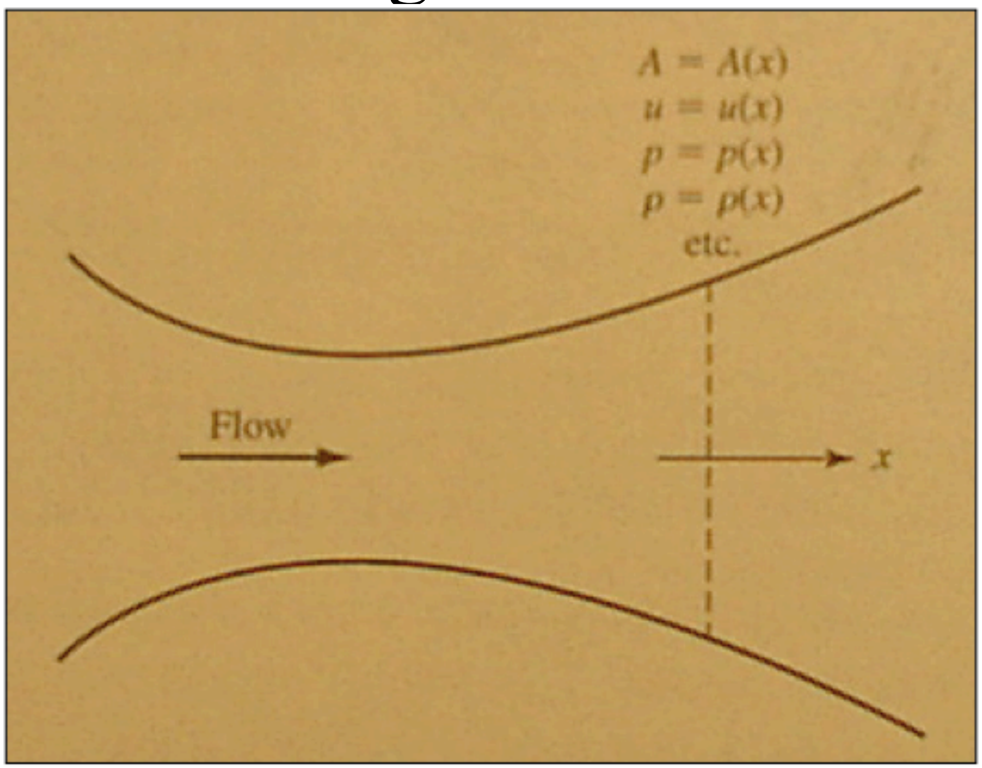
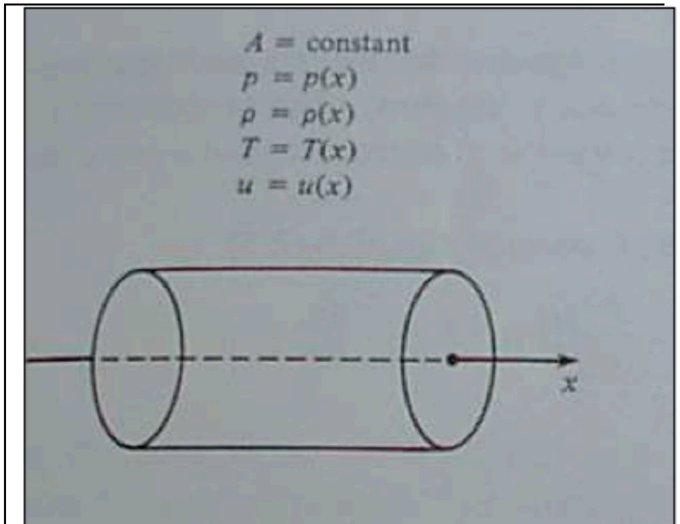
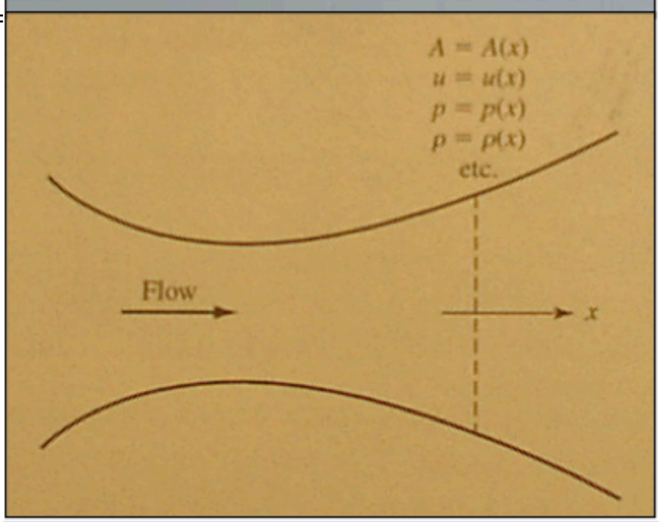
Section 4: Lecture 1
Quasi-One-Dimensional Flow in Convergent/ Divergent Nozzles



Anderson: Chapter 5 pp. 191-226

Distinction Between True 1-D Flow and Quasi 1-D Flow

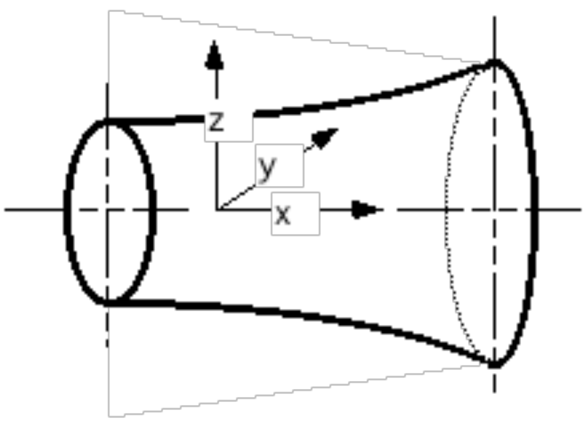




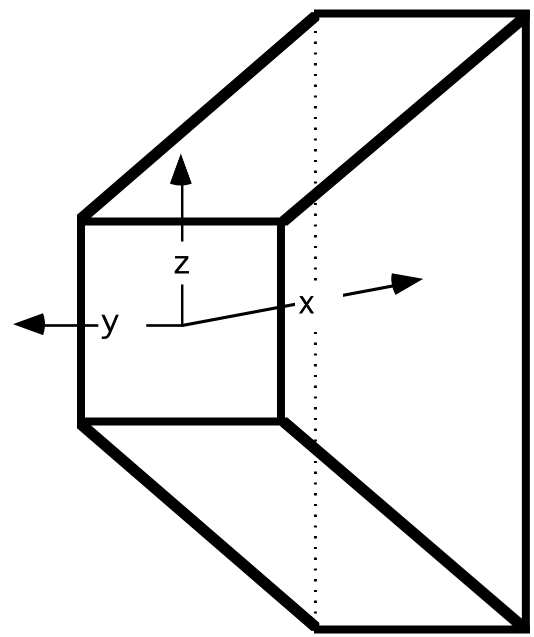
- In “true” 1-D flow Cross sectional area is strictly constant
- In quasi-1-D flow, cross section varies as a Function of the longitudinal coordinate, x
- Flow Properties are assumed constant across any cross-section
- Analytical simplification very useful for evaluating Flow properties in Nozzles, tubes, ducts, and diffusers Where the cross sectional area is large when compared to length

Distinction between True 1-D Flow and Quasi 1-D Flow (cont'd)

• One-D simplification requires axial symmetry in z and y directions
To insure that flow properties are constant across cross section



True Axi-Symmetric Duct
(Circular Cross-section)



Quasi-1-D Rectangular Duct

Review: Governing Equations

- Inviscid, No Body Forces, Steady Flow (As derived in section 3)

- Continuity

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) = 0$$

- Momentum

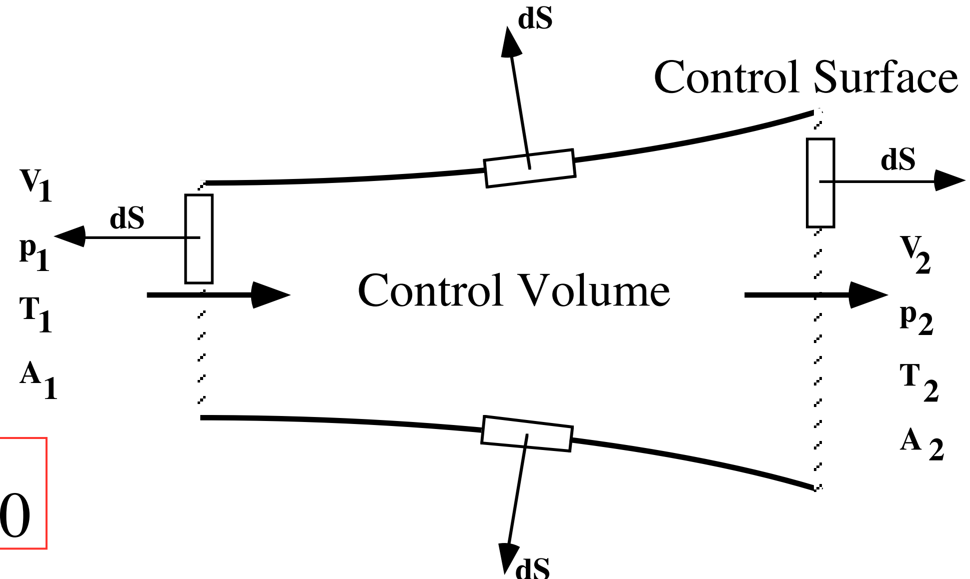
$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} = -\iint_{C.S.} (p) \vec{dS}$$

- Energy

$$\dot{Q} - \iint_{C.S.} (p d\vec{S}) \bullet \vec{V} = \iint_{C.S.} \rho \left(e + \frac{\left\| V^2 \right\|}{2} \right) \vec{V} \bullet d\vec{S}$$

Review: Continuity Equation for Quasi 1-D Control Volume

$$\iint_{C.S.}\left(\rho \vec{V} \bullet \vec{ds}\right) = 0$$



• Upper and Lower Surfaces … no flow across boundary

$$\vec{V} \bullet \vec{ds} = 0$$

• Inlet (properties constant across Cross section) -->

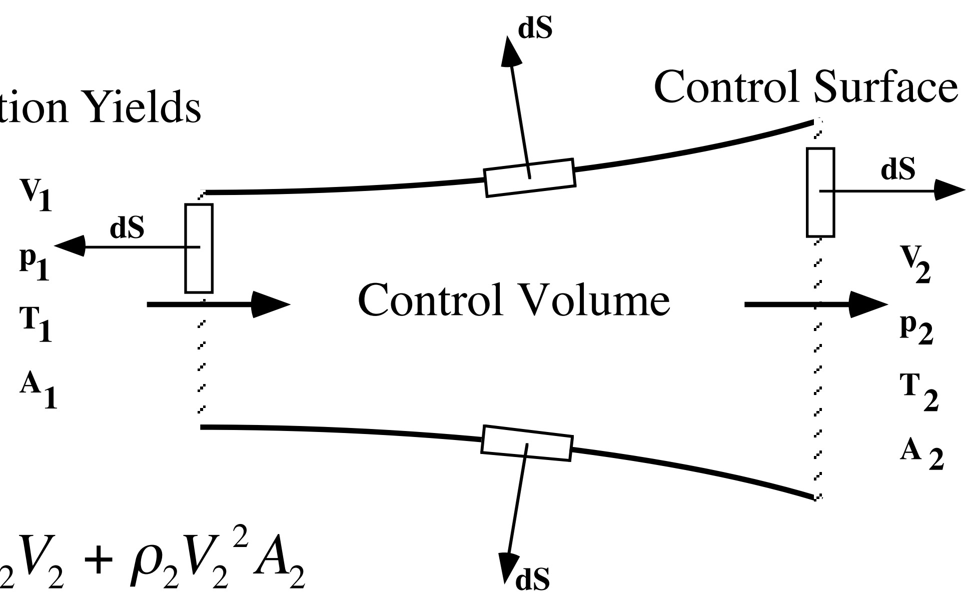
$$\iint_1\left(\rho \vec{V} \bullet \vec{ds}\right) = \rho_1 \vec{V_1} \bullet \iint_1 \vec{ds} = \rho_1 V_1 \cos(180^o) \iint_1 ds = -\rho_1 V_1 A_1$$

• Inlet (properties constant across Cross section) -->

$$\iint_2\left(\rho \vec{V} \bullet \vec{ds}\right) = \rho_2 \vec{V_2} \bullet \iint_2 \vec{ds} = \rho_2 V_2 \cos(0^o) \iint_2 ds = \rho_2 V_2 A_2$$

Review: Momentum Equation for Quasi 1-D Control Volume

- Similar Analysis for Momentum Equation Yields



$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} = -\iint_{C.S.} (p) \vec{dS} \rightarrow$$

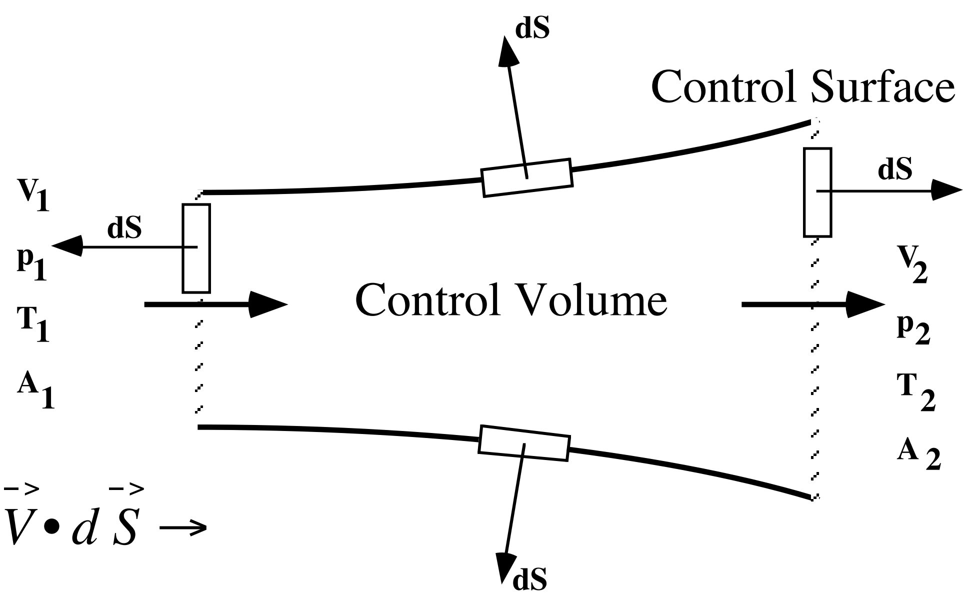
$$p_1 A_1 V_1 + \rho_1 V_1^{\,2} A_1 + \int_1^2 p \, \vec{ds} \bullet \vec{i}_x = p_2 A_2 V_2 + \rho_2 V_2^{\,2} A_2$$

- Because of duct symmetry the "Z-axis" Component of pressure integrated to zero

$\vec{i}_x$	"Unit vector" x-direction

Review: Energy Equation for Quasi 1-D Control Volume

- As derived in section 3



$$\dot{Q} - \iint_{C.S.} (p\, d\vec{S}) \bullet \vec{V} = \iint_{C.S.} \rho \left(e + \frac{\left\| V^2 \right\|}{2} \right) \vec{V} \bullet d\vec{S} \rightarrow$$

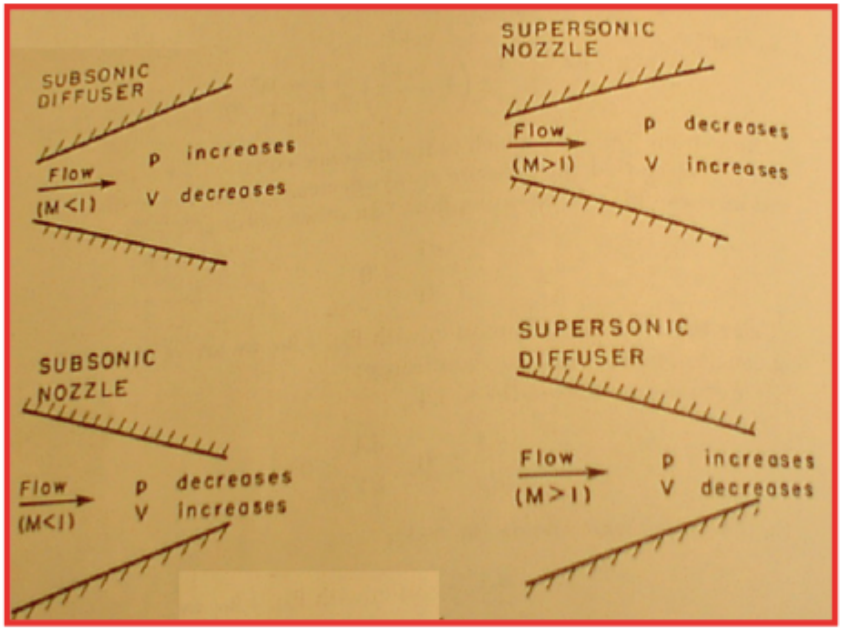
$$h_1 + \frac{V_1^{\,2}}{2} = h_2 + \frac{V_2^{\,2}}{2} = Const$$

$\vec{i}_x$ "Unit vector" x-direction

Review: Area/Velocity and Area Pressure Relationships

$$\left(\frac{dV}{dA}\right) = \frac{1}{\left(M^2 - 1\right)}\frac{V}{A}$$

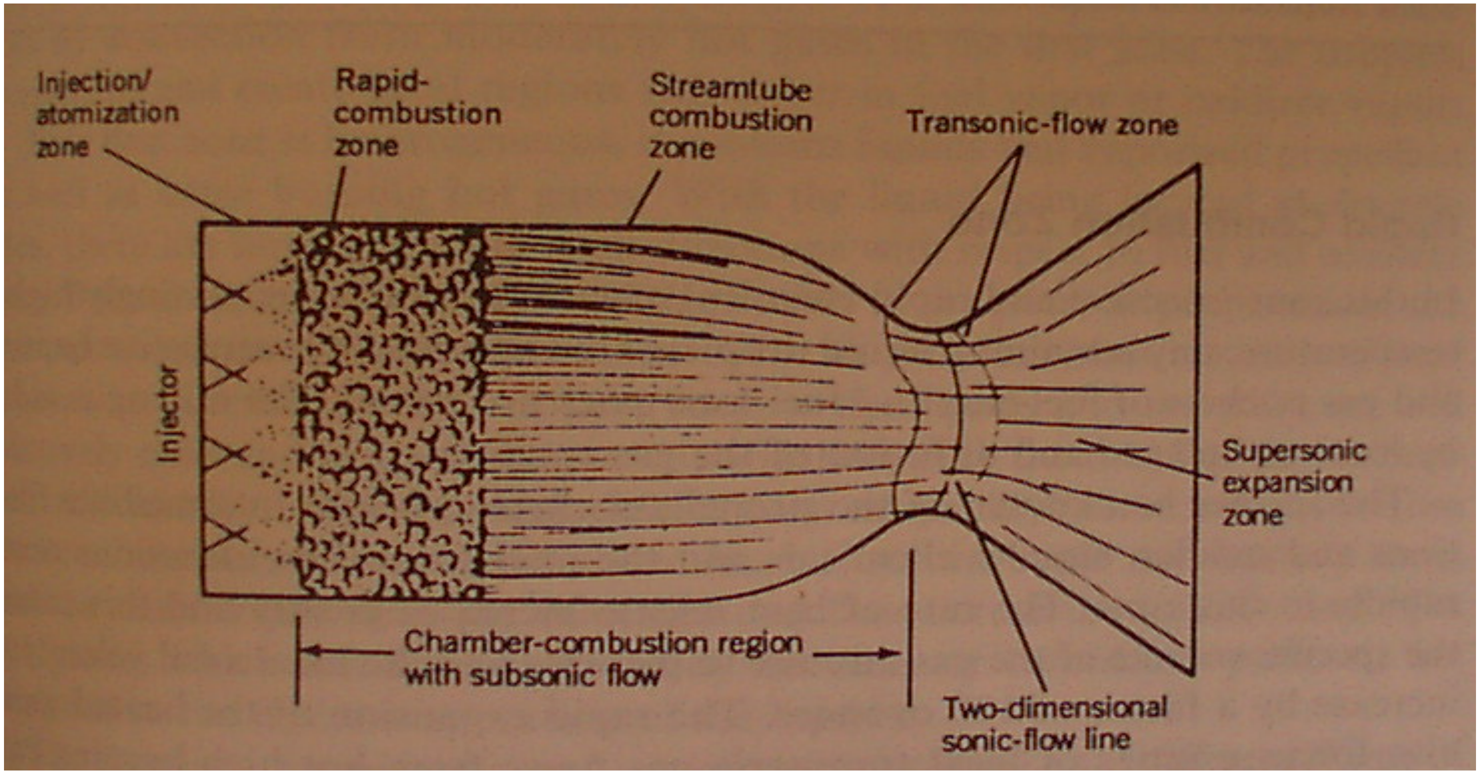
$$\frac{dp}{dA} = \frac{-1}{\left(M^2 - 1\right)}\frac{\rho V^2}{A}$$



$$M < 1 \rightarrow \left(\frac{dV}{dA}\right) < 0 \rightarrow \left(\frac{dp}{dA}\right) > 0$$

$$M > 1 \rightarrow \left(\frac{dV}{dA}\right) > 0 \rightarrow \left(\frac{dp}{dA}\right) < 0$$

What happens in combustion chamber?



P_0, T_0 ~ constant downstream of chamber

What happens in combustion chamber? (2)

- Combustion Produces High temperature gaseous By-products
- Gases Escape Through Nozzle Throat
- Nozzle Throat Chokes (maximum mass flow)
- Since Gases cannot escape as fast as they are produced
  … Pressure builds up
- As Pressure Builds .. Choking mass flow grows
- Eventually Steady State Condition is reached

Choking Massflow per Unit Area

- maximum Massflow/area Occurs when When M=1
- Effect known as *Choking* in a Duct or Nozzle
- i.e. nozzle will Have a mach 1 throat

$$\left(\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}\right)_{\max} = \left(\frac{\dot{m}}{A^*}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}\right) =$$

$$\frac{\sqrt{\gamma}}{\left[1+\frac{(\gamma-1)}{2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} = \sqrt{\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \rightarrow$$

$$\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\,\frac{p_0}{\sqrt{T_0}}$$

Review: Mass flow per Unit Area (1)

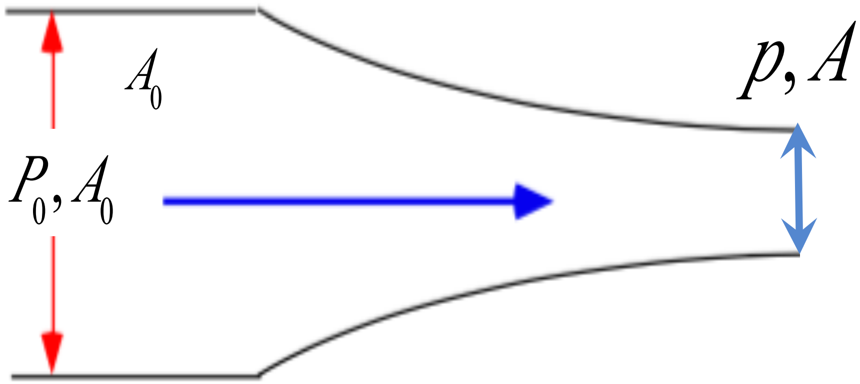
$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R_g}} \frac{p_0}{\sqrt{T_0}} \frac{M}{\left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

From your homework!

- maximum value when M=1

$$\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \frac{p_0}{\sqrt{T_0}}$$

1-D Compressible Mass Flow Equations,



Unchoked Flow $\rightarrow \dfrac{P_0}{p} < \left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$

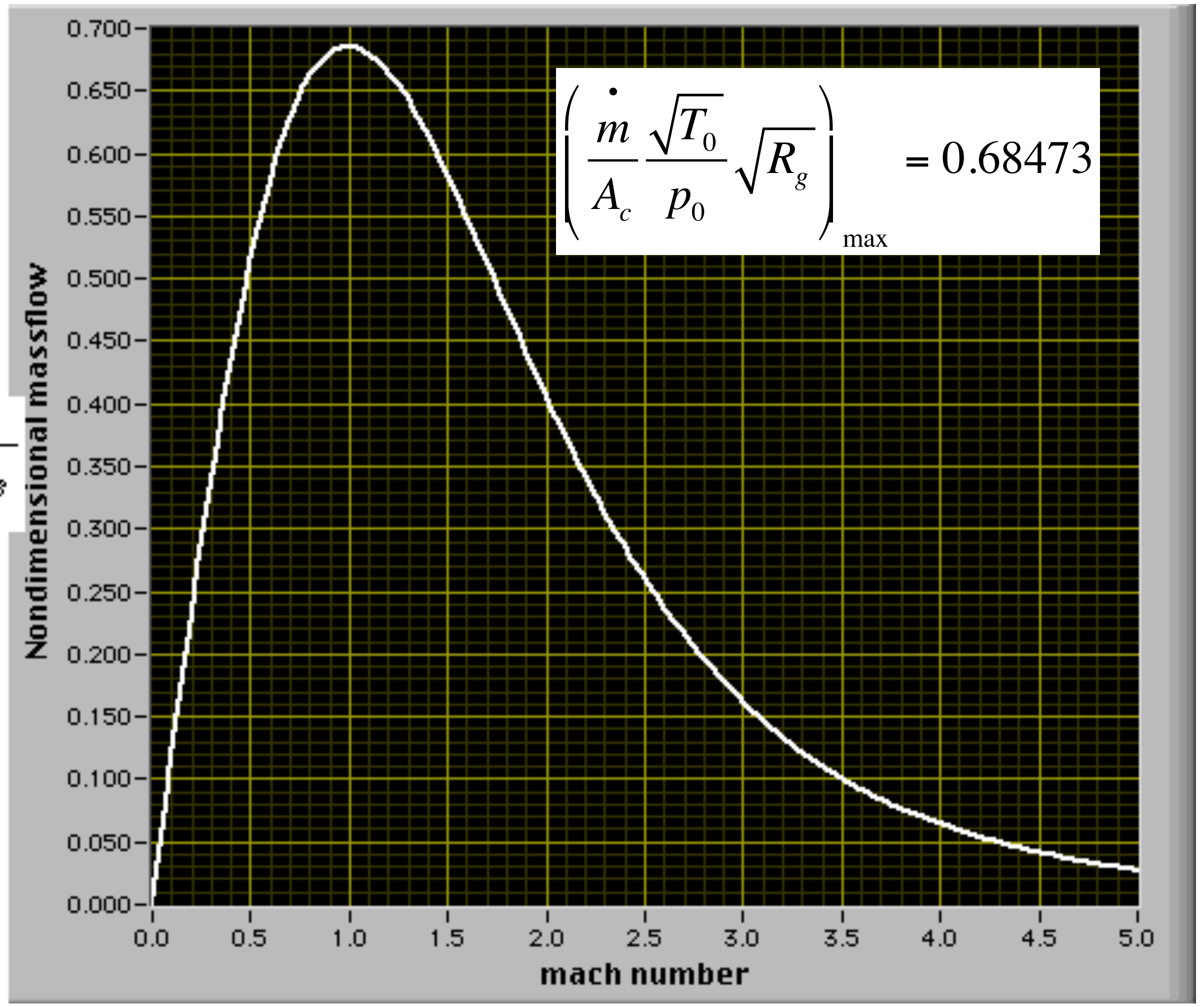
$$\left(\dot{m}\right)_{unchoked} = A \cdot P_0 \sqrt{\frac{2\gamma}{(\gamma-1) R_g T_0}\left[\left(\frac{p}{P_0}\right)^{\frac{2}{\gamma}} - \left(\frac{p}{P_0}\right)^{\frac{\gamma+1}{\gamma}}\right]}$$

Choked Flow $\rightarrow \dfrac{P_0}{p} \geq \left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$

$$\left(\dot{m}\right)_{choked} = A^* \cdot \frac{P_0}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

Review: Mass flow per Unit Area (2)

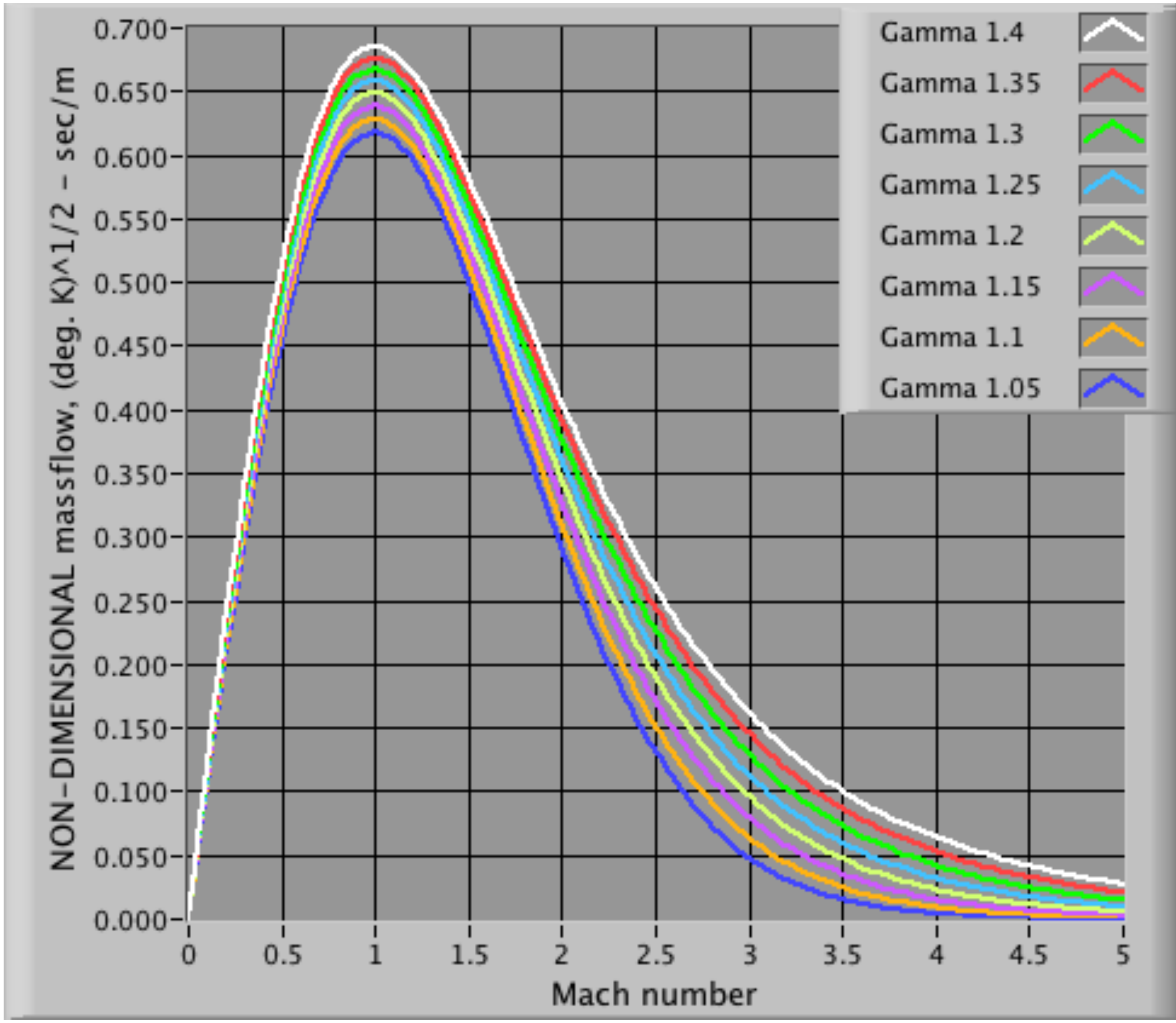
$$\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}$$



$$\left(\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}\right)_{max} = 0.68473$$

- maximum Massflow/area Occurs when When M=1

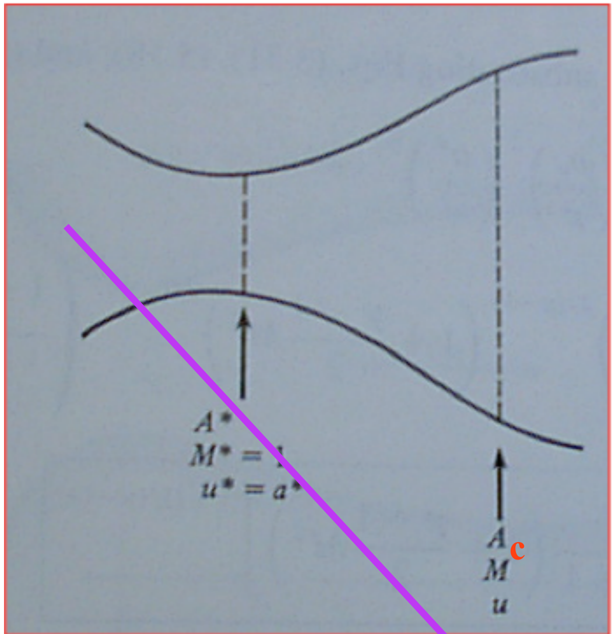
Review: Mass flow per Unit Area (3)



- maximum Massflow/area Occurs when When M=1

Increasing γ -- increasing choking massflow available

Nozzle Flow (1)



- Consider a "choked-flow" Nozzle … (i.e. M=1 at Throat)

- Then comparing the massflow /unit area at throat to some Downstream station

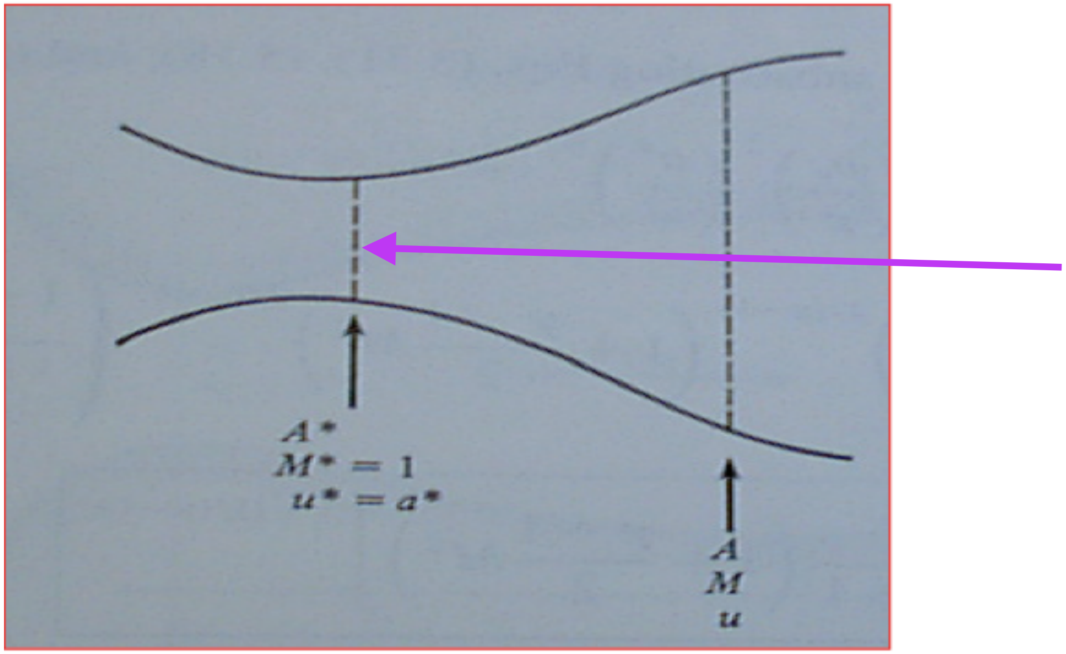
$$\frac{\dfrac{\dot{m}}{A^*}\dfrac{\sqrt{T_0}}{p_0}\sqrt{R_g}}{\dfrac{\dot{m}}{A_c}\dfrac{\sqrt{T_0}}{p_0}\sqrt{R_g}} = \frac{A_c}{A^*} = \frac{\sqrt{\left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}{\dfrac{1}{\left[1+\dfrac{(\gamma-1)}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Nozzle Flow (2)

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

T0

P0



- ***One of the most import expressions in Rocketry!***
- ***Area Ratio determines mach number! in Insentropic Nozzle***

SSME Computational Example

- Space Shuttle Main Engine …

• Unlike other propellants, the optimum mixture ratio for liquid oxygen and liquid hydrogen is not necessarily that which will produce the maximum specific impulse. Because of the extremely low density of liquid hydrogen, the propellant volume decreases significantly at higher mixture ratios.

• Maximum specific impulse typically occurs at a mixture ratio of around 3.5, however by increasing the mixture ratio to, say, 5.5 the storage volume is reduced by one-fourth. This results in smaller propellant tanks, lower vehicle mass, and less drag, which generally offsets the loss in performance that comes with using the higher mixture ratio. In practice, most liquid oxygen/liquid hydrogen engines typically operate at mixture ratios from about 5 to 6.

What is the Stoichiometric Mixture Ratio of LOX/LH_2?

$$2H_2 + O_2 \rightleftharpoons 2H_2O$$

$$M_w LH_2 \rightarrow 2.016_{kg/kg-mol}$$

$$M_w LO_2 \rightarrow 31.999_{kg/kg-mol}$$

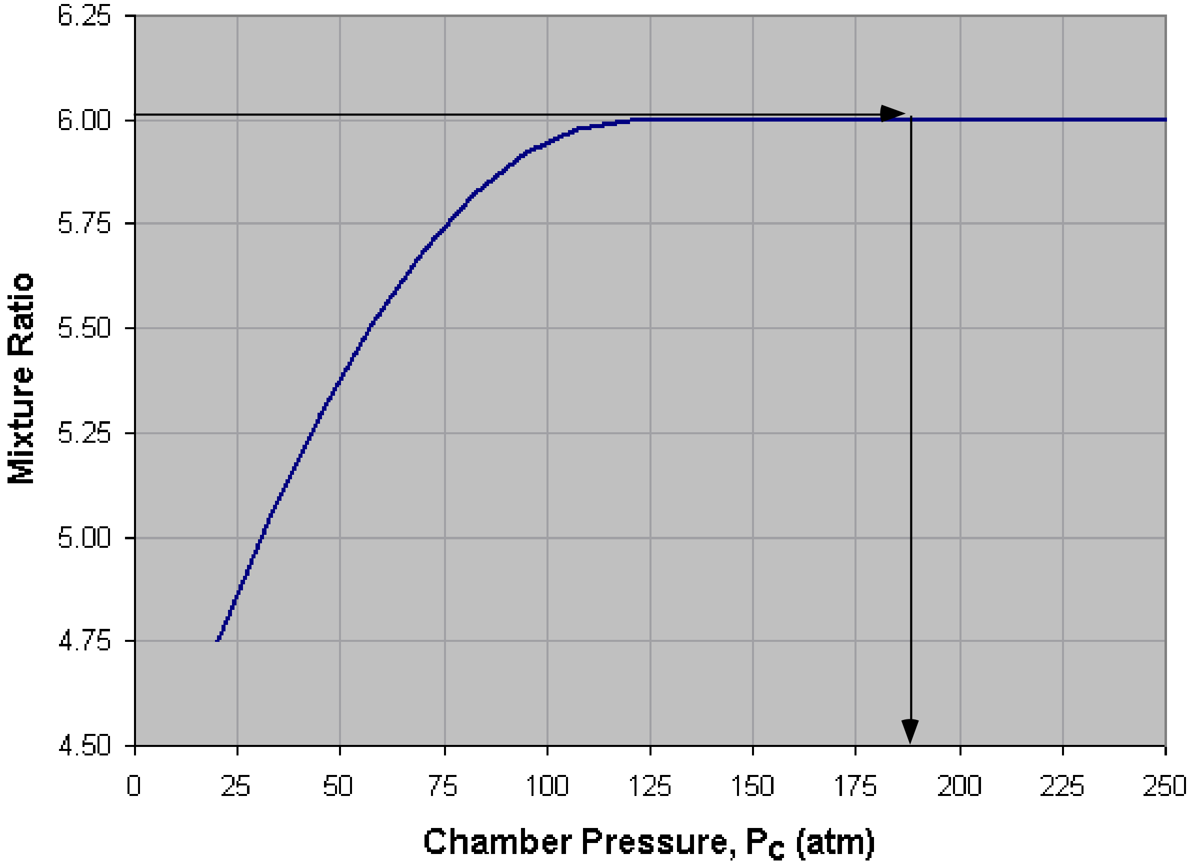
$$MR = \frac{1_{mol} LO_2 \times M_w LO_2}{2_{mol} LH_2 \times M_w LH_2} = \frac{31.999}{2 \times 2.016} = 7.936$$

MR=6.0 (What the shuttle operates at) --> “Rich Mixture”

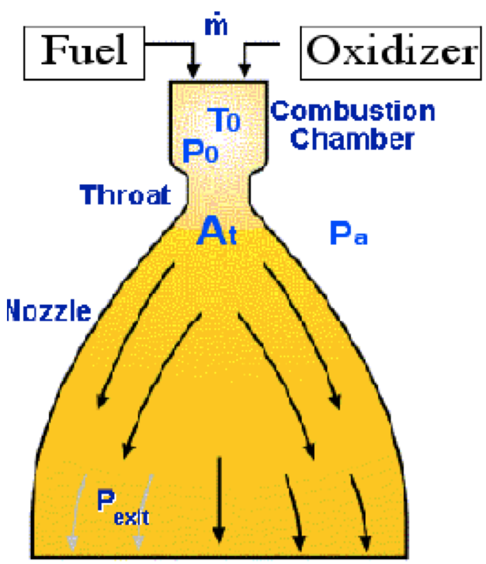
SSME Computational Example (cont'd)

- Space Shuttle Main Engine …

- LOX/LH2 Propellants, 6.03: 1 Mixture ratio



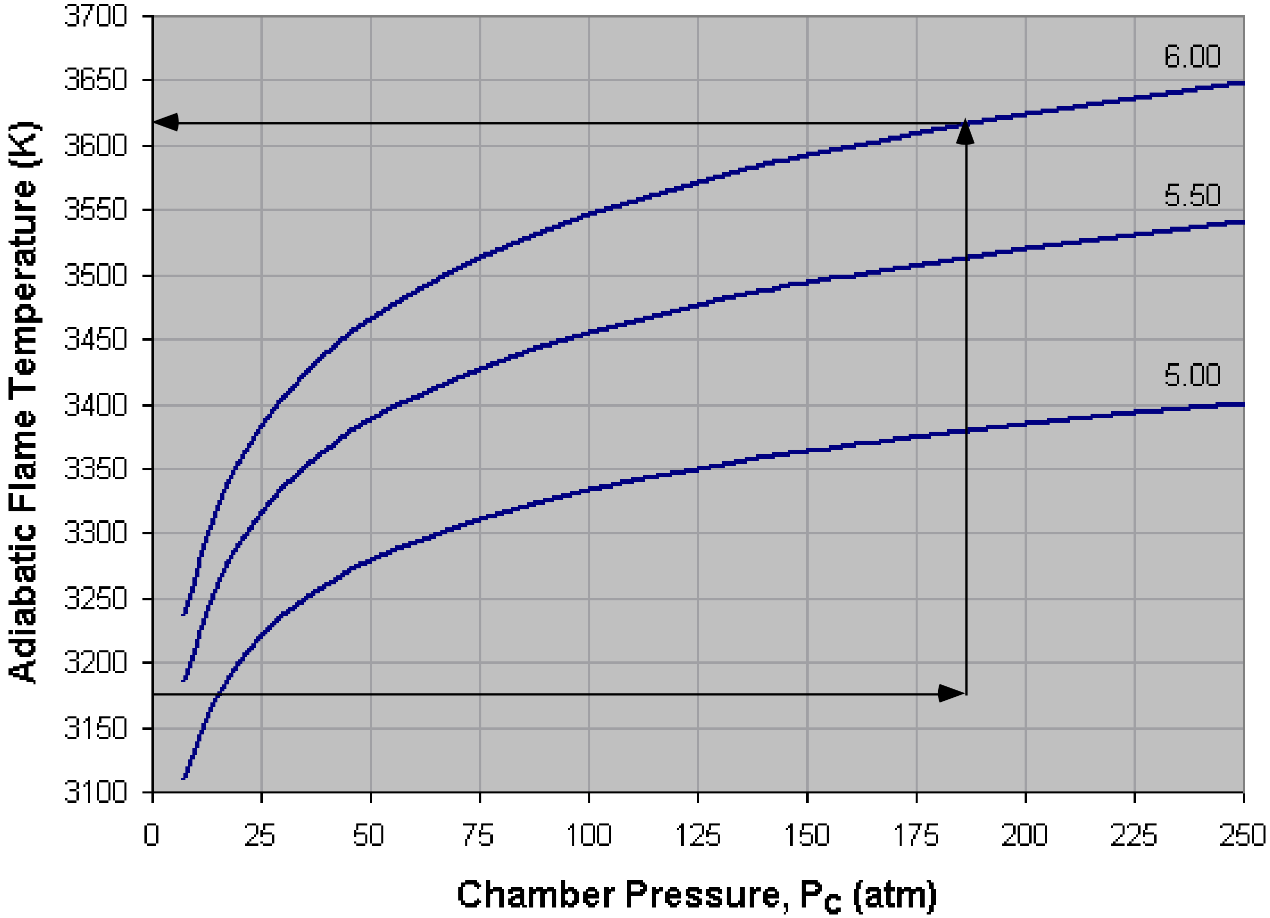
P_0=186.92 atm
=18940 Kpa



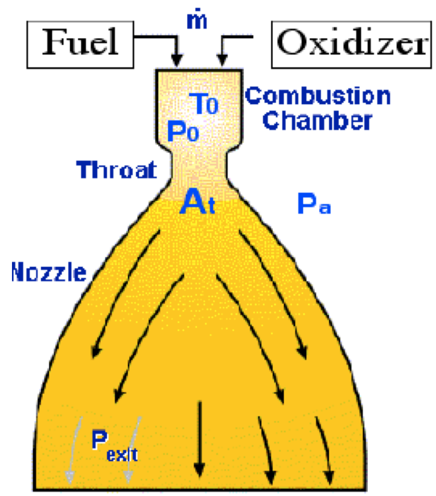
Ref: http://www.braeunig.us/space/

SSME Computational Example (cont'd)

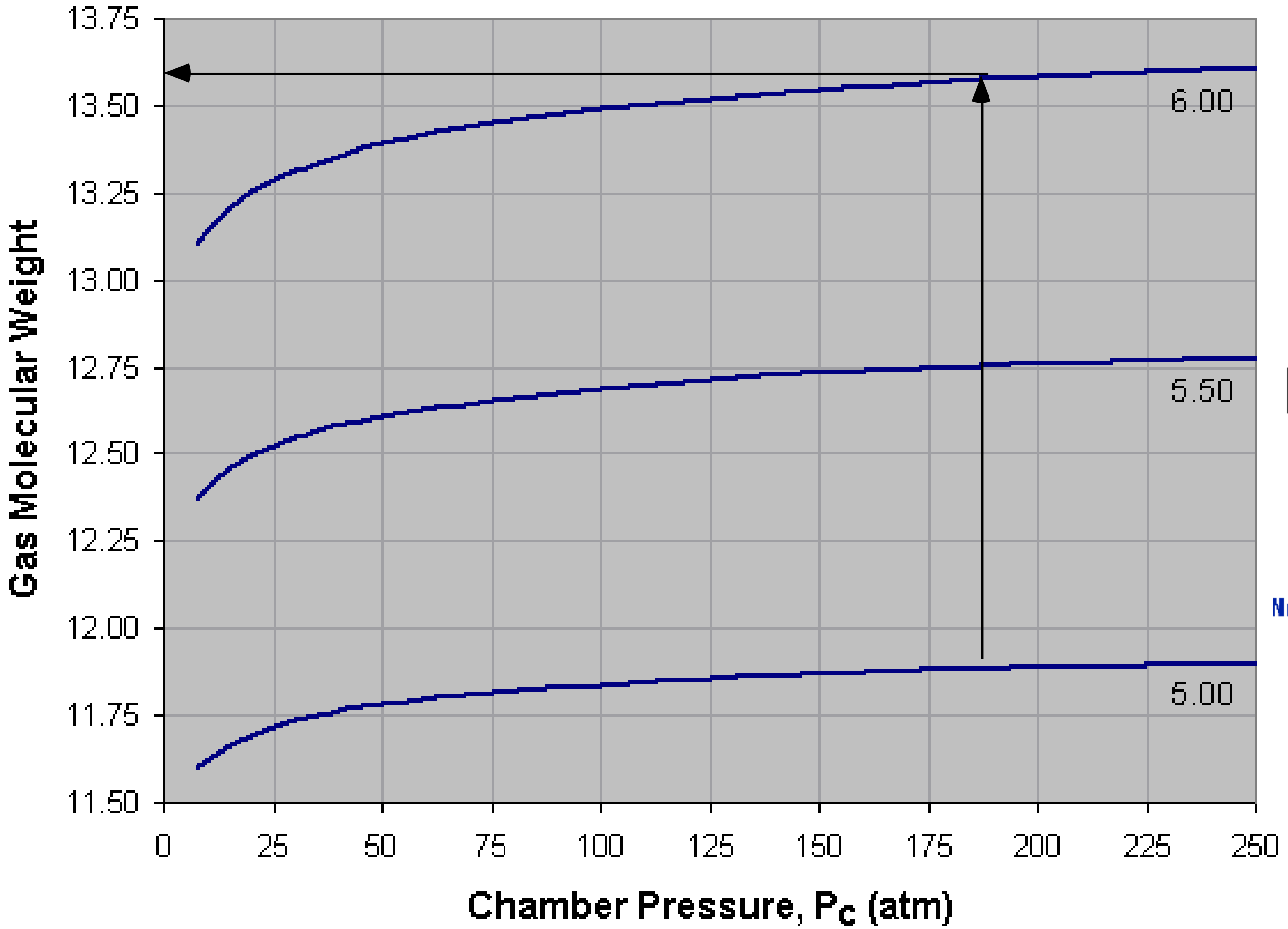
- Space Shuttle Main Engine …



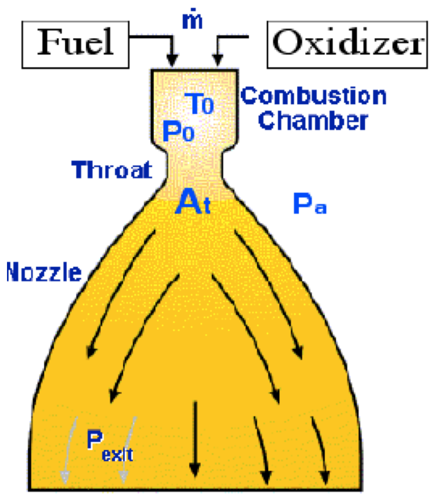
$T_0 \sim 3615°K$



SSME Computational Example (cont'd)



$M_w \sim$ 13.6 kg/kg-mol



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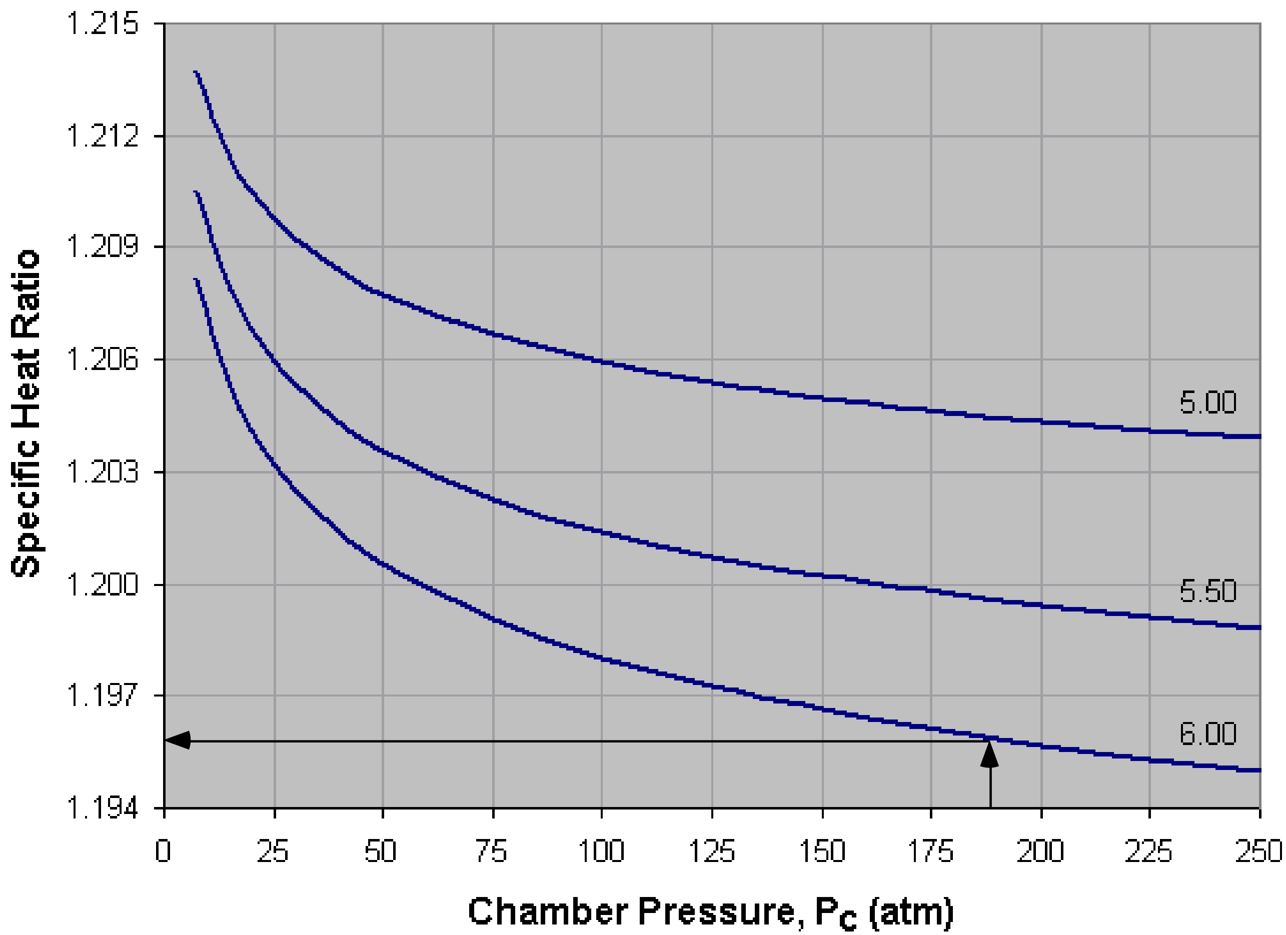
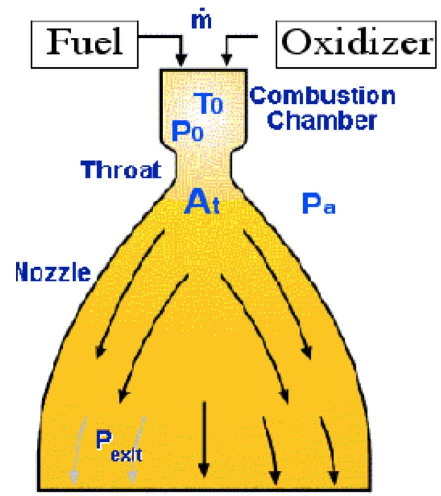
SSME Computational Example (cont'd)

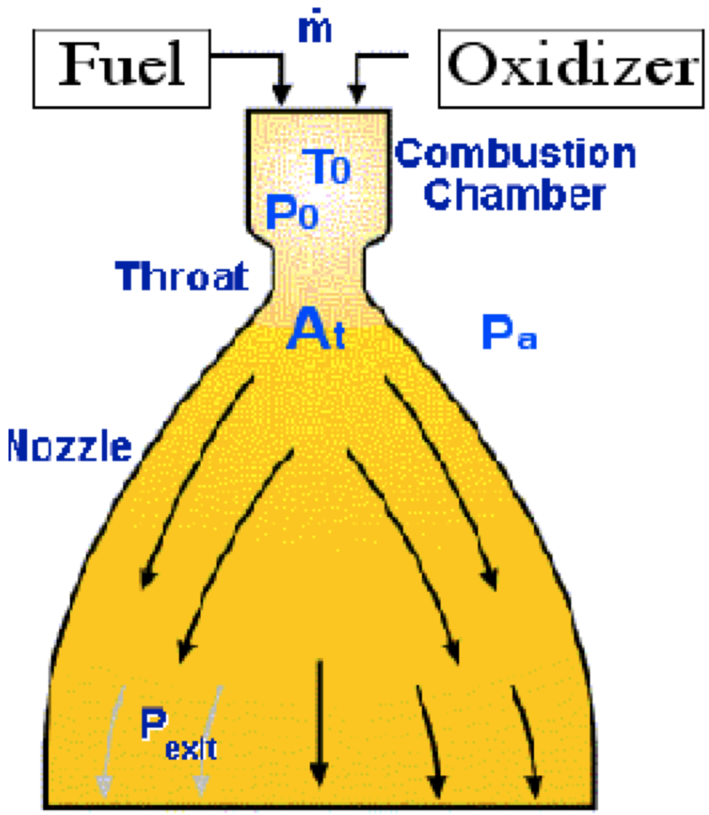
- Space Shuttle Main Engine …



$\gamma \sim 1.196$



Example: SSME Rocket Engine



- The Space Shuttle Main Engines Burn LOX/LH2 for Propellants with A ratio of LOX:LH2 =6:1
- The Combustor Pressure, p_0 Is 18.94 Mpa, combustor temperature, T_0 is 3615°K, throat diameter is 27.2 cm
- What propellant mass flow rate is required for choked flow in the Nozzle?
- Assume no heat transfer Thru Nozzle no frictional losses, γ=1.196

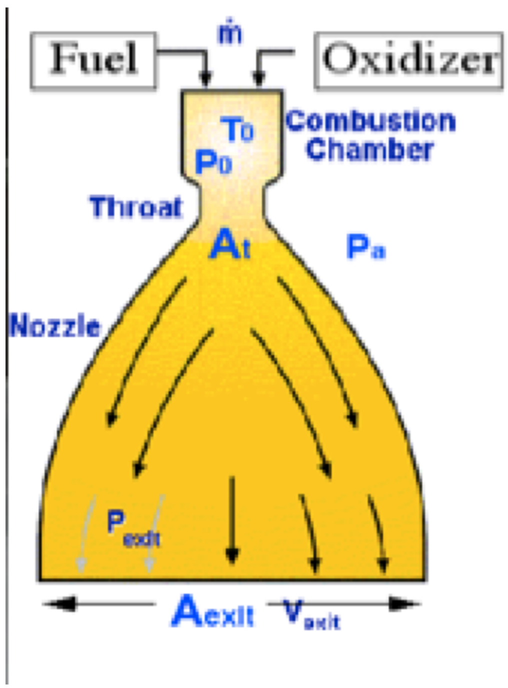
Rocket Thrust Equation, revisited

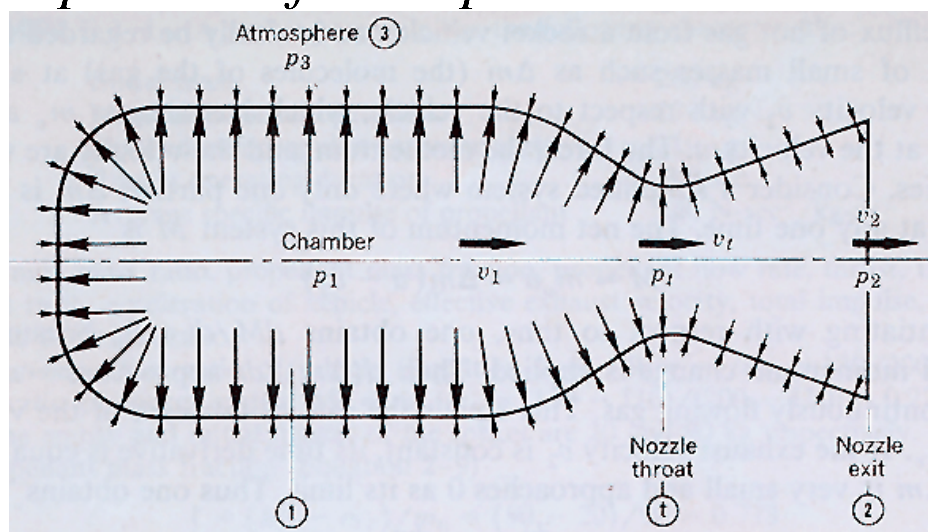
$$Thrust = \dot{m}_e V_e + \left(p_e A_e - p_\infty A_e \right)$$

$$\dot{m}_i = 0$$

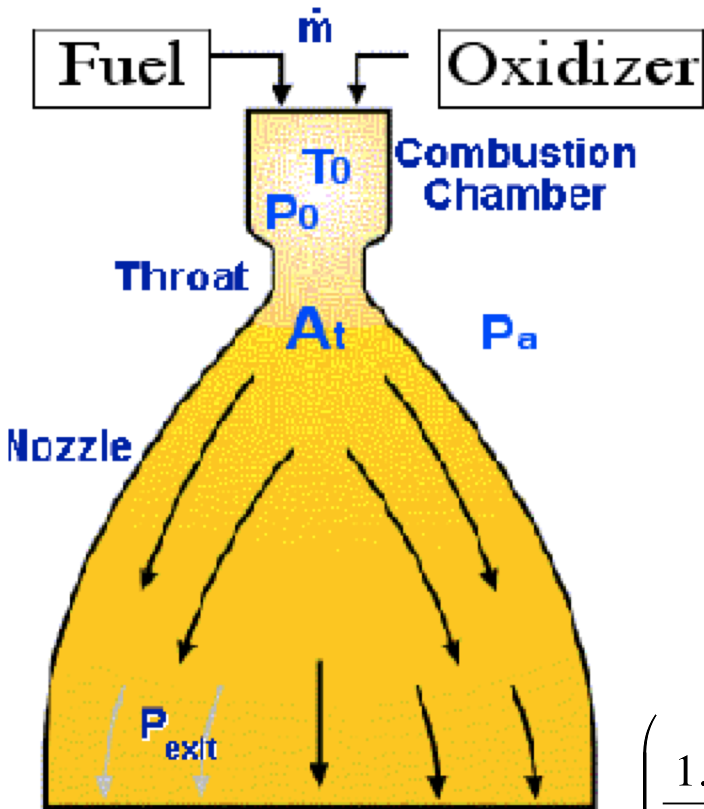
- Thrust + Oxidizer enters combustion Chamber at ~0 velocity, combustion Adds energy … High Chamber pressure Accelerates flow through Nozzle

Resultant pressure forces produce thrust





Example: SSME Rocket Engine (cont'd)



-- By product ~ Burns rich, byproduct is water vapor + GH_2

$M_W \sim 13.6$ kg/kg-mole

-- $R_g = 8314.4126 / 13.6 = 611.35$ J/°K-kg

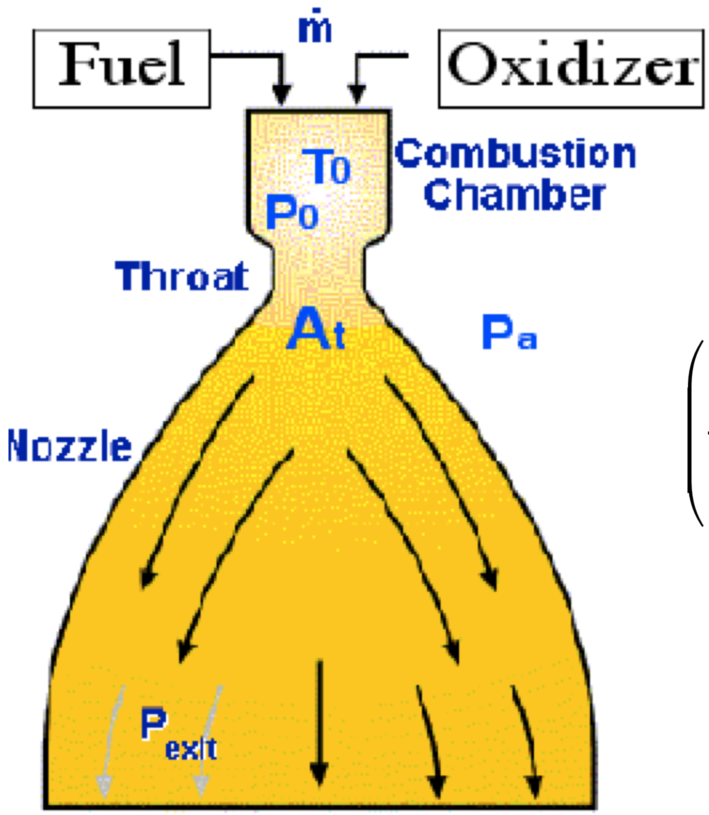
$$\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\frac{p_0}{\sqrt{T_0}} =$$

$$\left(\frac{1.196}{611.35}\left(\frac{2}{1.196+1}\right)^{\frac{(1.196+1)}{1.196-1}}\right)^{0.5}\frac{18.94\cdot 10^6}{(3615)^{0.5}}$$

$= 8252.59$ kg/sec-m²

Example: SSME Rocket Engine (cont'd)

Massflow rate



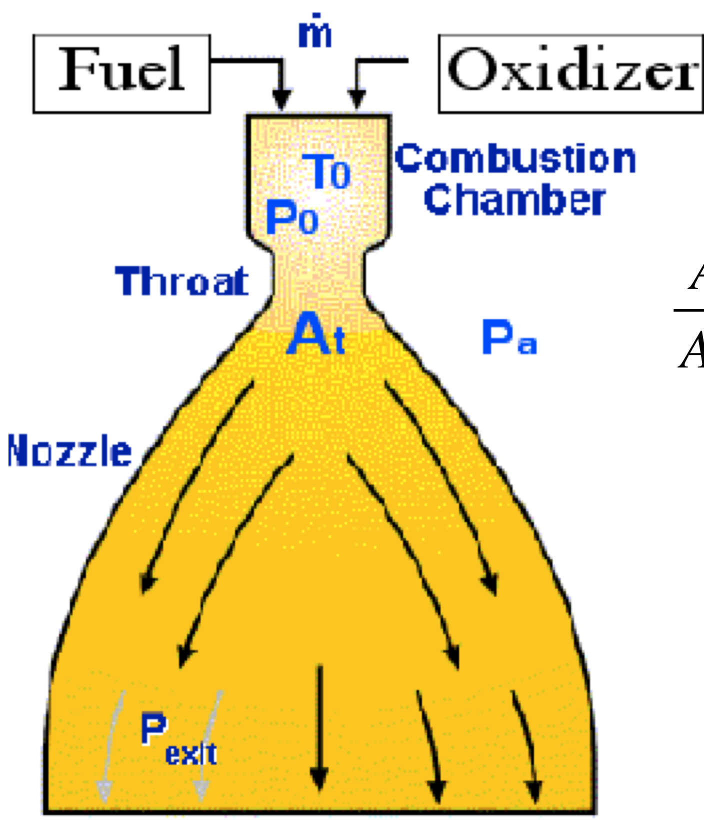
- Compute Throat Area

$$\left(\frac{27.2}{100}\right)^2 \frac{\pi}{4} = 0.0581069 \text{ m}^2$$

- Mass flow =

$$\left(\frac{\dot{m}}{A^*}\right) \times A^* = 8252.59 \times 0.0581069 = 479.532_{kg/sec}$$

Example: SSME Rocket Engine (concluded)



- The nozzle expansion ratio is 77.5 -- what is the exit mach number

$$\frac{A}{A^*} = 77.5 = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

- Non -linear function of mach number

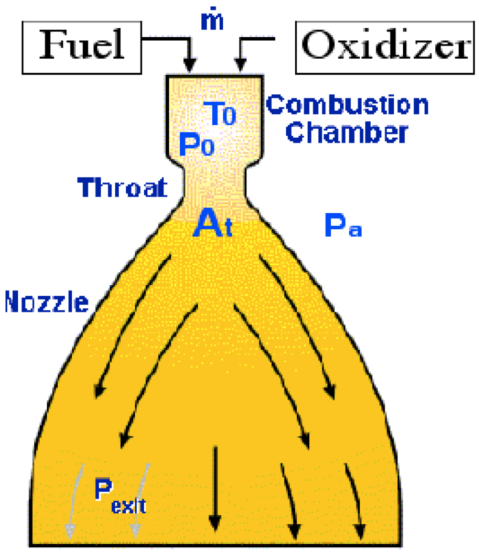
- Solution methods
  - i) Plot A/A* versus mach
  - ii) Numerical Solution

Example: SSME Rocket Engine (cont'd)

Compute Exit Mach Number

Expansion ratio = 77.5

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} =$$



$$\frac{\left(\left(\frac{2}{1.196+1}\right)\left(1+\frac{1.196-1}{2}(4.677084^2)\right)\right)^{\frac{1.196+1}{2(1.196-1)}}}{4.677084}$$

$= 77.49998$ ----> $M_{exit} = 4.677084$

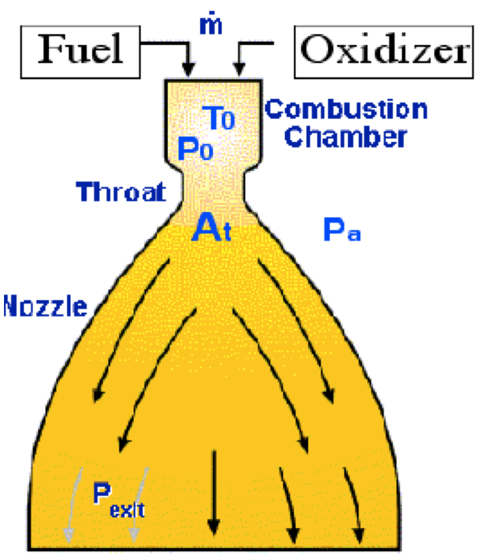
Newton Solver comes in handy here!
More on this method later

Example: SSME Rocket Engine (cont'd)

Compute Exit Temperature

$M_{exit} = 4.677084$

$$\frac{T_0}{T} = 1 + \frac{(\gamma - 1)}{2} M^2 \longrightarrow$$



$$T_{exit} = \frac{T_0}{1 + \frac{(\gamma - 1)}{2} M_{exit}^2} =$$

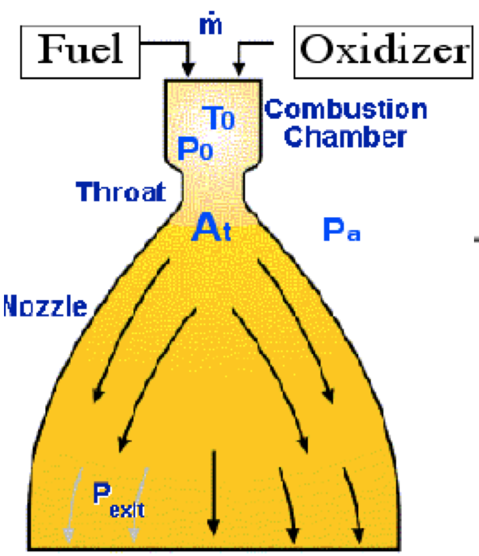
$$3615\left(1 + \frac{1.196 - 1}{2}(4.677084^2)\right)^{-1} = 1149.90\ °K$$

Example: SSME Rocket Engine (cont'd)

Compute Exit Pressure

$M_{exit} = 4.677084$

$$p_{exit} = \frac{P_0}{\left(1+\frac{\gamma-1}{2}M_{exit}^2\right)^{\gamma/(\gamma-1)}} =$$



$$\frac{18.93\cdot 10^3}{\left(1+\frac{1.196-1}{2}4.677084^2\right)^{\frac{1.196}{(1.196-1)}}} = 17.4459 \; kPa$$

Example: SSME Rocket Engine (cont'd)

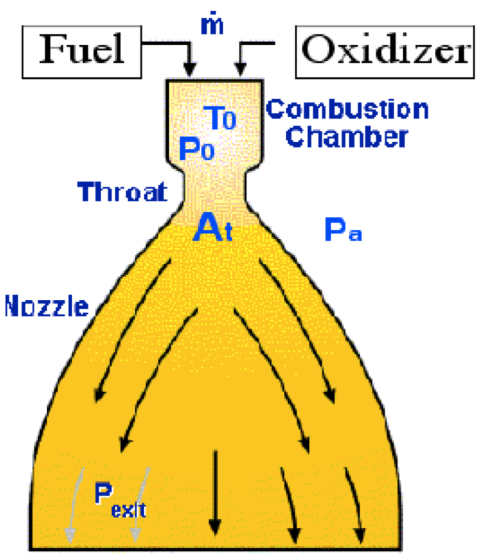
Compute Exit Velocity

$M_{exit} = 4.677084$

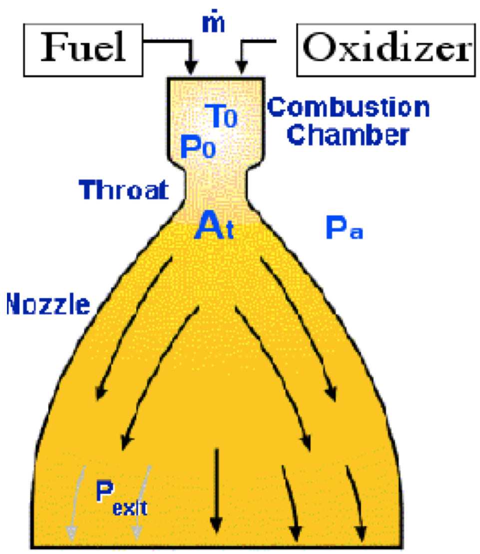
$$V_{exit} = M_{exit}\sqrt{\gamma R_g T_{exit}} =$$

$$4.677084\,(1.196 \cdot 611.35 \cdot 1149.9)^{0.5}$$

$$= 4288.61 \text{ m/sec}$$



Example: SSME Rocket Engine (cont'd)



Compute Thrust (Vacuum)

$$Thrust_{vac} = \dot{m} V_{exit} + \frac{A_{exit}}{A^*} A^* (p_{exit} - 0) =$$

$$\frac{479.532 \cdot 4288.61 + 77.5 \cdot 0.0581069\,(17.4459 \cdot 10^3)}{10^6}$$

$= 2.13509$ M Nt

Compute True I_{sp} (Vacuum)

$$I_{sp_{vac}} = \frac{Thrust_{vac}}{g_0 \cdot \dot{m}} =$$

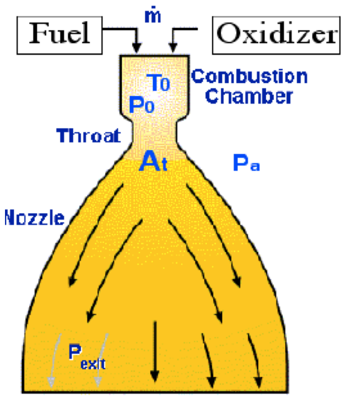
$$\frac{(479.532 \cdot 4288.61 + 77.5 \cdot 0.0581069\,(17.4459 \cdot 10^3))}{9.8067 \cdot 479.532} = 454.021$$

seconds

Example: SSME Rocket Engine (cont'd)

Compute Thrust (Sea level)

$P_{Sea\ Level} = 101.325$ kPa



$$Thrust_{\substack{sea\\level}} = Thrust_{vac} - \frac{A_{exit}}{A^*} A^* \cdot p_{\substack{sea\\level}} =$$

$$2.13509 - \frac{77.5 \cdot 0.0581069 \cdot 101325}{10^6} = 1.67879 \text{ kNt}$$

Compute Isp (Sea level)

$$I_{sp_{\substack{sea\\level}}} = I_{sp_{vac}} - \frac{\frac{A_{exit}}{A^*} \cdot A^* \cdot p_{exit}}{g_0 \cdot \dot{m}} = 454.021 - \frac{77.5 \cdot 0.0581069 \cdot 101325}{9.8067 \cdot 479.532} = 356.991 \text{ seconds}$$

Example: SSME Rocket Engine (cont'd)

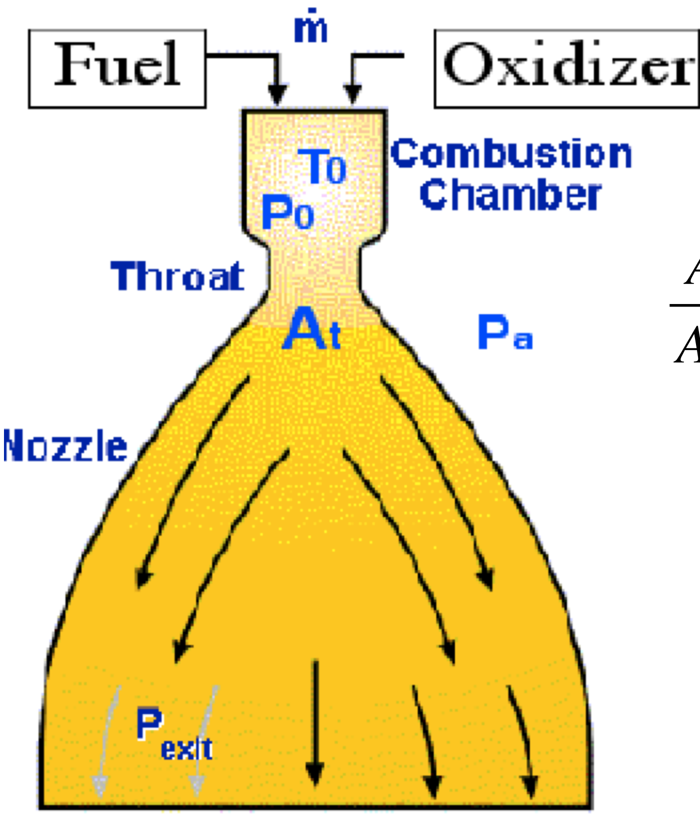
Summary:

	Calc. Vac.	Calc. S.L.	Actual Vac.	Actual S.L
I_{sp} (sec):	454.021	356.99	452.5	363
Thrust: (MNt)	2.135	1.679	2.100	1.670

- Our rough estimate using isentropic 1-D flow is within ……. 1.7% worst case error .. Pretty darn good!

https://www.rocket.com/rs-25-engine (SSME)

Example: SSME Rocket Engine (concluded)



• Revisit solution to

$$\frac{A}{A^*} = 77.5 = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

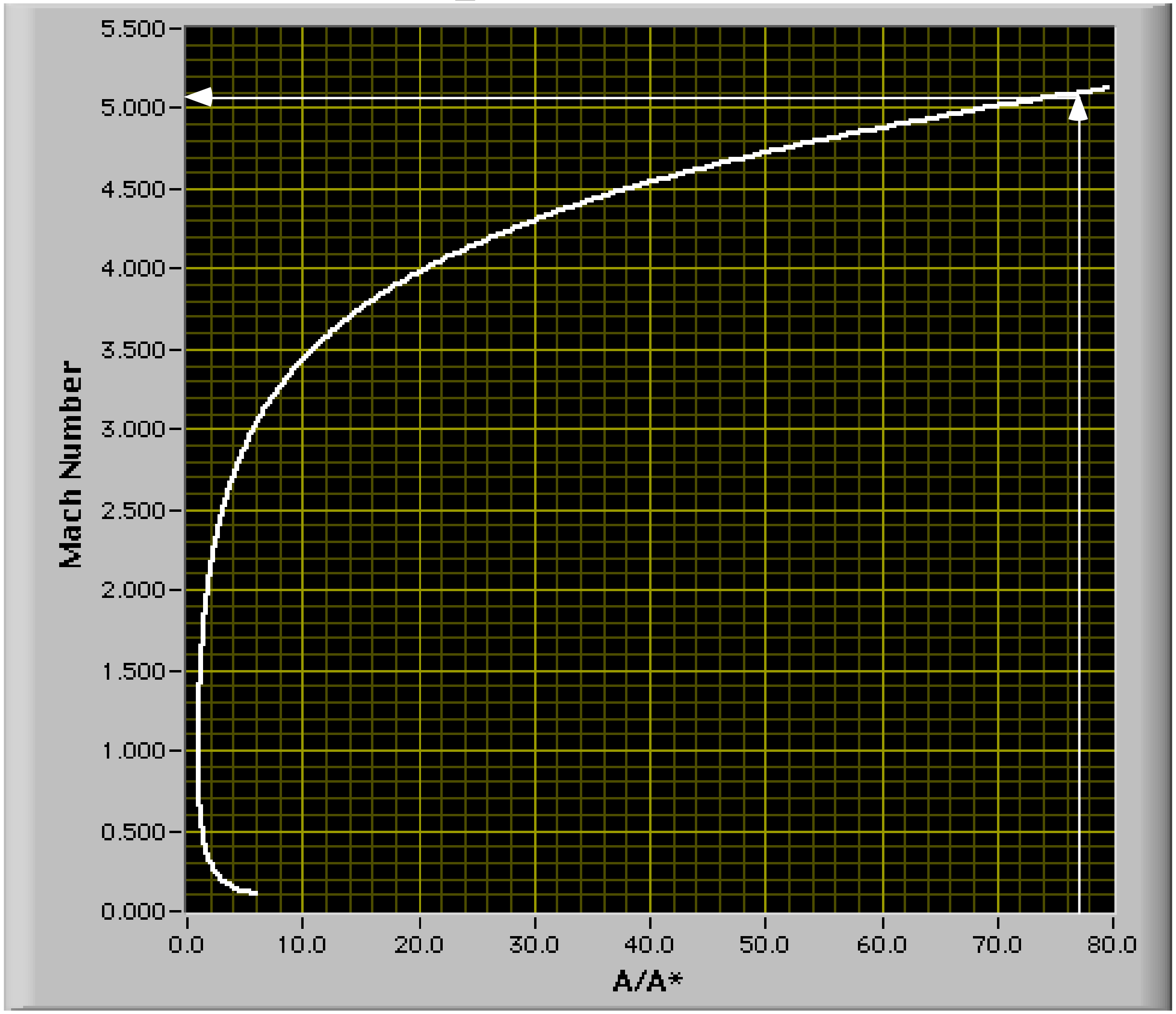
• Non -linear function of mach number

• Solution methods

i) Plot A/A* versus mach

ii) Numerical Solution

Graphical Solution for Mach



- Graphical Solution

$M_{exit} \sim 5.095$

Numerical Solution for Mach

- Graphical Solutions are good for "sanity check" but really Need automated solver to allow for iterative design, trade studies, sensitivity analyses, etc.

- Use "Newton's Method" to extract numerical solution

- Define: $F(M) \equiv \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} - \frac{A}{A^*}$

- At correct Mach number (for given A/A*) … $F(M) = 0$

Numerical Solution for Mach (cont'd)

- Expand F(M) is Taylor's series about some arbitrary Mach number $M_{(j)}$

$$F(M) = F(M_{(j)}) + \left(\frac{\partial F}{\partial M}\right)_{(j)} \left(M - M_{(j)}\right) + \frac{\left(\frac{\partial^2 F}{\partial M^2}\right)_{(j)} \left(M - M_{(j)}\right)^2}{2} + \ldots O\left(M - M_{(j)}\right)^3$$

- Solve for M

$$M = M_{(j)} + \frac{F(M) - F(M_{(j)}) - \left[\frac{\left(\frac{\partial^2 F}{\partial M^2}\right)_{(j)} \left(M - M_{(j)}\right)^2}{2} + \ldots O\left(M - M_{(j)}\right)^3\right]}{\left(\frac{\partial F}{\partial M}\right)_{(j)}}$$

Numerical Solution for Mach (cont'd)

- From Earlier Definition $F(M) = 0$, thus

$$M = M_{(j)} - \frac{F(M_{(j)}) + \left[\frac{\left(\frac{\partial^2 F}{\partial M^2}\right)_{(j)} \left(M - M_{(j)}\right)^2}{2} + ...O\left(M - M_{(j)}\right)^3 \right]}{\left(\frac{\partial F}{\partial M}\right)_{(j)}}$$

Still exact expression

- if $M_{(j)}$ is chosen to be "close" to M $\left(M - M_{(j)}\right)^2 << \left(M - M_{(j)}\right)$

And we can truncate after the first order terms with "little" Loss of accuracy

Numerical Solution for Mach (cont'd)

- First Order approximation of solution for M

$$\hat{M} = M_{(j)} - \frac{F(M_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{(j)}}$$

"Hat" indicates that solution is no longer exact

- However; one would anticipate that $\left\| M - \hat{M} \right\| < \left\| M - M_{(j)} \right\|$

"estimate is closer than original guess"

Numerical Solution for Mach (cont'd)

- If we substitute $\hat{M}$ back into the approximate expression

$$\hat{\hat{M}} = \hat{M} - \frac{F(\hat{M})}{\left(\frac{\partial F}{\partial M}\right)_{|\hat{M}}}$$

- And we would anticipate that $\left\| M - \hat{\hat{M}} \right\| < \left\| M - \hat{M} \right\|$

"refined estimate" …. Iteration 1

Numerical Solution for Mach

- Abstracting to a "jth" iteration

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{(j)}}$$

Iterate until convergence
j={0,1,….}

- Drop from loop when

$$\frac{\left\| \frac{1}{\hat{M}_{(j+1)}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\hat{M}_{(j+1)}^{2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} - \frac{A}{A^*} \right\|}{\frac{A}{A^*}} < \varepsilon$$

Numerical Solution for Mach (cont'd)

$$F(\hat{M}_{(j)}) = \frac{1}{\hat{M}_{(j)}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\hat{M}_{(j)}^{2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} - \frac{A}{A^{*}}$$

$$\left(\frac{\partial F}{\partial M}\right)_{|(j)} = \frac{\partial}{\partial \hat{M}_{(j)}}\left(\frac{1}{\hat{M}_{(j)}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\hat{M}_{(j)}^{2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right) =$$

$$\boxed{\left(2^{\left(\frac{1-3\gamma}{2-2\gamma}\right)}\right)\frac{\left(\hat{M}_{(j)}^{2}-1\right)}{\hat{M}_{(j)}^{2}\left[2+\hat{M}_{(j)}^{2}(\gamma-1)\right]}\left(\frac{1+\frac{(\gamma-1)}{2}\hat{M}_{(j)}^{2}}{\gamma+1}\right)^{\left(\frac{\gamma+1}{2(\gamma-1)}\right)}}$$

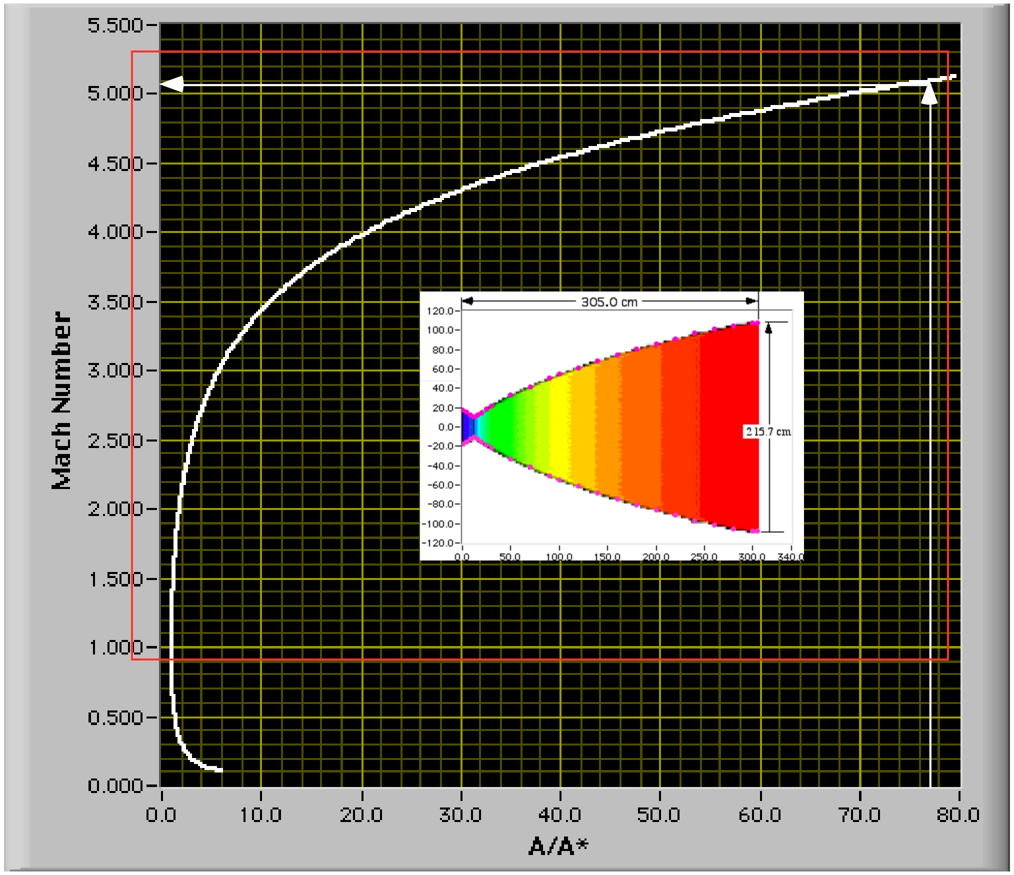
Numerical Solution for Mach (cont'd)

- Numerical Solution (Newton's Method)

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$

- Example $\frac{A}{A^*} = 77.5$
- Starting mach ->3.0
- Allowable Error, 0.001%
- $\gamma = 1.25$

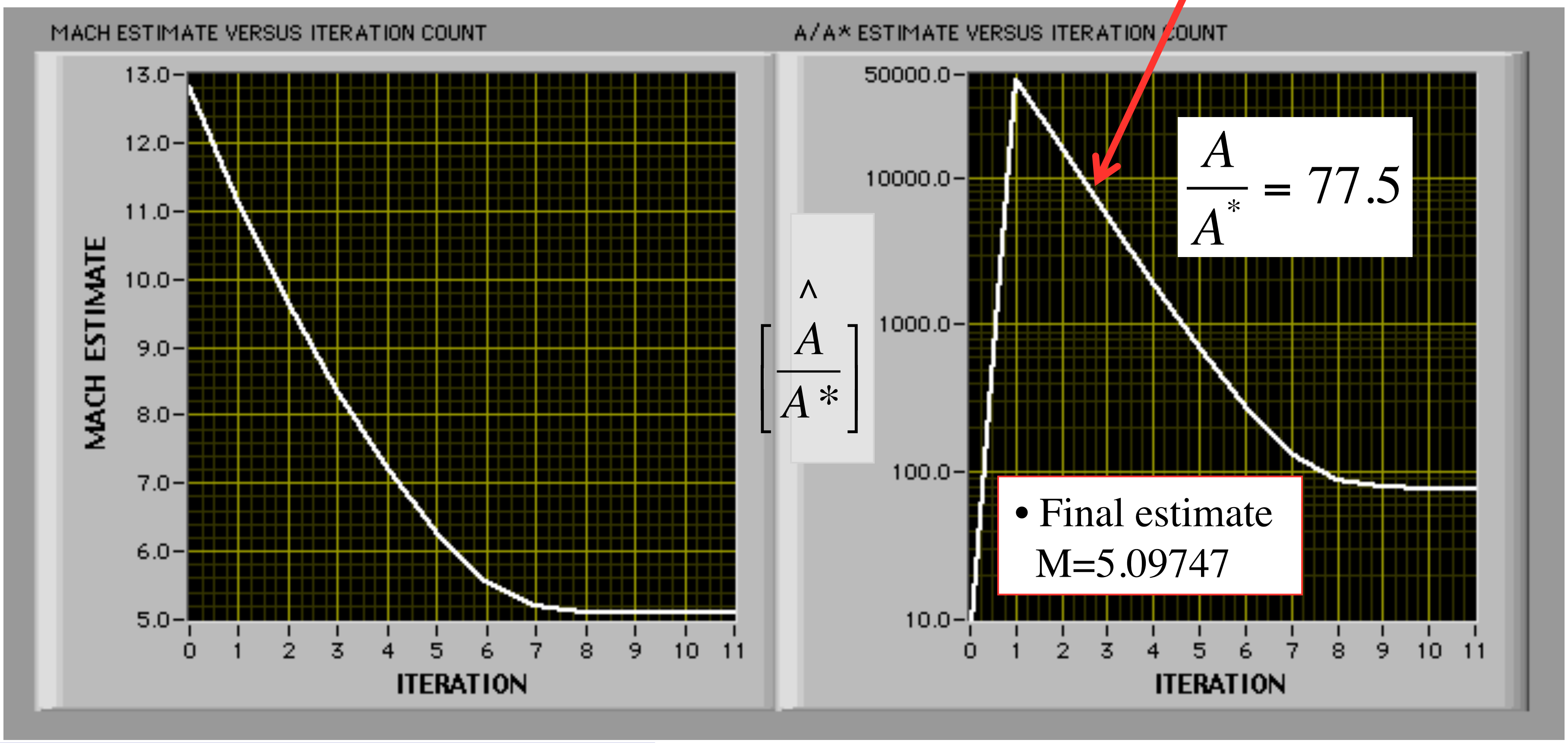
- Solves for Supersonic Branch of Curve



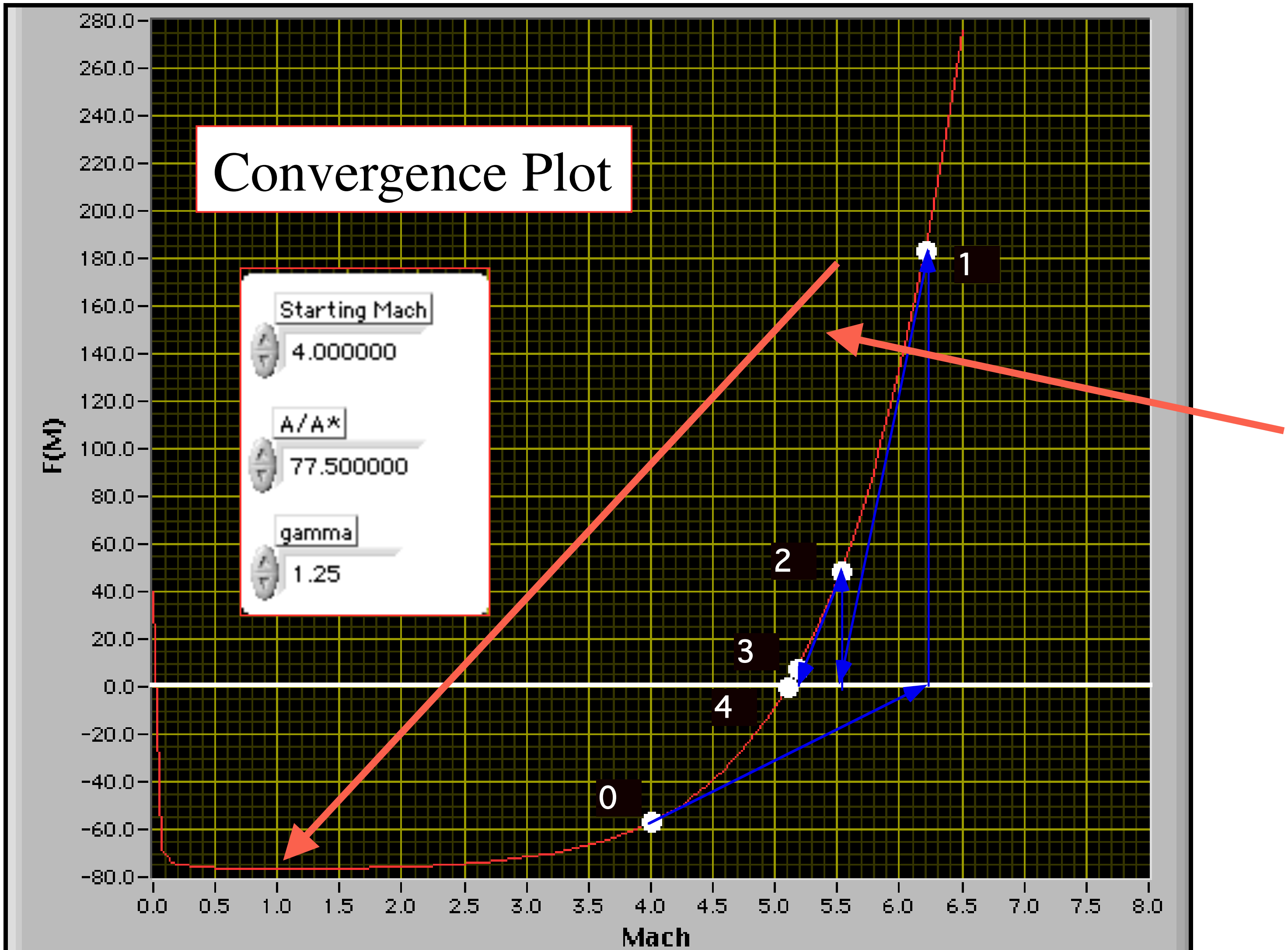
Numerical Solution for Mach (concluded)

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$

$$\frac{1}{\hat{M}_{(j)}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\hat{M}_{(j)}^{2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$



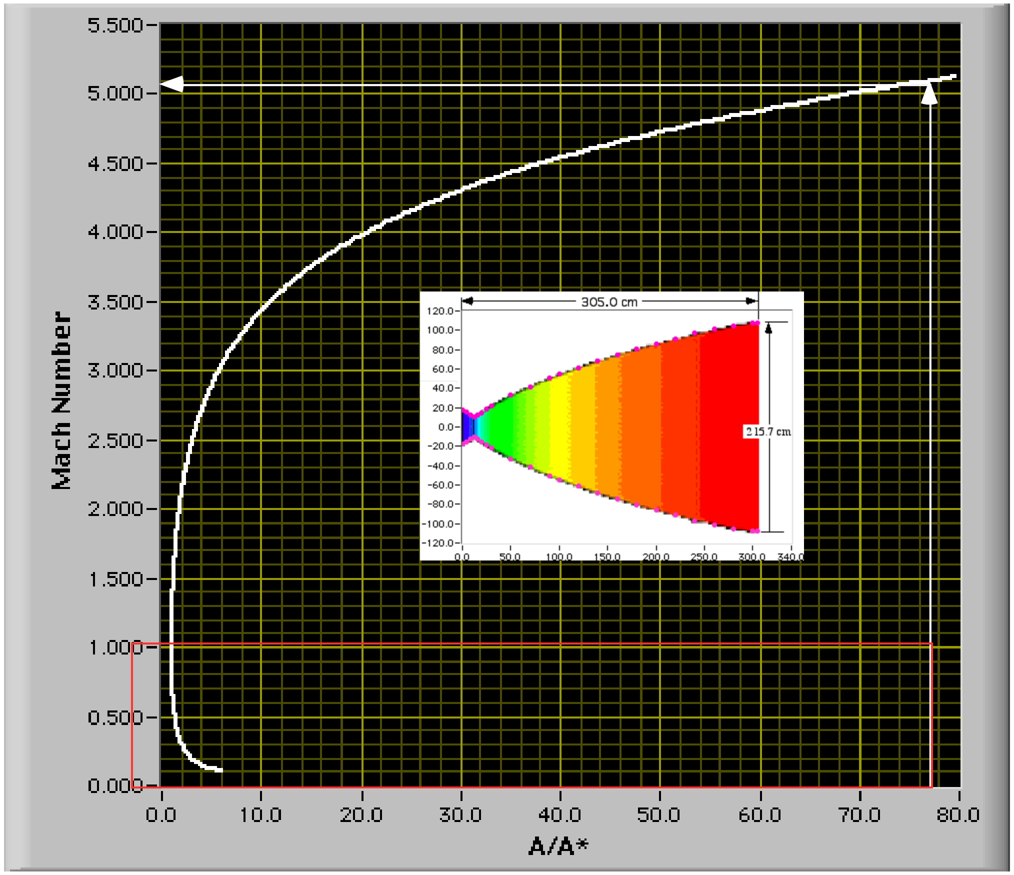
Numerical Solution for Mach (concluded)



Effect of Startup Condition

- Example $\frac{A}{A^*} = 77.5$
- Starting mach ->0.01
- Allowable Error, 0.001%
- $\gamma = 1.25$

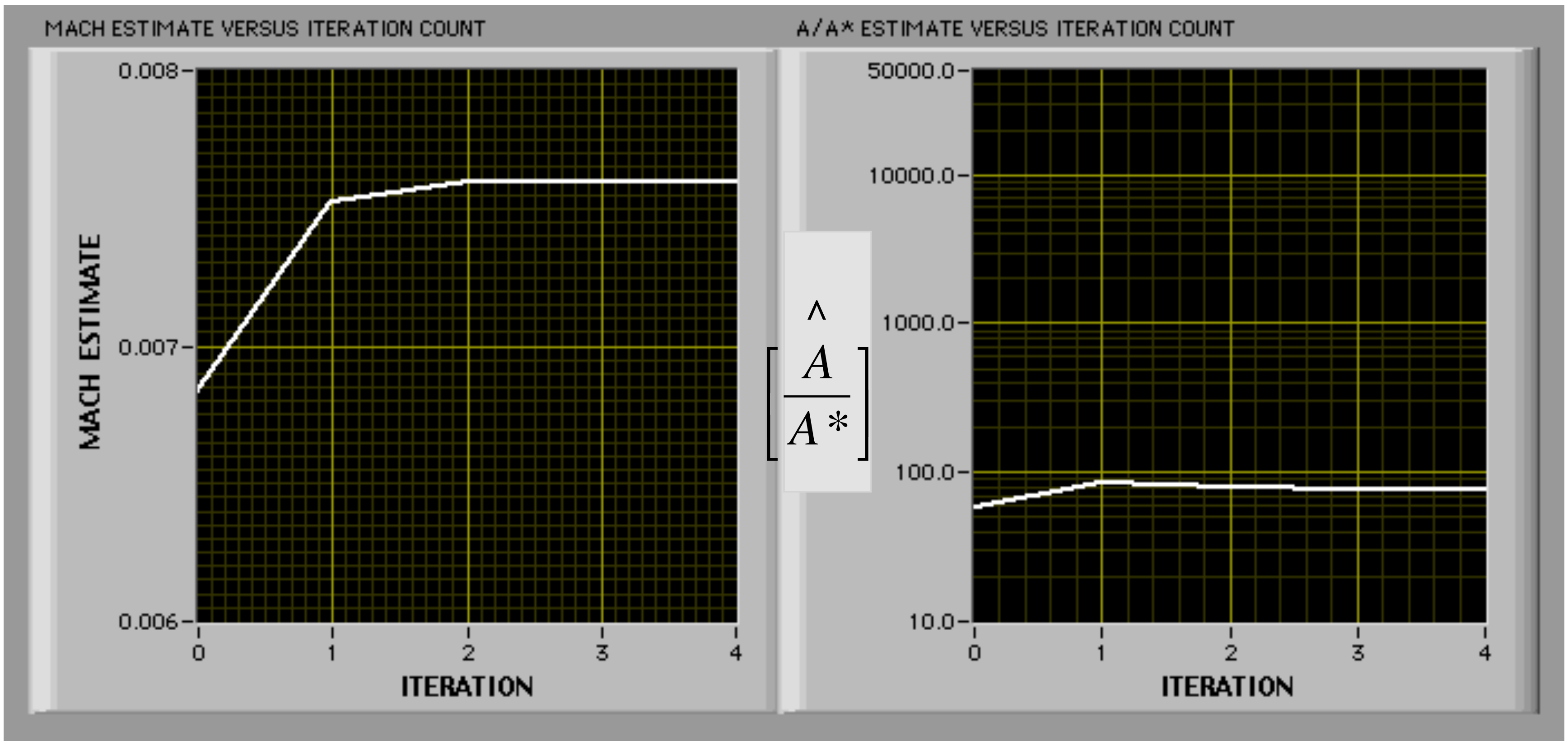
- Solves for Subsonic Branch Of Curve



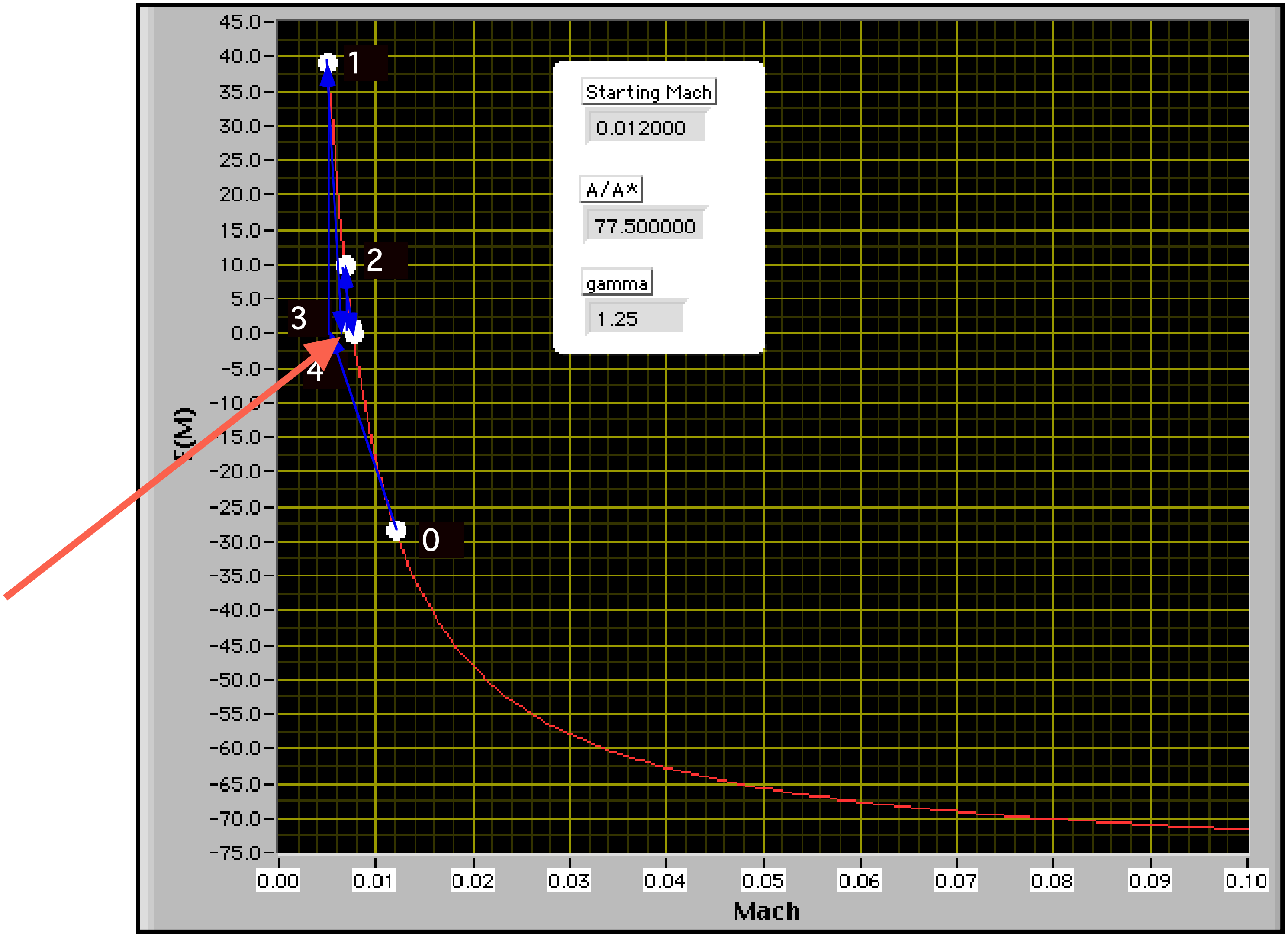
Effect of Startup Condition (concluded)

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$

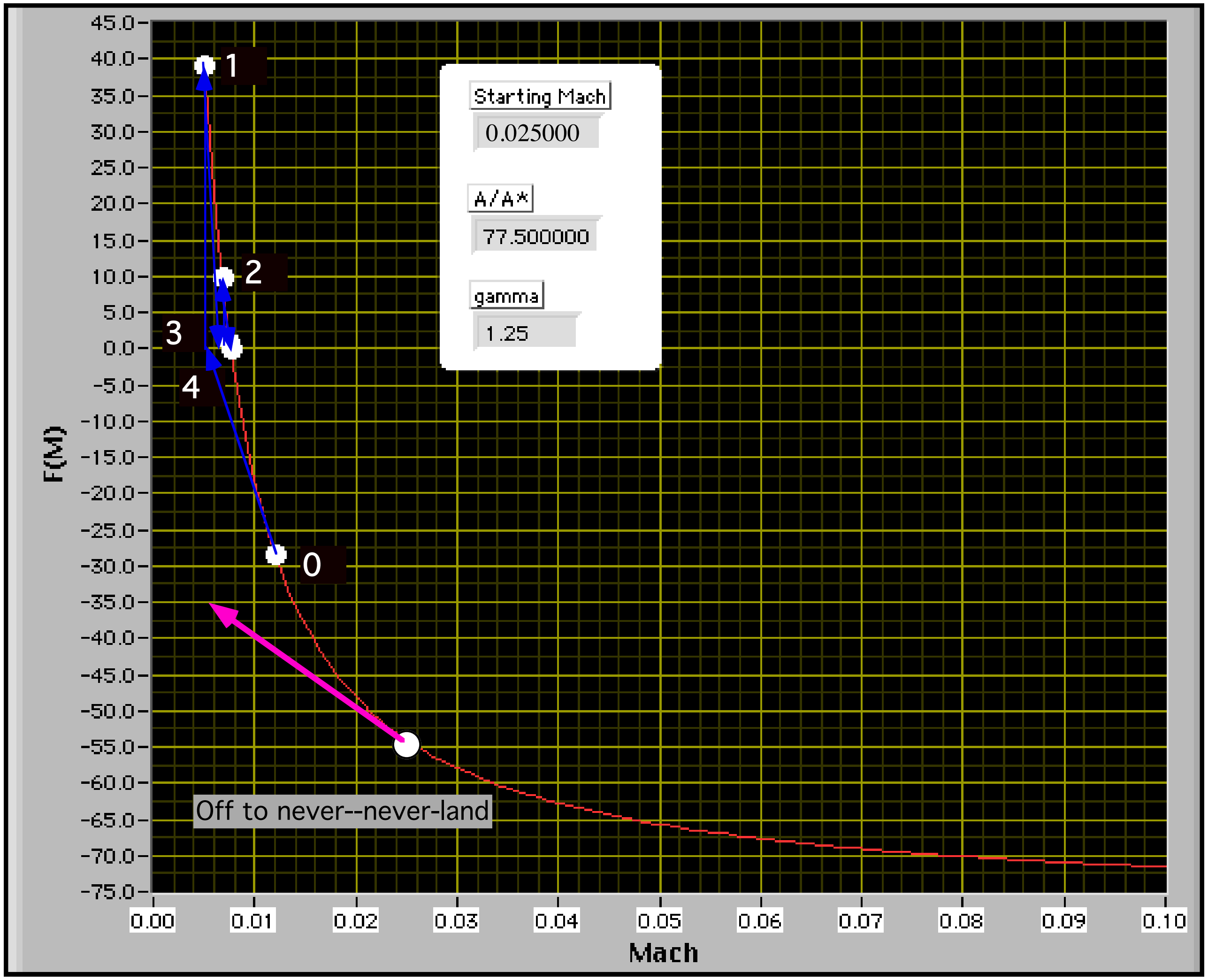
- Final estimate
  M=0.00759



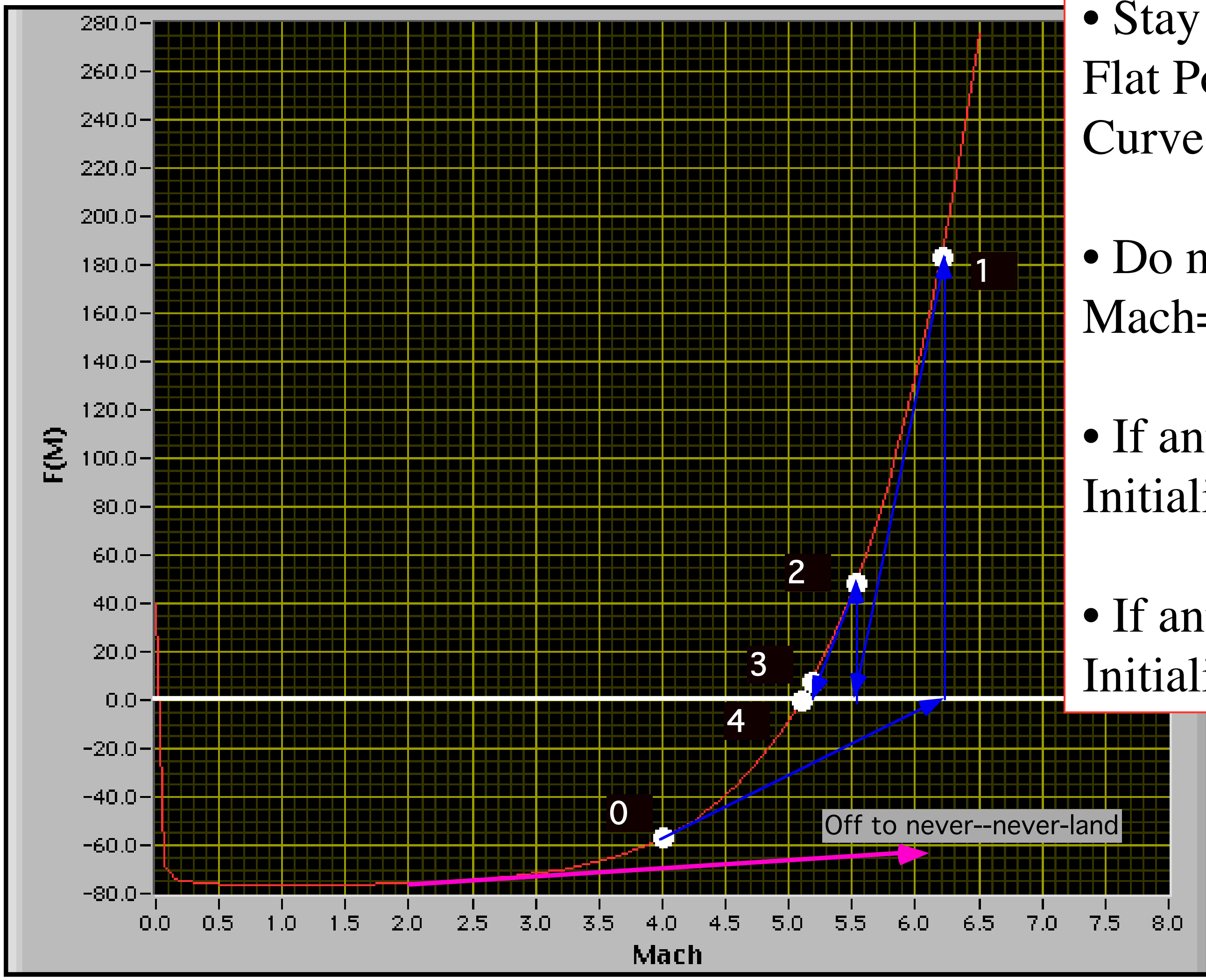
Convergence Plot



Be Careful About Startup Condition



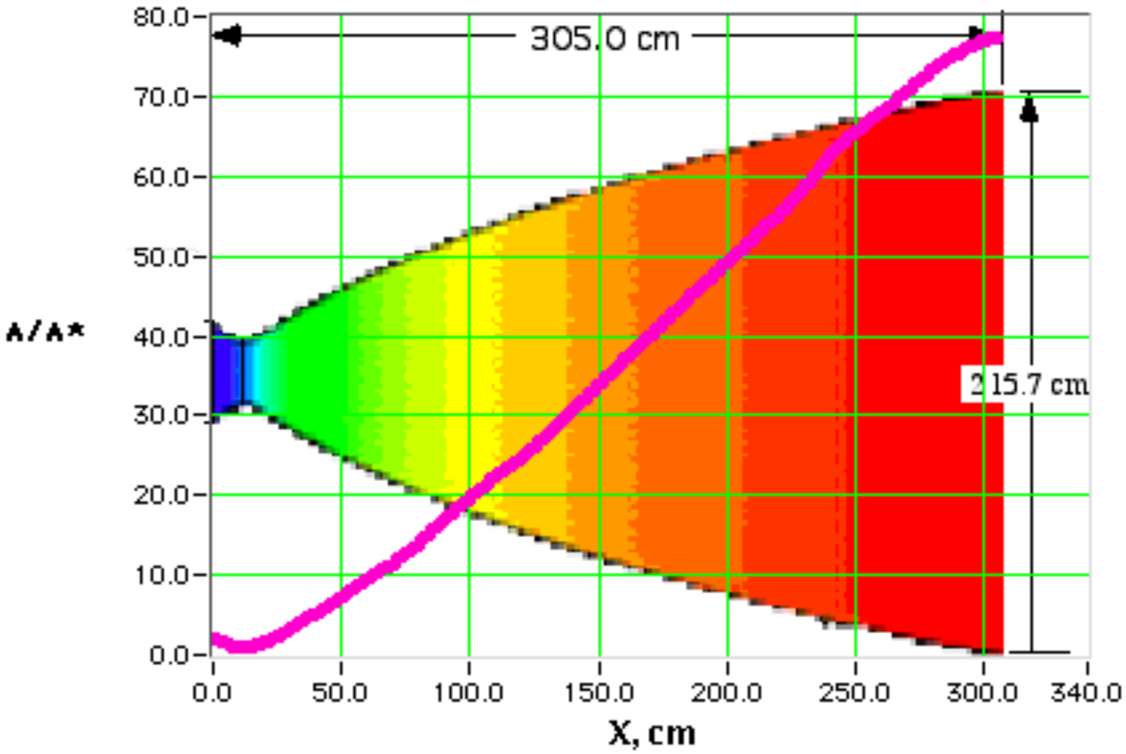
Be Careful About Startup Condition (cont'd)



- Stay Off of Flat Portion of Curve at Startup
- Do not start at Mach=0, Mach 1
- If anticipate Mach < 1 Initialize at M slightly > 0
- If anticipated Mach > 1 Initialize at Mach > 3

Plot Flow Properties Along SSME Nozzle Length

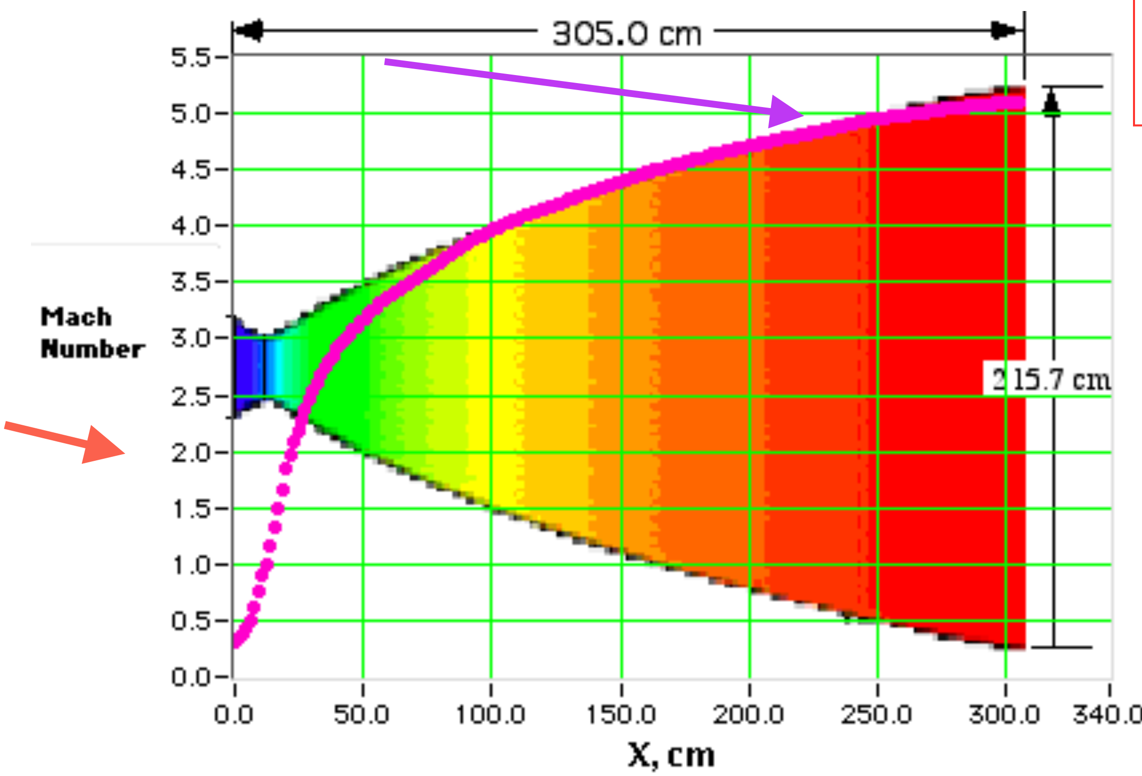
- A/A^*



Plot Flow Properties Along SSME Nozzle Length (cont'd)

- Mach Number

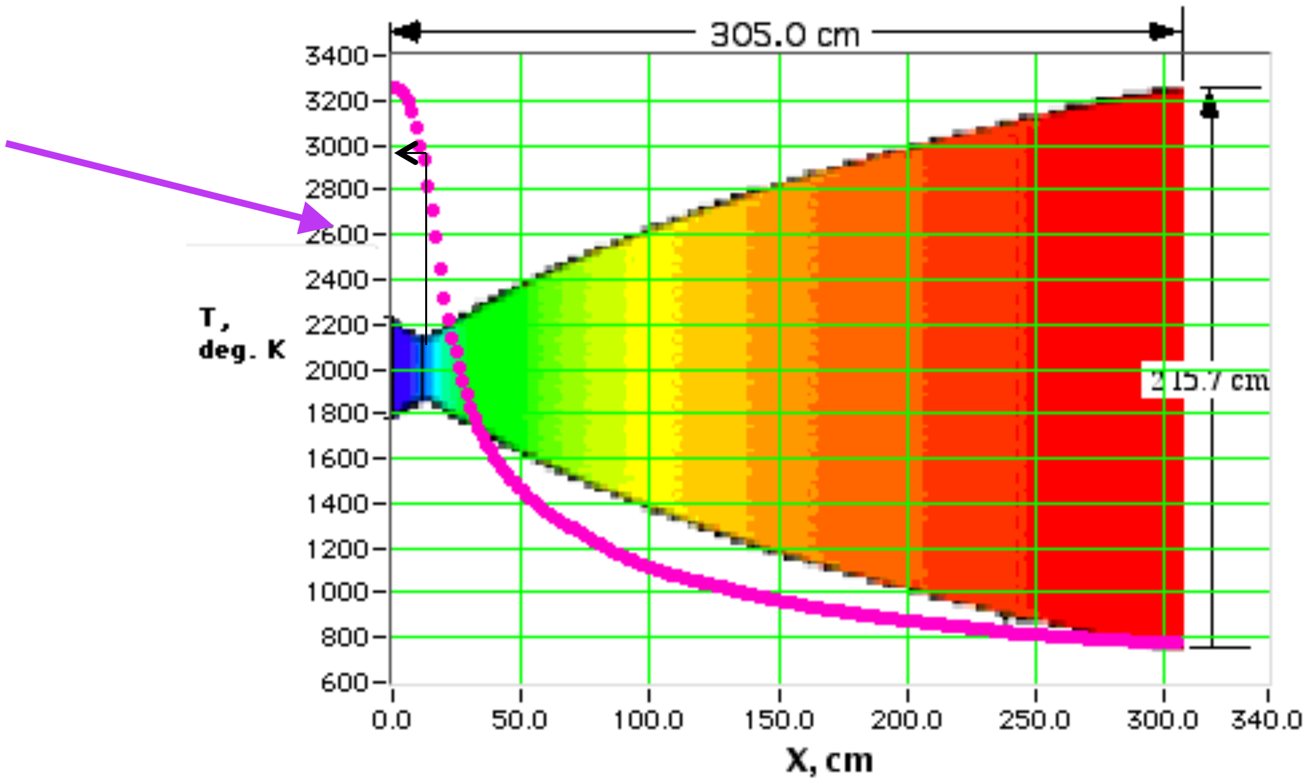
$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$



Plot Flow Properties Along SSME Nozzle Length (cont'd)

- Temperature

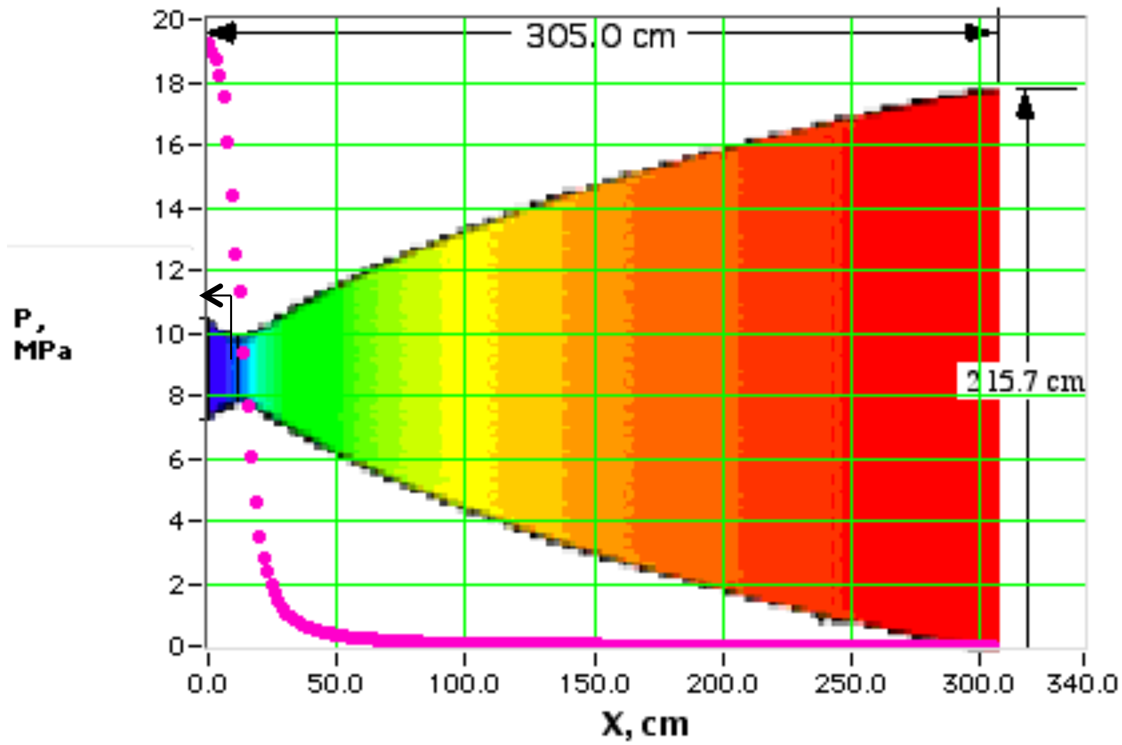
$$T(x) = \frac{T_0}{1 + \frac{\gamma - 1}{2} M(x)^2}$$



Plot Flow Properties Along SSME Nozzle Length (concluded)

- Pressure

$$P(x) = \frac{P_0}{\left(1 + \frac{\gamma - 1}{2} M(x)^2\right)^{\frac{\gamma}{\gamma - 1}}}$$

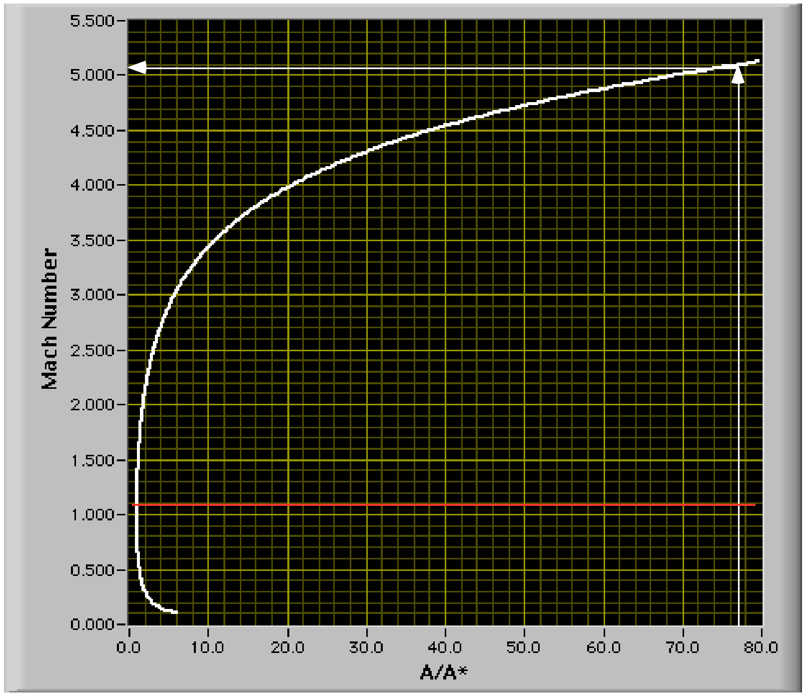


Summary

- A/A^* Directly related to Mach number

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma + 1}\right)\left(1 + \frac{(\gamma - 1)}{2} M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

- "Two-Branch solution: Subsonic, Supersonic



- Nonlinear Equation required Numerical Solution

- "Newton's Method"

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$

Summary (concluded)

- In Isentropic Nozzle, T_0, P_0 are constant

$$T(x) = \frac{T_0}{1 + \frac{\gamma - 1}{2} M(x)^2} \qquad P(x) = \frac{P_0}{\left(1 + \frac{\gamma - 1}{2} M(x)^2\right)^{\frac{\gamma}{\gamma - 1}}}$$

- Mass flow tuned with T_0, P_0 to give sonic velocity At Throat …

- Next: Effect of Exit Pressure Ratio on Nozzle Flow

Section 4 Homework

- Develop an iterative solver for

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

- Use any programming language you prefer i.e. Fortran, C++, MATLAB, …

- Clearly comment and document your code

Section 4 Homework (cont'd)

- Plot the Mach number, pressure, and temperature distribution along the SSME Nozzle Profile for each of the following conditions

P_e=16.8727 KPa, P_0=20.4 Mpa, T_0=3500°K

Gas Properties:

, MW=22

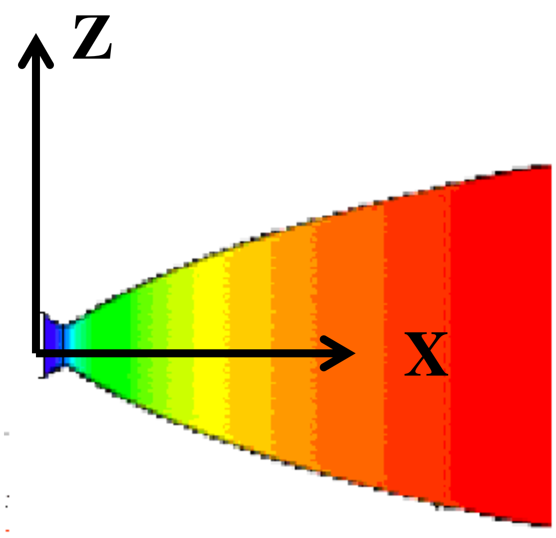
$\gamma = 1.22$

- Hint: make sure nozzle is choked first, then program all functions you might need

- ***Solve for Thrust and Isp in Vacuum***

Section 4 Homework (cont'd)

- SSME Nozzle profile



- **Note: Changes to original geometry from lecture notes**

SSME Nozzle Profile

X, cm	+Z, cm	-Z, cm
0.00	17.50	-17.50
4.00	15.50	-15.50
8.00	13.00	-13.00
12.00	12.25	-12.25
16.00	13.00	-13.00
20.00	15.50	-15.50
25.00	18.50	-18.50
30.00	22.00	-22.00
50.00	33.00	-33.00
70.00	41.50	-41.50
90.00	50.50	-50.50
100.00	54.50	-54.50
120.00	61.00	-61.00
140.00	68.00	-68.00
160.00	74.50	-74.50
180.00	80.50	-80.50
200.00	86.00	-86.00
220.00	91.00	-91.00
240.00	97.00	-97.00
260.00	101.00	-101.00
280.00	105.00	-105.00
300.00	107.50	-107.50
305.00	107.85	-107.85