Section 3: Lecture 2
Alternative Forms of the One-Dimensional Energy Equation

Anderson: Chapter 3 pp. 71-86
Review: 1-D, Steady, Flow: Collected Equations

• Continuity

\[ \rightarrow \rho_i V_i = \rho_e V_e \]

• Momentum

\[ \rightarrow p_i + \rho_i V_i^2 = p_e + \rho_e V_e^2 \]

• Energy

\[ \rightarrow q + h_i + \frac{V_i^2}{2} = h_e + \frac{V_e^2}{2} \]
Review:
Effect of Mach Number on Flow Properties

Mach number is a measure of the ratio of the fluid
Kinetic energy to the fluid internal energy (direct motion
To random thermal motion of gas molecules)

\[
\frac{V^2}{2} = \frac{V^2}{2c_vT} = \frac{V^2}{2\left(\frac{R_g}{\gamma - 1}\right)T} = \gamma \frac{V^2}{2\left(\frac{R_g}{\gamma - 1}\right)T} = \gamma \left(\frac{\gamma - 1}{2}\right) \frac{V^2}{\gamma R_g T} = \frac{\gamma (\gamma - 1)}{2} M^2
\]
Review:
Effect of Mach Number on Flow Properties

\[
\left(\frac{dV}{dA}\right) = \frac{1}{(M^2 - 1)} \frac{V}{A}
\]

\[
dp = \frac{-1}{(M^2 - 1)} \frac{\rho V^2}{A}
\]

\[
M < 1 \rightarrow \left(\frac{dV}{dA}\right) < 0 \rightarrow \left(\frac{dp}{dA}\right) > 0
\]

\[
M > 1 \rightarrow \left(\frac{dV}{dA}\right) > 0 \rightarrow \left(\frac{dp}{dA}\right) < 0
\]
Review:
Fundamental Properties of Supersonic and Supersonic Flow
Stagnation Temperature for the Adiabatic Flow of a Calorically Perfect Gas

- Consider an adiabatic flow field with a local gas Temperature $T(x)$, pressure $p(x)$, and a velocity $V(x)$

- Since the Flow is adiabatic

$$h(x) + \frac{V(x)^2}{2} = c_p T(x) + \frac{V(x)^2}{2} = Const$$
Stagnation Temperature for the Adiabatic Flow of a Calorically Perfect Gas
(cont’d)

• Now define a condition at some location \( x' \), within this flow field where the gas velocity is reduced to zero with no heat loss

\[
T_0(x') = x' - P_0(x') - V_0(x') = 0
\]

• Since the flow is adiabatic

\[
c_p T(x) + \frac{V(x)^2}{2} = c_p T_0 \rightarrow T_0 = T(x) + \frac{V(x)^2}{2c_p}
\]
Stagnation Temperature for the Adiabatic Flow of a Calorically Perfect Gas
(cont’d)

• From Earlier Analysis

\[
\frac{V^2}{2c_vT} = \frac{\gamma}{2} (\gamma - 1) M^2 \rightarrow \\
\frac{V^2}{2\gamma c_v} = \frac{V^2}{2c_p c_v} = \frac{V^2}{2c_p} = T \left(\frac{\gamma - 1}{2}\right) M^2
\]

• Therefore

\[
T_o = T(x) + T(x) \left(\frac{\gamma - 1}{2}\right) M(x)^2
\]

Holds anywhere Within an adiabatic Flow field
Stagnation Temperature for the Adiabatic Flow of a Calorically Perfect Gas (cont’d)

- In general for an adiabatic Flow Field the Stagnation Temperature is defined by the relationship

\[
\frac{T_0}{T} = 1 + \frac{(\gamma - 1)}{2} M^2
\]

- Stagnation Temperature is Constant Throughout An adiabatic Flow Field

- \(T_0\) is also sometimes referred to at Total Temperature

- \(T\) is sometimes referred to as Static Temperature
Stagnation Temperature for the Adiabatic Flow of a Calorically Perfect Gas (cont’d)

- Stagnation temperature is a measure of the Kinetic Energy of the flow Field.

- Largely responsible for the high Level of heating that occurs on high speed aircraft or reentering space Vehicles …

\[ T_0 = T \cdot \left(1 + \frac{\gamma - 1}{2} \cdot M^2 \right) \]
Hypersonic Flow of a “Real Gas”

What Characterizes Hypersonic Flow?

- As the Mach number increases, the shock angle becomes smaller, and distance between surface and shock wave decreases with increasing speed.

Let’s Revisit Hypersonic Flow
Hypersonic Flow of a Real Gas (cont’d)

• For a hypersonic body, shock distance from surface can become very small over a large portion of the body
  -- resulting flowfield between the surface and shock is often referred to as a shock layer.

• Thin layer can produce many complications in vehicle design, e.g. the shock layer may merge with the boundary layer at low Reynolds numbers to form a fully viscous shock layer.

• In the limit as Mach number goes to infinity, the shock layer forms an infinitely thin, infinitely dense sheet, or, essentially, a flat plate.
Hypersonic Flow of a Real Gas (cont’d)
Hypersonic Flow of a Real Gas (cont’d)
Hypersonic Flow of a Real Gas (cont’d)

• Hypersonic vehicles dissipate so much kinetic energy and produce such high temperatures due to shock wave that they cause chemical changes to occur in the fluid through which they fly.

<table>
<thead>
<tr>
<th>Temperature [K]</th>
<th>Chemical Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>800</td>
<td>Molecular vibration</td>
</tr>
<tr>
<td>2000</td>
<td>Oxygen molecules (O₂) dissociate</td>
</tr>
<tr>
<td>4000</td>
<td>Nitrogen molecules (N₂) dissociate</td>
</tr>
<tr>
<td></td>
<td>Nitric oxide (NO) forms</td>
</tr>
<tr>
<td>9000</td>
<td>Oxygen and nitrogen atoms ionize</td>
</tr>
</tbody>
</table>

• Gas chemistry effects are important factor in dissipation of heat on hypersonic vehicles
Hypersonic Flow of a Real Gas (cont’d)

- Across a Hypersonic Shock Wave, Temperature Rises Dramatically

- Thermal Properties ($c_p, c_v, \gamma$) of Gas Change

- $T_0$ not constant across shock

- Gas Dissociation, chemical reaction, molecular Vibration Significantly lower the Stagnation temperature Behind the shock wave when compared To “calorically perfect” gas

- In general Enthalpy is implicit function of Pressure and temperature

\[ h = \eta(T,P) \Rightarrow \text{"non–analytical–function"} \]

\[ c_p = \frac{\partial h}{\partial T} \]

\[ \rightarrow \{c_p, c_v, \gamma\} \neq \text{Const} \]
Hypersonic Flow of a Real Gas (cont’d)

\[ c_p = \frac{\partial h}{\partial T} \]

\[ \rightarrow \left\{ c_p, c_v, \gamma \right\} \neq \text{Const} \]

![Graph showing the ratio of specific heats with labels for Vibration, Dissociation, and Ionization.]
Hypersonic Flow of a Real Gas (cont’d)

Stagnation Temperature

- Constant Density
- Variable Density
- Supersonic
- Hypersonic
- Vibrational Modes
- Dissociation
- Plasma Ionization
- Re-Entry

Mach
Hypersonic Flow of a Real Gas (cont’d)

Vibration
Species remain intact

Dissociation
Creates different gas species

Ionization
Creates different, electrically-charged gas species

Bond Stretching
Valence angle bending
Out-of-plane wagging
What was Stagnation Temperature At Columbia Breakup?

Loss Of Signal at:
61.2 km altitude
~18.0 Mach Number

\[ T_\infty \sim 243 \, ^\circ K \]

\[ \frac{T_0}{T} = 1 + \frac{(\gamma - 1)}{2} M^2 \]

\[ \sim 16,000 \, K \! (15,726 \, ^\circ C) \]

- We’ll revisit this Again later when we talk About Normal Shockwaves And Hypersonic Flow

Shockwaves ar generally “Non-Isentropic”

Hypersonic Shockwaves are “calorically imperfect”
What was Stagnation Temperature

Wow! Hot! At Columbia Breakup (cont’d)

- Real Gas Shockwaves Are “non-adiabatic”

Vibration mode absorbs significant energy

Real gas effects make human spaceflight possible!

Ideal gas $\gamma = 1.4$

Variable $\gamma$

Real gas
Stagnation Pressure for the Isentropic Flow of a Calorically Perfect Gas

- Now consider an isentropic flow field with a local gas temperature $T(x)$, pressure $p(x)$, and a velocity $V(x)$.

Since the flow is isentropic, from Section 1

\[
\frac{p_0}{p} = \left[ \frac{T_0}{T} \right]^\frac{\gamma}{\gamma - 1} = \left[ 1 + \frac{(\gamma - 1)}{2} M^2 \right]^\frac{\gamma}{\gamma - 1} \quad \text{("stagnation" total pressure: Constant throughout isentropic flow field)}
\]

Also, analogous value for density

\[
\frac{\rho_0}{\rho} = \left[ 1 + \frac{(\gamma - 1)}{2} M^2 \right]^\frac{1}{\gamma - 1}
\]
Characteristic or “Sonic” Flow Parameters for Isentropic Flow Fields

- Stagnation properties of a flow field result when flow is isentropically slowed from local velocity to zero.

\[ T_0(x'), \quad P_0(x'), \quad V(x') = 0 \]
Characteristic or “Sonic” Flow Parameters for Isentropic Flow Fields (cont’d)

- Define a new set of parameters where flow field is Either decelerated (supersonic) or accelerated (subsonic)
  - From local velocity to sonic velocity
  - \( \{ T^*, P^*, \rho^* \ldots \} \)
Characteristic or “Sonic” Flow Parameters for Isentropic Flow Fields (cont’d)

\[
\frac{T_0}{T} = 1 + \frac{(\gamma - 1)}{2} M^2 \rightarrow \\
T^* \frac{1}{T_0} = \frac{1}{1 + \frac{(\gamma - 1)}{2} (1)^2} = \frac{2}{\gamma + 1}
\]

\[
\frac{\rho_0}{\rho} = \left[ 1 + \frac{(\gamma - 1)}{2} M^2 \right]^{\frac{1}{\gamma - 1}} \\
\frac{\rho^*}{\rho_0} = \left[ \frac{2}{\gamma + 1} \right]^{\frac{1}{\gamma - 1}}
\]

\[
\frac{p^*}{p_0} = \frac{1}{\left[ 1 + \frac{(\gamma - 1)}{2} (1)^2 \right]^{\frac{\gamma}{\gamma - 1}}} = \left[ \frac{2}{\gamma + 1} \right]^{\frac{\gamma}{\gamma - 1}}
\]
Sonic Velocity, Mach Number Based on $T^*$

- From the definition: 
  \[ T^* = \frac{2}{\gamma + 1} T_0 \]

- Useful to define fictitious parameters
  
  \[ c^* = \sqrt{\gamma R_g T^*} \rightarrow M^* = \frac{V}{\sqrt{\gamma R_g T^*}} \]

- Intermediate analysis tools to be used for normal shock equations
Sonic Velocity, Mach Number Based on T*
(cont’d)

- From Energy equation for adiabatic flow

\[ c_p T + \frac{V^2}{2} = c'_p T^* + \frac{c^*^2}{2} \rightarrow \]

\[ \frac{\gamma}{\gamma - 1} R g T + \frac{V^2}{2} = \frac{\gamma}{\gamma - 1} R g T^* + \frac{c^*^2}{2} \rightarrow \]

\[ \frac{c^2}{\gamma - 1} + \frac{V^2}{2} = \frac{c^*^2}{\gamma - 1} + \frac{c^*^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} c^*^2 \]
Sonic Velocity, Mach Number Based on $T^*$

(continued)

• Reorganizing terms

\[
2 \left( \frac{c}{V} \right)^2 + 1 = \frac{\gamma + 1}{(\gamma - 1)} \left( \frac{c^*}{V} \right)^2 \rightarrow \\
\frac{2}{M^2 (\gamma - 1)} = \frac{\gamma + 1}{M^*^2 (\gamma - 1)} \rightarrow \\
\frac{1}{M^2} = \frac{1}{2} \frac{\gamma + 1}{M^*^2} - \frac{(\gamma - 1)}{2} \rightarrow \\
M^2 = \frac{1}{\frac{1}{\gamma + 1} - \frac{(\gamma - 1)}{2 \ M^*^2}} = \frac{2 M^*^2}{\gamma + 1 - (\gamma - 1) M^*^2}
\]

• $M^* =$ “characteristic Mach number”
Characteristic Mach Number ($M^*$)

- One undesirable characteristic of the true Mach number is that it is not directly indicative of the local flow velocity.
  \(\rightarrow\) Sonic velocity itself is also function of temperature.

- Consider two fluid particles that are proceeding within the first stage of a compressor and that of a turbine section with the same velocity.
  \(\rightarrow\) Because the compressor particle is exposed to a much lower temperature than the turbine particle, the sonic speeds at these locations will be significantly different, and so will the Mach numbers.

- Sometimes it is advantageous to replace the Mach number with an alternative nondimensional-velocity ratio
  \(\rightarrow\) the magnitude of $M^*$ is more directly indicative of the local velocity.
  \(\rightarrow\) $M^*$ approaches a constant value at high velocities
Characteristic Sonic Velocity ($C^*$)
Characteristic Sonic Velocity ($C^*$)
Summary

• Adiabatic Flow Property: \( T_0 = \text{Const} \) (stagnation temperature)

\[
\frac{T_0}{T} = 1 + \frac{(\gamma - 1)}{2} M^2
\]

• Isentropic Flow Property: \( P_0 = \text{Const} \) (stagnation pressure)

\[
\frac{P_0}{p} = \left[ 1 + \frac{(\gamma - 1)}{2} M^2 \right]^{\frac{\gamma}{\gamma - 1}}
\]
Summary (concluded)

• “Characteristic” Flow Properties

\[ T^* = \frac{2}{\gamma + 1} T_0 \rightarrow 0.8333 T_0 \]

\[ p^* = \left[ \frac{2}{\gamma + 1} \right]^{\frac{\gamma}{\gamma - 1}} p_0 \rightarrow 0.5283 T_0 \]

\[ c^* = \sqrt{\gamma R_g T^*} = \sqrt{R_g \frac{2\gamma}{\gamma + 1} T_0} \rightarrow 18.3002 \frac{m}{\sec \sqrt{^o K}} \sqrt{T_o} \]

\[ M^2 = \frac{2M^*^2}{\gamma + 1 - (\gamma - 1)M^*^2} \rightarrow \frac{2M^*^2}{2.4 - 0.4M^*^2} \]

“air at normal temperatures”
Revisit De Laval Nozzle

• What Pressure Ratio “Choked” De Laval’s Nozzles?

Assume \( V_I \approx 0 \)

For \( M_t = 1 \rightarrow p_t = p^* \frac{p_I}{(p_t)^{\frac{\gamma}{\gamma+1}}} \rightarrow \text{steam} \equiv \)

\[
\gamma \approx 1.286 \\
\frac{p_I}{p_t} = 1.824
\]
De Laval Nozzle (cont’d)

• Once $M_t = 1$ throat is “Choked”

Exit pressure no longer has any influence on the massflow through the nozzle, only the upstream pressure value
Homework 3

• Solve for $M^*$ in terms of $M$  $\rightarrow$ Plot result with $M$ as independent variable

• Solve for the Mass Flow per Unit area in a 1-D, steady, isentropic duct flow field as function of $T_0$, $P_0$, $M$, $\gamma$, $R_g$ (hint start with continuity)

\[ \frac{m}{A_c} = \rho V \quad \text{then let} \quad \rho = \frac{\gamma P}{\gamma R_g T} \]
Homework 3 (cont’d)

i.e. … Show that …. for Quasi 1-D isentropic flow

\[
\frac{m}{A} = \sqrt{\frac{\gamma}{R_g}} \frac{P_0}{\sqrt{T_0}} \frac{M}{\left[1 + \frac{(\gamma - 1)}{2} M^2 \right]^\frac{\gamma + 1}{2(\gamma - 1)}}
\]

- Allowing that for isentropic flow .. Also show that for quasi 1-D flow

\[
T_0 = \left(1 + \frac{\gamma - 1}{2} M^2 \right) T \Rightarrow \frac{m}{A} = P_0 \sqrt{\frac{2\gamma}{(\gamma - 1)(R_g \cdot T_0)} \left[\left(\frac{p}{P_0}\right)^\frac{2}{\gamma} - \left(\frac{p}{P_0}\right)^\frac{\gamma + 1}{\gamma}\right]}
\]

\[
P_0 = \left(1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma}{\gamma - 1}}
\]
Homework 3 (cont’d)

Show that …. In general for Quasi 1-D flow

\[ \frac{m}{A} = \sqrt{\frac{\gamma}{R_g}} \frac{p_0}{\sqrt{T_0}} \frac{M}{\gamma + 1} \left[ 1 + \frac{(\gamma - 1)}{2} M^2 \right] \]

• show that massflow per unit area has a maximum value when \( M = 1 \)

\[ \frac{m}{A^*} = \sqrt{\frac{\gamma}{R_g} \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{\gamma - 1}}} \frac{p_0}{\sqrt{T_0}} \]

\[ \text{set} \ldots \frac{\partial}{\partial M} \left( \frac{m}{A} \frac{\sqrt{R_g T_0}}{p_0} \right) = \quad \text{Hint:} \]

\[ \frac{\partial}{\partial M} \left( \frac{\sqrt{\gamma} \cdot M}{\left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma + 1}{2(\gamma - 1)}}} \right) = 0 \ldots & \text{solve for } M \]
Homework (cont’d)

• Plot \( \frac{m}{A_c} \frac{\sqrt{T_0}}{p_0} \sqrt{R_g} \) as a function of mach number

.. Does this plot agree with result on previous page?

• At what mach number does \( \frac{m}{A_c} \frac{\sqrt{T_0}}{p_0} \sqrt{R_g} \) have the greatest value

• What does this result imply?

Assume \( \gamma = 1.4, \ R_g = 287.056 \) j/kg-°K