Section 3: Lecture 1
Introduction to One-Dimensional Compressible Flow

Anderson: Chapter 3 pp. 71-86
Review, General Integral Form of Equations of Compressible Flow

- **Continuity** (conservation of mass)

\[
- \iiint_{C.V} \left( \rho \mathbf{V} \cdot d\mathbf{s} \right) = \frac{\partial}{\partial t} \left( \iiint_{C.V} \rho \mathbf{V} \cdot d\mathbf{s} \right)
\]

- **Newton’s Second law** -- Time rate of change of momentum

Equals integral of external forces

\[
\iiint_{C.V} \rho \mathbf{f}_b \, d\mathbf{v} - \iint_{C.S.} (p) \, d\mathbf{s} + \iint_{C.S.} \mathbf{f} \times d\mathbf{S} \, \frac{d\mathbf{S}}{S_{C.S.}} = \\
\iiint_{C.S.} \left( \rho \mathbf{V} \cdot d\mathbf{s} \right) \mathbf{V} + \iiint_{C.V.} \frac{\partial}{\partial t} \left( \rho \mathbf{V} \right) d\mathbf{v}
\]
Review (concluded)

• Conservation of Energy--

\[
\frac{\partial}{\partial t} \iiint_{C.V.} \left( \rho \left( e + \frac{V^2}{2} \right) \right) dv + \iint_{C.S.} \rho \vec{V} \cdot d\vec{S} \left( e + \frac{V^2}{2} \right) = \\
\iiint_{C.V.} (\rho f \, d\vec{v}) \cdot \vec{V} - \iint_{C.S.} (pd \, d\vec{S}) \cdot \vec{V} + \iiint_{C.V.} \left( \rho \left( q \right) \right) dv
\]
One Dimensional Flow Approximations

• Many Useful and practical Flow Situations can be Approximated by one-dimensional flow analyses

• “Air-goes-in, Air-goes-out, or both”
Control Volume for 1-D Flow

- Between C.V. entry (1) and exit (2), there could be
  1) Normal Shock wave (supersonic engine duct),
  2) Heat could be added or subtracted, (heat exchanger), or
  3) There could we work performed (turbine element)

Flow Characterized by motion only along longitudinal axis
Continuity for Steady 1-D Flow

- The general equation for Continuity greatly simplifies for steady 1-D flow

\[- \iint_{C.S.} (\rho \mathbf{V} \cdot \mathbf{d}s) = \frac{\partial}{\partial t} \left( \iiint_{c.v.} \rho d\mathbf{v} \right)\]

\[\iint_{C.S.} (\rho \mathbf{V} \cdot \mathbf{d}s) = 0\]
Continuity for Steady 1-D Flow (cont’d)

- Evaluating surface integral across the C.S.

$$\int \int C.S. \left( \rho \vec{V} \cdot \vec{d}s \right) = 0$$

$$\int \int C.S. \left( \rho \vec{V} \cdot \vec{d}s \right) = \rho V A_e$$

$$\int \int C.S. \left( \rho \vec{V} \cdot \vec{d}s \right) = -\rho V A_i$$

$$\int \int C.S. \left( \rho \vec{V} \cdot \vec{d}s \right) = 0$$
Continuity for Steady 1-D Flow (concluded)

- For true 1-D flow $A_e = A_i$

\[ \Rightarrow \rho_i V_i = \rho_e V_e \]
Momentum for Steady 1-D Flow

• General equation for Momentum greatly simplifies if we assume inviscid Flow with no body forces

\[
\iiint \rho \mathbf{f} \cdot d\mathbf{v} - \iint (p) dS + \iiint \mathbf{f} \times d\mathbf{S} \cdot \frac{d\mathbf{S}}{S_{c.s.}} = \iint (\rho \mathbf{V} \cdot d\mathbf{S}) \mathbf{V} + \iiint \frac{\partial}{\partial t} (\rho \mathbf{V}) d\mathbf{v}
\]

\[
\iiint (\rho \mathbf{V} \cdot d\mathbf{S}) \mathbf{V} = -\iint (p) dS
\]
Momentum for Steady 1-D Flow (cont’d)

• General equation for Momentum greatly simplifies if we assume inviscid Flow with no body forces

• Evaluating momentum surface integral across the C.S.

\[ \iiint (\rho \vec{V} \cdot d\vec{S}) dV = 0 \]

\[ \iiint (\rho \vec{V} \cdot d\vec{S}) dV = \rho_e V_e^2 A_e \]

\[ \iiint (\rho \vec{V} \cdot d\vec{S}) dV = -\rho_i V_i^2 A_i \]

\[ \iiint (\rho \vec{V} \cdot d\vec{S}) dV = 0 \]
Momentum for Steady 1-D Flow (concluded)

- Evaluating pressure surface integral across the C.S. Because of flow “symmetry” upper and lower flow surfaces pressure forces “cancel out”

\[-\int \int (p) dS = -p_e A_e\]

- Summing up terms and once again enforcing, \(A_i = A_e\)

\[p_i + \rho_i V_i^2 = p_e + \rho_e V_e^2\]
Energy Equation for Steady 1-D Flow

• Assume no body forces, and steady flow

\[
\frac{\partial}{\partial t} \iint_{C.V.} \left( \rho \left( e + \frac{V^2}{2} \right) \right) dv + \iiint_{C.S.} \rho \vec{V} \cdot d\vec{S} \left( e + \frac{V^2}{2} \right) = \\
\iiint_{C.V.} (\rho f) dv \cdot \vec{V} - \iiint_{C.S.} (pd\vec{S}) \cdot \vec{V} + \iiint_{C.V.} \left( \rho \left( q \right) \right) dv
\]

• Let \( \iiint_{C.V.} \left( \rho \left( q \right) \right) dv = \dot{Q} \)
Energy Equation for Steady 1-D Flow (cont’d)

- Energy equation reduces to

\[
\dot{Q} - \oint (pdS) \cdot \vec{V} = \oint \rho \left( e + \frac{\|V^2\|}{2} \right) \vec{V} \cdot d\vec{S}
\]

- Evaluating Surface Integrals

\[
-\oint (pdS) \cdot \vec{V} = p_i V_i A_i
\]

\[
\oint (pdS) \cdot \vec{V} = 0
\]

\[
-\oint (pdS) \cdot \vec{V} = -p_e V_e A_e
\]

\[
\oint (pdS) \cdot \vec{V} = 0
\]

\[
-\oint (pdS) \cdot \vec{V} = p_i V_i A_i
\]
Energy Equation for Steady 1-D Flow (cont’d)

- Continuing with Surface Integrals

\[ \iint_{c.s.} \rho \left( e + \frac{\| V \|^2}{2} \right) \cdot V \, d\hat{S} \equiv 0 \]

\[ \iint_{c.s.} \rho \left( e + \frac{\| V \|^2}{2} \right) \cdot V \, d\hat{S} \equiv \rho_e \left( e_e + \frac{\| V_e \|^2}{2} \right) V_e \cdot A_e \]

\[ \iint_{c.s.} \rho \left( e + \frac{\| V_i \|^2}{2} \right) \cdot V_i \, d\hat{S} \equiv \rho_i \left( e_i + \frac{\| V_i \|^2}{2} \right) V_i \cdot A_i \]
Energy Equation for Steady 1-D Flow (cont’d)

• Collecting Terms and enforcing $A_i = A_e = A_c$

\[
\dot{Q} \frac{p_i}{A_c} V_i + \rho_i \left( e_i + \frac{V_i^2}{2} \right) V_i = p_e V_e + \rho_e \left( e_e + \frac{V_e^2}{2} \right) V_e
\]

• Dividing thru by $\rho_i V_i$

\[
\dot{Q} \frac{p_i}{\rho_i V_i A_c} + \frac{p_i}{\rho_i} + \frac{V_i^2}{2} = \frac{p_e}{\rho_e} + e_e + \frac{V_e^2}{2} = \dot{Q} \frac{m}{\rho_i} + e_i + \frac{V_i^2}{2} = \frac{p_e}{\rho_e} + e_e + \frac{V_e^2}{2}
\]
Energy Equation for Steady 1-D Flow (concluded)

- Recalling from the basic thermodynamics lecture (definition of Enthalpy)

\[ h = e + pv = e + \frac{p}{\rho} \]

- The energy equation for 1-D steady, inviscid flow becomes

\[ q + h_i + \frac{V_i^2}{2} = h_e + \frac{V_e^2}{2} \]

\[ \frac{Q}{m} = q \]
1-D, Steady, Flow: Collected Equations

- **Continuity**
  \[ \rightarrow \rho_i V_i = \rho_e V_e \]

- **Momentum**
  \[ \rightarrow p_i + \rho_i V_i^2 = p_e + \rho_e V_e^2 \]

- **Energy**
  \[ \rightarrow q + h_i + \frac{V_i^2}{2} = h_e + \frac{V_e^2}{2} \]
Mach Number: Revisited

• From the definition of $c_v$

\[ c_v = c_p - R_g \rightarrow \]

\[ c_v = R_g \left( \frac{c_p}{R_g} - 1 \right) = R_g \left( \frac{c_p}{c_p - c_v} - 1 \right) = \]

\[ R_g \left( \frac{c_p - c_v + c_v}{c_p - c_v} \right) = R_g \left( \frac{c_v}{c_p - c_v} \right) = \left( \frac{R_g}{\gamma - 1} \right) \]
Mach Number: Revisited (cont’d)

• Recall from the fundamental definition

\[ M = \frac{V}{c} = \frac{V}{\sqrt{\gamma R_g T}} \]

• But if we take the ratio or kinetic to internal energy of a fluid element

\[ \frac{V^2}{2e} = \frac{V^2}{c_v T} \]

• calorically perfect gas
Mach Number: Revisited (cont’d)

• Then

\[
\frac{V^2}{2} = \frac{V^2}{\frac{e}{c_v T}} = \frac{V^2}{\frac{R_g}{\gamma - 1} T}
\]

\[
\gamma \frac{V^2}{2} = \gamma (\gamma - 1) \frac{V^2}{\frac{R_g T}{\gamma - 1}} = \gamma (\gamma - 1) \frac{V^2}{\gamma R_g T} = \gamma (\gamma - 1) M^2
\]

i.e. Mach number is a measure of the ratio of the fluid Kinetic energy to the fluid internal energy (direct motion
To random thermal motion of gas molecules)
Mach Number: Revisited  (cont’d)

• Look at the Steady Flow Energy equation With no heat addition

\[ h_i + \frac{V_i^2}{2} = h_e + \frac{V_e^2}{2} \rightarrow h + \frac{V^2}{2} = \text{const} \]

• Look at Differential form

\[ d \left( h + \frac{V^2}{2} \right) = 0 \rightarrow dh + VdV = 0 \]
Mach Number: Revisited (cont’d)

• For a reversible process

\[ Tds = dh - vdp \]

• With no heat addition, \( ds = 0 \)

\[ dh = vdp = \frac{dp}{\rho} \]

• Subbing into previous

\[ dp = -\rho V dV \] ("euler’s equation")
Mach Number: Revisited (cont’d)

• Now look at Continuity equation

\[ \rho_i V_i = \rho_e V_e \rightarrow \rho AV = \text{const} \]

\[ \frac{d(\rho AV)}{\rho AV} = 0 = \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} \]

• Substituting in Euler’s Equation

\[ dp = -\rho V dV \]

\[ \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{-\rho V}{V} = 0 \]
Mach Number: Revisited (cont’d)

• Solving for $dA/A$

$$\frac{dA}{A} = \frac{dp}{\rho} \left( \frac{1}{V^2} - \frac{d\rho}{\rho} \right) = \frac{dp}{\rho} \left( \frac{1}{V^2} - \frac{d\rho}{dp} \right)$$

• But since we are considering a process with $ds=0$

$$\frac{d\rho}{dp} = \frac{1}{c^2} \quad \text{and} \quad \frac{dA}{A} = \frac{dp}{\rho} \left( \frac{1}{V^2} - \frac{1}{c^2} \right) = \frac{(1 - M^2)}{\rho V^2} dp$$

• Substituting in Euler’s Equation $dp = -\rho V dV$

$$\frac{dA}{A} = \frac{(1 - M^2)}{\rho V^2} (-\rho V dV) = (M^2 - 1) \left( \frac{dV}{V} \right)$$
Mach Number: Revisited (cont’d)

- Rearranging gives the TWO relationships
  
  Whose ramifications are fundamental to this class

\[
\left( \frac{dV}{dA} \right) = \frac{1}{(M^2 - 1)} \frac{V}{A}
\]

\[
\frac{d\rho}{dA} = \frac{-1}{(M^2 - 1)} \frac{\rho V^2}{A}
\]
Mach Number: Revisited (cont’d)

\[
\left( \frac{dV}{dA} \right) = \frac{1}{\left(M^2 - 1\right)} \frac{V}{A}
\]

\[
\frac{dp}{dA} = \frac{-1}{\left(M^2 - 1\right)} \frac{\rho V^2}{A}
\]

\[
M < 1 \rightarrow \left( \frac{dV}{dA} \right) < 0 \rightarrow \left( \frac{dp}{dA} \right) > 0
\]

\[
M > 1 \rightarrow \left( \frac{dV}{dA} \right) > 0 \rightarrow \left( \frac{dp}{dA} \right) < 0
\]
Fundamental Properties of Supersonic and Supersonic Flow

Subsonic Diffuser
Flow \((M<1)\)
- \(p\) increases
- \(v\) decreases

Subsonic Nozzle
Flow \((M<1)\)
- \(p\) decreases
- \(v\) increases

Supersonic Nozzle
Flow \((M>1)\)
- \(p\) decreases
- \(v\) increases

Supersonic Diffuser
Flow \((M>1)\)
- \(p\) increases
- \(v\) decreases
... Hence the shape of the rocket Nozzle
Why does a rocket nozzle look like this?

Expanding/accelerating flow

\[ M < 1 \]

\[ \left( \frac{dA}{A} \right) < 0 \]

\[ \left( \frac{dV}{V} \right) > 0 \]

\[ \left( \frac{dA}{A} \right) = 0 \quad \ldots \quad M = 1 \]

\[ M > 1 \]

\[ \left( \frac{dA}{A} \right) > 0 \]

\[ \left( \frac{dV}{V} \right) > 0 \]
What is a NOZZLE

- FUNCTION of rocket nozzle is to convert thermal energy in propellants into kinetic energy as efficiently as possible.
- Nozzle is substantial part of the total engine mass.
- Many of the historical data suggest that 50% of solid rocket failures stemmed from nozzle problems.

The design of the nozzle must trade off:
1. Nozzle size (needed to get better performance) against nozzle weight penalty.
2. Complexity of the shape for shock-free performance vs. cost of fabrication.
Temperature/Entropy
Diagram for a Typical Nozzle

\[ q + h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \]

\[ c_p = \left( \frac{dh}{dT} \right)_p \]

\[ Tds = \delta q + ds_{irrev} \]
• But De Laval Discovered that when the Nozzle throat Area was adjusted downward until so that the pressure ratio \( \frac{p_t}{p_I} < 0.5484 \) -> then the exit Pressure dropped dramatically And the exit velocity rose significantly … Which is counter to What Bernoulli’s law predicts … he had inadvertently Generated supersonic flow! …