_2^{\,2}}{2\left(\frac{7}{2}R_g\right)} + \frac{2}{7}\frac{T_r}{\left\{e^{\frac{T_r}{T_2}} - 1\right\}} = T_1 + \frac{V_1^{\,2}}{2\left(\frac{7}{2}R_g\right)} + \frac{2}{7}\frac{T_r}{\left\{e^{\frac{T_r}{T_1}} - 1\right\}}$$

- But

$$c_{p1} = R_g + c_v = R_g\left[\frac{7}{2} + \left(\frac{T_r}{T}\right)^2 \frac{e^{\frac{T_r}{T_1}}}{\left\{e^{\frac{T_r}{T_1}} - 1\right\}^2}\right] \Rightarrow T_1 << T_r \Rightarrow c_{p1} = \frac{7}{2}R_g$$

$$\frac{2}{7} = \frac{R_g}{c_{p1}} = \left(\frac{\gamma_1 - 1}{\gamma_1}\right)$$

Evaluate Integral

(cont'd)

- Finally!

$$T_2 + \frac{V_2^{\,2}}{2c_{p1}} + \left(\frac{\gamma_1 - 1}{\gamma_1}\right)\frac{T_r}{\left\{e^{\frac{T_r}{T_2}} - 1\right\}} = T_1 + \frac{V_1^{\,2}}{2c_{p1}} + \left(\frac{\gamma_1 - 1}{\gamma_1}\right)\frac{T_r}{\left\{e^{\frac{T_r}{T_1}} - 1\right\}}$$

Ideal gas terms

"T_{01}"

- Implicit function of {T, V} … two variables, need another equation

Evaluate Integral

(cont'd)

$$T_{0_2} = T_{0_1} + \left(\frac{\gamma_1 - 1}{\gamma_1}\right) T_r \left[\frac{e^{\frac{T_r}{T_2}} - e^{\frac{T_r}{T_1}}}{\left\{ e^{\frac{T_r}{T_2}} - 1 \right\}\left\{ e^{\frac{T_r}{T_1}} - 1 \right\}} \right]$$

- Vibrational correction to ideal gas equations

Continuity and Momentum Across Shock Wave

- 1-D momentum equation

$$p_1 + \rho_1 V_1^{\,2} = p_2 + \rho_2 V_2^{\,2} \downarrow$$

$$p_1 + \left(\frac{p_1}{R_g T_1}\right) V_1^{\,2} = p_2 + \left(\frac{p_2}{R_g T_2}\right) V_2^{\,2} \downarrow$$

$$p_1 \left[1 + \gamma_1 M_1^{\,2}\right] = p_2 \left[1 + \gamma_2 M_2^{\,2}\right] \downarrow$$

$$\frac{p_2}{p_1} = \frac{\left[1 + \gamma_1 M_1^{\,2}\right]}{\left[1 + \gamma_2 M_2^{\,2}\right]}$$

Continuity and Momentum Across Shock Wave

- Apply 1-D continuity equation and equation of state

$$\rho_1 V_1 = \rho_2 V_2 \rightarrow \frac{\rho_1}{\rho_2} = \frac{V_2}{V_1} \downarrow$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \frac{p_2}{p_1}\frac{V_2}{V_1} = \frac{\left[1+\gamma_1 M_1{}^2\right]}{\left[1+\gamma_2 M_2{}^2\right]}\frac{V_2}{V_1}$$

Continuity and Momentum Across Shock Wave

- Rearrange terms

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \frac{p_2}{p_1}\frac{V_2}{V_1} = \frac{\left[1+\gamma_1 {M_1}^2\right]}{\left[1+\gamma_2 {M_2}^2\right]}\frac{V_2}{V_1} =$$

$$\frac{\left[1+\frac{\gamma_1 {V_1}^2}{\gamma_1 R_g T_1}\right]}{\left[1+\frac{\gamma_2 {V_2}^2}{\gamma_2 R_g T_2}\right]}\frac{V_2}{V_1} = \frac{\left[1+\frac{{V_1}^2}{R_g T_1}\right]}{\left[1+\frac{{V_2}^2}{R_g T_2}\right]}\frac{V_2}{V_1}$$

Continuity and Momentum Across Shock Wave

- Solve for T_2 in terms of T_1

$$T_2 + \frac{V_2^{\,2}}{R_g} = \left[T_1 + \frac{V_1^{\,2}}{R_g} \right] \frac{V_2}{V_1}$$

- second equation allows numerical solution

Numerical Solution Via 2-D Newton Method

- Define

$$T_2 + \frac{V_2^{\,2}}{2c_{p1}} + \left(\frac{\gamma_1 - 1}{\gamma_1}\right)\frac{T_r}{\left\{e^{\frac{T_r}{T_2}} - 1\right\}} - \left\{T_{0_1} + \left(\frac{\gamma_1 - 1}{\gamma_1}\right)\frac{T_r}{\left\{e^{\frac{T_r}{T_1}} - 1\right\}}\right\} = f(T_2, V_2) = 0$$

$$T_2 + \frac{V_2^{\,2}}{R_g} - \left[T_1 + \frac{V_1^{\,2}}{R_g}\right]\frac{V_2}{V_1} = g(T_2, V_2) = 0$$

- System of equations of the form: $\begin{bmatrix} f(T_2, V_2) \\ g(T_2, V_2) \end{bmatrix} = 0$

Numerical Solution Via 2-D Newton Method
(cont'd)

- Expand in Taylor Series (higher dimensional)

$$\begin{bmatrix} f(T_2,V_2) \\ g(T_2,V_2) \end{bmatrix} = \begin{bmatrix} f(T_2,V_2) \\ g(T_2,V_2) \end{bmatrix}_{|0} + \nabla_{\begin{bmatrix} T_2 \\ V_2 \end{bmatrix}} \left\{ \begin{bmatrix} T_2 \\ V_2 \end{bmatrix} - \begin{bmatrix} T_2 \\ V_2 \end{bmatrix}_{|o} \right\} + ...$$

$$\nabla_{\begin{bmatrix} T_2 \\ V_2 \end{bmatrix}} = \begin{bmatrix} \frac{\partial f}{\partial T_2} & \frac{\partial f}{\partial V_2} \\ \frac{\partial g}{\partial T_2} & \frac{\partial g}{\partial V_2} \end{bmatrix}$$

But …

$$\begin{bmatrix} f(T_2,V_2) \\ g(T_2,V_2) \end{bmatrix} = 0$$

Numerical Solution Via 2-D Newton Method

(cont'd)

- Truncate after first order, solve for $[T_2, V_2]$

$$\begin{bmatrix} T_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} T_2 \\ V_2 \end{bmatrix}_{|o} - \nabla_{\begin{bmatrix} T_2 \\ V_2 \end{bmatrix}}^{-1} \left\{ \begin{bmatrix} f(T_2, V_2) \\ g(T_2, V_2) \end{bmatrix}_{|0} \right\}$$

$$\nabla_{\begin{bmatrix} T_2 \\ V_2 \end{bmatrix}}^{-1} = \begin{bmatrix} \frac{\partial f}{\partial T_2} & \frac{\partial f}{\partial V_2} \\ \frac{\partial g}{\partial T_2} & \frac{\partial g}{\partial V_2} \end{bmatrix}^{-1} \rightarrow Cramer's.Rule = \frac{\begin{bmatrix} \frac{\partial g}{\partial V_2} & -\frac{\partial f}{\partial V_2} \\ -\frac{\partial g}{\partial T_2} & \frac{\partial f}{\partial T_2} \end{bmatrix}}{\left[\frac{\partial f}{\partial T_2} \frac{\partial g}{\partial V_2} - \frac{\partial g}{\partial T_2} \frac{\partial f}{\partial V_2} \right]}$$

$$\begin{bmatrix} T_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} T_2 \\ V_2 \end{bmatrix}_{|o} - \begin{bmatrix} \frac{\partial g}{\partial V_2} & -\frac{\partial f}{\partial V_2} \\ -\frac{\partial g}{\partial T_2} & \frac{\partial f}{\partial T_2} \end{bmatrix} \frac{\left\{ \begin{bmatrix} f(T_2, V_2) \\ g(T_2, V_2) \end{bmatrix}_{|0} \right\}}{\left[\frac{\partial f}{\partial T_2} \frac{\partial g}{\partial V_2} - \frac{\partial g}{\partial T_2} \frac{\partial f}{\partial V_2} \right]}$$

Numerical Solution Via 2-D Newton Method

(cont'd)

- 2-D extension of the earlier 1-D Newton method

$$T_2^{(n+1)} = T_2^{(n)} - \frac{f(T_2^{(n)},V_2^{(n)})\left[\frac{\partial g(T_2,V_2)}{\partial V_2}\right]_{|n} - \left[\frac{\partial f(T_2,V_2)}{\partial V_2}\right]_{|n} g(T_2^{(n)},V_2^{(n)})}{\left[\frac{\partial f(T_2^{(n)},V_2^{(n)})}{\partial T_2}\right]_{|n}\left[\frac{\partial g(T_2,V_2)}{\partial V_2}\right]_{|n} - \left[\frac{\partial f(T_2,V_2)}{\partial V_2}\right]_{|n}\left[\frac{\partial g(T_2,V_2)}{\partial T_2}\right]_{|n}}$$

$$V_2^{(n+1)} = V_2^{(n)} - \frac{\left[\frac{\partial f(T_2^{(n)},V_2^{(n)})}{\partial T_2}\right]_{|n} g(T_2^{(n)},V_2^{(n)}) - f(T_2^{(n)},V_2^{(n)})\left[\frac{\partial g(T_2,V_2)}{\partial T_2}\right]_{|n}}{\left[\frac{\partial f(T_2^{(n)},V_2^{(n)})}{\partial T_2}\right]_{|n}\left[\frac{\partial g(T_2,V_2)}{\partial V_2}\right]_{|n} - \left[\frac{\partial f(T_2,V_2)}{\partial V_2}\right]_{|n}\left[\frac{\partial g(T_2,V_2)}{\partial T_2}\right]_{|n}}$$

Numerical Solution Via 2-D Newton Method

(cont'd)

- Where the Jacobean derivatives are

$$\frac{\partial f(T_2,V_2)}{\partial T_2} = \left(\frac{\gamma_1 - 1}{\gamma_1}\right)\frac{e^{\frac{T_r}{T_2}}\left[\frac{T_r}{T_2}\right]^2}{\left\{e^{\frac{T_r}{T_2}} - 1\right\}} + 1 \qquad \frac{\partial f(T_2,V_2)}{\partial V_2} = \frac{V_2}{c_{p1}}$$

$$\frac{\partial g(T_2,V_2)}{\partial T_2} = 1 \qquad \frac{\partial g(T_2,V_2)}{\partial V_2} = \frac{2V_2}{R_g} - \frac{T_1 + \frac{V_1^{\,2}}{R_g}}{V_1}$$

Numerical Solution Via 2-D Newton Method

(cont'd)

- Iterate to convergence

$$\sqrt{\left(\frac{V_2^{(n+1)} - V_2^{(n)}}{V_2^{(n)}}\right)^2 + \left(\frac{T_2^{(n+1)} - T_2^{(n)}}{T_2^{(n)}}\right)^2} < \varepsilon$$

- Using Solution for T_2, V_2 $\longrightarrow$

$$(c_v)_2 = R_g\left[\frac{5}{2} + \left(\frac{T_r}{T_2}\right)^2 \frac{e^{\frac{T_r}{T_2}}}{\left\{e^{\frac{T_r}{T_2}} - 1\right\}^2}\right] \Leftrightarrow (c_p)_2 = R_g\left[\frac{7}{2} + \left(\frac{T_r}{T_2}\right)^2 \frac{e^{\frac{T_r}{T_2}}}{\left\{e^{\frac{T_r}{T_2}} - 1\right\}^2}\right] \Leftrightarrow (\gamma)_2 = \frac{(c_p)_2}{(c_v)_2}$$

The Rest of the Solution

- Calculate M_2

$$M_2 = \frac{V_2}{\sqrt{\gamma_2 R_g T_2}}$$

- Calculate p_2

$$\frac{p_2}{p_1} = \frac{\left[1+\gamma_1 M_1{}^2\right]}{\left[1+\gamma_2 M_2{}^2\right]}$$

The Rest of the Solution (continued)

- Calculate P_{0_2}

$$P_{02} = p_2\left[1 + \frac{\gamma_2 - 1}{2} M_2^{\,2}\right]^{\frac{\gamma_2}{\gamma_2 - 1}}$$

- Calculate T_{0_2}

$$T_{02} = T_2\left[1 + \frac{\gamma_2 - 1}{2} M_2^{\,2}\right]$$

The Rest of the Solution (concluded)

- Calculate $C_{p_{Max}}$

$$Cp_{\max} = \frac{P_{02} - p_1}{\gamma_0 p_1 M_1^{\ 2}}$$

- Calculate Density

$$\rho_2 = \frac{p_2}{R_g T_2}$$

Example Calculation

- Theoretically valid solution for mach numbers up to ~10.0 where **gas dissociation/ionization & reaction begin to occur**

$$e = \frac{3}{2}R_g T + R_g T + \frac{\dfrac{h\nu}{\kappa T}}{\left\{ e^{\frac{h\nu}{\kappa T}} - 1 \right\}} R_g T + e_{electronic}$$

Rotational energy

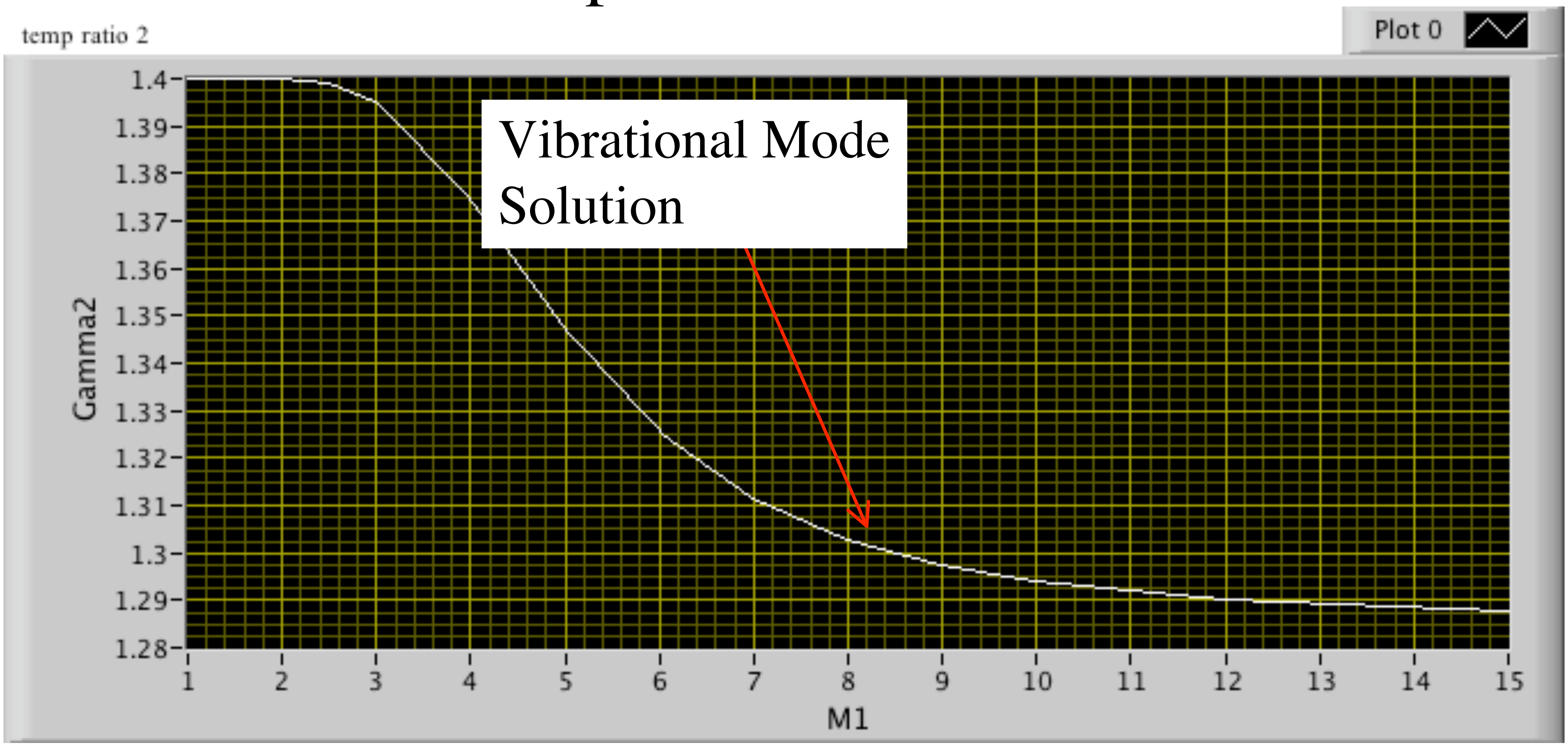
Ionization/
Dissociation energy

Translational energy

Vibrational energy

- Effect of Vibrational mode excitation is to lower temperature rise / increase Mach loss across shock wave

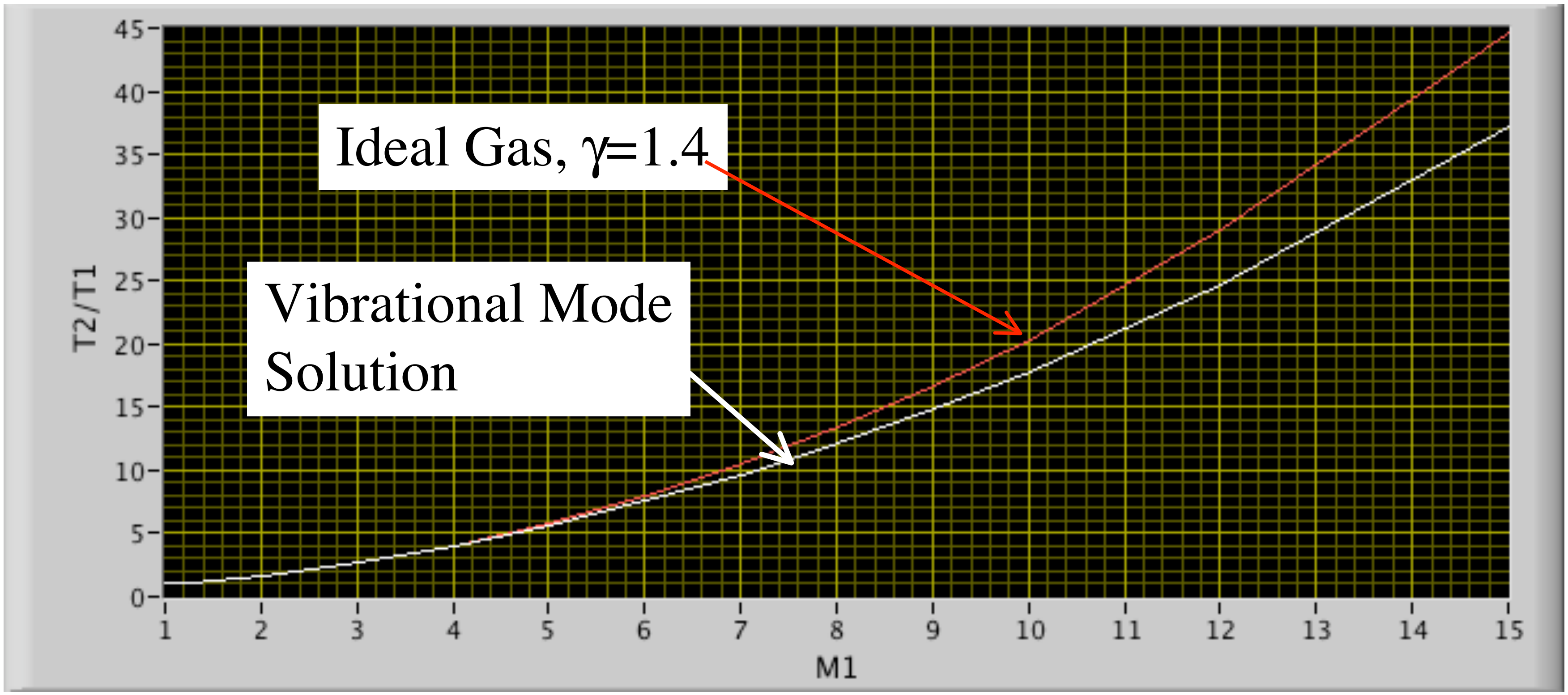
Example Calculation (cont'd)



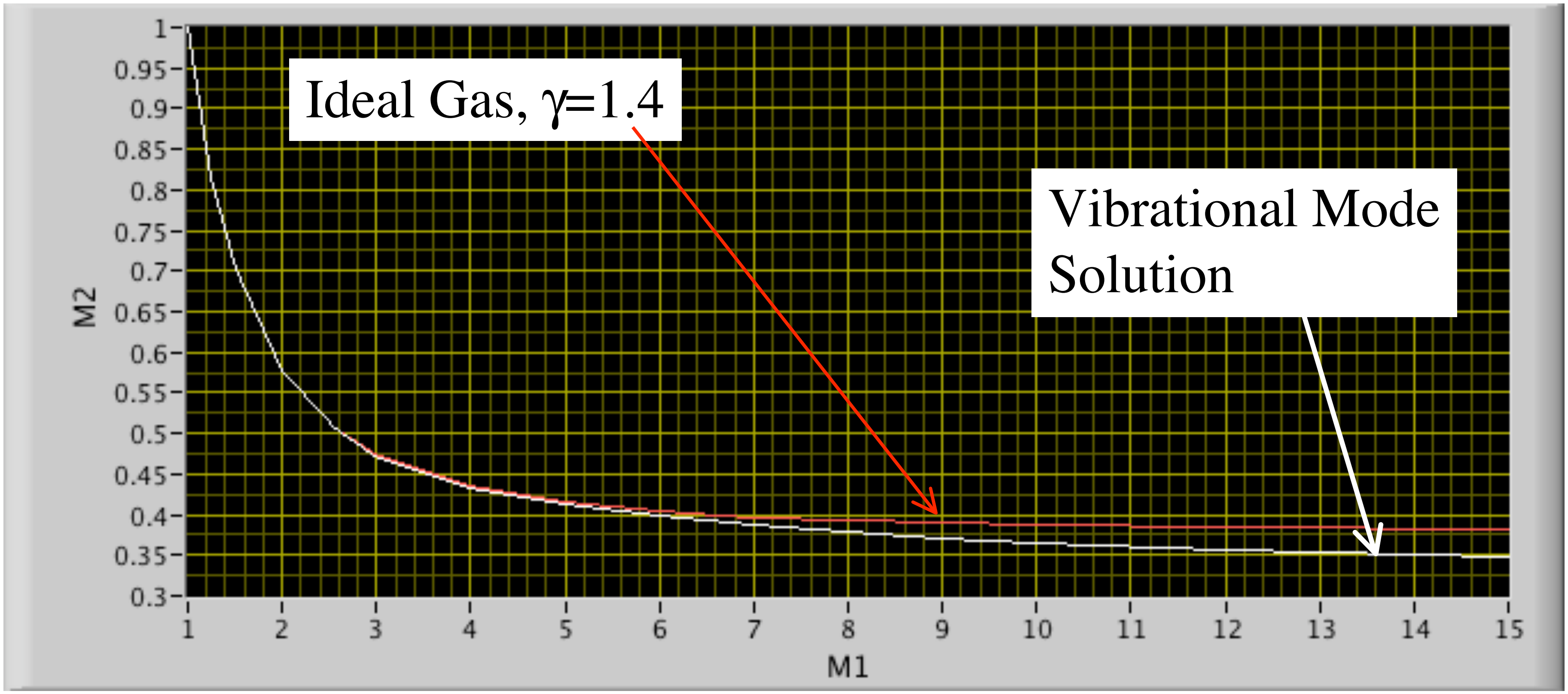
• Specific heat calculation

Example Calculation (cont'd)

- Temperature ratio Calculation

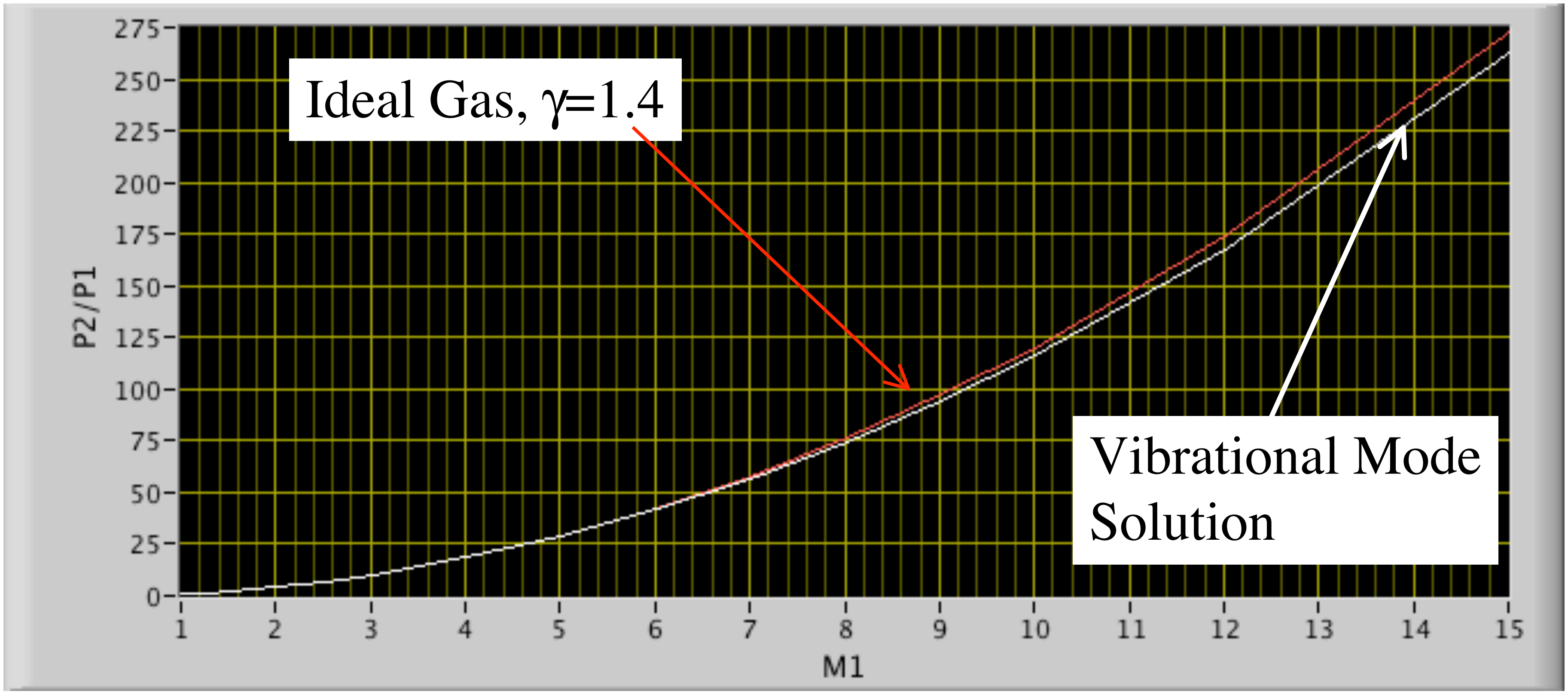


Example Calculation (cont'd)



- M_2 Calculation

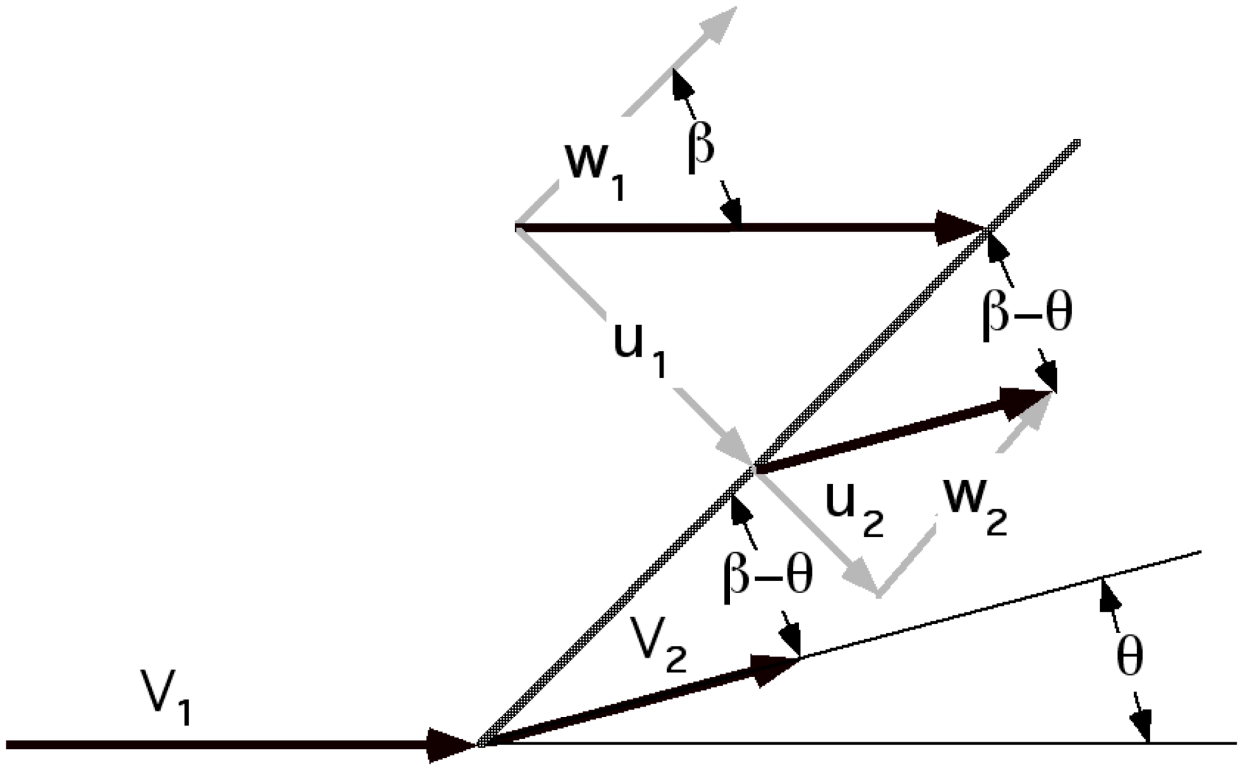
Example Calculation (concluded)



- Pressure Ratio Calculation

-- Pressure, being a mechanical quantity is not as greatly effected by variable gamma

Application to Hypersonic Oblique Shock Wave



Repeat analysis using, u_1 instead of V_1

Application to Hypersonic Oblique Shock Wave (cont'd)

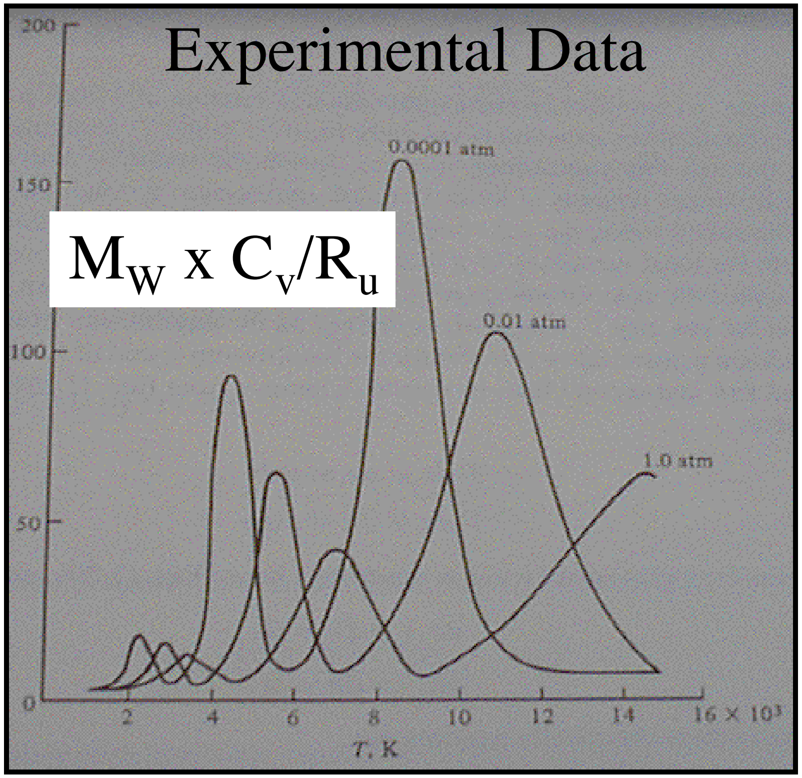
$$\left[\begin{array}{ll} T_2 + \frac{u_2^{\,2}}{R_g} = \left[T_1 + \frac{\left[V_1 \sin(\beta)\right]^2}{R_g}\right]\frac{u_2}{V_1 \sin(\beta)} & (a \\ T_2 + \frac{u_2^{\,2}}{2c_{p1}} + \left(\frac{\gamma_1 - 1}{\gamma_1}\right)\frac{T_r}{\left\{e^{\frac{T_r}{T_2}} - 1\right\}} = & \\ T_1 + \frac{\left[V_1 \sin(\beta)\right]^2}{2c_{p1}} + \left(\frac{\gamma_1 - 1}{\gamma_1}\right)\frac{T_r}{\left\{e^{\frac{T_r}{T_1}} - 1\right\}} & (b \\ \tan(\theta) = \frac{\left[V_1 \sin(\beta) - u_2\right]}{\left[V_1 \cos(\beta) + u_2 \tan(\beta)\right]} & (c \end{array}\right]$$

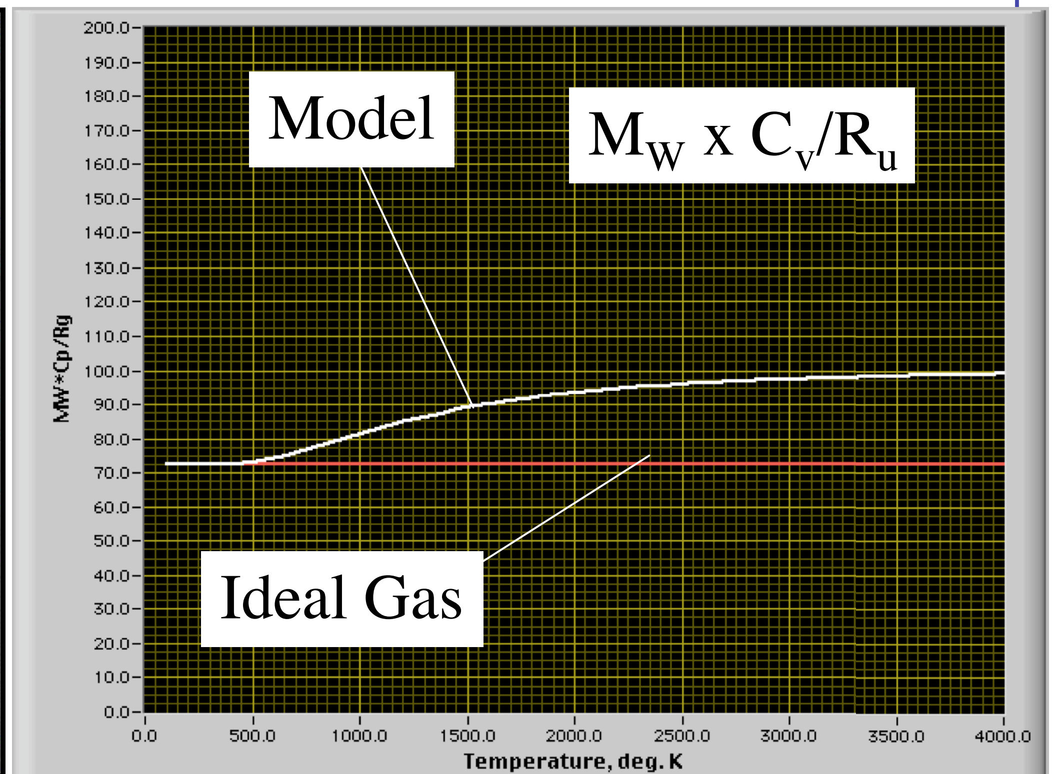
$$V_2 = \frac{u_2}{\sin(\beta - \theta)}$$

$$M_2 = \frac{V_2}{\sqrt{\gamma_2 R_g T_2}}$$

$$\frac{p_2}{p_1} = \frac{\left[1 + \gamma_1 (M_1 \sin\beta)^2\right]}{\left[1 + \gamma_2 \left(M_2 \sin(\beta - \theta)\right)^2\right]}$$

Compare Model with Real Data?





- Obviously there is a lot going on that Statistical Mechanics Doesn't Account for … **gas dissociation/ionization …reaction**

- Exact Solution but Upper Limit on Validity ~ Mach 10

"Engineering" Model # 2
Enthalpy Balance Across Shock Wave

- 2) Assume calorically perfect gas on each side of shock wave

$$c_{p1}T_1 + \frac{V_1^{\,2}}{2} = c_{p2}T_2 + \frac{V_2^{\,2}}{2} \Rightarrow c_{p1}\left[T_1 + \frac{V_1^{\,2}}{2c_{p1}}\right] = c_{p2}\left[T_2 + \frac{V_2^{\,2}}{2c_{p2}}\right] \Rightarrow$$

$$c_{p1}T0_1 = c_{p2}T0_2 \Rightarrow \frac{c_{p1}}{R_g}T0_1 = \frac{c_{p2}}{R_g}T0_2 \Rightarrow \frac{c_{p1}}{c_{p1}-c_{v1}}T0_1 = \frac{c_{p2}}{c_{p2}-c_{v2}}T0_2 \Rightarrow$$

$$\boxed{\frac{\gamma_1}{\gamma_1-1}T0_1 = \frac{\gamma_2}{\gamma_2-1}T0_2} \Rightarrow \boxed{\frac{\gamma_1}{\gamma_1-1}T_1\left[1+\frac{\gamma_1-1}{2}M_1^{\,2}\right] = \frac{\gamma_2}{\gamma_2-1}T_2\left[1+\frac{\gamma_2-1}{2}M_2^{\,2}\right]}$$

"Engineering" #2 Model (cont'd)

$$\frac{\gamma_1}{\gamma_1-1}T_1\left[1+\frac{\gamma_1-1}{2}M_1^{\,2}\right]=\frac{\gamma_2}{\gamma_2-1}T_2\left[1+\frac{\gamma_2-1}{2}M_2^{\,2}\right]$$

$$\frac{T_2}{T_1}=\frac{\frac{\gamma_1}{\gamma_1-1}\left[1+\frac{\gamma_1-1}{2}M_1^{\,2}\right]}{\frac{\gamma_2}{\gamma_2-1}\left[1+\frac{\gamma_2-1}{2}M_2^{\,2}\right]}$$

Continuity and Momentum Across Non-ideal Shock Wave

- 1-D momentum equation

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \downarrow$$

$$p_1 + \left(\frac{p_1}{R_g T_1}\right) V_1^2 = p_2 + \left(\frac{p_2}{R_g T_2}\right) V_2^2 \downarrow$$

$$p_1\left[1 + \gamma_1 M_1^2\right] = p_2\left[1 + \gamma_2 M_2^2\right] \downarrow$$

$$\frac{p_2}{p_1} = \frac{\left[1 + \gamma_1 M_1^2\right]}{\left[1 + \gamma_2 M_2^2\right]}$$

Continuity and Momentum Across Non-ideal Shock Wave (cont'd)

- Apply 1-D continuity equation and equation of state

$$\rho_1 V_1 = \rho_2 V_2 \rightarrow \frac{\rho_1}{\rho_2} = \frac{V_2}{V_1} \downarrow$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \frac{p_2}{p_1}\frac{V_2}{V_1} = \frac{\left[1+\gamma_1 M_1^{\ 2}\right]}{\left[1+\gamma_2 M_2^{\ 2}\right]}\frac{V_2}{V_1}$$

$$\frac{T_2}{T_1} = \frac{\gamma_2}{\gamma_1}\left\{\frac{\left[1+\gamma_1 M_1^{\ 2}\right]}{\left[1+\gamma_2 M_2^{\ 2}\right]}\frac{M_2}{M_1}\right\}^2$$

Continuity and Momentum Across Non-ideal Shock Wave (cont'd)

- But from earlier

$$\frac{T_2}{T_1} = \frac{\dfrac{\gamma_1}{\gamma_1 - 1}\left[1 + \dfrac{\gamma_1 - 1}{2} M_1^{\ 2}\right]}{\dfrac{\gamma_2}{\gamma_2 - 1}\left[1 + \dfrac{\gamma_2 - 1}{2} M_2^{\ 2}\right]}$$

- and

$$\frac{\gamma_2}{\gamma_1}\left\{\frac{\left[1 + \gamma_1 M_1^{\ 2}\right]}{\left[1 + \gamma_2 M_2^{\ 2}\right]}\frac{M_2}{M_1}\right\}^2 = \frac{\dfrac{\gamma_1}{\gamma_1 - 1}\left[1 + \dfrac{\gamma_1 - 1}{2} M_1^{\ 2}\right]}{\dfrac{\gamma_2}{\gamma_2 - 1}\left[1 + \dfrac{\gamma_2 - 1}{2} M_2^{\ 2}\right]}$$

Continuity and Momentum Across Non-ideal Shock Wave (cont'd)

- But from earlier

$$\frac{\gamma_2 M_2{}^2\left[1+\frac{\gamma_2-1}{2}M_2{}^2\right]}{\left[1+\gamma_2 M_2{}^2\right]^2}=\frac{\frac{\gamma_1}{\gamma_1-1}}{\frac{\gamma_2}{\gamma_2-1}}\times\frac{\gamma_1 M_1{}^2\left[1+\frac{\gamma_1-1}{2}M_1{}^2\right]}{\left[1+\gamma_1 M_1{}^2\right]^2}\rightarrow\frac{T_{0_2}}{T_{0_1}}=\frac{\frac{\gamma_1}{\gamma_1-1}}{\frac{\gamma_2}{\gamma_2-1}}$$

$$\frac{\gamma_2 M_2{}^2\left[1+\frac{\gamma_2-1}{2}M_2{}^2\right]}{\left[1+\gamma_2 M_2{}^2\right]^2}=\frac{T_{0_2}}{T_{0_1}}\times\frac{\gamma_1 M_1{}^2\left[1+\frac{\gamma_1-1}{2}M_1{}^2\right]}{\left[1+\gamma_1 M_1{}^2\right]^2}$$

WE HAVE SEEN THIS BEFORE!

UtahState UNIVERSITY

Mechanical & Aerospace Engineering

Solve for M_2

$$\frac{M_2^{\;2}\left[1+\frac{\gamma_2-1}{2}M_2^{\;2}\right]}{\left[1+\gamma_2 M_2^{\;2}\right]^2}=\left[\frac{T_{02}}{T_{0_1}}\right]\left[\frac{\gamma_1}{\gamma_2}\right]\frac{M_1^{\;2}\left[1+\frac{\gamma_1-1}{2}M_1^{\;2}\right]}{\left[1+\gamma_1 M_1^{\;2}\right]^2}$$

- Let

$$F(M_1)=\left[\frac{T_{02}}{T_{0_1}}\right]\left[\frac{\gamma_1}{\gamma_2}\right]\frac{M_1^{\;2}\left[1+\frac{\gamma_1-1}{2}M_1^{\;2}\right]}{\left[1+\gamma_1 M_1^{\;2}\right]^2}$$

M_2 Solution (continued)

- Regroup in terms of M^2

$$\frac{M_2^{\,2}\left[1+\frac{\gamma_2-1}{2}M_2^{\,2}\right]}{\left[1+\gamma_2 M_2^{\,2}\right]^2}=\frac{\gamma_1}{\gamma_2}\left[\frac{T_{02}}{T_{0_1}}\right]\frac{M_1^{\,2}\left[1+\frac{\gamma_1-1}{2}M_1^{\,2}\right]}{\left[1+\gamma_1 M_1^{\,2}\right]^2}\downarrow$$

$$M_2^{\,2}\left[1+\frac{\gamma_2-1}{2}M_2^{\,2}\right]-F(M_1)\left[1+\gamma_2 M_2^{\,2}\right]^2=0$$

$$F(M_1)=\left[\frac{T_{02}}{T_{0_1}}\right]\left[\frac{\gamma_1}{\gamma_2}\right]\frac{M_1^{\,2}\left[1+\frac{\gamma_1-1}{2}M_1^{\,2}\right]}{\left[1+\gamma_1 M_1^{\,2}\right]^2}$$

Engi (cont'd)

- Sub in and collect terms in powers of M_2

$$\frac{\gamma_2 - 1}{2} M_2^{\ 4} + M_2^{\ 2} - F(M_1)\left[1 + \gamma_2 M_2^{\ 2}\right]^2 = 0$$

$$\frac{\gamma_2 - 1}{2} M_2^{\ 4} + M_2^{\ 2} - F(M_1)\left[1 + 2\gamma_2 M_2^{\ 2} + \gamma_2^{\ 2} M_2^{\ 4}\right] = 0$$

$$\left[\frac{\gamma_2 - 1}{2} - \gamma_2^{\ 2} F(M_1)\right] M_2^{\ 4} + \left[1 - F(M_1) 2\gamma_2\right] M_2^{\ 2} - F(M_1) = 0$$

- **Quartic Equation, … but quadratic in M_2^2**

Enthalpy Balance Model

- ***Calculation procedure:***
- Given $M_{1,}\ T_{1,}\ p_1$
- 1) Assume γ_1, γ_2
- 2) Compute T_{0_1}

$$\rightarrow T_{0_1} = T_1\left[1+\frac{\gamma-1}{2}M_1^{\;2}\right]$$

- 3) Compute T_{0_2} / T_{0_1}

$$\rightarrow \frac{T_{0_2}}{T_{0_1}} = \frac{\dfrac{\gamma_1}{\gamma_1-1}}{\dfrac{\gamma_2}{\gamma_2-1}}$$

- 4) Compute M_2 from quartic

$$\left[\frac{\gamma_2-1}{2}-\gamma_2^{\;2}F(M_1)\right]M_2^{\;4}+\left[1-F(M_1)2\gamma_2\right]M_2^{\;2}-F(M_1)=0$$

Enthalpy Balance Model (cont'd)

- ***Calculation procedure:*** *cont'd*

- 5) Compute T_2

$$\rightarrow T_2 = \frac{T_{0_2}}{\left[1 + \frac{\gamma_2 - 1}{2} {M_2}^2\right]}$$

- assume adiabatic flow behind shock

- 6) Compute p_2

$$p_2 = p_1 \frac{\left[1 + \gamma_1 {M_1}^2\right]}{\left[1 + \gamma_2 {M_2}^2\right]}$$

Enthalpy Balance Model (cont'd)

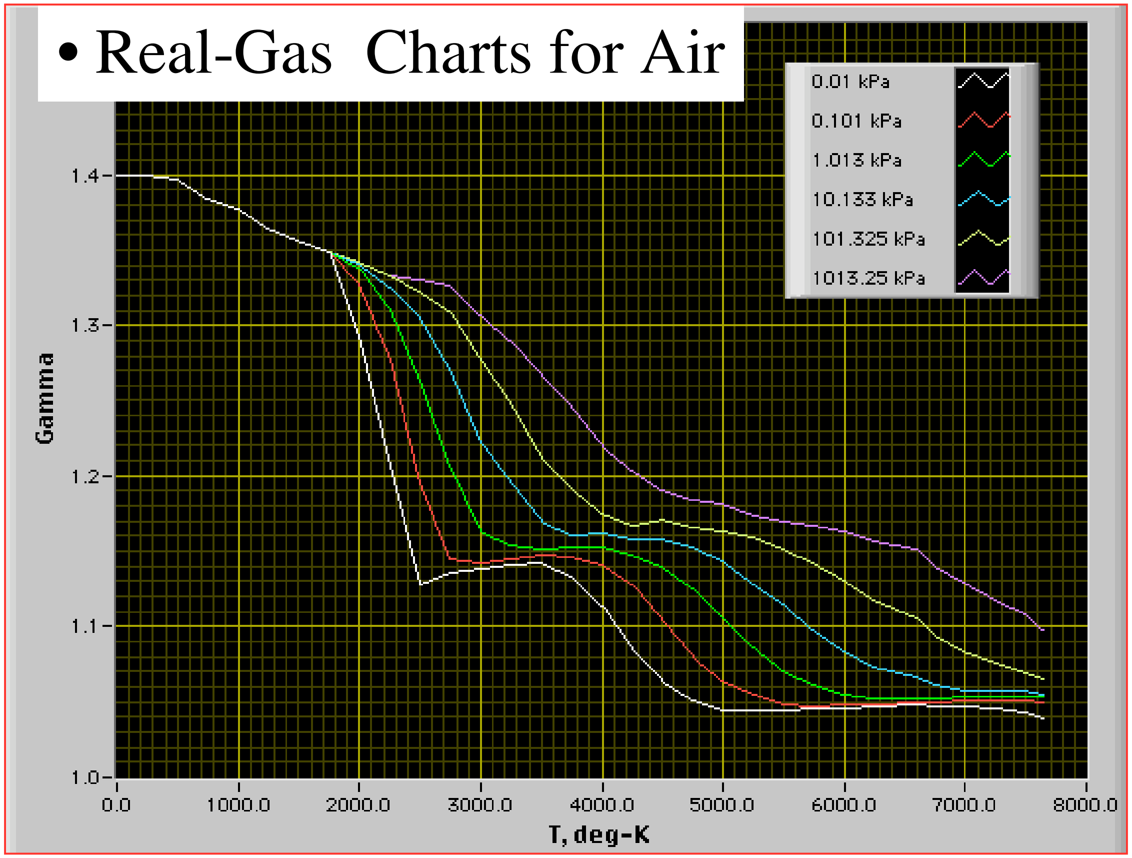
- ***Calculation procedure:*** *cont'd*

- 5) Look up Gamma from real gas tables

$$\left(\gamma_2\right)_{j+1} = \frac{\left(\gamma_2\right)_j + \gamma(p_2, T_2)}{2}$$

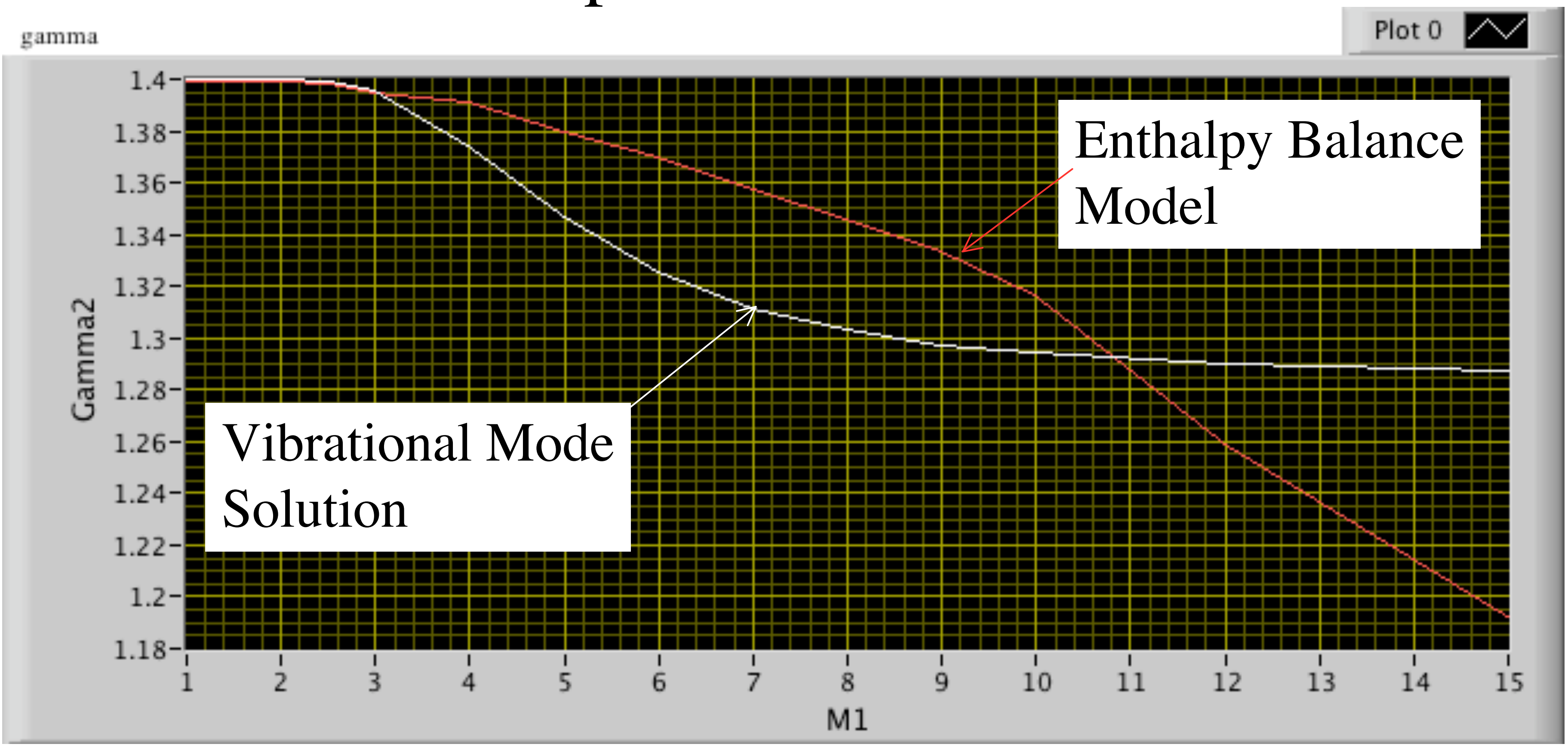
6) Repeat Steps 2-5 Until convergence j=0,1, ….

- Real-Gas Charts for Air



Hansen, C. Frederick, *Approximations for the Thermodynamic and Transport Properties of High-Temperature Air*, NASA TR R-50, 1959.

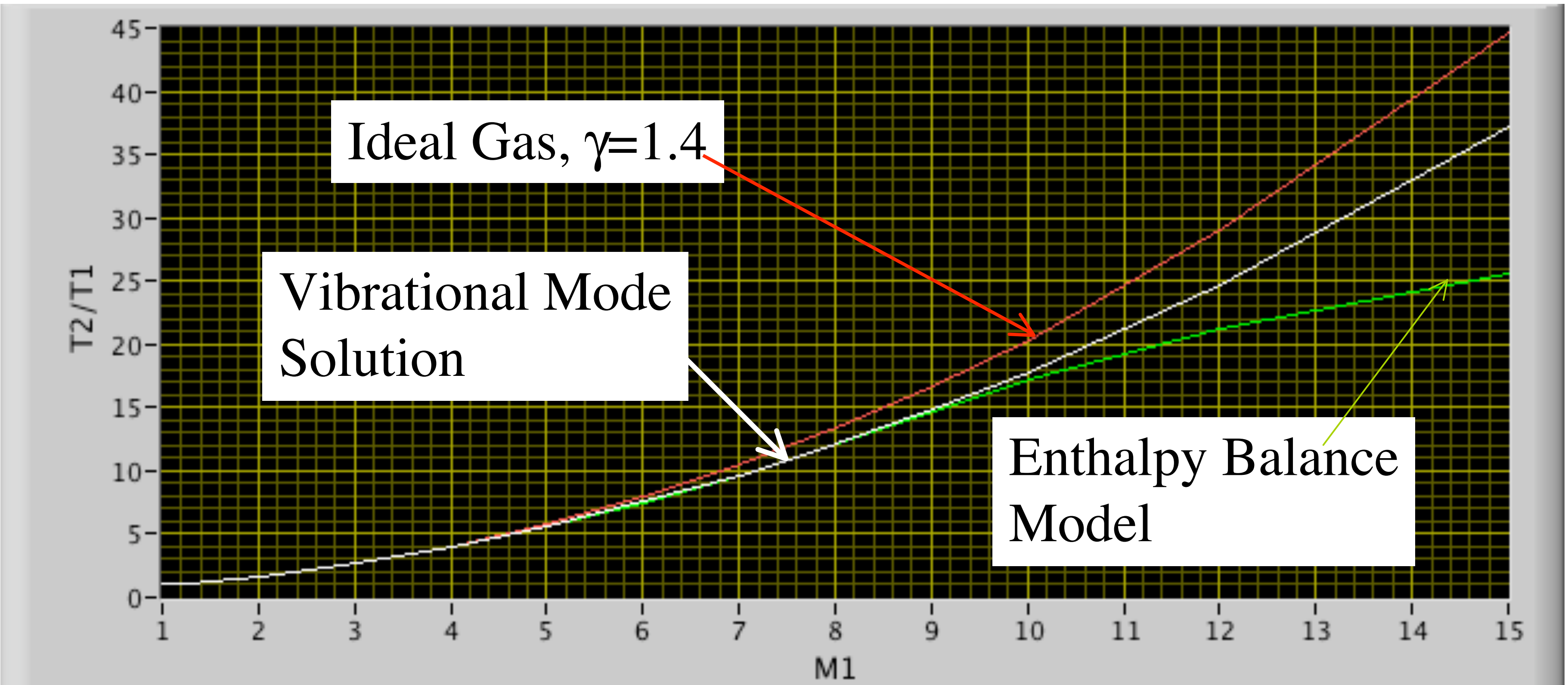
Example Calculation (cont'd)



- Specific heat calculation

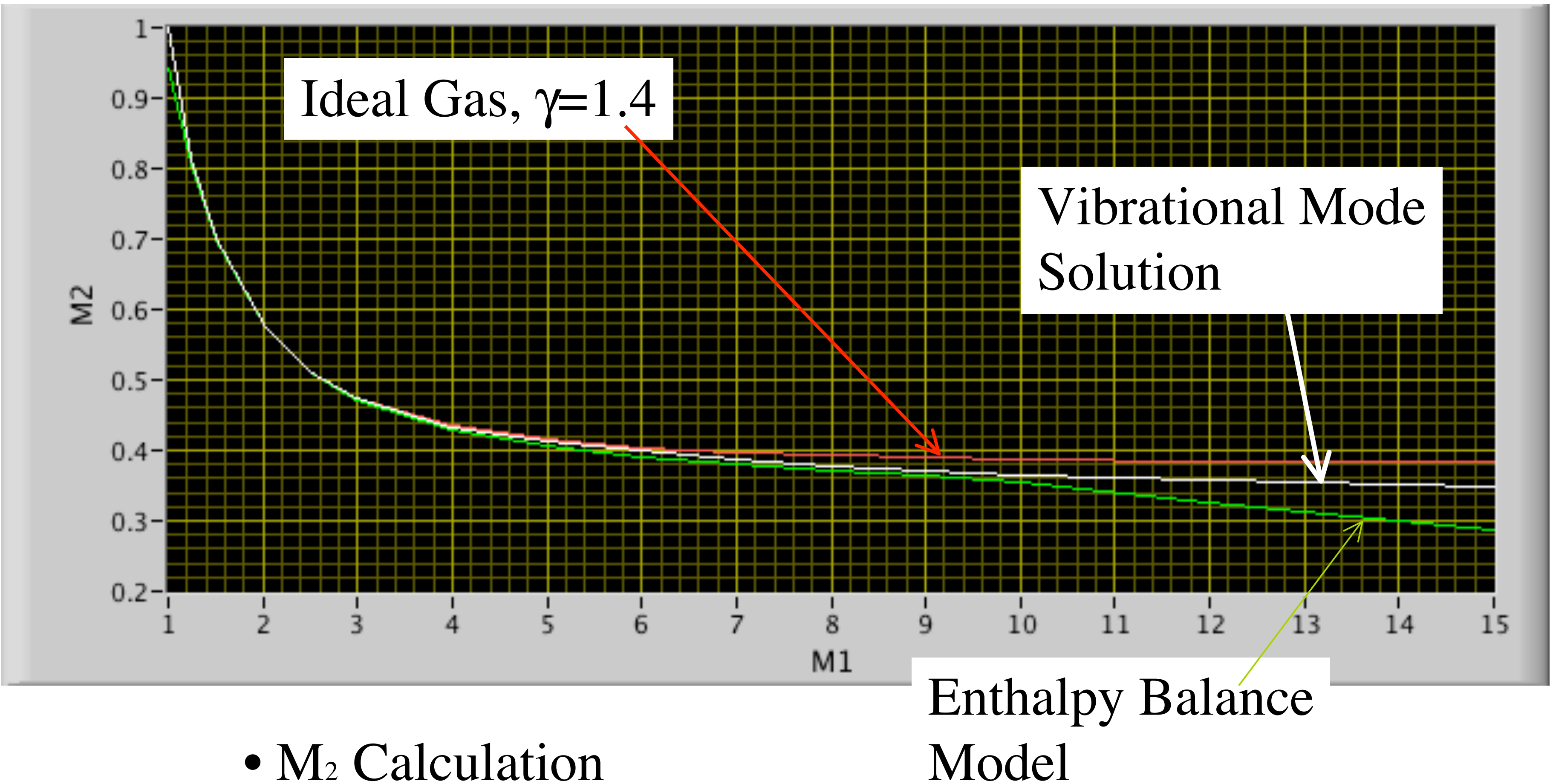
Example Calculation (cont'd)

- Temperature ratio Calculation



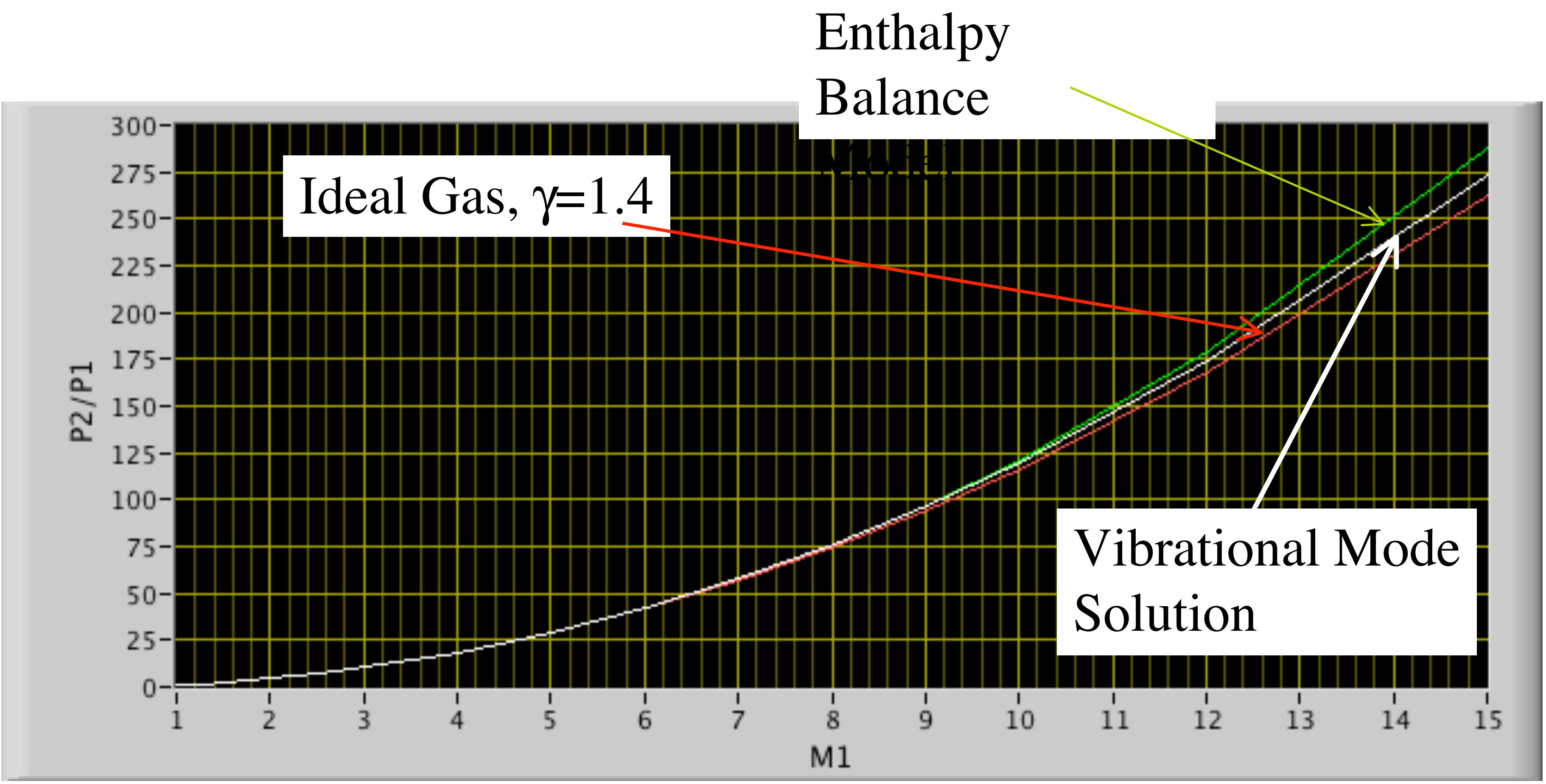
- Gas Chemistry is the only reason That Hypersonic flight is possible!

Example Calculation (cont'd)



- M_2 Calculation

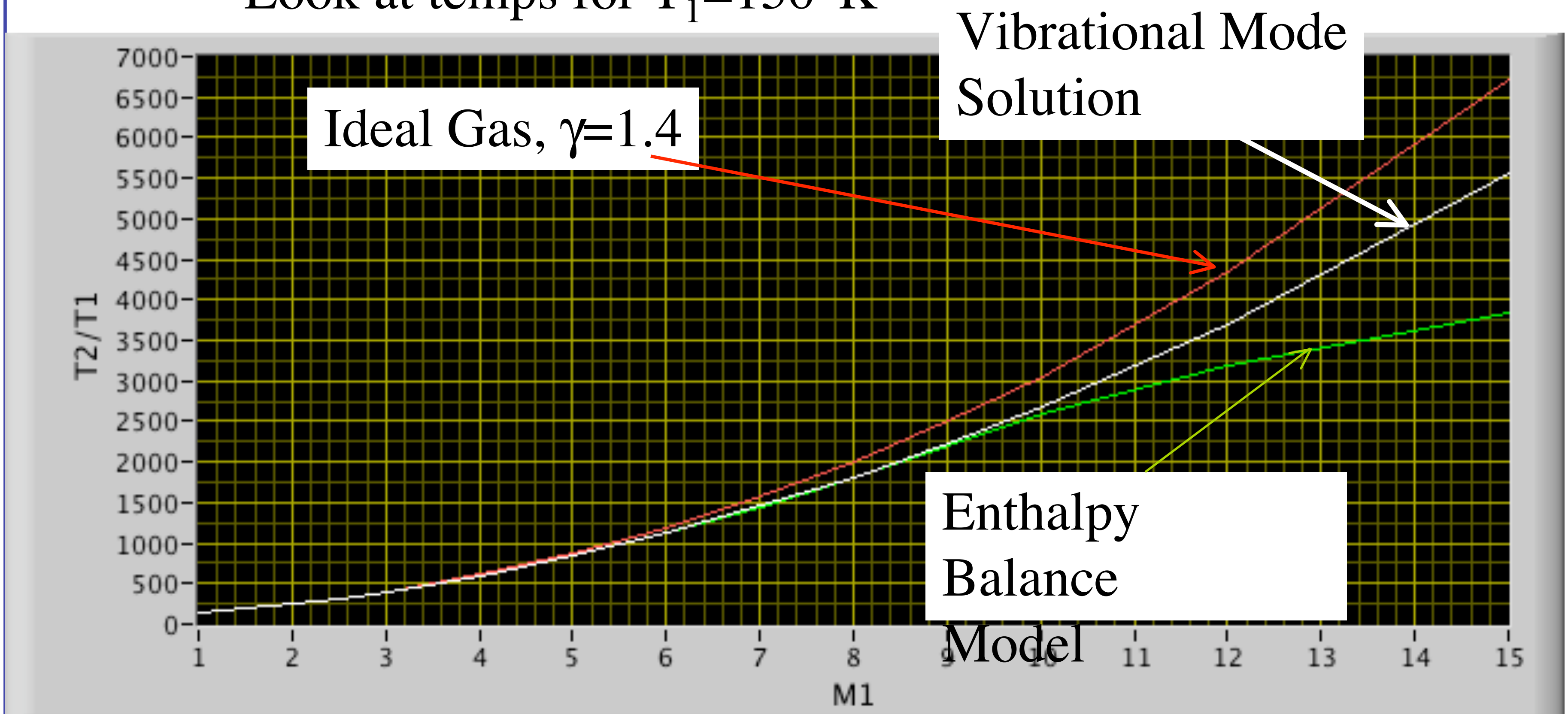
Example Calculation (continued)



- Pressure Ratio Calculation -- Pressure, being a mechanical quantity is not as greatly effected by variable gamma

Example Calculation (concluded)

- Look at temps for T_1=150°K



- Significant Differences!

- Gas Chemistry is the only reason That Hypersonic flight is possible!

Engineering Model # 3
Reference Temperature Method

- 1) "Smear" gas properties across shock wave

Since γ is not constant across the shockwave, an app roximation can be made by interpolating the real-gas specific heat ratio chart using Eckert's[i] empirical reference temperature $T_{(ref)}$

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[i] Mills, Anthony F., *Heat Transfer*, Irwin Publishing, Homewood, IL, 1992, pp. 377-379.

$$T_{(ref)} = T_\infty + \frac{1}{2}\left(T_2 - T_\infty\right) + 0.22\left(T_{02} - T_\infty\right)$$

Reference Temperature Method (cont'd)

- Example Computational Sequence:

$$\left\{M = 20, p_\infty = 0.005\, kPa, T_\infty = 150^0\, K\right\}$$

- For Calorically perfect gas with γ=1.4

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\,2} - 1\right)\right]\left[\frac{\left(2 + (\gamma-1)M_1^{\,2}\right)}{(\gamma+1)M_1^{\,2}}\right]$$

= 78.72 ---> 11808°K   Hot!

Reference Temperature Method (cont'd)

- For Calorically perfect gas with γ=1.4

$$T_{02} = T_{01} = T_{\infty}\left[1 + \frac{\gamma - 1}{2} M_1^{\ 2}\right]$$

=12150°K Hot!

- Compute reference temperature

$$T_{(ref)} = T_{\infty} + \frac{1}{2}\left(T_2 - T_{\infty}\right) + 0.22\left(T_{02} - T_{\infty}\right)$$

$$150 + \frac{(11808 - 150)}{2} + 0.22\,(12150 - 150)$$ =8619°K

Still hot, but not as hot

Reference Temperature Method (cont'd)

- Get Pressure Behind Shock wave (based on γ=1.4)

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\,2} - 1\right) \quad =466.500000$$

p2 = 2.33 kPa

- Interpolate Gas tables for new γ
  - γ ~ 1.0357

$$T_{02} = T_\infty\left[1 + \frac{\gamma_{ref} - 1}{2} M_1^{\,2}\right]$$

=12150°K Hot!

Reference Temperature Method (cont'd)

- Revaluate T_2 based on new γ

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\,2} - 1\right)\right]\left[\frac{\left(2 + (\gamma - 1)M_1^{\,2}\right)}{(\gamma+1)M_1^{\,2}}\right]$$

$=11.731226 \text{ ---> } T_2=1759.6$

- Get New Reference temperature

$$150 + \frac{1759.6 - 150}{2} + 0.22\,(12150 - 150) \quad = 3594.8°K$$

Reference Temperature Method (cont'd)

- Get Pressure Behind Shock wave (based on γ=1.0537)

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\ 2} - 1\right) \quad =410.4$$

p2 = 2.05 kPa

- Interpolate Gas tables for new γ

    - γ ~ 1.1535

Reference Temperature Method (cont'd)

- Revaluate T_2 based on new γ

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\,2} - 1\right)\right]\left[\frac{\left(2 + (\gamma-1)M_1^{\,2}\right)}{(\gamma+1)M_1^{\,2}}\right]$$

=31.533 ---> T_2=4729.6

- Get New Reference temperature

$$150 + \frac{1759.6 - 150}{2} + 0.22\,(12150 - 150) \quad = 5089.8°K$$

Reference Temperature Method (cont'd)

- Finally … after iterating to convergence

- Based on real γ, T_{ref}

$T_2 = 4751.98$ °K

$p_2 = 2.14$ kPa

$T_{ref} = 3468.99$ °K

$M_2 = 0.263$

$\gamma = 1.154$ $T_{02} = 4777.26$ °K

- Based on real $\gamma = 1.4$

$T_2 = 11808$ °F

$p_2 = 2.33$ kPa

$T_{ref} = 3254.9$ °K

$M_2 = 0.265$ $T_{02} = 12150$ °K

- Significant Differences!

- Gas Chemistry is the only reason That Hypersonic flight is possible!

Reference Temperature Method (concluded)

- Why is pressure not affected as much as temperature?

- Based on real γ, T_{ref}

$T_2 = 4751.98\ °K$

$p_2 = 2.14$ kPa

$T_{ref} = 3468.99\ °K$

$M_2 = 0.263$

$\gamma = 1.154 \quad T_{02} = 4777.26\ °K$

- Based on real $\gamma = 1.4$

$T_2 = 11808\ °K$

$p_2 = 2.33$ kPa

$T_{ref} = 3254.9\ °K$

$M_2 = 0.380$

$T_{02} = 12150\ °K$

- Pressure is a Mechanical and Not thermodynamic parameter and as such is affected more by Newton (momentum) than Boltzmann (thermo)