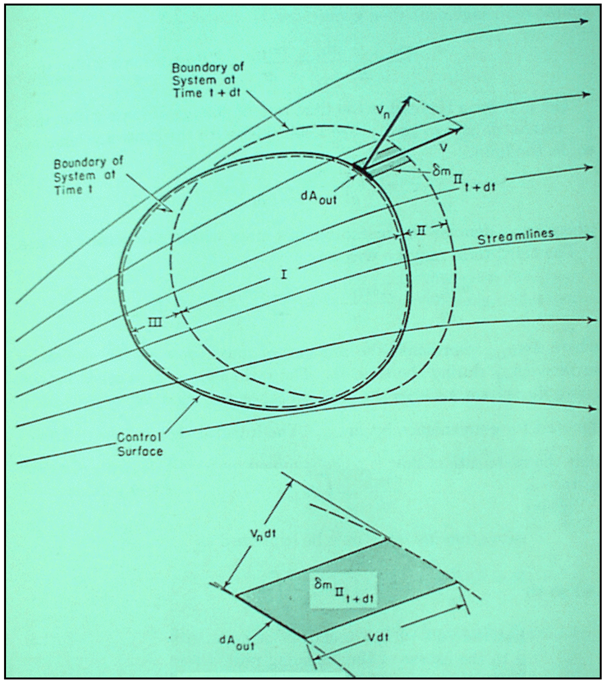
Section 2: Lecture 1
Integral Form of the Conservation Equations for Compressible Flow



Anderson: Chapter 2 pp. 41-54

Section 1 Review

- Equation of State: $p = \rho R_g T \rightarrow R_g = \frac{R_u}{M_w}$
  - R_u = 8314.4126 J/°K-(kg-mole)
  - $R_{g\ (air)}$ = 287.056 J/°K-(kg-mole)

- Relationship of R_g to specific heats, $\gamma = \frac{c_p}{c_v}$

$$c_p = c_v + R_g$$

- Internal Energy and Enthalpy

$$h = e + Pv$$

$$c_v = \left(\frac{de}{dT}\right)_v$$

$$c_p = \left(\frac{dh}{dT}\right)_p$$

Chapter 1 Review (cont'd)

- First Law of Thermodynamics, *reversible process*

$$de = dq - pdv$$

$$dh = dq + vdp$$

- First Law of Thermodynamics, *isentropic process*

(adiabatic, reversible)

$$de = -pdv$$

$$dh = vdp$$

Chapter 1 Review (cont'd)

- Second Law of Thermodynamics, nonadiabatic, *reversible process*

$$s_2 - s_1 = c_p \ln\left[\frac{T_2}{T_1}\right] - R_g \ln\left[\frac{p_2}{p_1}\right]$$

- Second Law of Thermodynamics, *isentropic process* (adiabatic, reversible)

$$\frac{p_2}{p_1} = \left[\frac{T_2}{T_1}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\left[\frac{p_2}{p_1}\right] = \left[\frac{\rho_2}{\rho_1}\right]^{\gamma}$$

Chapter 1 Review (concluded)

- Speed of Sound for calorically Perfect gas

$$c = \sqrt{\left[\frac{\partial p}{\partial \rho}\right]_{\Delta s = 0}}$$

$$c = \sqrt{\gamma R_g T}$$

- Mathematic definition of Mach Number

$$M = \frac{V}{\sqrt{\gamma R_g T}}$$

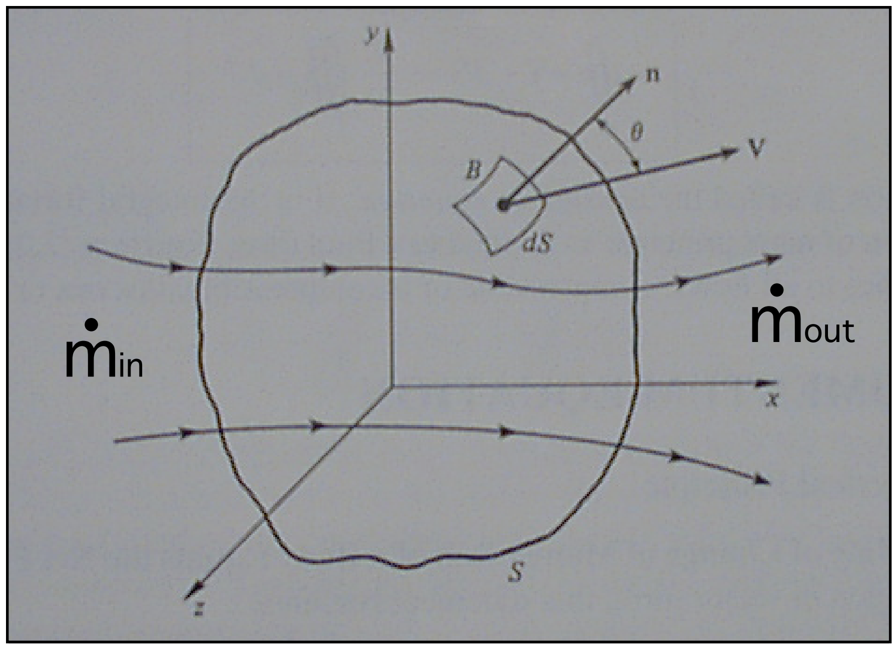
Concept of a *Control Volume*

- Arbitrary Volume (C.V), fixed in space, with fluid flux across its boundaries

- Outer surface of C.V is known as *Control Surface* (C.S.)

- *System* defined as the Mass Within the control volume at Some Instant in time *t*

- *Mass within control volume Must OBEY Laws of mass And momentum conservation*



Conservation of Mass

- At any instant in time, the mass contained within the C.V. is conserved

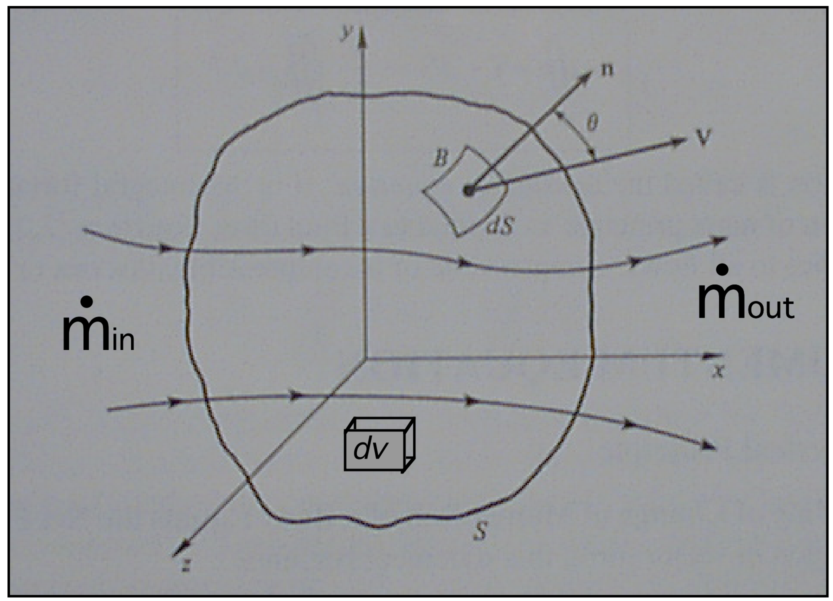
$$\frac{\delta}{\delta t}\left(m_{C.V}\right)=\iiint_{C.V.}\dot{m}_{in}-\iiint_{C.V.}\dot{m}_{out}$$

- Within an elemental piece *dv* of the control volume

$$dm_{C.V}=\rho dv$$

- And the total Contained mass is

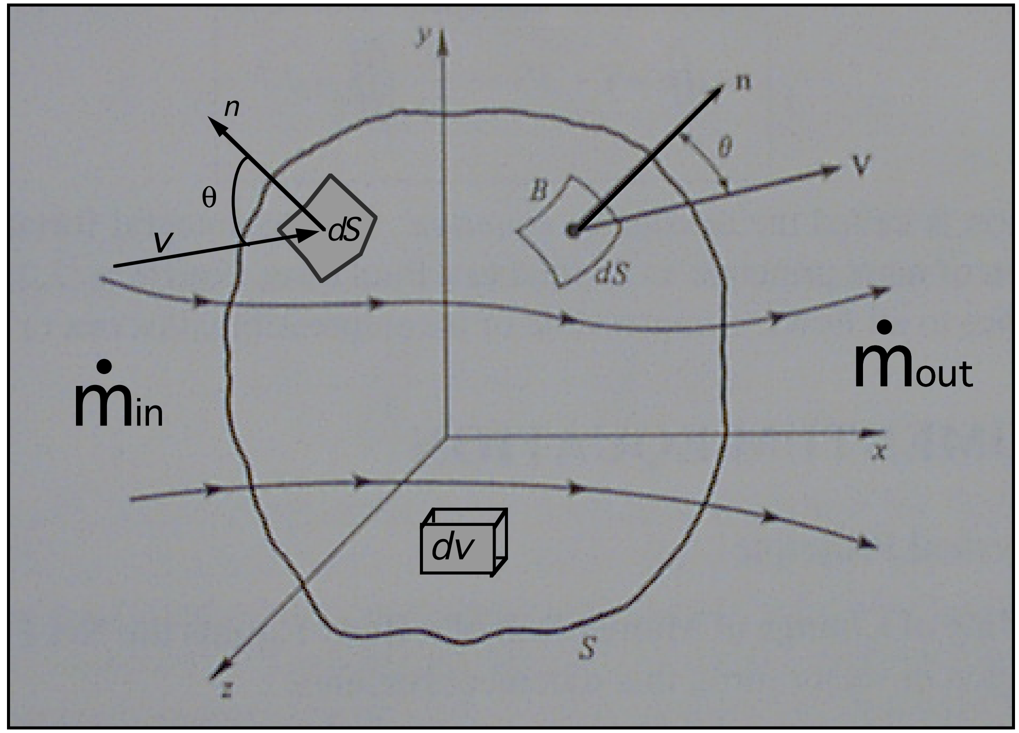
$$m_{C.V}=\iiint_{c.v.}\rho dv$$



Conservation of Mass (continued)

• The mass flow out of the C.V. across an elemental piece of the control surface, *ds*, is

$$d\left(\dot{m}_{out}\right) = \rho V_{out}\cos(\theta)\,ds$$



• And the incremental mass-flow into the C.V. is:

$$d\left(\dot{m}_{in}\right) = -\rho V_{in}\cos(\theta)\,ds$$

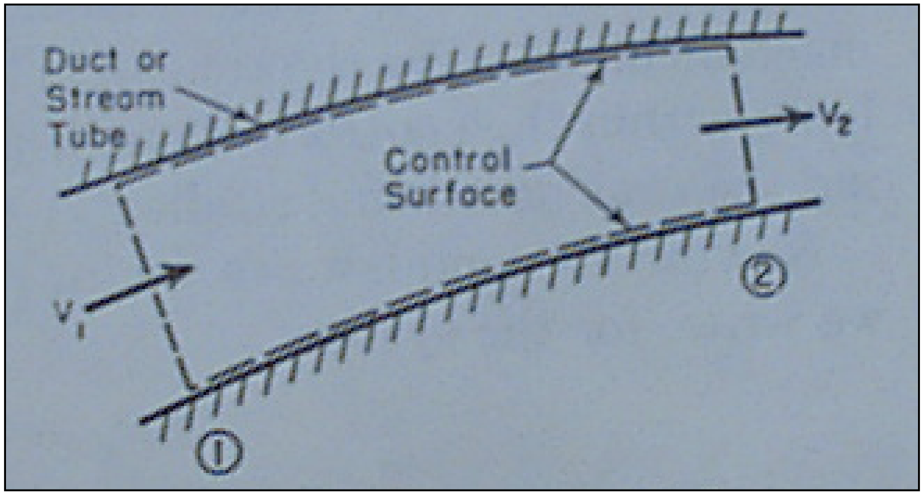
• Integrating over the entire control surface

$$\dot{m}_{C.V.} = -\iint_{C.S.}\left(\rho\vec{V}\bullet\vec{ds}\right)$$

$$\longrightarrow \quad -\iint_{C.S.}\left(\rho\vec{V}\bullet\vec{ds}\right) = \frac{\partial}{\partial t}\left(\iiint_{c.v.}\rho\,dv\right)$$

Conservation of Mass (concluded)

• One Dimensional Steady Flow *(mass flow in = mass flow out)*



$$\frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho dv\right) = 0 = -\iint_{C.S.}\left(\rho \vec{V} \bullet \vec{ds}\right) \rightarrow \rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

"continuity equation"

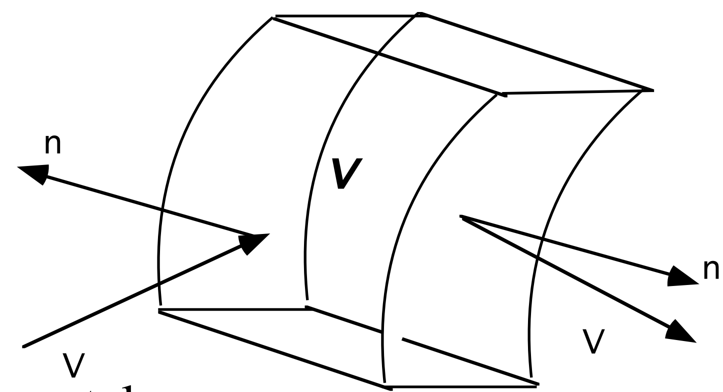
Conservation of Momentum

- Newton's second law applied to control volume

$$\sum \vec{F} = \frac{d}{dt}\left(m\vec{V} \right) =$$ Momentum flow across C.S. + Rate of Change of Momentum within C.V.

- *Momentum flow across control surface*

$$\iint_{C.S.} \left(\dot{m} \right) \vec{V}$$



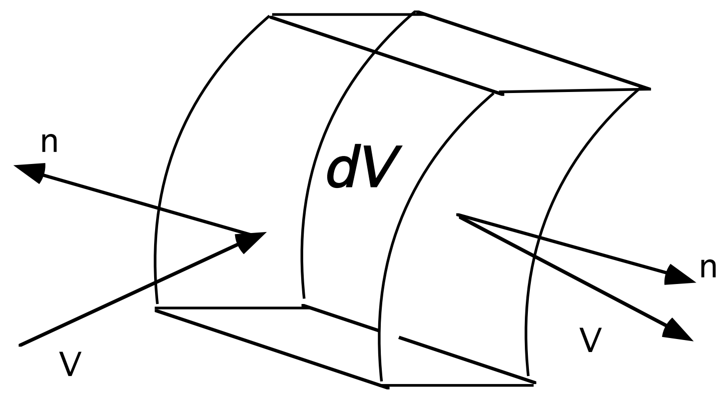
- But from earlier analysis, across an incremental Surface area

$$\dot{m} = \rho \vec{V} \bullet \vec{ds} \rightarrow \iint_{C.S.} \left(\dot{m} \right) \vec{V} = \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V}$$

Conservation of Momentum (cont'd)

• Rate of Momentum Change within Control Volume

$$\frac{\partial}{\partial t}(m\vec{V})$$



• But from earlier (continuity) analysis
For an incremental volume, *dv*

$$m = \rho dv$$

• And total net rate of momentum change within the control volume is

$$\frac{\partial}{\partial t}(m\vec{V})_{C.V.} = \frac{\partial}{\partial t}\iiint_{C.V.}\left(\rho dv\,\vec{V}\right) = \iiint_{C.V.}\frac{\partial}{\partial t}\left(\rho\vec{V}\right)dv$$

Conservation of Momentum (cont'd)

- Then Newton's second law becomes

$$\sum \vec{F} = \frac{d}{dt}\left(m\vec{V} \right) = \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} + \iiint_{C.V.} \frac{\partial}{\partial t}\left(\rho \vec{V} \right) dv$$

- Resolution of Forces Acting on Control Volume

- Body Forces (electo-magnetic, gravitational, buoyancy etc.)

$$\left(\sum \vec{F} \right)_{body} = \iiint_{C.V.} \rho \vec{f_b}\, dv$$

- Pressure Forces

$$\left(\sum \vec{F} \right)_{pressure} = -\iint_{C.S.} (p) \vec{dS}$$

(Minus sign Because pressure Acts inward)

- Viscous or frictional Forces

$$\left(\sum \vec{F} \right)_{fric} = \iint_{C.S.} \left| \left(\vec{f} \times \vec{dS} \right) \right| \frac{\vec{dS}}{S_{C.S.}}$$

Conservation of Momentum (continued)

- Collected terms for conservation of Momentum

$$\iiint_{C.V.} \rho \vec{f}_b \, dv - \iint_{C.S.} (p) \vec{dS} + \iint_{C.S.} \left| \left(\vec{f} \times \vec{dS} \right) \right| \frac{\vec{dS}}{S_{C.S.}} =$$

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} + \iiint_{C.V.} \frac{\partial}{\partial t} \left(\rho \vec{V} \right) dv$$

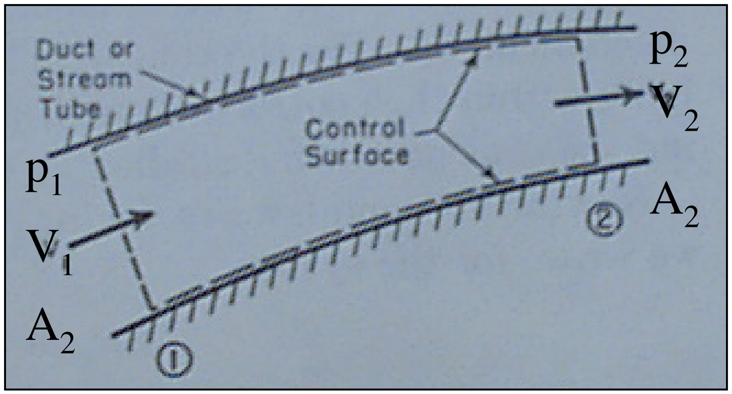
- Steady, In-viscous
Flow, Body Forces negligible

$$\iiint_{C.V.} \rho \vec{f}_b \, dv - \iint_{C.S.} (p) \vec{dS} + \iint_{C.S.} \left| \left(\vec{f} \times \vec{dS} \right) \right| \frac{\vec{dS}}{S_{C.S.}} =$$

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} + \iiint_{C.V.} \frac{\partial}{\partial t} \left(\rho \vec{V} \right) dv \longrightarrow -\iint_{C.S.} (p) \vec{dS} = \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V}$$

Conservation of Momentum (cont'd)

- One-Dimensional Flow



$$-\iint_{C.S.} (p)\vec{dS} = \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds}\right)\vec{V}$$

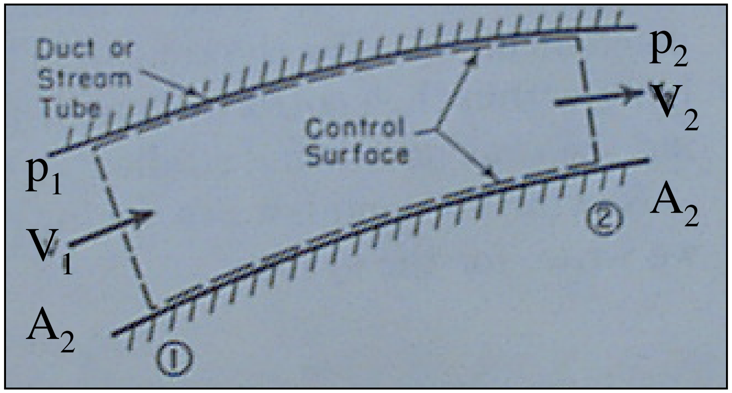
- No flow across solid boundary

$$-\left(p_2 A_2 - p_1 A_1\right)\bar{i}_x = \rho_2 V_2 A_2 V_2 \,\bar{i}_x - \rho_1 V_1 A_1 V_1 \,\bar{i}_x$$

$$\rightarrow \boxed{\left(p_1 + \rho_1 V_1^{\,2}\right)A_1 = \left[p_2 + \rho_2 V_2^{\,2}\right]A_2}$$

Conservation of Momentum (concluded)

- One-Dimensional Flow



$$-\iint_{C.S.} (p)\vec{dS} = \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds}\right)\vec{V}$$

- No flow across solid boundary

$$\left(p_1 + \rho_1 V_1^{\,2}\right)A_1 = \left[p_2 + \rho_2 V_2^{\,2}\right]A_2 \rightarrow$$

$$\rho_j = \frac{p_j}{R_g T_j} = \frac{\gamma p_j}{\gamma R_g T_j} = \frac{\gamma p_j}{c_j^{\,2}} \rightarrow p_1\left(1 + \gamma \frac{V_1^{\,2}}{c_1^{\,2}}\right)A_1 = p_2\left(1 + \gamma \frac{V_2^{\,2}}{c_2^{\,2}}\right)A_2$$

$$\boxed{\frac{p_2}{p_1} = \frac{A_1}{A_2}\frac{\left(1 + \gamma M_1^{\,2}\right)}{\left(1 + \gamma M_2^{\,2}\right)}}$$

Conservation of Energy

- From First law of thermodynamics (reversible process) (corresponding to inviscid fluid)

$$\frac{dE}{dt} = \frac{dQ}{dt} + \frac{dW}{dt}$$

$$\frac{dE}{dt} \rightarrow$$ Rate of change of fluid energy as it flows thru C.V.

$$\frac{dQ}{dt} \rightarrow$$ Rate of external heat addition to C.V.

$$\frac{dW}{dt} \rightarrow$$ Rate of work done to fluid inside of C.V.

Conservation of Energy (cont'd)

- $\frac{dE}{dt} \rightarrow$ Rate of change of fluid energy as it flows thru C.V.

Earlier thermodynamic discussions were for static fluid… flow thru C.V. is not static … Kinetic Energy terms must also Be captured … following process similar to Momentum flow through C.V.

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\iiint_{C.V.}\left(\rho\left(e + \frac{\left\|V^2\right\|}{2}\right)dv\right) + \iint_{C.S.}\rho \vec{V} \bullet d\vec{S}\left(e + \frac{\left\|V^2\right\|}{2}\right)$$

Rate of change of energy with C.V

Energy Flow Across C.S.

Conservation of Energy (cont'd)

- $\frac{dQ}{dt} \longrightarrow$ Rate of heat addition to C.V. $\iiint_{C.V.} \left(\rho \left(\dot{q} \right) dv \right)$

- $\frac{dW}{dt} \longrightarrow$ Rate of work done to fluid inside of C.V.

$$\frac{dW}{dt} \rightarrow \frac{d}{dt}(F \bullet dx) = F \bullet V \rightarrow dx = direction - of - motion$$

Consider: 1) Pressure Forces, 2) Body Forces

Conservation of Energy (cont'd)

1) Pressure Forces

$$-\iint_{C.S.} (p\, d\vec{S}) \bullet \vec{V}$$

2) Body Forces

$$\iiint_{C.V.} (\rho \vec{f}\, dv) \bullet \vec{V}$$

Conservation of Energy (cont'd)

- Collected Energy Equation

Scalar equation

$$\frac{\partial}{\partial t}\iiint_{C.V.}\left(\rho\left(e+\frac{\left\|V^2\right\|}{2}\right)dv\right)+\iint_{C.S.}\rho\vec{V}\bullet d\vec{S}\left(e+\frac{\left\|V^2\right\|}{2}\right)=$$

$$\iiint_{C.V.}(\rho\vec{f}\,dv)\bullet\vec{V}-\iint_{C.S.}(pd\vec{S})\bullet\vec{V}+\iiint_{C.V.}\left(\rho\left(\dot{q}\right)dv\right)$$

Conservation of Energy (concluded)

- Steady, Inviscid quasi 1-D Flow, Body Forces negligible

$$\frac{\partial}{\partial t}\iiint_{C.V.}\left(\rho\left(e+\frac{\left\|V^2\right\|}{2}\right)dv\right)+\iint_{C.S.}\rho\vec{V}\bullet d\vec{S}\left(e+\frac{\left\|V^2\right\|}{2}\right)=$$

$$\iiint_{C.V.}(\rho\vec{f}dv)\bullet\vec{V}-\iint_{C.S.}(pd\vec{S})\bullet\vec{V}+\iiint_{C.V.}\left(\rho\left(\dot{q}\right)dv\right)$$

1-D eqn.

$$\dot{m}=\rho\vec{V}\bullet d\vec{s}\rightarrow\iint_{C.S.}\rho\vec{V}\bullet d\vec{S}\left(e+\frac{\left\|V^2\right\|}{2}\right)=$$

$$\dot{m}\left(e_2+\frac{\left\|V_2{}^2\right\|}{2}\right)-\dot{m}\left(e_1+\frac{\left\|V_1{}^2\right\|}{2}\right)$$

Conservation of Energy (concluded)

- Steady, Inviscid quasi-1-D Flow, Body Forces negligible

$$\frac{\partial}{\partial t}\iiint_{C.V.}\left(\rho\left(e+\frac{\left\|V^2\right\|}{2}\right)dv\right)+\iint_{C.S.}\rho\vec{V}\bullet d\vec{S}\left(e+\frac{\left\|V^2\right\|}{2}\right)=$$

$$\iiint_{C.V.}(\rho\vec{f}dv)\bullet\vec{V}-\iint_{C.S.}(pd\vec{S})\bullet\vec{V}+\iiint_{C.V.}\left(\rho\left(\dot{q}\right)dv\right)$$

$$-\iint_{C.S.}(pd\vec{S})\bullet\vec{V}=p_1V_1A_1-p_2V_2A_2$$

Conservation of Energy (concluded)

- Steady, Inviscid quasi 1-D Flow, Body Forces negligible

$$\frac{\partial}{\partial t}\iiint_{C.V.}\left(\rho\left(e+\frac{\left\|V^2\right\|}{2}\right)dv\right)+\iint_{C.S.}\rho\vec{V}\bullet d\vec{S}\left(e+\frac{\left\|V^2\right\|}{2}\right)=$$

$$\iiint_{C.V.}(\rho\vec{f}dv)\bullet\vec{V}-\iint_{C.S.}(pd\vec{S})\bullet\vec{V}+\iiint_{C.V.}\left(\rho\left(\dot{q}\right)dv\right)$$

$$\iiint_{C.V.}\left(\rho\left(\dot{q}\right)dv\right)=M_{c.v.}\dot{q}=\frac{d}{dt}Q$$

Conservation of Energy (concluded)

- Steady, Inviscid quasi 1-D Flow, Body Forces negligible, collected

$$\dot{Q} = \left(p_2 A_2 V_2 - p_1 A_1 V_1\right) + \dot{m}\left(e_2 + \frac{\left\|V_2^{\,2}\right\|}{2}\right) - \dot{m}\left(e_1 + \frac{\left\|V_1^{\,2}\right\|}{2}\right)$$

- Divide through by massflow

$$\rightarrow \frac{\dot{Q}}{\dot{m}} \equiv \Delta q = \left[\frac{p_2 A_2 V_2}{\dot{m}} + \left(e_2 + \frac{\left\|V_2^{\,2}\right\|}{2}\right)\right] - \left[\frac{p_1 A_1 V_1}{\dot{m}} + \left(e_1 + \frac{\left\|V_1^{\,2}\right\|}{2}\right)\right]$$

Conservation of Energy (concluded)

- but from continuity, equation of state

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \rightarrow \frac{pAV}{\dot{m}} = \frac{p}{\rho} = R_g T$$

$$\rightarrow \Delta q = \left[\frac{p_2}{\rho_2} + \left(e_2 + \frac{\left\|V_2^{\,2}\right\|}{2}\right)\right] - \left[\frac{p_1}{\rho_1} + \left(e_1 + \frac{\left\|V_1^{\,2}\right\|}{2}\right)\right]$$

$$\rightarrow \Delta q = \left[R_g T_2 + \left(e_2 + \frac{\left\|V_2^{\,2}\right\|}{2}\right)\right] - \left[R_g T_1 + \left(e_1 + \frac{\left\|V_1^{\,2}\right\|}{2}\right)\right]$$

Conservation of Energy (concluded)

- but from continuity, equation of state

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \rightarrow \frac{pAV}{\dot{m}} = \frac{p}{\rho} = R_g T$$

$$\rightarrow \Delta q = \left[\frac{p_2}{\rho_2} + \left(e_2 + \frac{\|V_2{}^2\|}{2}\right)\right] - \left[\frac{p_1}{\rho_1} + \left(e_1 + \frac{\|V_1{}^2\|}{2}\right)\right]$$

$$\rightarrow \Delta q = \left[R_g T_2 + \left(e_2 + \frac{\|V_2{}^2\|}{2}\right)\right] - \left[R_g T_1 + \left(e_1 + \frac{\|V_1{}^2\|}{2}\right)\right]$$

Combine internal energy and thermal energy into enthalpy

$$h = e + Pv = e + R_g T$$

$$\rightarrow \Delta q = \left[h_2 + \frac{\|V_2{}^2\|}{2}\right] - \left[h_1 + \frac{\|V_1{}^2\|}{2}\right]$$

Summary

- Continuity (conservation of mass)

$$-\iint_{C.S.}\left(\rho \vec{V} \bullet \vec{ds}\right) = \frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho dv\right)$$

- Steady quasi One-dimensional Flow

$$\boxed{\frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho dv\right) = 0 = -\iint_{C.S.}\left(\rho \vec{V} \bullet \vec{ds}\right) \rightarrow \rho_1 V_1 A_1 = \rho_2 V_2 A_2}$$

Summary (continued)

• Newton's Second law-- Time rate of change of momentum Equals integral of external forces

$$\iiint_{C.V.} \rho \vec{f_b}\, dv - \iint_{C.S.} (p)\vec{dS} + \iint_{C.S.} \left| \left(\vec{f} \times \vec{dS}\right) \right| \frac{\vec{dS}}{S_{C.S.}} =$$

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds}\right) \vec{V} + \iiint_{C.V.} \frac{\partial}{\partial t}\left(\rho \vec{V}\right) dv$$

• Steady, Inviscid quasi 1-D Flow, Body Forces negligible

$$p_1 A_1 + \rho_1 A_1 V_1^{\,2} = p_2 A_2 + \rho_2 A_2 V_2^{\,2}$$

$$\frac{p_2}{p_1} = \frac{A_1}{A_2} \frac{\left(1 + \gamma M_1^{\,2}\right)}{\left(1 + \gamma M_2^{\,2}\right)}$$

Summary (concluded)

- Conservation of Energy--

$$\dot{Q} = \left(p_2 A_2 V_2 - p_1 A_1 V_1\right) + \dot{m}_2\left(e_2 + \frac{\left\|V_2{}^2\right\|}{2}\right) - \dot{m}_1\left(e_1 + \frac{\left\|V_1{}^2\right\|}{2}\right)$$

- Steady, Inviscid quasi 1-D Flow, Body Forces negligible

$$\boxed{\rightarrow \Delta q = \left[h_2 + \frac{\left\|V_2{}^2\right\|}{2}\right] - \left[h_1 + \frac{\left\|V_1{}^2\right\|}{2}\right]}$$