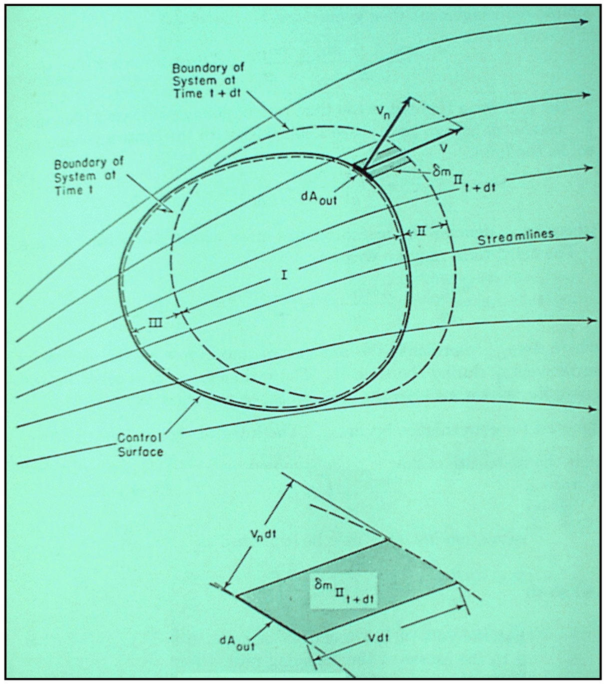
Section 2: Lecture 2
Simple Applications of Integral Equations of Motion



Anderson: Chapter 2 pp. 41-54

Review

- Continuity (conservation of mass)

$$-\iint_{C.S.}\left(\rho \vec{V} \bullet \vec{ds}\right) = \frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho dv\right)$$

- Steady One-dimensional Flow

$$\frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho dv\right) = 0 = -\iint_{C.S.}\left(\rho \vec{V} \bullet \vec{ds}\right) \rightarrow \rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

Review (continued)

• Newton's Second law-- Time rate of change of momentum Equals integral of external forces

$$\iiint_{C.V.} \rho \vec{f_b}\, dv - \iint_{C.S.} (p)\vec{dS} + \iint_{C.S.} \left| \left(\vec{f} \times \vec{dS} \right) \right| \frac{\vec{dS}}{S_{C.S.}} =$$

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} + \iiint_{C.V.} \frac{\partial}{\partial t} \left(\rho \vec{V} \right) dv$$

• Steady, Inviscid 1-D Flow, Body Forces negligible

$$p_1 A_1 - p_2 A_2 = \rho_2 V_2^{\,2} A_2 - \rho_1 V_1^{\,2} A_1$$

Review (concluded)

- Conservation of Energy--

$$\frac{\partial}{\partial t}\iiint_{C.V.}\left(\rho\left(e+\frac{\|V^2\|}{2}\right)dv\right)+\iint_{C.S.}\rho\vec{V}\bullet d\vec{S}\left(e+\frac{\|V^2\|}{2}\right)=$$

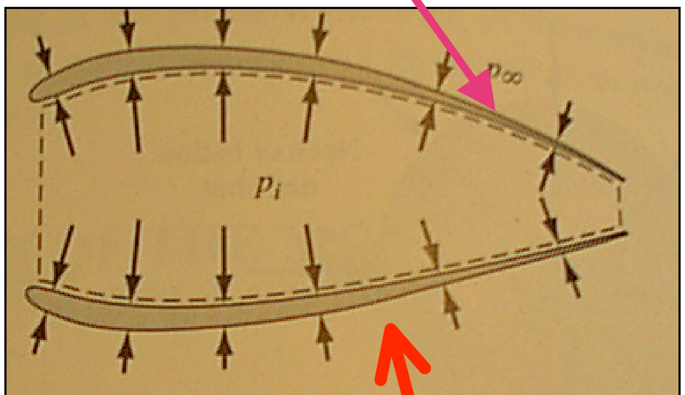
$$\iiint_{C.V.}(\rho\vec{f}\,dv)\bullet\vec{V}-\iint_{C.S.}(pd\vec{S})\bullet\vec{V}+\iiint_{C.V.}\left(\rho\left(\dot{q}\right)dv\right)$$

- Steady, Inviscid 1-D Flow, Body Forces negligible

$$\rightarrow \Delta q=\left[h_2+\frac{\|V_2{}^2\|}{2}\right]-\left[h_1+\frac{\|V_1{}^2\|}{2}\right]$$

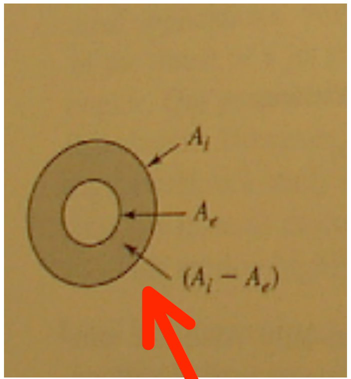
Example Usage: Engine Thrust Model

- Steady, Inviscid, quasi One-Dimensional Flow Through Ramjet



- *From a **balance of forces**, thrust of jet engine is **axial component** of the integral of internal pressure forces acting On wall minus integral of external Pressure forces acting on wall*

- ***Due to symmetry Non axial pressure forces Cancel out***



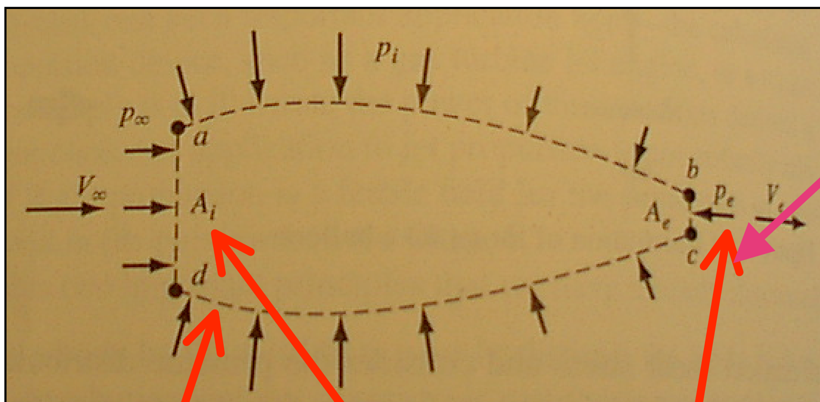
$$Thrust = \left[\iint_{wall} p_i dA_{wall} - \iint_{wall} p_\infty dA_{wall} \right]_x = \left[\iint_{wall} p_i dA_{wall} \right]_x - p_\infty \left(A_e - A_i \right)$$

Example Usage: Engine Thrust Model

- Apply Steady, Inviscid, quasi One-Dimensional Flow Through Ramjet

$$-\iint_{C.S.} (p)\,\vec{dS} = \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V}$$

Integrate in axial direction



- ***Due to symmetry Non axial pressure forces Cancel out***

$$\left[\iint_{wall} p_i dA_{wall} \right]_x + p_\infty A_i - p_e A_e = \rho_e V_e^2 A_e - \rho_i V_i^2 A_i = \dot{m}_e V_e - \dot{m}_i V_i$$

Example Usage: Engine Thrust Model (cont'd)

- Adding $\left(p_e A_e - p_\infty A_e\right)$ to Both Sides, and collecting terms

$$\left[\iint_{wall} p_i dA_{wall}\right]_x + p_\infty A_i - p_e A_e + \left(p_e A_e - p_\infty A_e\right) = \dot{m}_e V_e - \dot{m}_i V_i + \left(p_e A_e - p_\infty A_e\right) \rightarrow$$

$$\left[\iint_{wall} p_i dA_{wall}\right]_x + p_\infty\left(A_i - A_e\right) = \dot{m}_e V_e - \dot{m}_i V_i + \left(p_e A_e - p_\infty A_e\right)$$

- But from balance of forces

$$Thrust = \left[\iint_{wall} p_i dA_{wall}\right]_x - p_\infty\left(A_e - A_i\right)$$

Integrated Pressure Forces Acting on External + Internal Surface of Engine Wall = *Thrust*

$$Thrust = \dot{m}_e V_e - \dot{m}_i V_i + \left(p_e A_e - p_\infty A_e\right)$$

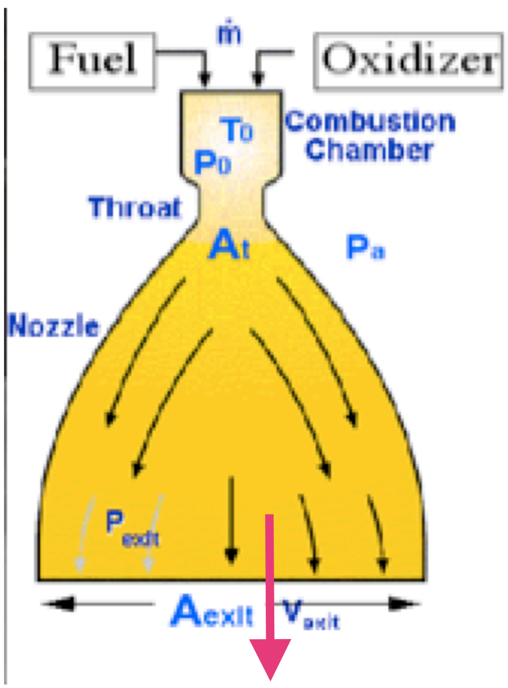
Rocket Thrust Equation

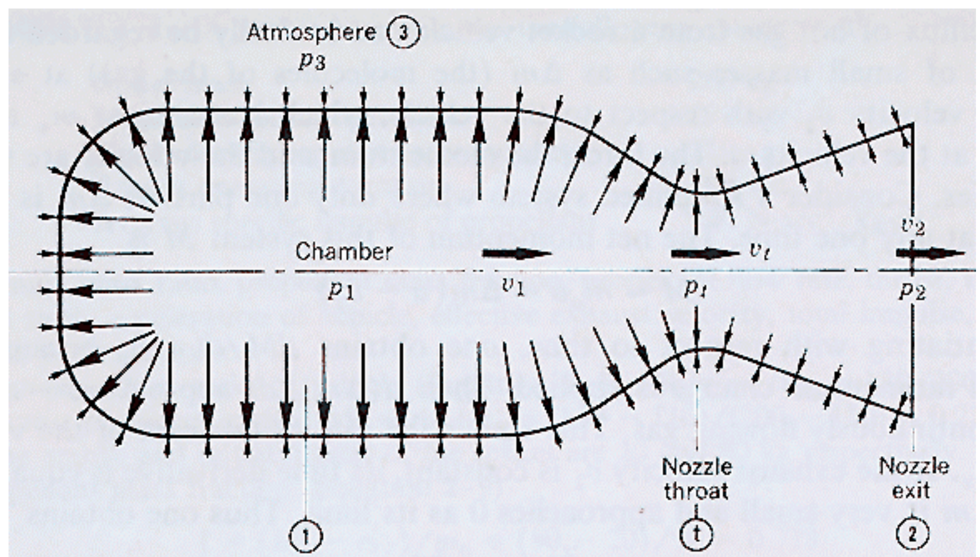
$$Thrust = \dot{m}_e V_e + \left(p_e A_e - p_\infty A_e\right)$$

$$\dot{m}_i = 0$$

- Thrust + Oxidizer enters combustion Chamber at ~0 velocity, combustion Adds energy … High Chamber pressure Accelerates flow through Nozzle

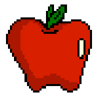
Resultant pressure forces produce thrust





Thrust Equation, Alternate Formulation

Newton



"The acceleration produced by a force is directly proportional to the force and inversely proportional to the mass which is being accelerated"

Newton's Second law $\quad \bar{F} = m\,\bar{a} = m\,\dfrac{d\bar{V}}{dt}$

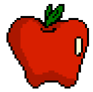
- But what happens when the mass is no longer constant?

- Newton recognized that the early formulation of second law was incomplete and modified the formulation accordingly

$$\bar{F} = \frac{d\left[m\bar{V}\right]}{dt} = \frac{d\left[\bar{P}\right]}{dt} \Rightarrow \bar{P} \equiv m\bar{V} \left[\text{"momentum vector"}\right]$$

Thrust Equation, Alternate Formulation (cont'd)

Newton



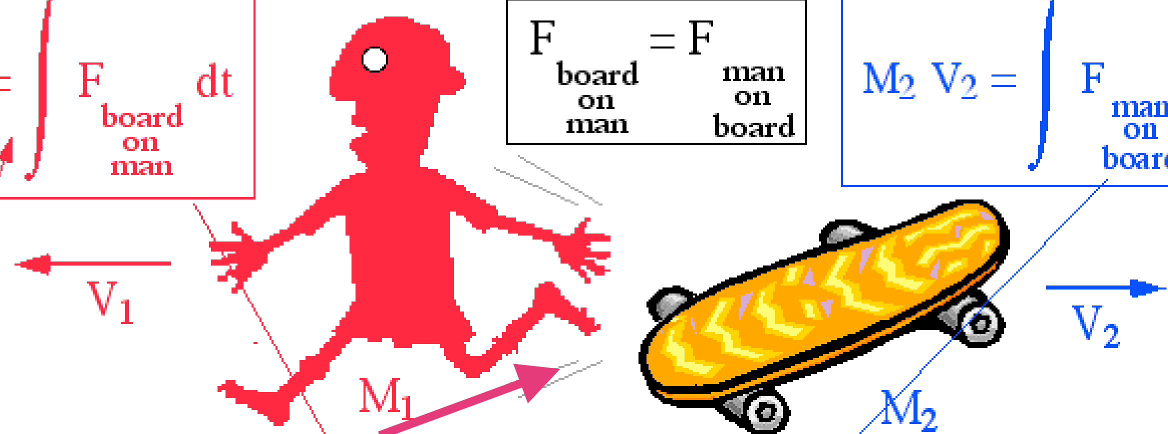
Conservation of momentum

- *For every action, there is an equal and oppposite RE-action*

$$M_1 V_1 = \int F_{\text{board on man}} \, dt$$

$$F_{\text{board on man}} = F_{\text{man on board}}$$

$$M_2 \, V_2 = \int F_{\text{man on board}} \, dt$$



Man jumps off skate board ... initially at rest

$$\bar{F} = \frac{d\,[\,m\bar{V}\,]}{dt}$$

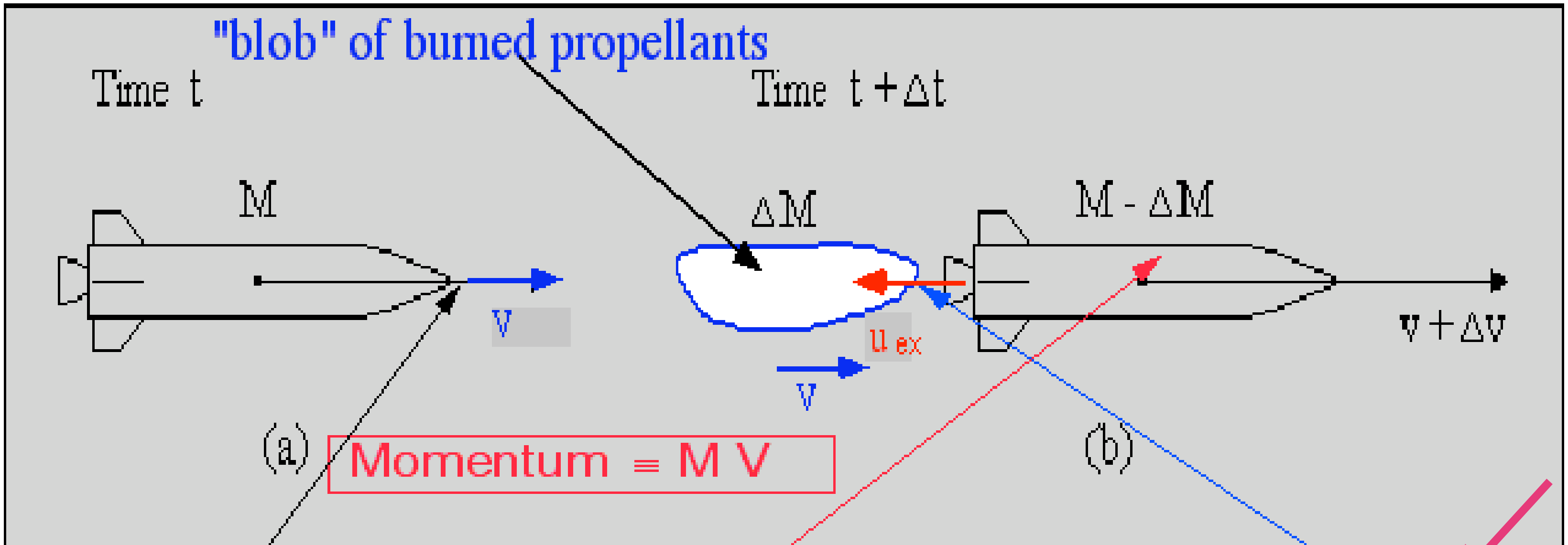
NewtonIII

$$M_1 V_1 = M_2 V_2$$

Thrust Equation, Alternate Formulation (cont'd)

Conservation of momentum

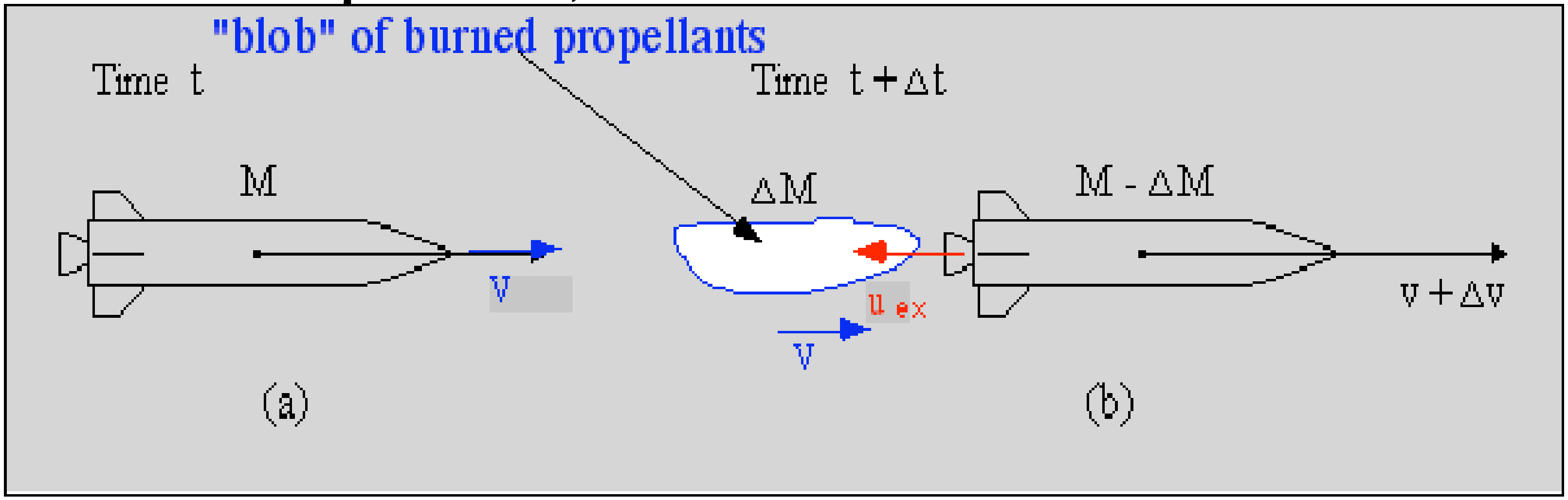
Look at a rocket in horizontal flight



$$M V = \left[M - \Delta M\right]\left[V + \Delta V\right] + \Delta M\left[V - U_{ex}\right]$$

$$\sum_{external} \bar{F} = \frac{d\bar{P}}{dt} = \frac{d\left[M\bar{V}\right]}{dt}$$

Thrust Equation, Alternate Formulation (cont'd)



$$M\,V = \left[M - \Delta M\right]\left[V + \Delta V\right] + \Delta M\left[V - U_{ex}\right]$$

$$\Downarrow$$

$$\underline{M\,V} = \underline{M\,V} - \underline{\Delta M\,V} + M\,\Delta V - \underline{\Delta M\,\Delta V} + \underline{\Delta M\,V} - \Delta M\,U_{ex}$$

$$\Downarrow$$

$$M\,\Delta V = \Delta M\,U_{ex} + \Delta M\,\Delta V$$

Thrust Equation, Alternate Formulation (cont'd)

- **Dividing by Δt and evaluating limit {ΔM , ΔV , Δt} -> 0**

$$\left[\frac{M\dfrac{\Delta V}{\Delta t} = \dfrac{\Delta M}{\Delta t} U_{ex} + \Delta M \dfrac{\Delta V}{\Delta t}}{\lim \{\Delta M, \Delta V, \Delta t\} \Rightarrow 0}\right] =$$

We shrink time
As small as possible

$$\Downarrow$$

$$F = M\frac{dV}{dt} = \frac{dM}{dt} U_{ex}$$

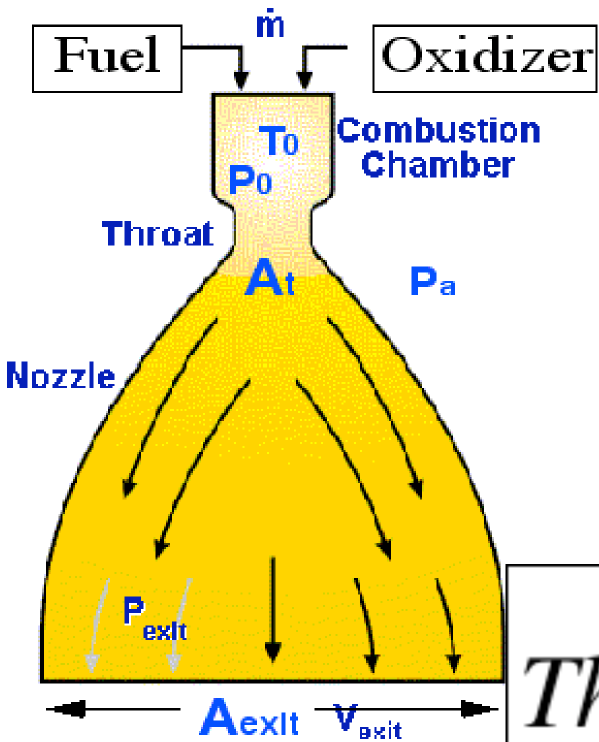
Engine massflow

Effective exhaust velocity

Engine thrust equation

Thrust Equation, Alternate Formulation (cont'd)

what happens here?



- Propellants combine and burn in combustion chamber
- Combustion products exhaust via throat
- Nozzle expands combustion products, increasing velocity & decreasing pressure
- Propellants are initially at zero velocity relative to motor, thus thrust produced by the motor = rate of momentum change + residual pressure force difference

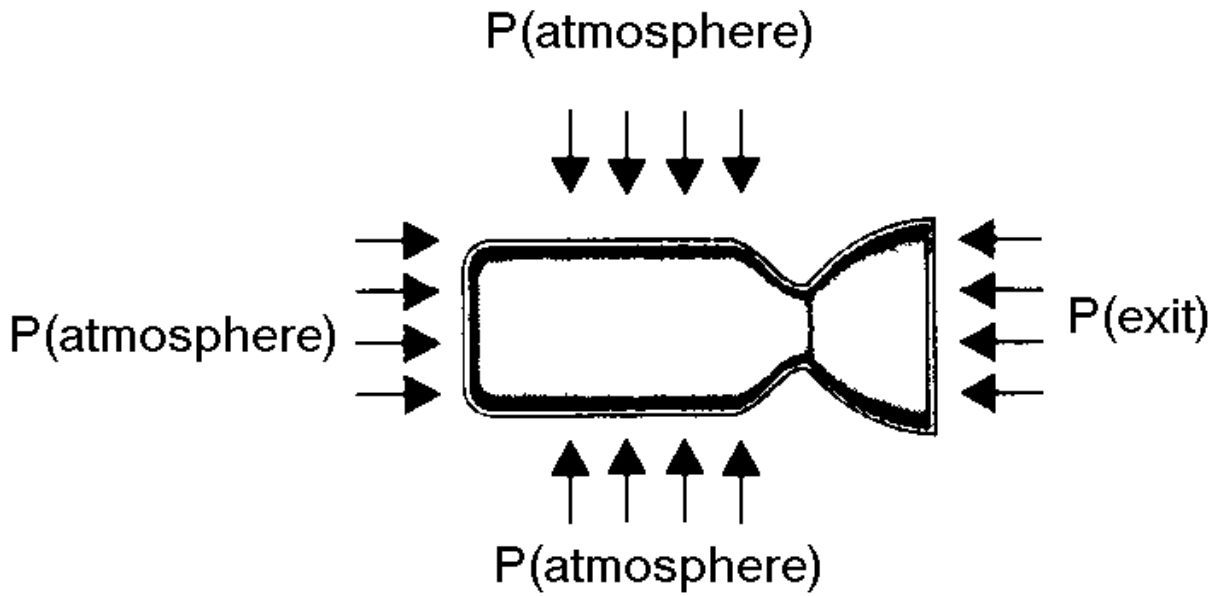
$$Thrust = \dot{m}_e V_e + \left(p_e A_e - p_\infty A_e\right)$$

$$Thrust = \dot{m}\,\boldsymbol{Ce} \rightarrow \boldsymbol{Ce} = V_e + \frac{p_e A_e - P_\infty A_e}{\dot{m}}$$

Engine thrust equation

Ce —> "Effective Exhaust Velocity"

Pressure Thrust



- Pressure is identical from all directions except for the Area of the exit nozzle. This pressure difference produces a thrust (which may be negative or positive.)

Specific Impulse

- Specific Impulse is a scalable characterization of a rocket's Ability to deliver a certain *(specific)* impulse for a given weight of propellant

$$\rightarrow g_0 = 9.8067 \frac{m}{\sec^2} (mks)$$

$$I_{sp} = \frac{\mathrm{I}_{mpulse}}{g_0 M_{propellant}} = \frac{\int_0^t F_{thrust} dt}{g_0 \int_0^t \dot{m}_{propellant}\, dt}$$

Mean specific impulse

- At a constant altitude, with Constant mass flow through engine

$$I_{sp} = \frac{\mathrm{I}_{mpulse}}{g_0 M_{propellant}} = \frac{\int_0^t F_{thrust} dt}{g_0 \int_0^t \dot{m}_{propellant}\, dt} = \frac{F_{thrust}}{g_0 \dot{m}_{propellant}}$$

- *Instantaneous specific impulse*

Specific Impulse (cont'd)

• Historically, I_{sp} was measured in units of *seconds*

$$I_{sp} = \frac{|\overline{F}|}{\dot{m}_p} \Rightarrow \text{(English Units)} \frac{\text{lbf}}{\text{lbm/sec}} \approx \text{seconds, right?}$$

Wrong! *lbms* are not a fundamental unit for mass
(Slugs are the fundamental english unit of mass)

$$I_{sp} = \frac{|\overline{F}|}{\dot{m}_p} \Rightarrow \text{(MKS units)} \frac{\text{Nt}}{\text{kg/sec}} \approx \frac{\text{kg-m/sec}^2}{\text{kg/sec}} \approx \frac{\text{m}}{\text{sec}}$$

Goddard got it wrong!

• Since most engine manufacturers still give I_{sp} in *seconds* -- we correct for this by letting

$$I_{sp} \equiv \frac{|\overline{F}|}{g_0 \, \dot{m}_p} \Rightarrow g_0 \approx 9.81 \frac{\text{m}}{\text{sec}^2}$$ [*acceleration of gravity at sea level*]

English Units -- use *slugs* not *lbms* !

$$\text{(MKS units)} \frac{\frac{\text{Nt}}{\text{kg/sec}}}{\frac{\text{m}}{\text{sec}^2}} \approx \frac{\frac{\text{kg-m/sec}^2}{\text{kg/sec}}}{\frac{\text{m}}{\text{sec}^2}} \approx \text{sec}$$

Specific Impulse *(cont'd)*

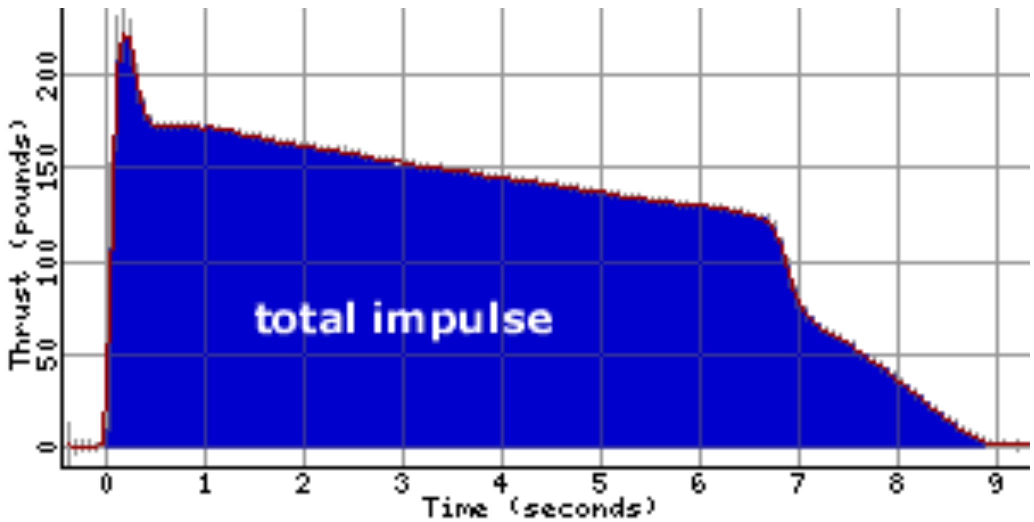
$$I_{sp} = \frac{1}{g_0} \frac{F_{thrust}}{\dot{m}_{propellant}} \rightarrow \dot{m}_e \equiv \dot{m}_{propellant} \rightarrow$$

$$I_{sp} = \frac{1}{g_0}\left[V_e + \frac{p_e A_e - p_\infty A_e}{\dot{m}_e}\right] \equiv \frac{C_e}{g_0}$$

"Units ~ seconds"

Specific Impulse (cont'd)

- Look at total impulse for a rocket

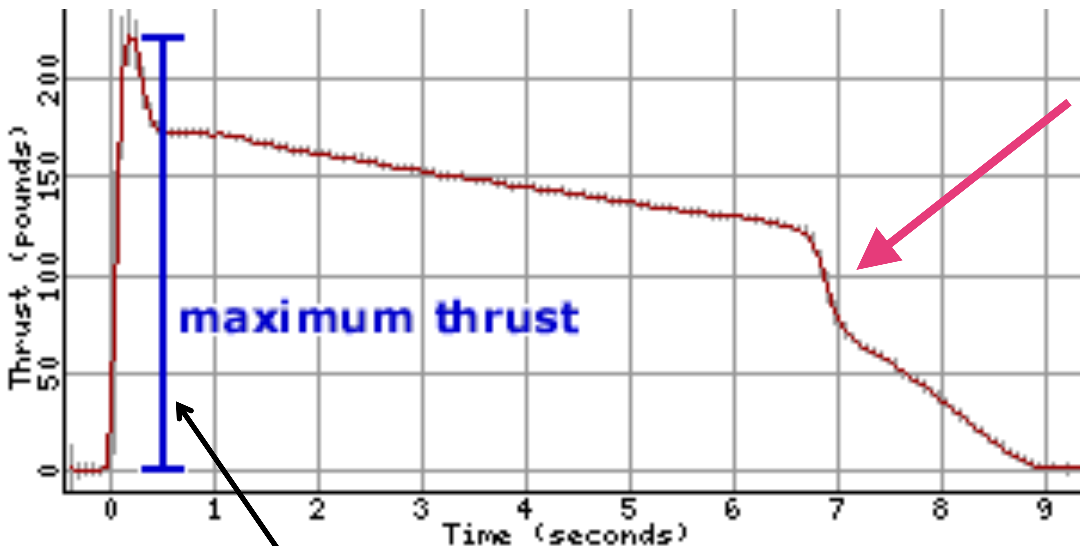


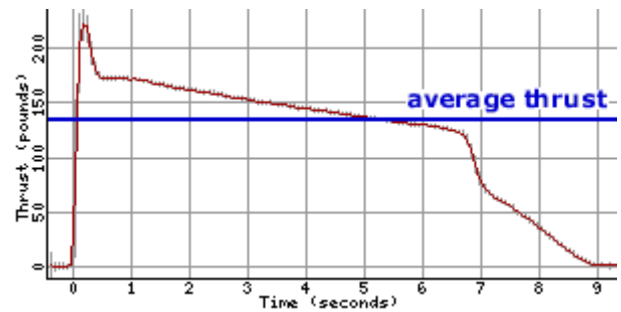
- Mean I_{sp}

$$I_{sp} = \frac{I_{mpulse}}{g_0 M_{propellant}} = \frac{\int_0^t F_{thrust}\, dt}{g_0 \int_0^t \dot{m}_{propellant}\, dt}$$

Specific Impulse (cont'd)

- Look at instantaneous impulse for a rocket





Instantaneous $\dot{m}_{propellant}$

$$I_{sp} = \frac{1}{g_0}\frac{F_{thrust}}{\dot{m}_{propellant}}$$

- *Not necessarily the same*

Example 2

- A man is sitting in a rowboat throwing bricks over the stern. Each brick weighs 5 lbs, he is throwing six bricks per minute, at a velocity of 32 fps. What is his thrust and I_{sp}?

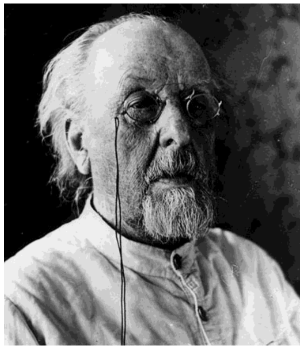
$$F = \dot{m}_{propellant} C_e = \frac{6_{bricks}}{1\,\text{min}} \times 5\,\frac{lbm}{brick} \times 32\,\frac{ft}{\text{sec}} \times \frac{1\,\text{min}}{60\,\text{sec}} =$$

$$\frac{6 \times 5 \times 32}{60}\,\frac{lbm - ft}{\text{sec}^2} \quad ...ooops...need....g_c$$

That is why we prefer metric Units!

$$F = \frac{6 \times 5 \times 32}{60}\,\frac{lbm - ft}{\text{sec}^2} \times \frac{1}{32.1742\,\frac{lbm - ft}{lbf - \text{sec}^2}} =$$

$$I_{sp} = \frac{F}{\dot{m}_{propellant}\, g_0} = \frac{0.497\,lbf \times 32.1742\,\frac{ft}{\text{sec}^2}}{\frac{6_{bricks}}{1\,\text{min}} \times 5\,\frac{lbm}{brick} \times \frac{1\,\text{min}}{60\,\text{sec}} \times 32.1742\,\frac{lbm - ft}{lbf - \text{sec}^2}} = 0.994\,\text{sec}$$



How Much Fuel? "The Rocket Equation"

Conservation of momentum leads to the so-called rocket equation, which trades off exhaust velocity with payload fraction. Based on the assumption of short impulses with coast phases between them, it applies to chemical and nuclear-thermal rockets. First derived by Konstantin Tsiolkowsky in 1895 for straight-line rocket motion with constant exhaust velocity, it is also valid for elliptical trajectories with only initial and final impulses.

Re-visit
Newton's Third Law (cont'd)

- Dividing by Δt and evaluating limit {ΔM , ΔV , Δt} -> 0

$$\left[\frac{M \frac{\Delta V}{\Delta t} = \frac{\Delta M}{\Delta t} U_{ex} + \Delta M \frac{\Delta V}{\Delta t}}{\lim \{ \Delta M, \Delta V, \Delta t \} \Rightarrow 0} \right] =$$

$$\Downarrow$$

$$M \frac{dV}{dt} = \frac{dM}{dt} U_{ex} = \dot{m}_{propellant} C_e$$

Rocket Equation (cont'd)

$$M\frac{dV}{dt} = \dot{m}_{propellant} C_e = g_0 I_{sp} \dot{m}_{propellant} \rightarrow \dot{m}_{propellant} = -\frac{dM}{dt}$$

$$\frac{dV}{dt} = g_0 I_{sp} \frac{-\frac{dM}{dt}}{M} \rightarrow \boxed{dV = -g_0 I_{sp} \frac{dM}{M}}$$

M -> rocket mass

- Assuming constant I_{sp} and burn rate …. integrating over a burn time t_{burn}

$$V_{final} - V_0 = -g_0 I_{sp} \ln\left[M_{final}\right] + g_0 I_{sp} \ln\left[M_0\right] = g_0 I_{sp} \ln\left[\frac{M_0}{M_{final}}\right]$$

$$V_{final} = V_0 + g_0 I_{sp} \ln\left[\frac{M_0}{M_{final}}\right]$$

Rocket Equation (cont'd)

- Consider a rocket burn of duration t_{burn}

"the Rocket Equation"

$$V_{final} = V_0 + g_0 I_{sp} \ln\left[\frac{M_0}{M_{final}}\right]$$

Initial Mass

Initial Velocity

Final Mass

Final Velocity

$$M_{final} = M_0 - \dot{m}_{propellant} \times t_{burn}$$

Rocket Equation (cont'd)

- Continuing with the Rocket Equation, Solve for Initial Mass

$$\ln\left(\frac{M_0}{M_{final}}\right) = \frac{V_{final} - V_0}{g_0 \cdot I_{sp}} \rightarrow \frac{M_0}{M_{final}} = e^{\frac{V_{final} - V_0}{g_0 \cdot I_{sp}}} = \frac{M_{final} + M_{\substack{propellant \\ burned}}}{M_{final}} = 1 + \frac{M_{\substack{propellant \\ burned}}}{M_{final}}$$

- Solve for Propellant Burned

$$M_{\substack{propellant \\ burned}} = M_{final} \cdot \left(e^{\frac{V_{final} - V_0}{g_0 \cdot I_{sp}}} - 1\right) \rightarrow \begin{array}{c} \Delta V \equiv V_{final} - V_0 \\ M_{final} = M_{"dry"} + M_{payload} \end{array}$$

$$M_{\substack{propellant \\ burned}} = \left(M_{"dry"} + M_{payload}\right) \cdot \left(e^{\frac{\Delta V_0}{g_0 \cdot I_{sp}}} - 1\right)$$

Mass Budget Equation

Specific Impulse (revisited)

- For chemical Rockets, I_{sp} depends on the type of fuel/oxydizer used

Vacuum Isp

Fuel	Oxidizer	Isp (s)
Liquid propellants		
Hydrogen (LH2)	Oxygen (LOX)	450
Kerosene (RP-4)	Oxygen (LOX)	260
Monomethyl hydrazine	Nitrogen Tetraoxide	310
Solid propellants		
Powered Al	Ammonium Perchlorate	270

450 sec is "best you can get" with chemical rockets

Homework, Section 2

• Specific Impulse is a commonly used measure of performance For Rocket Engines,and for steady state-engine operation is defined As:

$$I_{sp} = \frac{1}{g_0}\frac{F_{thrust}}{\dot{m}_{propellant}} \rightarrow g_0 = 9.8067 \ \frac{m}{\sec^2} \ (mks)$$

• At 100% Throttle a Solid Rocket Motor has the Following performance characteristics

F_{vac}	=	90,000 Nt
F_{sl}	=	60,000 Nt
Isp_{vac}	=	280 sec.

Homework, Section 2

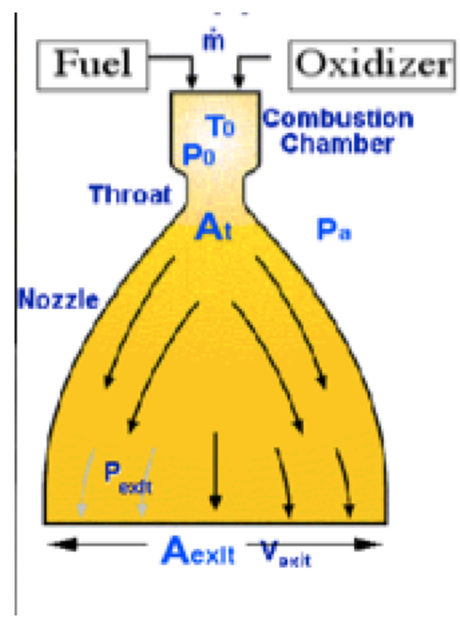
- Additional data

p_∞ Sea level	-- 101.325 kpa
p_∞ 20km altitude	-- 5.4748 kpa
p_e	-- 35 .000 kpa

- Exit nozzle gas has a molecular weight of 19.4831 kg/kg-mole

C_p=1649.18 J/kg-K, …… T_{exit} = 1800 °K

- What is the diameter of the Nozzle exit?
- What is the Nozzle Exit Mach Number
- What is the Specific Impulse and Thrust of the Rocket motor at 20 km altitude?



Homework, Section 2, Concluded

- ***If our Rocket Weighs 10,000 kg Dry, how Much Propellant is Needed to Accelerate by 1 km/sec @ 20,000 meters altitude***

change in velocity