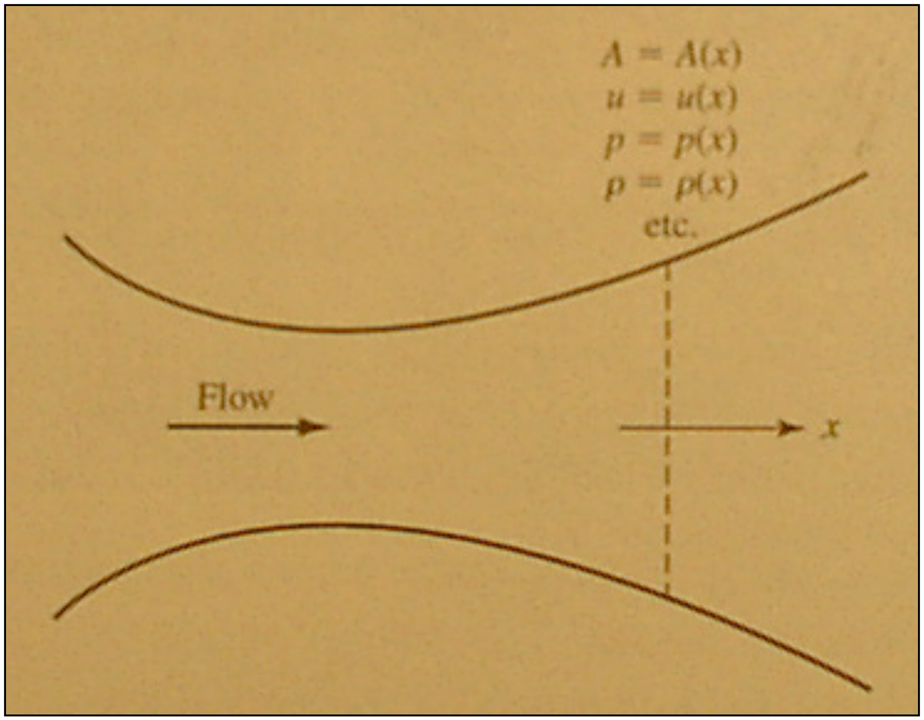
Section 4: Lecture 2
Effect of Pressure Ratio on Nozzle Performance

P0



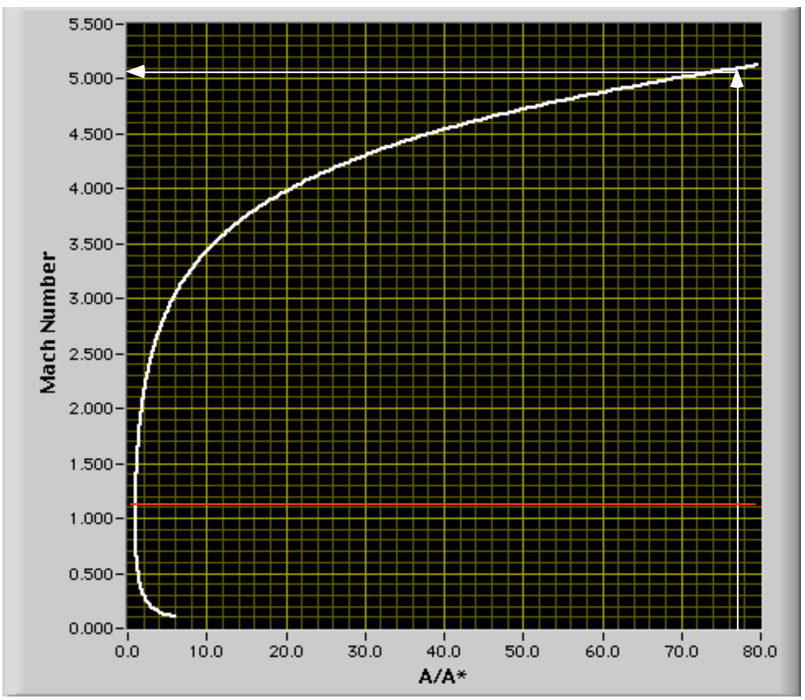
Anderson: Chapter 5 pp. 206-218

Review

- A/A* Directly related to Mach number

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

- "Two-Branch solution: Subsonic, Supersonic

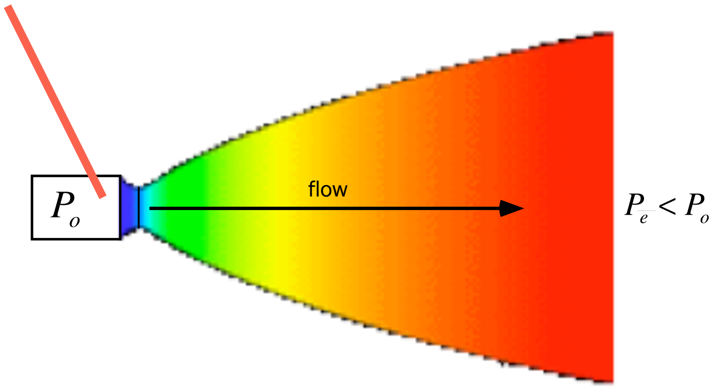


- For Isentropic Flow

$$P(x) = \frac{P_0}{\left(1+\frac{\gamma-1}{2}M(x)^2\right)^{\frac{\gamma}{\gamma-1}}}$$

Case 1: Isentropic Flow in Nozzle *(subsonic)*

• What do we need to "do" to get Isentropic Flow



• For Isentropic Flow P_0~constant and

$$P_e = \frac{P_0}{\left(1+\frac{\gamma-1}{2}M_e^{\;2}\right)^{\frac{\gamma}{\gamma-1}}} \rightarrow$$

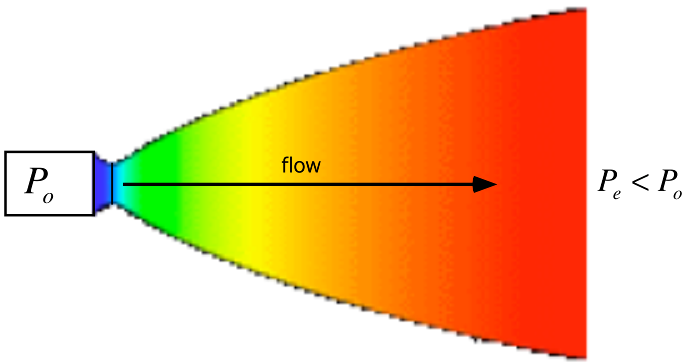
$$M_e = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_0}{P_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

• Consider a Nozzle at Ambient Temperature an Pressure

• Raise Chamber (inlet) Pressure Slightly So that $P_0 > P_e$ (to get flow started)

Case 1: Isentropic Flow in Nozzle (cont'd)

- Let P_e=101.325 KPa, P_0=101.326 Kpa, γ=1.4



SSME Nozzle Geometry

$$M_e = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{P_0}{P_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} =$$

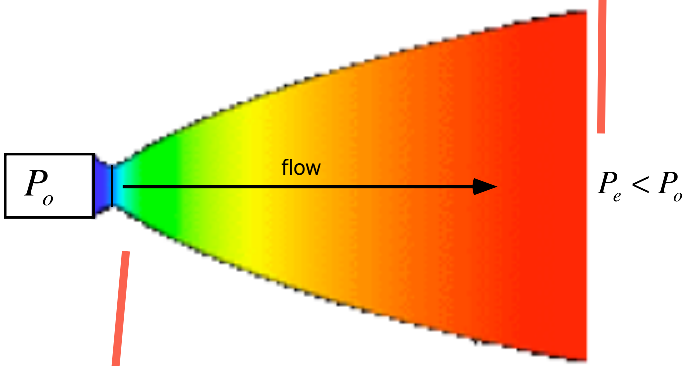
$$\left(\left(\frac{2}{1.4-1}\right)\left(\left(\frac{101.326}{101.325}\right)^{\frac{1.4-1}{1.4}} - 1\right)\right)^{0.5} =$$

$$0.003755$$

- i.e. exit flow is subsonic
- Throat may or may not be sonic

Case 1:Isentropic Flow in Nozzle (cont'd)

- Compute ratio of geometric throat area to Sonic throat area



SSME Nozzle Geometry

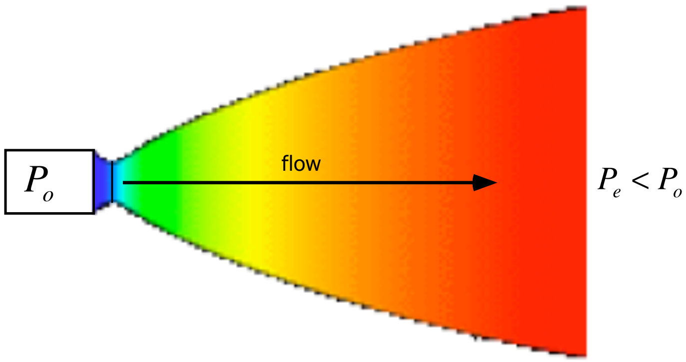
$A_e/A_t = 77.5$

$$\frac{A_t}{A^*} = \left(\frac{A_t}{A_e}\right)\left(\frac{A_e}{A^*}\right) = \left(\frac{A_t}{A_e}\right)\left[\frac{1}{M_e}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M_e^{\,2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right] =$$

$$\frac{1}{77.5}\left(\left(\frac{1}{0.003755}\right)\left(\left(\frac{2}{1.4+1}\right)\left(1+\left(\frac{1.4-1}{2}\right)0.003755^2\right)\right)^{\left(\frac{1.4+1}{2\,(1.4-1)}\right)}\right) = 1.9886$$

Case 1:Isentropic Flow in Nozzle (cont'd)

- Use Iterative Solver to calculate Mach number at throat



SSME Nozzle Geometry

$A_t/A^* = 1.9866$

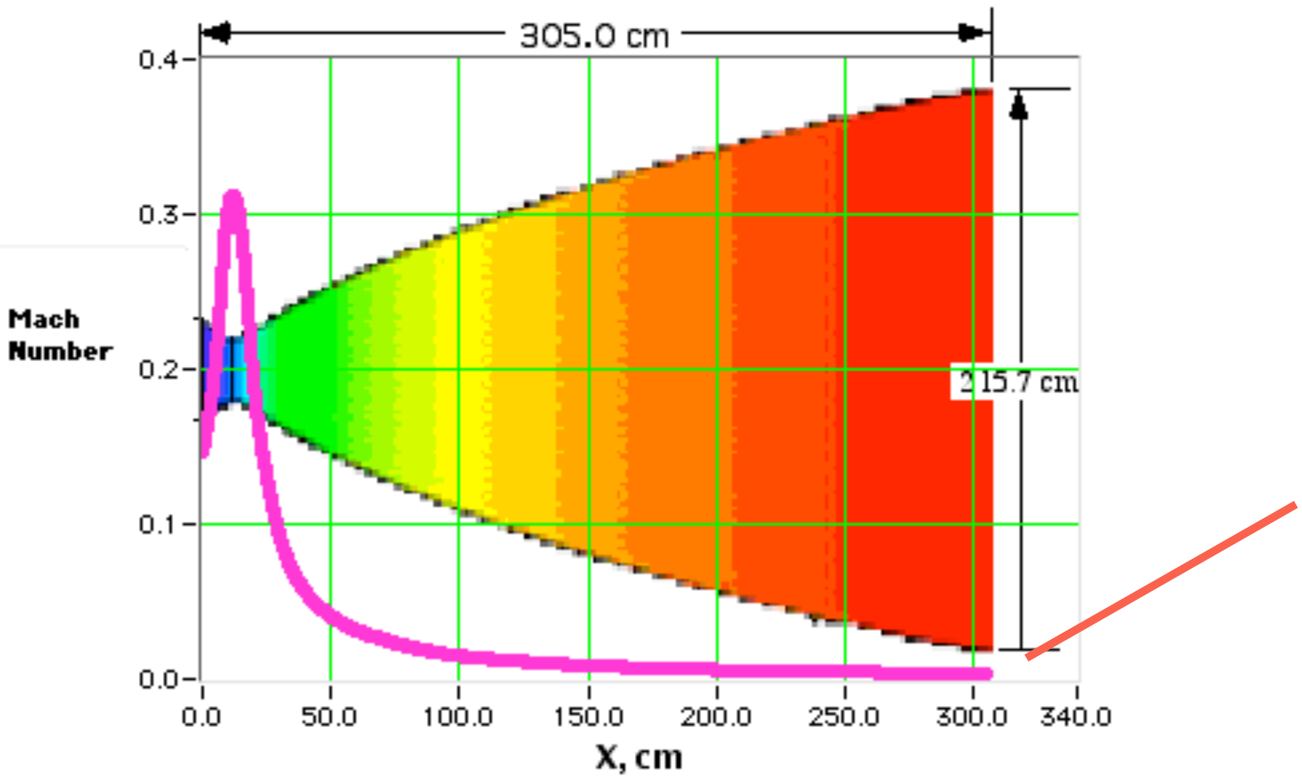
$$M_{(j+1)} = M_{(j)} + \frac{\frac{A}{A^*} - F(M_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{(j)}}$$

$M_t = 0.308226$

- Entire Nozzle is Subsonic

Case 1:Isentropic Flow in Nozzle (cont'd)

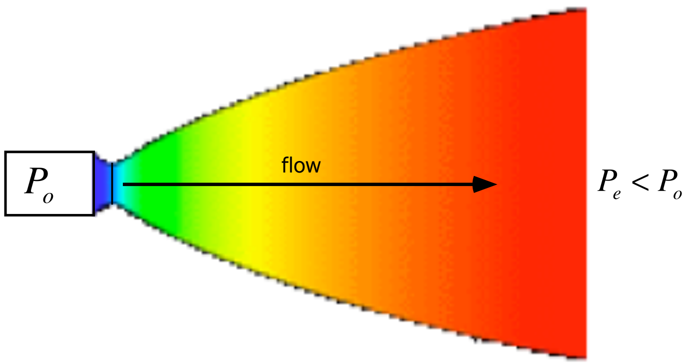
• Use Iterative Solver to calculate Mach number at throat



• Entire Nozzle is Subsonic

Case 2: Isentropic Flow in Nozzle *(subsonic)*

- Increase P_0 to 101.328 Kpa



SSME Nozzle Geometry

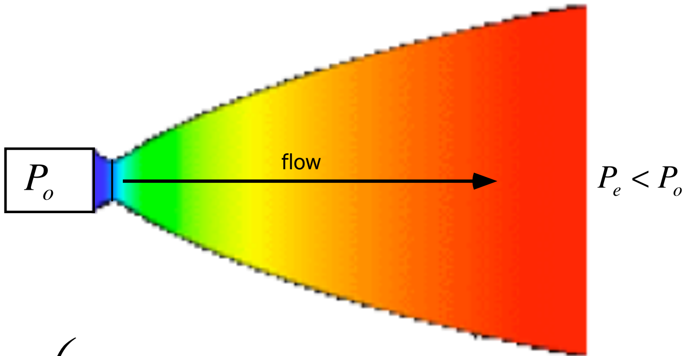
$$M_e = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{P_0}{P_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} =$$

$$\left(\left(\frac{2}{1.4-1}\right)\left(\left(\frac{101.328}{101.325}\right)^{\frac{(1.4-1)}{1.4}} - 1\right)\right)^{0.5} =$$

$$0.0065$$

Case 2:Isentropic Flow in Nozzle (cont'd)

- Compute ratio of geometric throat area to Sonic throat area



SSME Nozzle Geometry

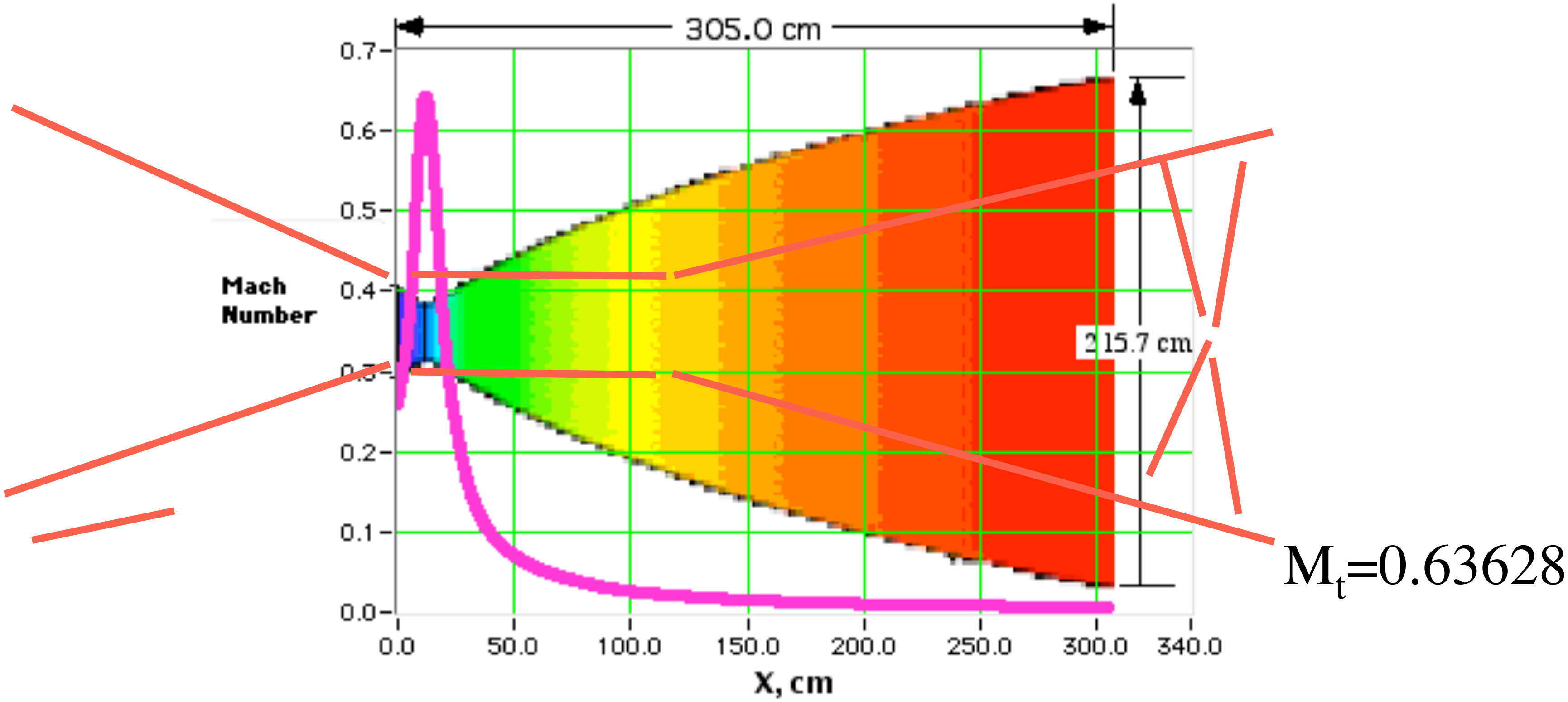
$A_e/A_t = 77.5$

$$\frac{\left(\left(\frac{1}{0.0065}\right)\left(\left(\frac{2}{1.4+1}\right)\left(1+\frac{(1.4-1)}{2}0.0065^2\right)\right)^{\frac{(1.4+1)}{2(1.4-1)}}\right)}{77.5}$$

$= 1.14882$

Case 2:Isentropic Flow in Nozzle (cont'd)

- Use Iterative Solver to calculate Mach number at throat



- Mach Number is Higher but Entire Nozzle is still Subsonic

Case 3: Isentropic Flow in Nozzle *(barely sonic)*

- What Pressure ratio causes sonic velocity at throat

$$\left[\frac{1}{M_e}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M_e^{\,2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right]=\left(\frac{A_e}{A_t}\right)\frac{A_t}{A^*}=77.5\times 1$$

$M_e -> 0.007467$ *(use iterative solver)*

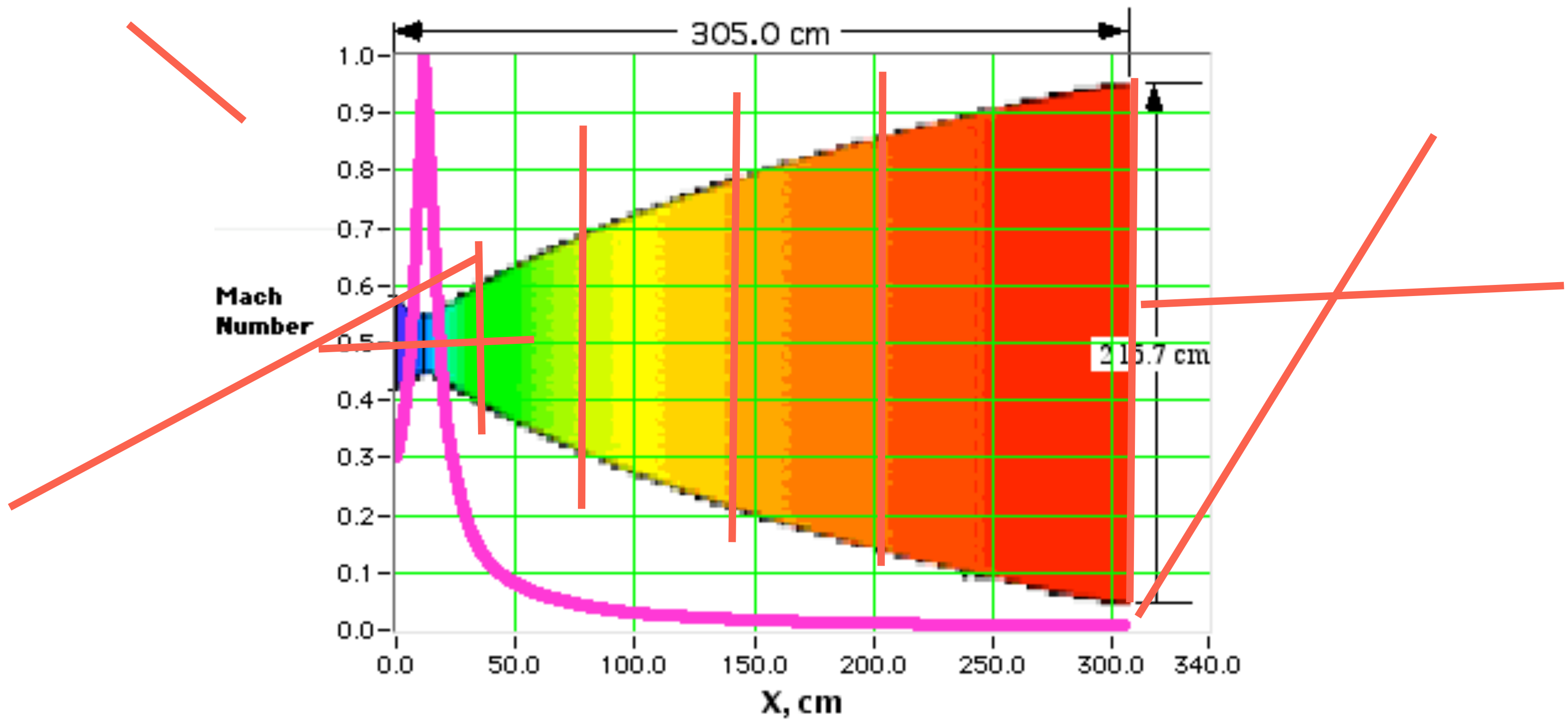
$$P_0 = P_e\left(1+\frac{\gamma-1}{2}M_e^{\,2}\right)^{\frac{\gamma}{\gamma-1}} =$$

$$101.325\left(1+\frac{(1.4-1)}{2}0.007467^2\right)^{\frac{1.4}{(1.4-1)}} = 101.328955 \text{ KPa}$$

$$\frac{p_e}{P_0} = \frac{101.325}{101.328995} = 0.99996$$

Case 3:Isentropic Flow in Nozzle (cont'd)

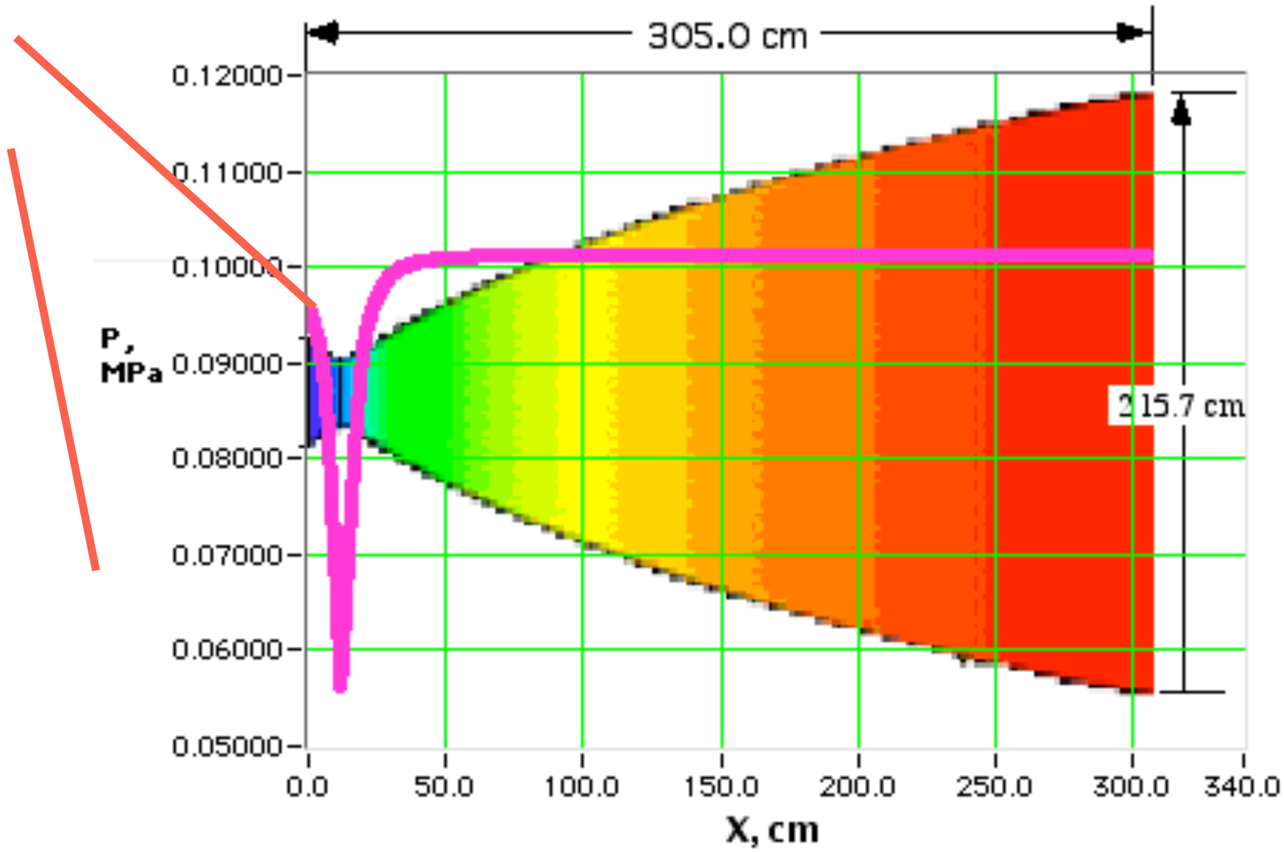
- Mach 1 at Throat ("Barely Choked Nozzle"



- Mach number profile

Case 3: Isentropic Flow in Nozzle (concluded)

- Mach 1 at Throat "Barely Choked Nozzle"



- Pressure Profile

Mass Flow per Area in Choked Nozzle

- Compute mass-flow/area at throat for the three cases
- From Homework 3

$$\frac{\dot{m}}{A_c} = \sqrt{\frac{\gamma}{R_g}}\frac{p_0}{\sqrt{T_0}}\frac{M}{\left[1+\frac{(\gamma-1)}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

- $R_g = 287.056$ J/°K-(kg)
- T_0=273 °K

Mass Flow per Area in Choked Nozzle (cont'd)

- Case 1, P_0=101.326 Kpa (M_t=0.308226)

$$\frac{\dot{m}}{A_c} = \sqrt{\frac{\gamma}{R_g}} \frac{p_0}{\sqrt{T_0}} \frac{M}{\left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}} =$$

$$\frac{\left(\frac{1.4}{287.056}\right)^{0.5} (101.326 \cdot 1000)}{273^{0.5}} (0.2913) = 124.76 \text{ kg/(sec-m}^2\text{)}$$

- R_g = 287.056 J/°K-(kg)
- T_0=273 °K

Mass Flow per Area in Choked Nozzle (cont'd)

- Case 2, P_0= 101.328 Kpa (M_t=0.63628)

$$\frac{\dot{m}}{A_c} = \sqrt{\frac{\gamma}{R_g}}\frac{p_0}{\sqrt{T_0}}\frac{M}{\left[1+\frac{(\gamma-1)}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}} =$$

$$\frac{\left(\frac{1.4}{287.056}\right)^{0.5}(101.328\cdot 1000)}{273^{0.5}}(0.50374) = 215.74 \text{ kg/(sec-m}^2\text{)}$$

- R_g = 287.056 J/°K-(kg)
- T_0=273 °K

Mass Flow per Area in Choked Nozzle (cont'd)

- Case 3, P_0= 101.328995 Kpa (M_t=1.0)

$$\frac{\dot{m}}{A_c} = \sqrt{\frac{\gamma}{R_g}}\frac{p_0}{\sqrt{T_0}}\frac{M}{\left[1+\frac{(\gamma-1)}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}} =$$

$$\frac{\left(\frac{1.4}{287.056}\right)^{0.5}(101.328995\cdot 1000)}{273^{0.5}}(0.5787) = 247.85 \text{ kg/(sec-m}^2\text{)}$$

- R_g = 287.056 J/°K-(kg)
- T_0=273 °K

Mass Flow per Area in Choked Nozzle (cont'd)

- Verify that maximum mass flow/area occurs in choked nozzle

$$\frac{\dot{m}}{A_t} = \sqrt{\frac{\gamma}{R_g}} \frac{p_0}{\sqrt{T_0}} \frac{M_t}{\left[1 + \frac{(\gamma-1)}{2} M_t^{\,2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

- Necessary Condition for maximum

$$\frac{\partial}{\partial M}\left[\frac{\dot{m}}{A_t}\right] = 0 = \frac{\partial}{\partial M}\left[\frac{M_t}{\left[1 + \frac{(\gamma-1)}{2} M_t^{\,2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}}\right] \rightarrow \frac{(2 - 2\,M^2)\;\left(1 + \frac{1}{2}\,M^2\,(-1+\gamma)\right)^{\frac{1+\gamma}{2-2\gamma}}}{2 + M^2\,(-1+\gamma)}$$

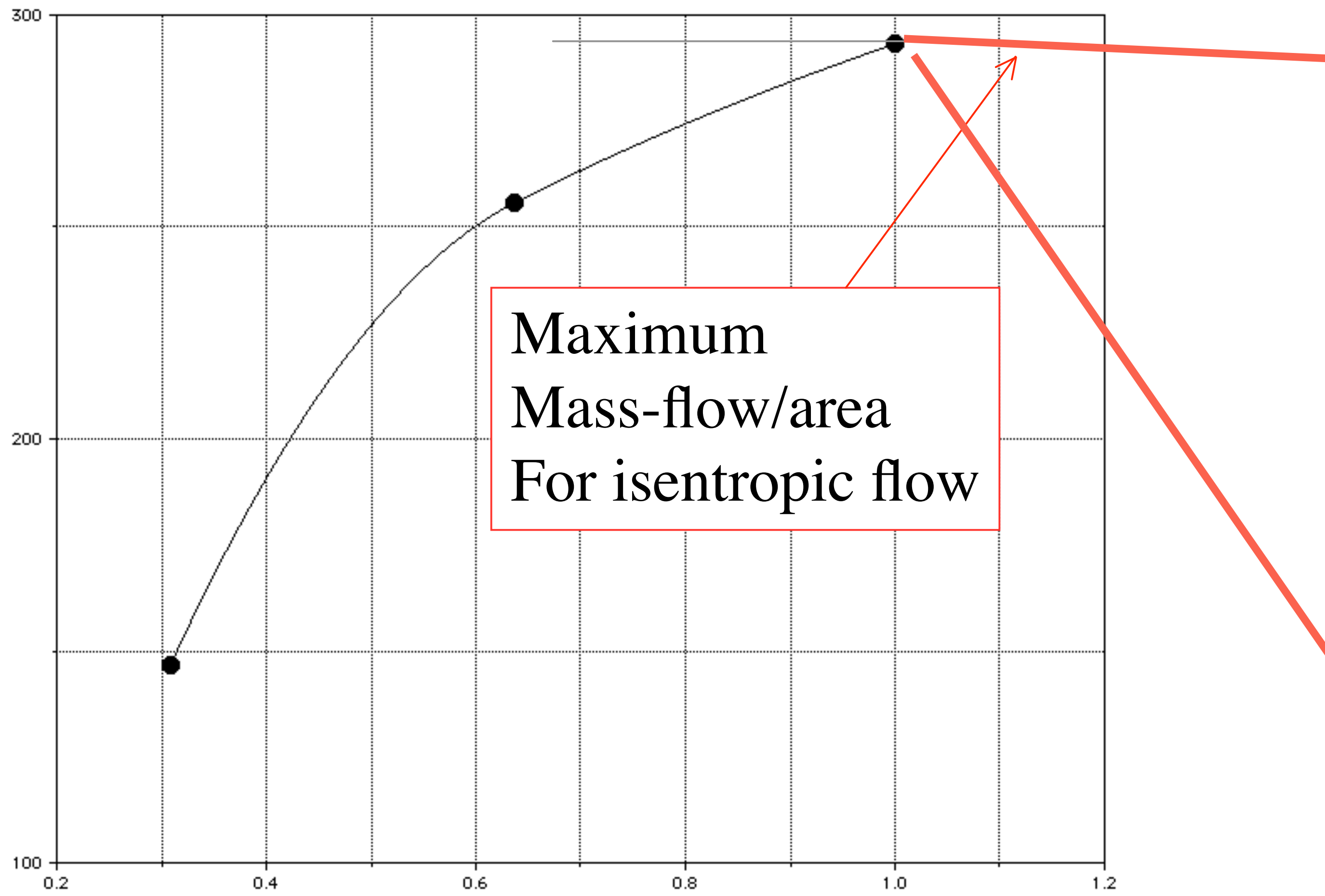
Mass Flow per Area in Choked Nozzle (cont'd)

- Verify that maximum mass flow occurs in choked nozzle

$$\frac{\partial}{\partial M}\left[\frac{\dot{m}}{A_t}\right] = \frac{\left(2 - 2\,M^2\right)\left(1 + \frac{1}{2}\,M^2\,(-1+\gamma)\right)^{\frac{1+\gamma}{2-2\gamma}}}{2 + M^2\,(-1+\gamma)}$$

$$M_t = 1 \longrightarrow \frac{\partial}{\partial M}\left[\frac{\dot{m}}{A_t}\right] = 0$$

Mass Flow per Area in Choked Nozzle (concluded)



$$\frac{\dot{m}}{A_t}$$

kg/(sec-m^2)

M_t

Case 4: Isentropic Flow in Nozzle *(supersonic)*

- Now assume that flow in throat is choked and $A_t = A^*$

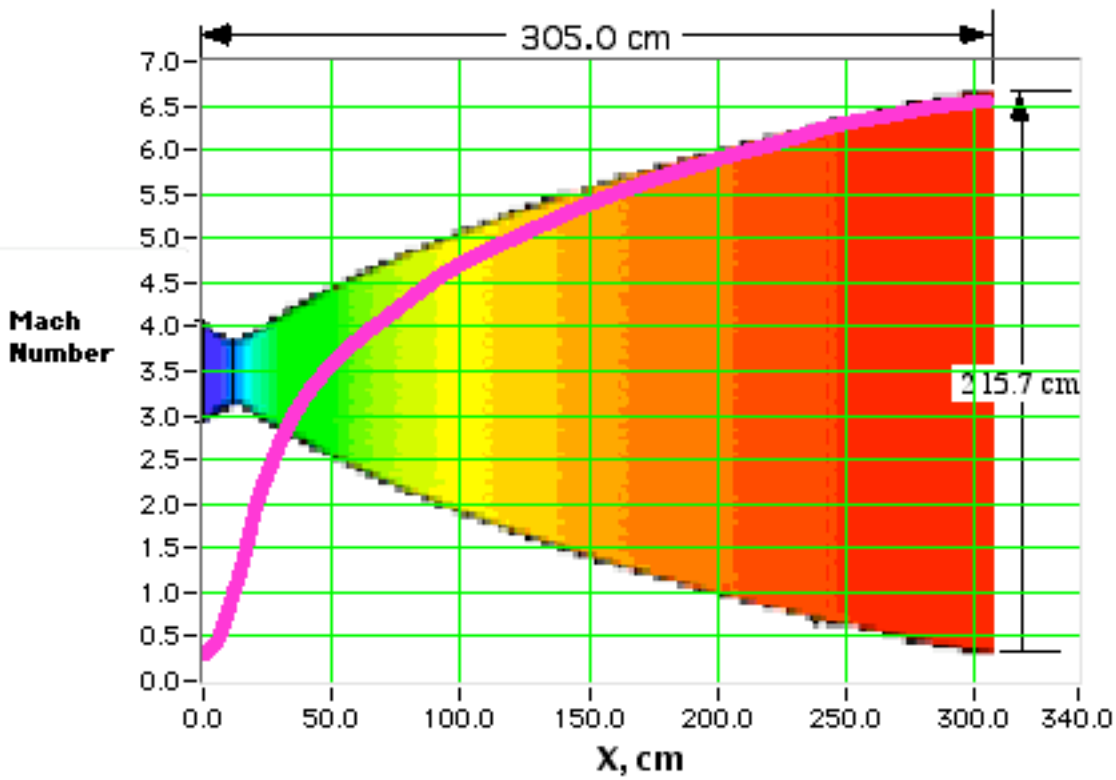
- Furthermore, $A_e/A_t = A_e/A_* = 77.5$

- Use iterative Solver to compute Isentropic Supersonic mach Number at Nozzle exit

$$\frac{A_e}{A^*} = \left[\frac{1}{M_e} \left[\left(\frac{2}{\gamma + 1} \right) \left(1 + \frac{(\gamma - 1)}{2} M_e{}^2 \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}} \right] \rightarrow M_e = 6.546284$$

Case 4: Isentropic Flow in Nozzle *(supersonic)*

- Mach Number profile



Case 4: Isentropic Flow in Nozzle *(supersonic)* *(cont'd)*

- Assume the nozzle exits at a pressure of 5 Kpa

Compute the required chamber pressure for isentropic flow in Nozzle, compute Pressure P_e/P_0

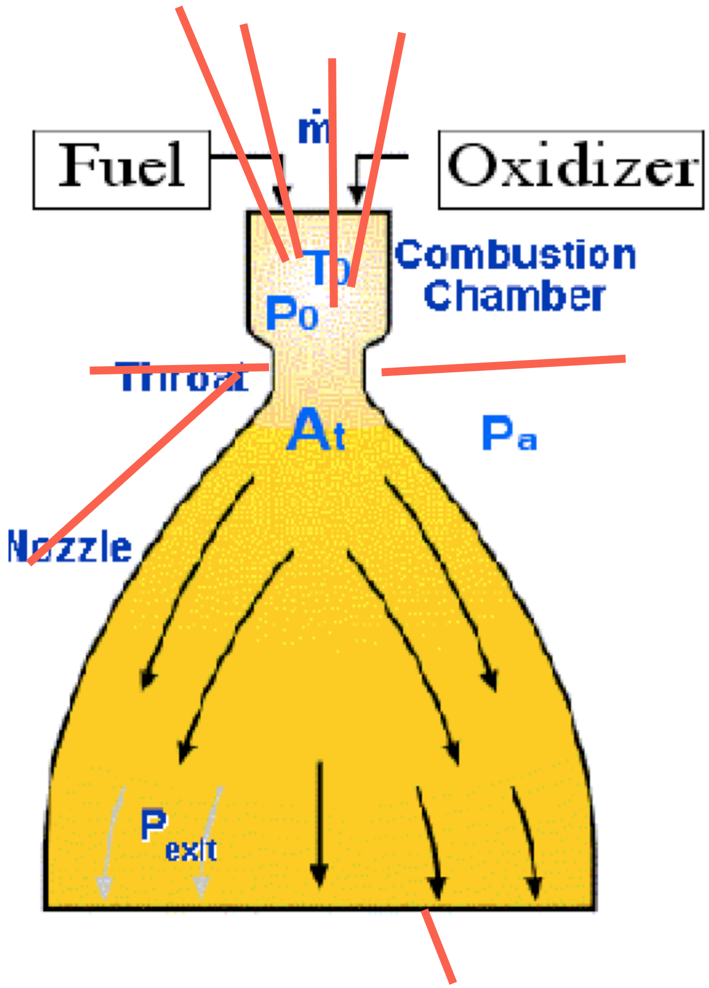
$$P_0 = P_e\left(1+\frac{\gamma-1}{2}M_e^{\ 2}\right)^{\frac{\gamma}{\gamma-1}} =$$

$$5\left(1+\left(\frac{1.4-1}{2}\right)6.546284^2\right)^{\left(\frac{1.4}{1.4-1}\right)} = 13560.7 \text{ Kpa}$$

Wow!

- 134 atmospheres .. That! Is a lot of pressure

Example: SSME Rocket Engine

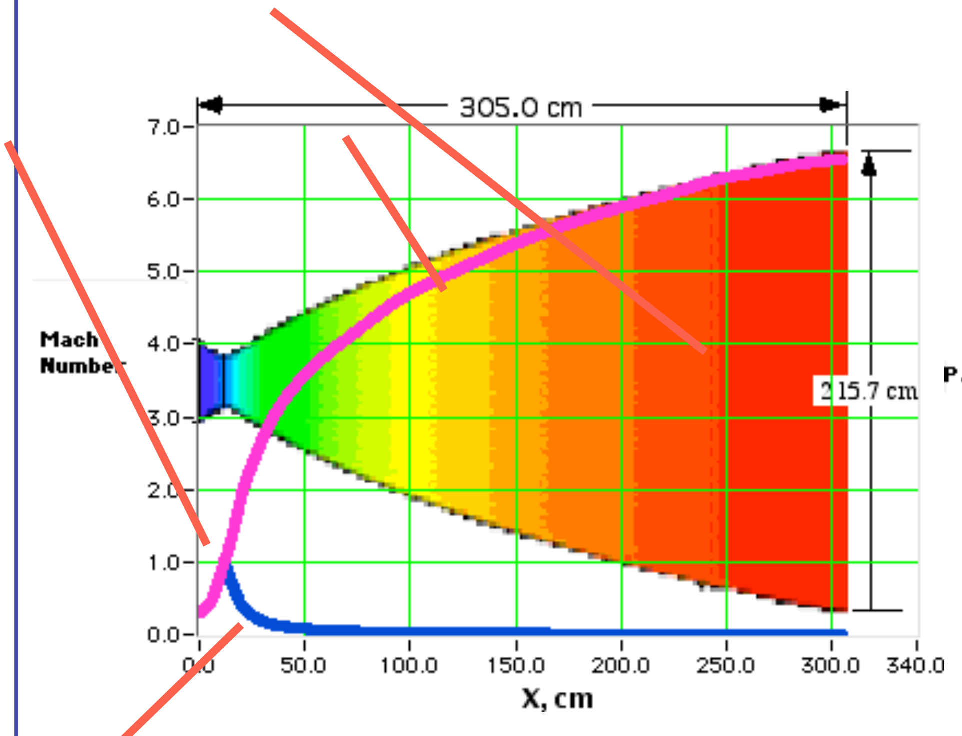


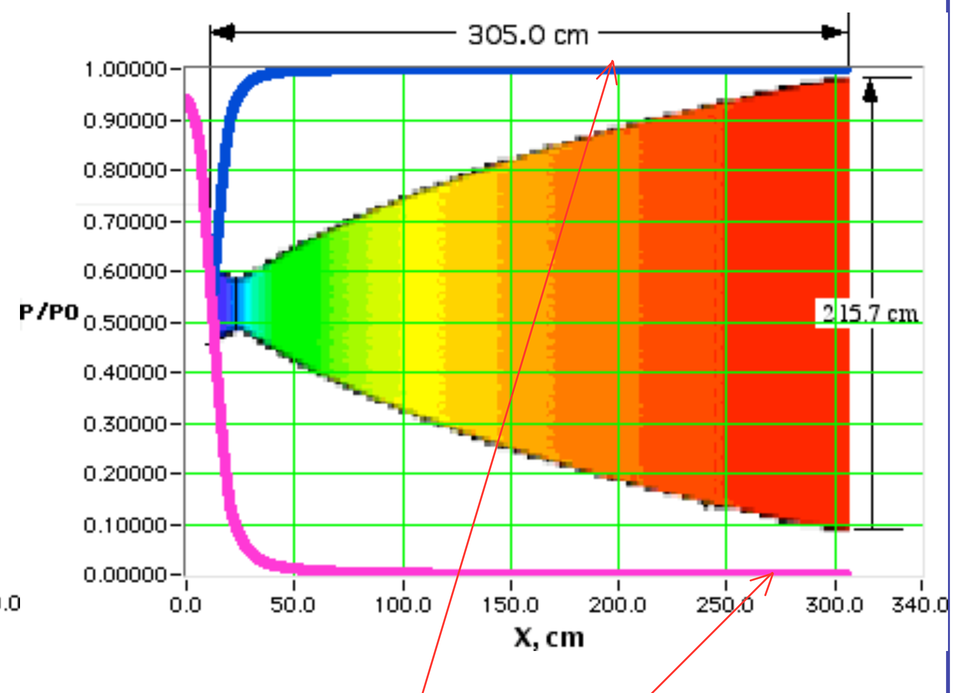
- The Space Shuttle Main Engines Burn LOX/LH2 for Propellants with A ratio of LOX:LH2 =6:1

- The Combustor Pressure, p_0 Is 18.94 Mpa, combustor temperature, T_0 is 3615°K, throat diameter is 26.0 cm

- OK so these are the chamber pressures we are dealing with for large rocket engines

Two Distinct Solutions for Isentropic Flow in Duct with (M=1) Choked Throat



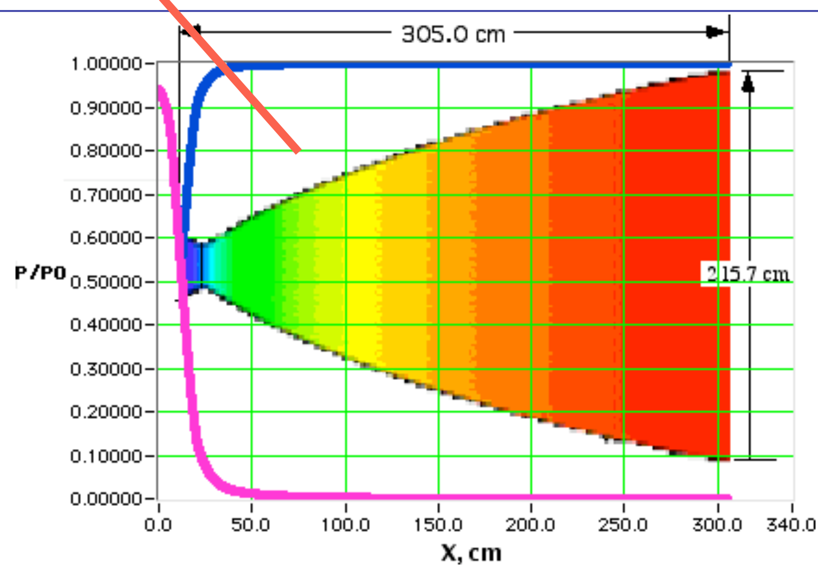


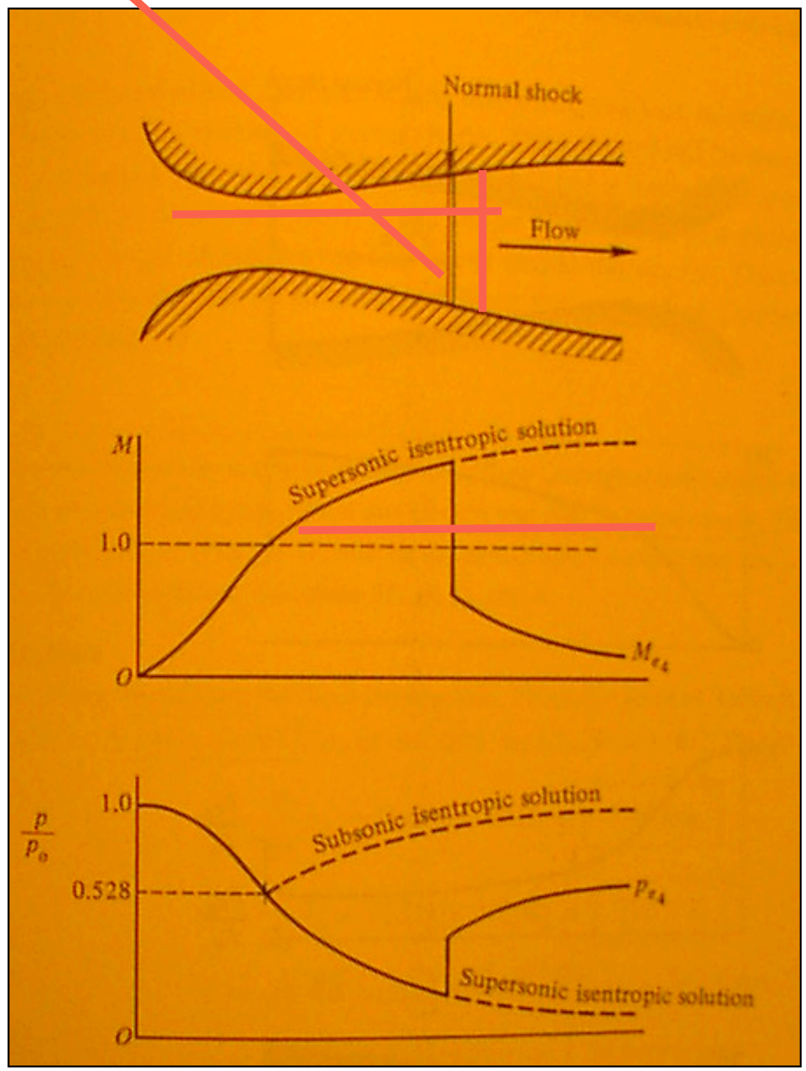
- What happens when pressure ratio across nozzle is between these two options?

Two Distinct Solutions for Isentropic Flow in Duct with (M=1) Choked Throat (cont'd)

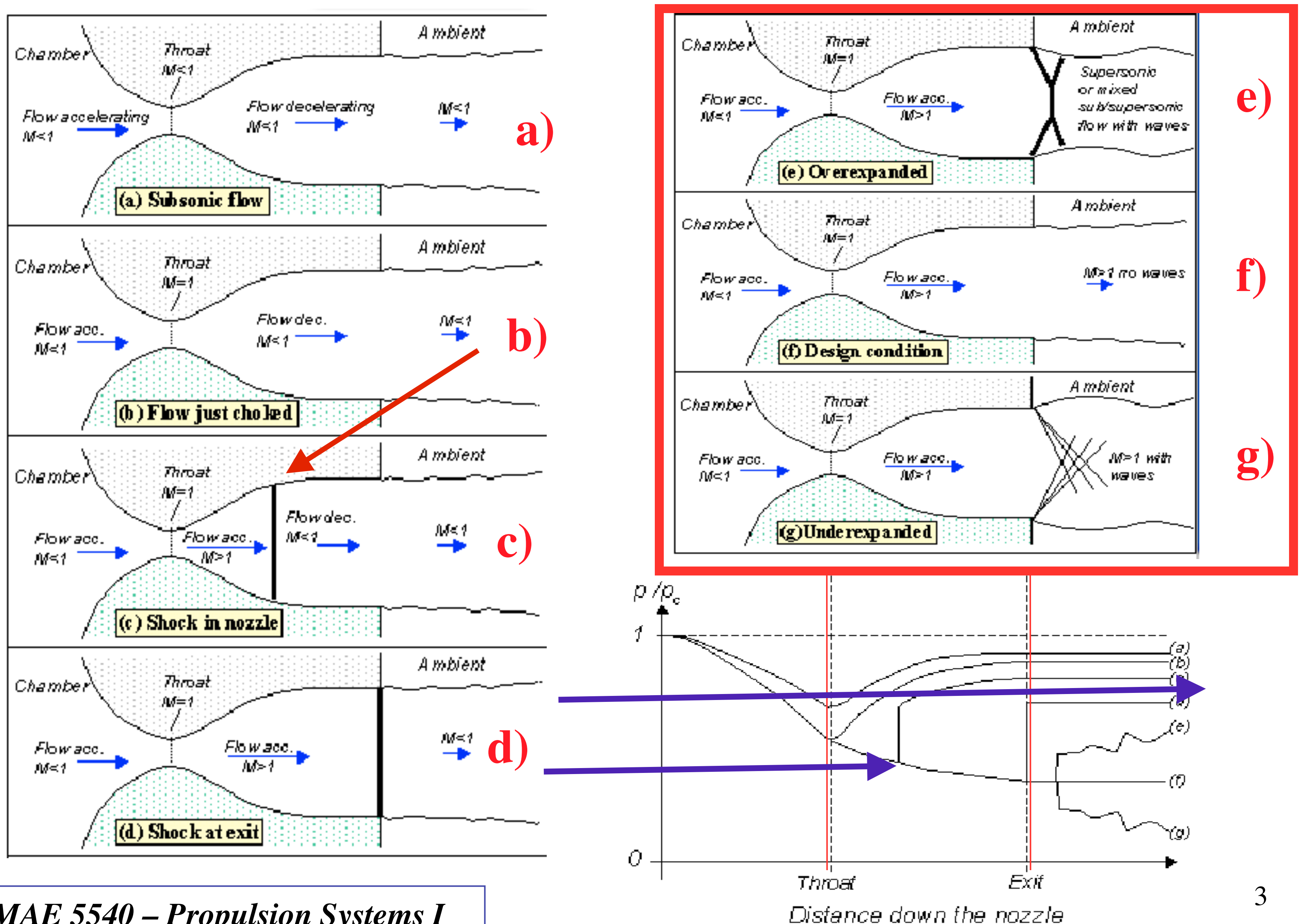
- What happens when pressure ratio across nozzle is between these two options?

- Flow in Nozzle is no longer isentropic?
- Normal shockwave appears in divergent section of Nozzle





Nozzle Flow Summary



Section 4 Homework

- Develop an iterative solver for

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

- Use any programming language you prefer
i.e. Fortran, C++, MATLAB, …

- Clearly comment and document your code

Section 4 Homework (cont'd)

- Plot the Mach number, pressure, and temperature distribution along the SSME Nozzle for each of the following conditions

P_e=16.8727 KPa, P_0=20.4 Mpa, T_0=3500°K

Gas Properties:

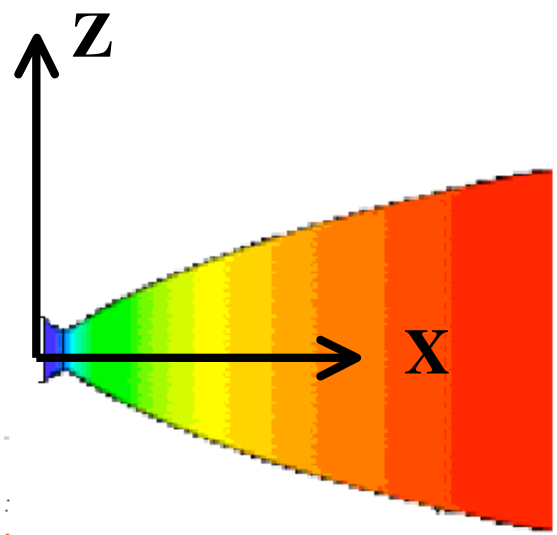
$$\gamma = 1.22$$, MW=22

- Hint: make sure nozzle is choked first, then program all functions you might need

- Solve for Thrust and Isp in Vacuum

Section 4 Homework (cont'd)

- SSME Nozzle profile



- **Note: Changes to original geometry**

SSME Nozzle Profile

X, cm	+Z, cm	-Z, cm
0.00	17.50	-17.50
4.00	15.50	-15.50
8.00	13.00	-13.00
12.00	12.25	-12.25
16.00	13.00	-13.00
20.00	15.50	-15.50
25.00	18.50	-18.50
30.00	22.00	-22.00
50.00	33.00	-33.00
70.00	41.50	-41.50
90.00	50.50	-50.50
100.00	54.50	-54.50
120.00	61.00	-61.00
140.00	68.00	-68.00
160.00	74.50	-74.50
180.00	80.50	-80.50
200.00	86.00	-86.00
220.00	91.00	-91.00
240.00	97.00	-97.00
260.00	101.00	-101.00
280.00	105.00	-105.00
300.00	107.50	-107.50
305.00	107.85	-107.85