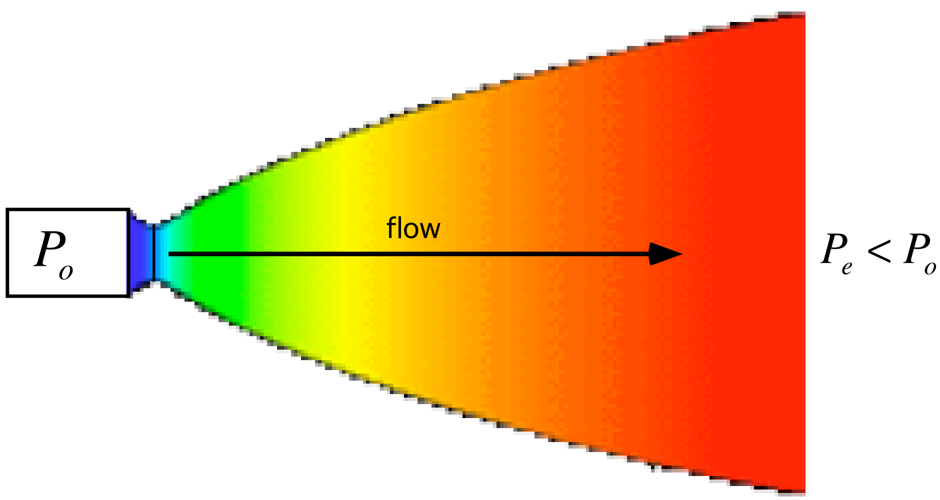
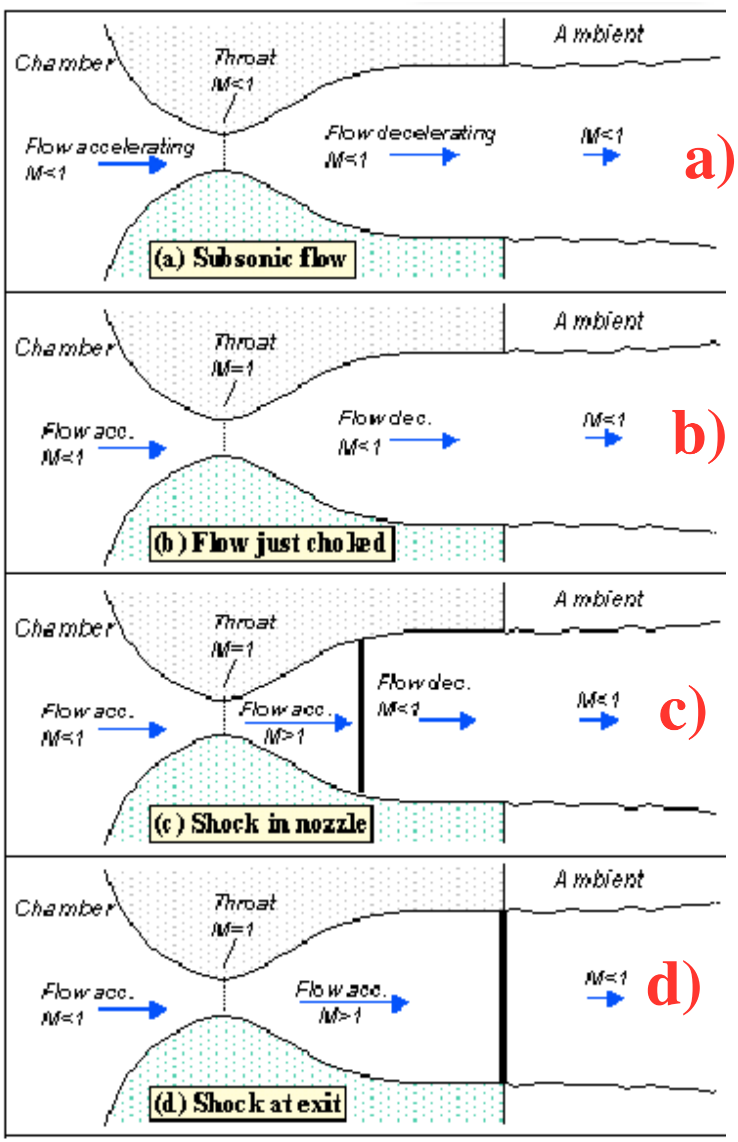
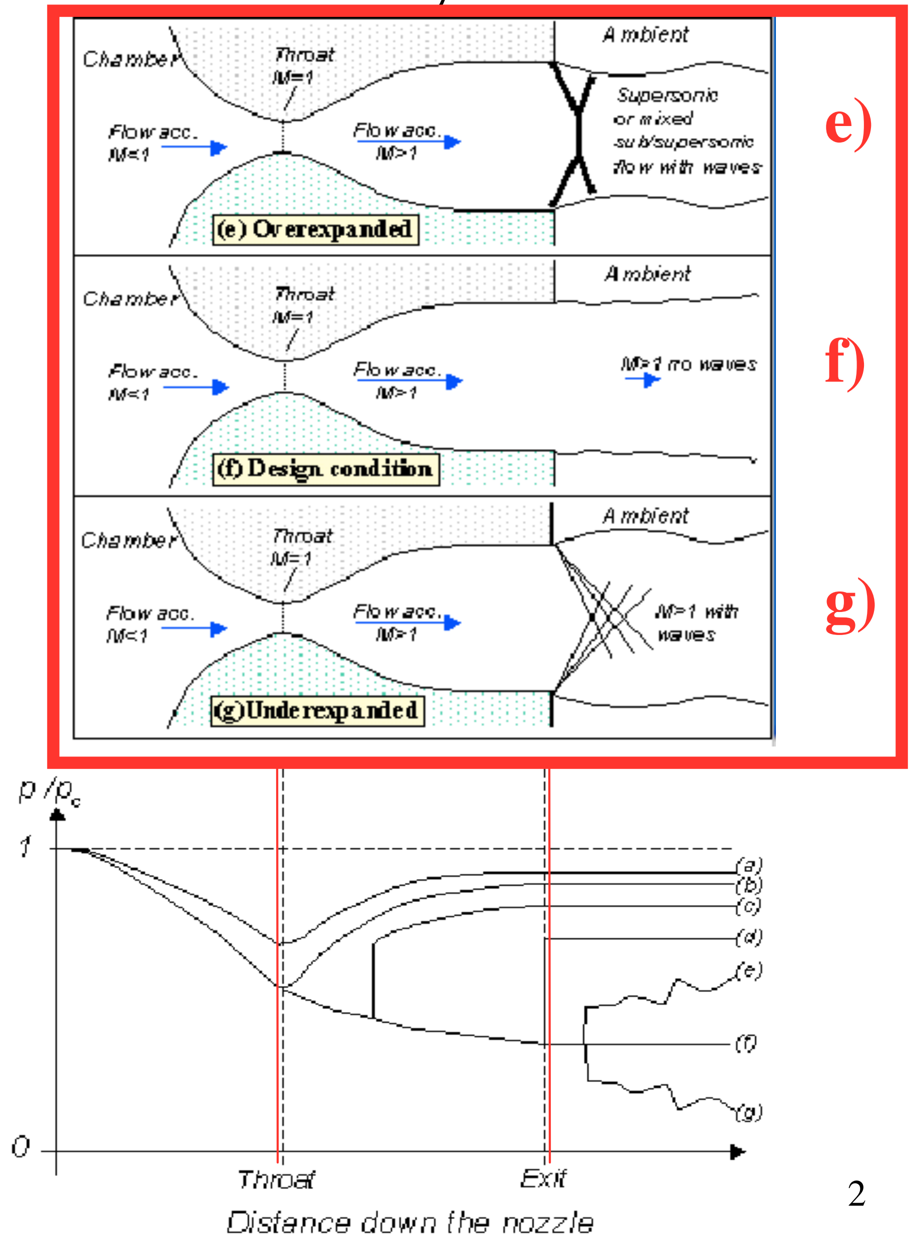
Section 5: Lecture 3
The Optimum Rocket Nozzle



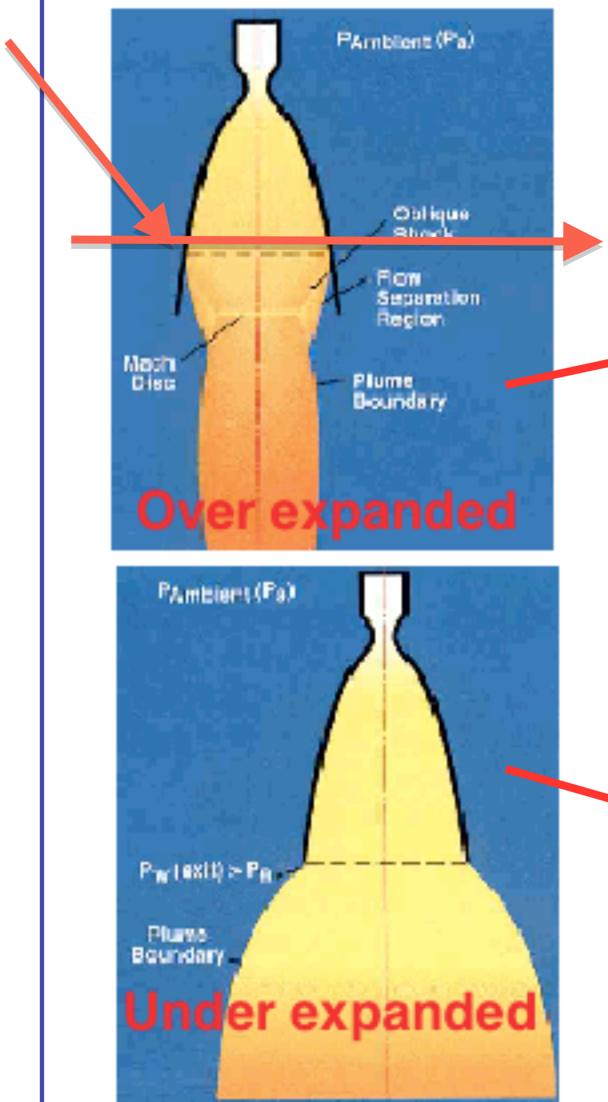
Not in Anderson

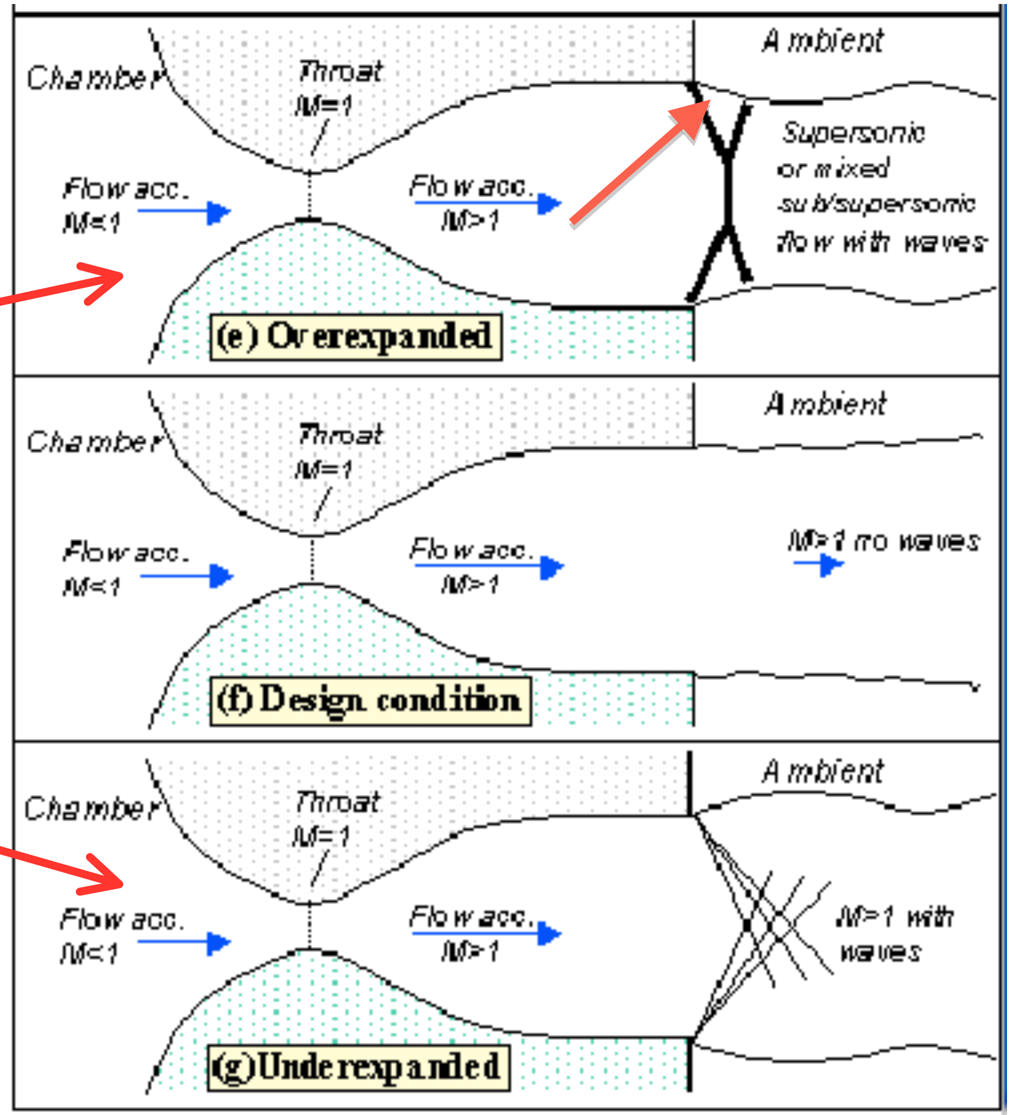
Nozzle Flow Summary



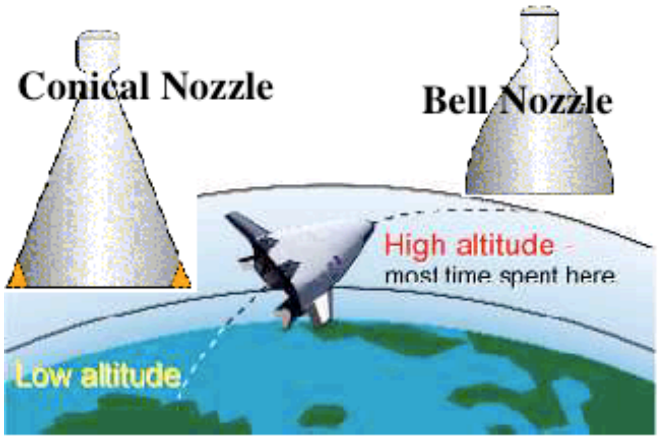


Next: The Optimum Nozzle (1)



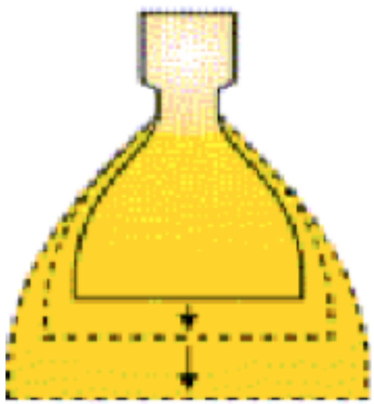


Next: The Optimum Nozzle (2)



$$Thrust = \dot{m} V_{exit} + A_{exit}(p_{exit} - p_{\infty})$$

$$for\ given\ \dot{m} \rightarrow \boxed{\begin{array}{c} V_{exit} \propto \dfrac{A_{exit}}{A^*} \\ \dfrac{1}{P_{exit}} \propto \dfrac{A_{exit}}{A^*} \end{array}}$$



$$\rightarrow both\ \{V_{exit}, P_{exit}\}\ contribute\ to\ thrust$$

$$\rightarrow what\ \frac{A_{exit}}{A^*}\ is\ "optimal"?$$

Rocket Thrust Equation

$$Thrust = \dot{m} V_{exit} + A_{exit}(p_{exit} - p_{\infty})$$

- Non dimensionalize as

$$\frac{Thrust}{P_0 A_{throat}} = \frac{\dot{m} V_{exit}}{P_0 A_{throat}} + \frac{A_{exit}}{A_{throat}} \frac{(p_{exit} - p_{\infty})}{P_0}$$

- For a choked throat

$$\frac{\dot{m}}{A^* P_0} = \frac{1}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \longrightarrow \frac{Thrust}{P_0 A^*} = \frac{V_{exit}}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} + \frac{A_e}{A^*} \frac{(p_{exit} - p_{\infty})}{P_0}$$

Rocket Thrust Equation (cont'd)

$$\frac{Thrust}{P_0 A^*} = \frac{V_{exit}}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} + \frac{A_e}{A^*}\frac{(p_{exit} - p_\infty)}{P_0}$$

- For isentropic flow

$$\frac{Thrust}{P_0 A^*} \equiv C_F \rightarrow \text{"Thrust Coefficent"}$$

$$V_{exit} = \sqrt{2c_p\left[T_{0_{exit}} - T_{exit}\right]} = \sqrt{2c_p T_{0_{exit}}}\left[1 - \frac{T_{exit}}{T_{0_{exit}}}\right]^{1/2}$$

- Also for isentropic flow

$$\frac{p_2}{p_1} = \left[\frac{T_2}{T_1}\right]^{\frac{\gamma}{\gamma-1}} \longrightarrow \frac{T_{exit}}{T_{0_{exit}}} = \left(\frac{p_{exit}}{P_{0_{exit}}}\right)^{\frac{\gamma-1}{\gamma}}$$

Rocket Thrust Equation (cont'd)

- Subbing into velocity equation

$$V_{exit} = \sqrt{2c_p\left[T_{0_{exit}} - T_{exit}\right]} = \sqrt{2c_p T_{0_{exit}}}\left[1-\left(\frac{p_{exit}}{P_{0_{exit}}}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}$$

- Subbing into the thrust equation

$$\frac{Thrust}{p_0 A^*} = \frac{\sqrt{2c_p T_{0_{exit}}}\left[1-\left(\frac{p_{exit}}{P_{0_{exit}}}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0} =$$

$$\left[1-\left(\frac{p_{exit}}{P_{0_{exit}}}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}\sqrt{\frac{2c_p\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0}$$

Rocket Thrust Equation (cont'd)

- Simplifying

$$\frac{2c_p\gamma}{R_g} = \frac{2c_p\gamma}{c_p - c_v} = \frac{2\gamma}{1-\frac{1}{\gamma}} = \frac{2\gamma^2}{\gamma-1}$$

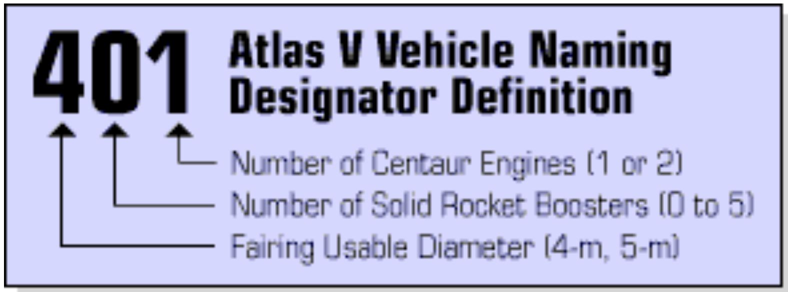
$$\frac{Thrust}{P_0 A^*} \equiv C_F \rightarrow \text{"Thrust Coefficent"}$$

- Finally, for an isentropic nozzle $P_{0_{exit}} = P_0$

$$\frac{Thrust}{P_0 A^*} = \gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0}$$

- **Non-dimensionalized *thrust coefficient* is a function of Nozzle pressure ratio and back pressure only**

Example: Atlas V 401 First Stage



- $Thrust_{vac}$ = 4152 kn
- $Thrust_{sl}$ = 3827 kn
- $I_{sp_{vac}}$ = 337.8 sec
- A_e/A_* = 36.87
- P_0 = 25.7 Mpa
- Lox/RP-1 Propellants

- γ =1.220122

- Plot Thrust Versus Altitude

Example: Atlas V 401 First Stage (cont'd)

- From Homework 2

p_∞ Sea level -- 101.325 kpa

$$F_{vac} - F_{sl} = \left[\dot{m}_e V_e + \left(p_e A_e\right)\right] - \left[\dot{m}_e V_e + \left(p_e A_e - p_{sl} A_e\right)\right] = p_{sl} A_e$$

$$A_e = \frac{F_{vac} - F_{sl}}{p_{sl}} = \frac{4152000_{\frac{kg-m}{\sec^2}} - 3827000_{\frac{kg-m}{\sec^2}}}{101325_{\frac{kg-m}{\sec^2}/m^2}} = 3.2705_{m^2}$$

$$A^* = A_{exit} \Big/ \frac{A_{exit}}{A^*} = \frac{3.2705}{36.87} = 0.0870_{m^2}$$

Compute Isentropic Exit Pressure

- User Iterative Solve to Compute Exit Mach Number

$$\frac{A_{exit}}{A^*} = 36.87 = \left[\frac{1}{M_{exit}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M_{exit}^{\;2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right] \Rightarrow$$

$$M_{exit} = 4.2954$$

- Compute Exit Pressure

$$p_{exit} = \frac{P_{0_{exit}}}{\left(1+\frac{\gamma-1}{2}M_{exit}^{\;2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}} = \frac{25.7\cdot 1000}{\left(1+\frac{1.220122-1}{2}4.2954^2\right)^{\left(\frac{1.220122}{1.220122-1}\right)}} = 55.06 \text{ kPa}$$

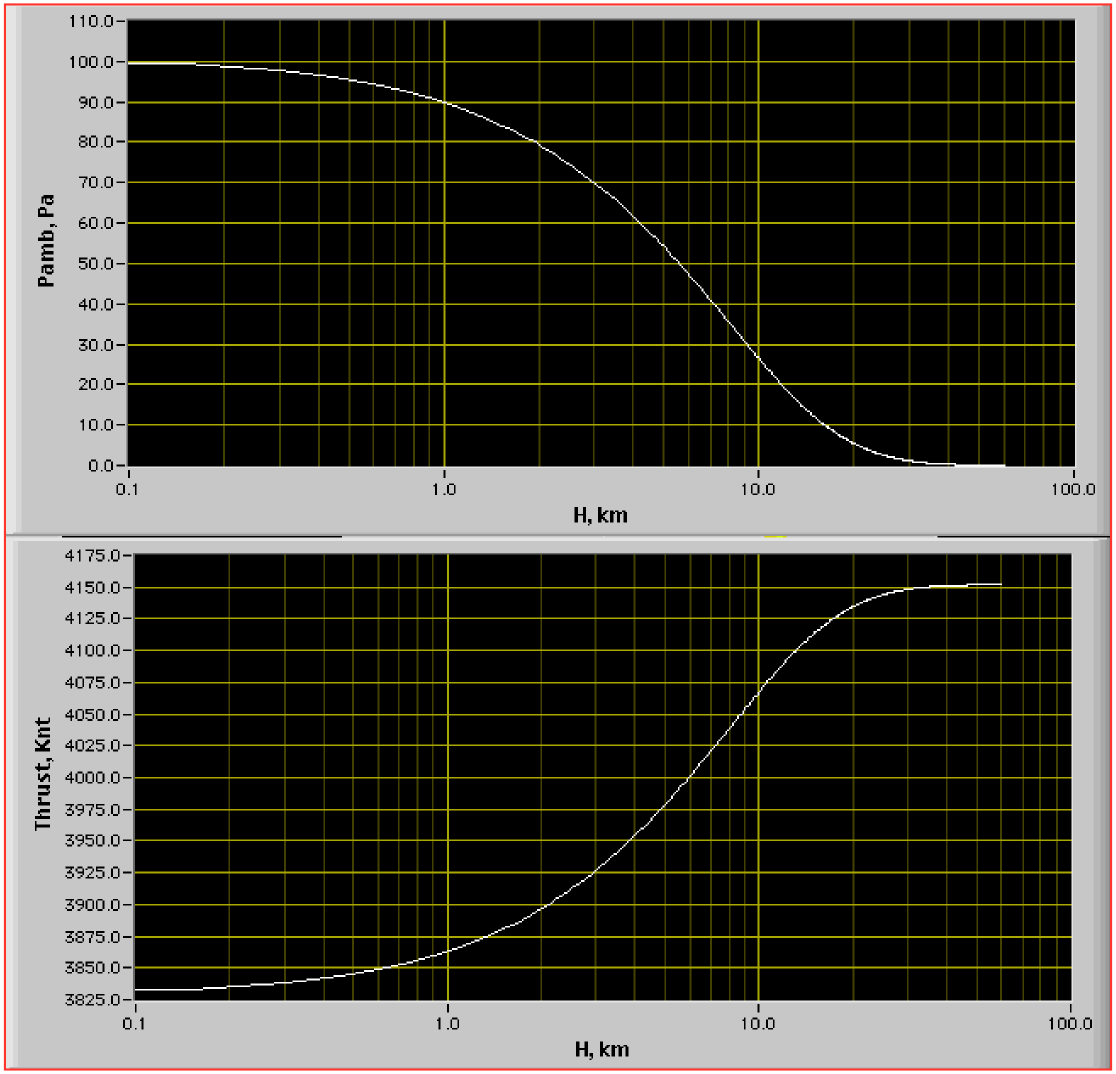
Look at Thrust as function of Altitude (p_∞)

- All the pieces we need now

$$Thrust = \gamma P_0 A^* \sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]}^{1/2} + A_{exit}(p_{exit}-p_\infty)$$

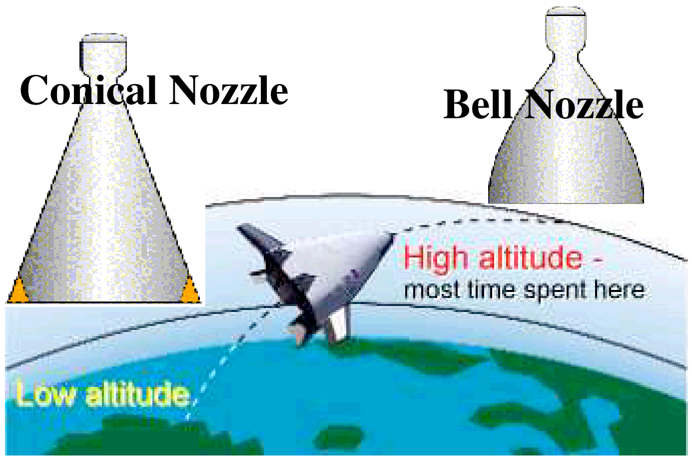
$$\begin{bmatrix} \gamma = 1.220122 \\ P_0 = 25.7_{Mpa} \\ p_{exit} = 55.06_{kPa} \\ A^* = 0.087_{m^2} \\ A_{exit} = 3.2705_{m^2} \end{bmatrix}$$

Look at Thrust as function of Altitude (p_∞) (cont'd)

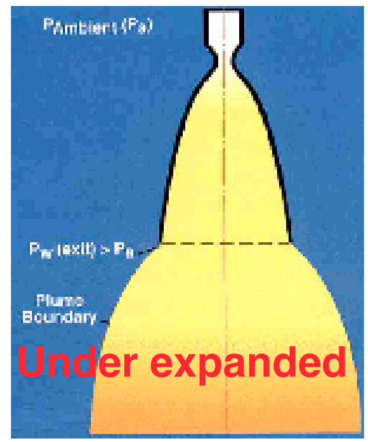


- Thrust increases logarithmically with altitude

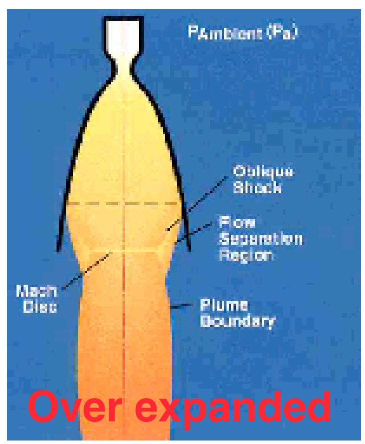
Exit Pressure has a dramatic effect on Nozzle performance



Vacuum (Space)



Lift off



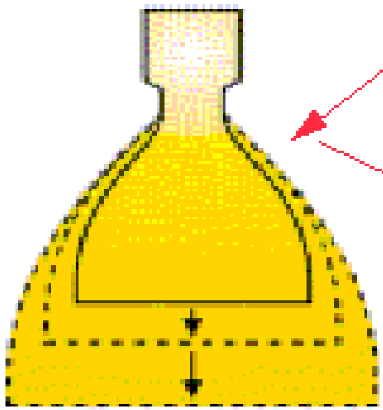
Large area ratio nozzles at sea level cause flow separation, performance losses, high nozzle structural loads

Bell constrains flow limiting performance

The "Optimum Nozzle"

- Expanding nozzle increases V_{exit}, but decreases P_{exit} -- there is trade-off here
- It can be shown using variational calculus on the relationships from the previous pages that the Optimum nozzle performance occurs when

$$\frac{A_{exit}}{A_t} \Rightarrow p_{exit} = p_a$$



"telescoping nozzle"

Unfeasible because of the large weight penalty and complexity of deployment mechanisms, also requires that nozzle expand to very large area ratios

Lets Do the Calculus

- Prove that Maximum performance occurs when

$$\frac{A_{exit}}{A^*} \quad \text{Is adjusted to give} \quad p_{exit} = p_\infty$$

Optimal Nozzle

- Show $\frac{A_{exit}}{A^*}$ is a function of $\frac{P_0}{p_{exit}}$

$$\frac{A_{exit}}{A^*} = \left[\frac{1}{M_{exit}} \left[\left(\frac{2}{\gamma+1} \right) \left(1 + \frac{(\gamma-1)}{2} M_{exit}^{\;2} \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}} \right] =$$

$$\frac{1}{M_{exit}} \sqrt{\left[\left[\left(\frac{2}{\gamma+1} \right) \left(1 + \frac{(\gamma-1)}{2} M_{exit}^{\;2} \right) \right]^{\frac{\gamma+1}{(\gamma-1)}} \right]}$$

Optimal Nozzle (cont'd)

$$M_{exit} = \sqrt{\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]} \Rightarrow$$

- Substitute in

$$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]\right)\right]^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]}} = \sqrt{\frac{\left[\left(\frac{2}{\gamma+1}\right)\left(\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}}\right)\right]^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]}} =$$

$$\sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}}\sqrt{\frac{\left(\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]}} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}}\sqrt{\frac{\left(\frac{P_0}{p_{exit}}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]}}$$

Optimal Nozzle (cont'd)

$$\frac{Thrust}{P_0 A^*} = \gamma \sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0}$$

$$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}} \sqrt{\frac{\left(\frac{P_0}{p_{exit}}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}}-1\right]}}$$

Optimal Nozzle (cont'd)

- Subbing into thrust coefficient equation

$$\frac{Thrust}{P_0 A^*} = \gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}}\sqrt{\frac{\left(\frac{P_0}{p_{exit}}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}}-1\right]}}\frac{(p_{exit}-p_\infty)}{P_0} =$$

$$\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\left\{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + \frac{1}{\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}\sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}}\sqrt{\frac{\left(\frac{P_0}{p_{exit}}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}}-1\right]}}\frac{(p_{exit}-p_\infty)}{P_0}\right\} =$$

$$\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\left\{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + \frac{\gamma-1}{2\gamma}\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\left[\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right]\right\}$$

Optimal Nozzle (cont'd)

- Necessary condition for Maxim (Optimal) Thrust

$$\frac{\partial\left(\frac{Thrust}{P_0 A^*}\right)}{\partial p_{exit}} =$$

$$\frac{\partial}{\partial p_{exit}}\left[\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\left\{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}+\frac{\gamma-1}{2\gamma}\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\left[\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right]\right\}\right] =$$

$$\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\frac{\partial}{\partial p_{exit}}\left[\left\{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}+\frac{\gamma-1}{2\gamma}\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\left[\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right]\right\}\right] = 0$$

Optimal Nozzle (cont'd)

- Evaluating the derivative

$$\frac{\partial}{\partial p_{exit}}\left[\left\{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}+\frac{\gamma-1}{2\gamma}\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\left[\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right]\right\}\right]=$$

$$\frac{(-1+\gamma)\left(\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right)\left(-\frac{(1+\gamma)\left(\frac{p_{exit}}{P_0}\right)^{-1-\frac{1+\gamma}{\gamma}}}{\gamma\left(-1+\left(\frac{p_{exit}}{P_0}\right)^{-\frac{-1+\gamma}{\gamma}}\right)P_0}+\frac{(-1+\gamma)\left(\frac{p_{exit}}{P_0}\right)^{-1-\frac{-1+\gamma}{\gamma}-\frac{1+\gamma}{\gamma}}}{\gamma\left(-1+\left(\frac{p_{exit}}{P_0}\right)^{-\frac{-1+\gamma}{\gamma}}\right)^2 P_0}\right)}{4\gamma\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{-\frac{1+\gamma}{\gamma}}}{-1+\left(\frac{p_{exit}}{P_0}\right)^{-\frac{-1+\gamma}{\gamma}}}}}+$$

$$\frac{(-1+\gamma)\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{-\frac{1+\gamma}{\gamma}}}{-1+\left(\frac{p_{exit}}{P_0}\right)^{-\frac{-1+\gamma}{\gamma}}}}}{2\gamma P_0}-\frac{(-1+\gamma)\left(\frac{p_{exit}}{P_0}\right)^{-1+\frac{-1+\gamma}{\gamma}}}{2\gamma\sqrt{1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{-1+\gamma}{\gamma}}}\,P_0}$$

Let's try to get rid of This term

Optimal Nozzle (cont'd)

- Look at the term

$$\frac{(-1+\gamma)\sqrt{\dfrac{\left(\frac{Pexit}{P0}\right)^{-\frac{1+\gamma}{\gamma}}}{-1+\left(\frac{Pexit}{P0}\right)^{-\frac{-1+\gamma}{\gamma}}}}}{2\,\gamma\,P0} - \frac{(-1+\gamma)\left(\frac{Pexit}{P0}\right)^{-1+\frac{-1+\gamma}{\gamma}}}{2\,\gamma\sqrt{1-\left(\frac{Pexit}{P0}\right)^{\frac{-1+\gamma}{\gamma}}}\;P0} =$$

$$\left(\frac{\gamma-1}{2\gamma P_0}\right)\left\{\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}} - \left(\frac{p_{exit}}{P_0}\right)^{\frac{1}{-\gamma}}\sqrt{\frac{1}{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{\gamma}}\right]}}\right\}$$

Optimal Nozzle (cont'd)

- Look at the term

Bring Inside

$$\left(\frac{\gamma-1}{2\gamma P_0}\right)\left\{\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}-\left(\frac{p_{exit}}{P_0}\right)^{\frac{1}{-\gamma}}\sqrt{\frac{1}{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{\gamma}}\right]}}\right\}=$$

$$\left(\frac{\gamma-1}{2\gamma P_0}\right)\left\{\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}-\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{1}{-\gamma}}\left(\frac{p_{exit}}{P_0}\right)^{\frac{1}{-\gamma}}}{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{\gamma}}\right]}}\right\}$$

Optimal Nozzle (cont'd)

- Look at the term

$$\left(\frac{\gamma-1}{2\gamma P_0}\right)\left\{\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}-\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{1}{-\gamma}}\left(\frac{p_{exit}}{P_0}\right)^{\frac{1}{-\gamma}}}{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{\gamma}}\right]}}\right\}=$$

Factor Out

$$\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}$$

$$\left(\frac{\gamma-1}{2\gamma P_0}\right)\left\{\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}-\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{1}{-\gamma}}\left(\frac{p_{exit}}{P_0}\right)^{\frac{1}{-\gamma}}\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\right\}$$

Optimal Nozzle (cont'd)

- Look at the term

$$\left(\frac{\gamma-1}{2\gamma P_0}\right)\left\{\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}-\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{1}{-\gamma}}\left(\frac{p_{exit}}{P_0}\right)^{\frac{1}{-\gamma}}\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\right\}=$$

Collect Exponents

$$\left(\frac{\gamma-1}{2\gamma P_0}\right)\left\{\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}-\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\right\}=0$$

Good!

Optimal Nozzle (cont'd)

- and the derivative reduces to

$$\frac{\partial}{\partial p_{exit}}\left[\left\{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}+\frac{\gamma-1}{2\gamma}\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\left[\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right]\right\}\right]=$$

$$\frac{(-1+\gamma)\left(\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right)\left(-\frac{(1+\gamma)\left(\frac{p_{exit}}{P_0}\right)^{-1-\frac{1+\gamma}{\gamma}}}{\gamma\left(-1+\left(\frac{p_{exit}}{P_0}\right)^{-\frac{-1+\gamma}{\gamma}}\right)P_0}+\frac{(-1+\gamma)\left(\frac{p_{exit}}{P_0}\right)^{-1-\frac{-1+\gamma}{\gamma}-\frac{1+\gamma}{\gamma}}}{\gamma\left(-1+\left(\frac{p_{exit}}{P_0}\right)^{-\frac{-1+\gamma}{\gamma}}\right)^2 P_0}\right)}{4\gamma\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{-\frac{1+\gamma}{\gamma}}}{-1+\left(\frac{p_{exit}}{P_0}\right)^{-\frac{-1+\gamma}{\gamma}}}}}$$

Optimal Nozzle (concluded)

- Find Condition where

$$\frac{(-1+\gamma)\left(\frac{P_{exit}}{P_0}-\frac{P_\infty}{P_0}\right)\left(-\frac{(1+\gamma)\left(\frac{P_{exit}}{P_0}\right)^{-1-\frac{1+\gamma}{\gamma}}}{\gamma\left(-1+\left(\frac{P_{exit}}{P_0}\right)^{-\frac{-1+\gamma}{\gamma}}\right)P_0}+\frac{(-1+\gamma)\left(\frac{P_{exit}}{P_0}\right)^{-1-\frac{-1+\gamma}{\gamma}-\frac{1+\gamma}{\gamma}}}{\gamma\left(-1+\left(\frac{P_{exit}}{P_0}\right)^{-\frac{-1+\gamma}{\gamma}}\right)^{2}P_0}\right)}{4\gamma\sqrt{\frac{\left(\frac{P_{exit}}{P_0}\right)^{-\frac{1+\gamma}{\gamma}}}{-1+\left(\frac{P_{exit}}{P_0}\right)^{-\frac{-1+\gamma}{\gamma}}}}}=0$$

$$\left[\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right]=0\Rightarrow\boxed{p_{exit}=p_\infty}$$

- Condition for Optimality *(maximum Isp)*

Optimal Thrust Equation

$$Thrust_{opt} = \gamma P_0 A^* \sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

$$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}}\sqrt{\frac{\left(\frac{P_0}{p_\infty}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_\infty}\right)^{\frac{(\gamma-1)}{\gamma}}-1\right]}} \Rightarrow forces...p_{exit} = p_\infty$$

Rocket Nozzle Design Point

$$Thrust_{opt} = \gamma P_0 A^* \sqrt{\frac{2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left[1 - \left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

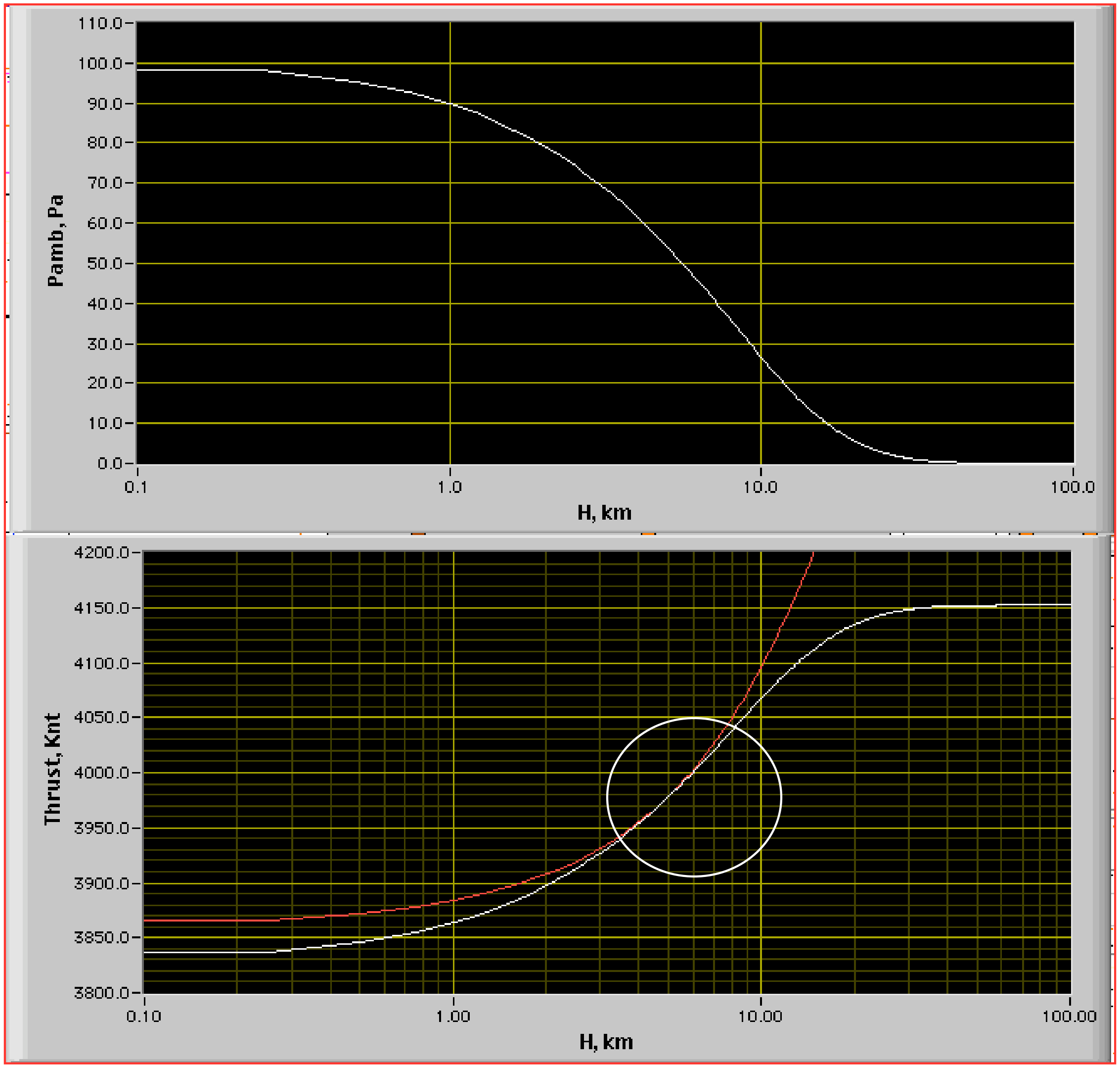
$$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}}\sqrt{\frac{\left(\frac{P_0}{p_\infty}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_\infty}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]}} \Rightarrow forces \ldots p_{exit} = p_\infty$$

Atlas V, Revisited

• Re-do the Atlas V plots for Optimal Nozzle
i.e. Let

$$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}} \sqrt{\frac{\left(\frac{P_0}{p_\infty}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_\infty}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]}} \Rightarrow forces...p_{exit} = p_\infty$$

Atlas V, Revisited (cont'd)



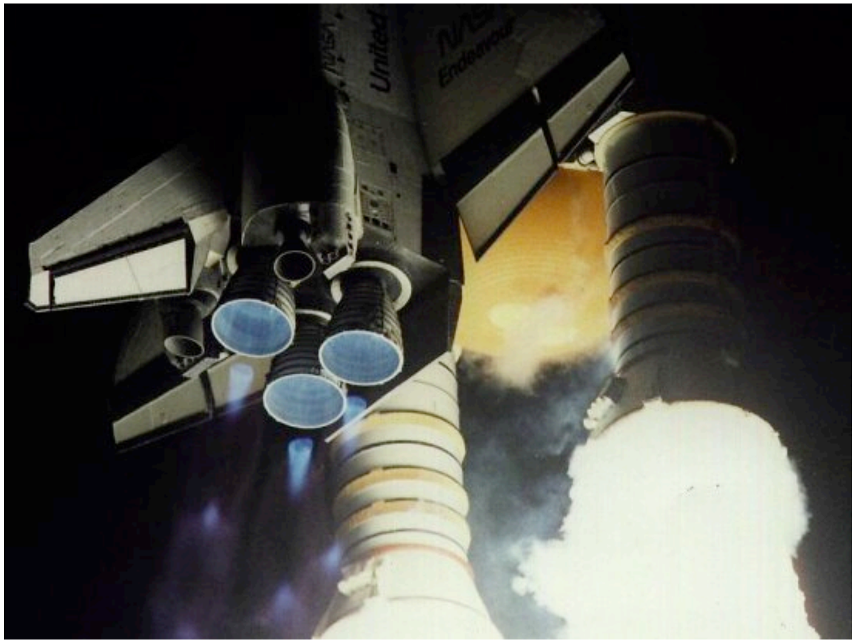
- ATLAS V First stage is Optimized for Maximum performance At~ 5k altitude

  16,404 ft.

How About Space Shuttle SSME

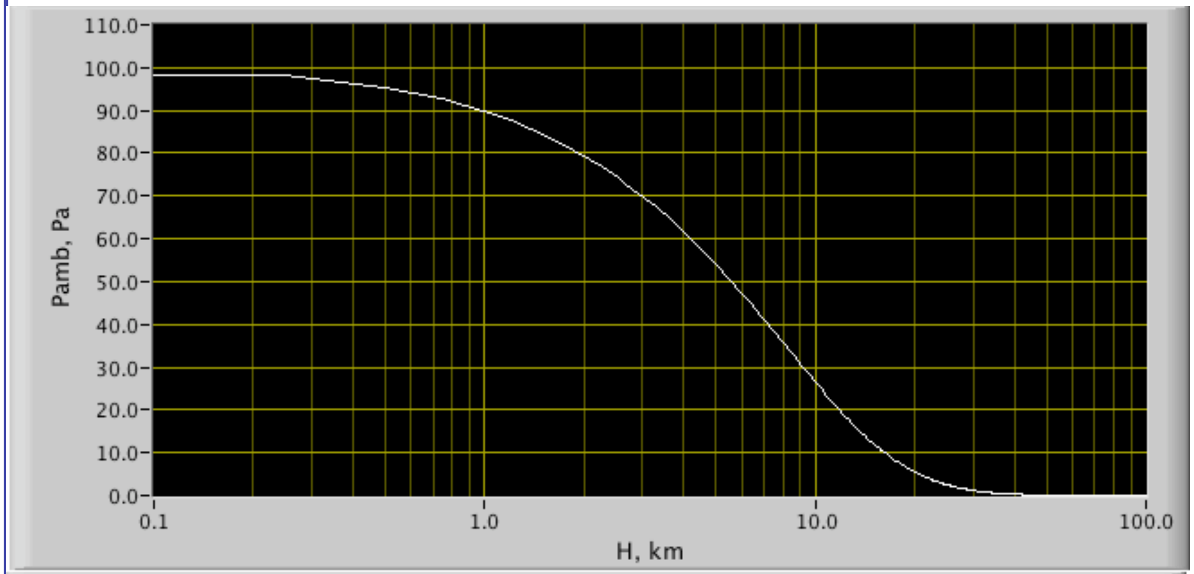
Per Engine (3)

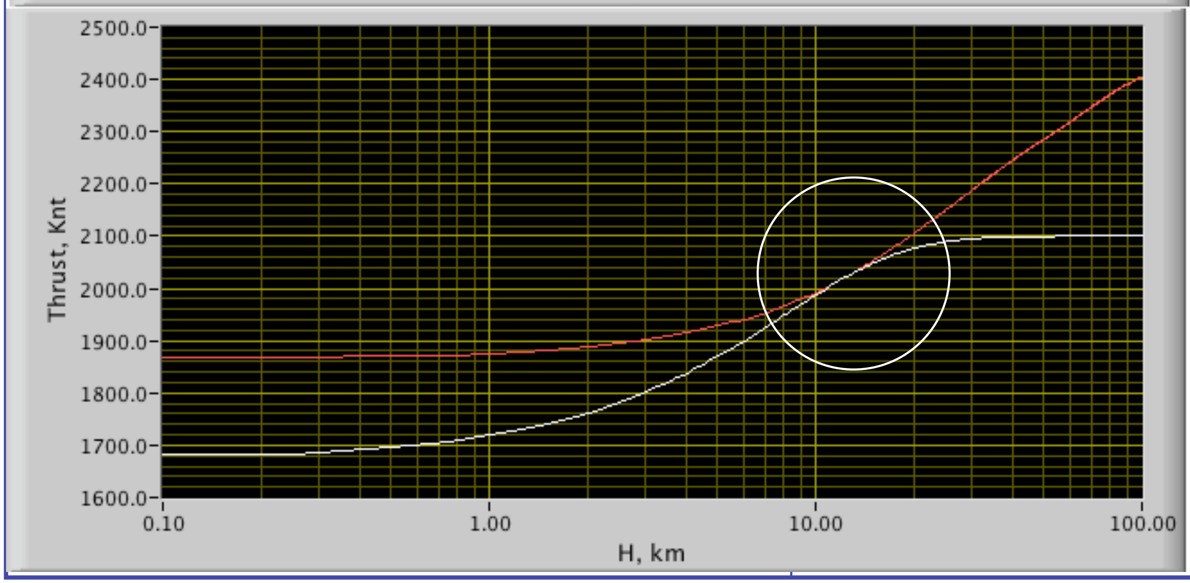
- $Thrust_{vac}$ = 2100.00 kn
- $Thrust_{sl}$ = 1670.00 kn
- $I_{sp_{vac}}$ = 452.55sec
- A_e/A_* = 77.52
- P_0 = 18.96Mpa
- Lox/LH2 Propellants



- γ =1.196

How About Space Shuttle SSME (cont'd)





- SSME is Optimized for Maximum performance At~ 12.5k Altitude

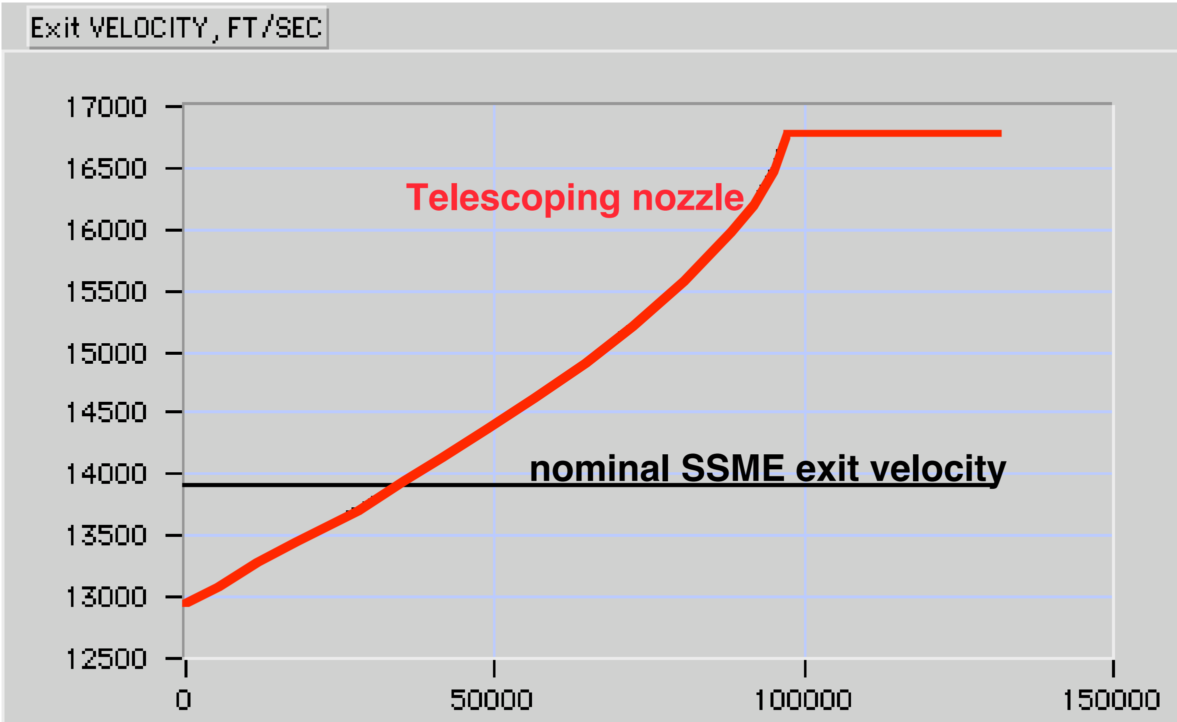
~ 40,000 ft

How About Space Shuttle SSME (cont'd)

"Optimum Nozzle" (cont'd)

- **Exit Velocity**

V_{exit} , $\frac{ft}{sec}$

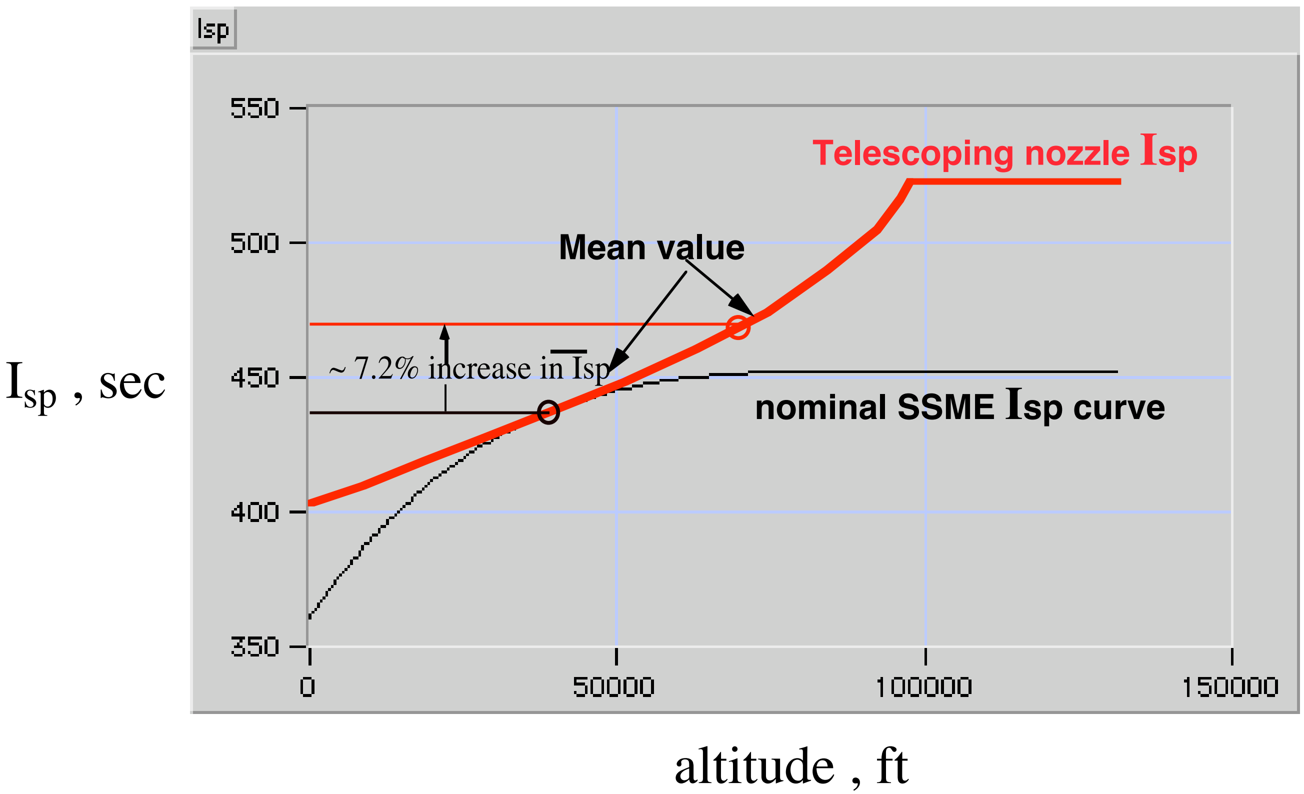


altitude , ft

How About Space Shuttle SSME (cont'd)

"Optimum Nozzle" (concluded)

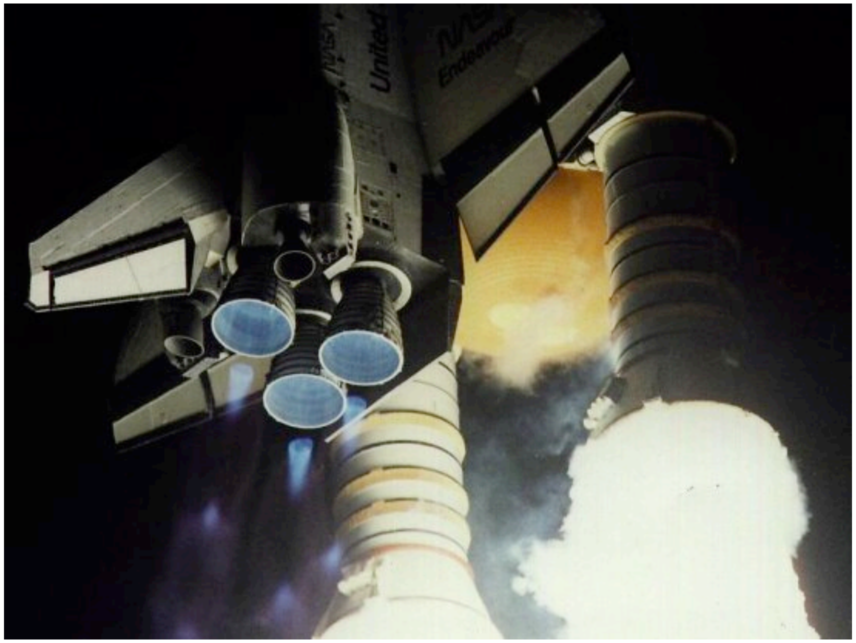
- **Isp**



How About Space Shuttle SRB

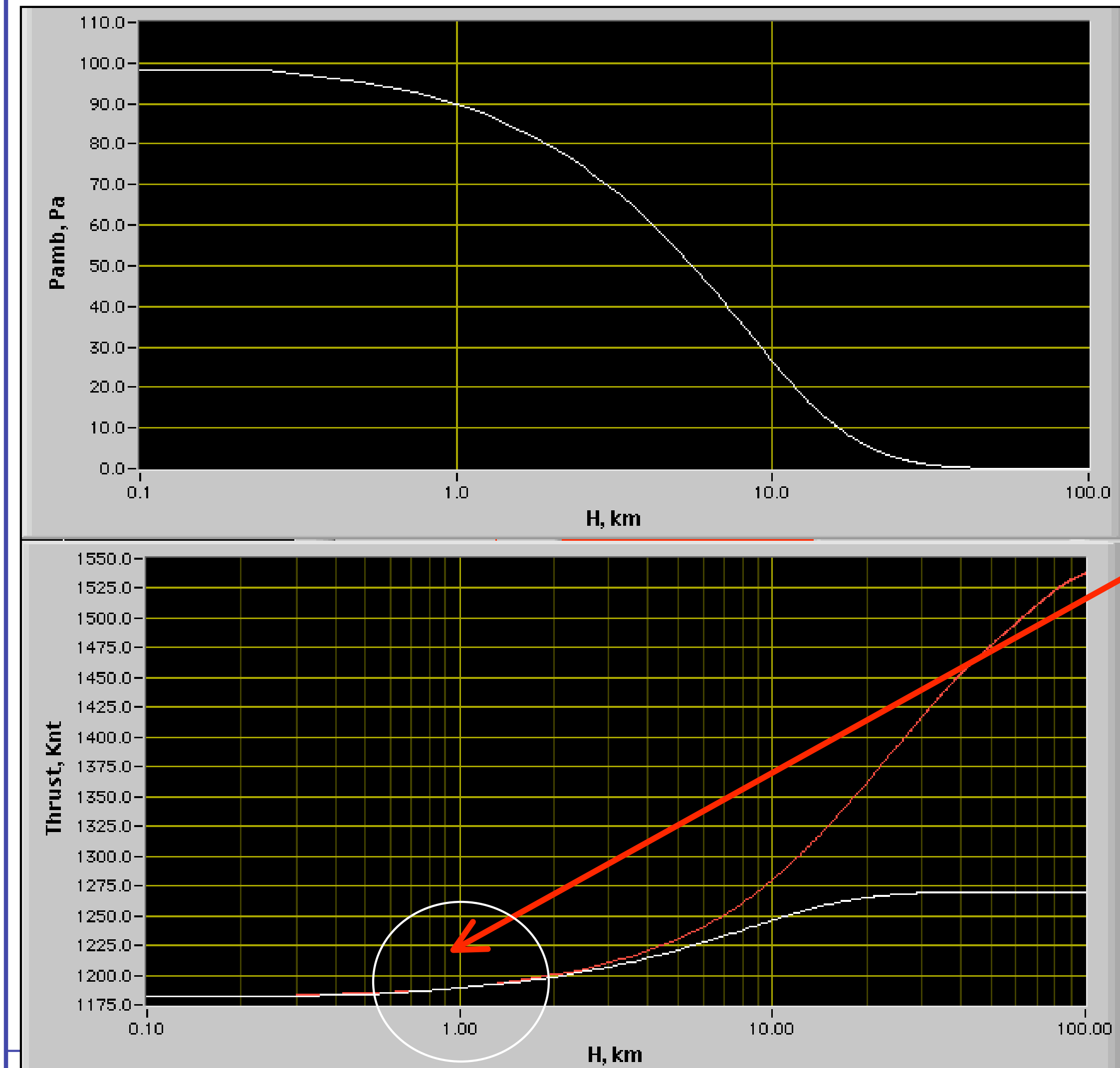
Per Motor (2)

- $Thrust_{vac}$ = 1270.00 kn
- $Thrust_{sl}$ = 1179.00 kn
- $I_{sp_{vac}}$ = 267.30 sec
- A_e/A_* = 7.50
- P_0 = 6.33 Mpa
- PABM (Solid) Propellant



- γ =1.262480

How About Space Shuttle SSME (cont'd)



- SRB is Optimized for Maximum performance At <1k altitude

3280 ft.

Solve for Design Altitude of Given Nozzle

$$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}} \sqrt{\frac{\left(\frac{P_0}{p_\infty}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_\infty}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]}} \Rightarrow rewrite...as$$

$$\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{P_0}{p_\infty}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]\left(\frac{A_{exit}}{A^*}\right)^2 - \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left(\frac{P_0}{p_\infty}\right)^{\frac{\gamma+1}{\gamma}} = 0$$

Solve for Design Altitude of Given Nozzle

(cont'd)

Factor out $\left(\frac{p_\infty}{P_0}\right)^{\frac{\gamma+1}{\gamma}}$

$$\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{P_0}{p_\infty}\right)^{\frac{(\gamma-1)}{\gamma}}-1\right]\left(\frac{A_{exit}}{A^*}\right)^2-\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left(\frac{P_0}{p_\infty}\right)^{\frac{\gamma+1}{\gamma}}=0$$

$$\left(\frac{2}{\gamma-1}\right)\left(\frac{p_\infty}{P_0}\right)^{\frac{\gamma+1}{\gamma}}\left[\left(\frac{P_0}{p_\infty}\right)^{\frac{(\gamma-1)}{\gamma}}-1\right]\left(\frac{A_{exit}}{A^*}\right)^2-\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}=0$$

$$\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{p_\infty}{P_0}\right)^{\frac{\gamma+1}{\gamma}}\left(\frac{p_\infty}{P_0}\right)^{-\frac{(\gamma-1)}{\gamma}}-\left(\frac{p_\infty}{P_0}\right)^{\frac{\gamma+1}{\gamma}}\right]\left(\frac{A_{exit}}{A^*}\right)^2-\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}=0$$

Solve for Design Altitude of Given Nozzle

(cont'd)

Simplify

$$\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{p_\infty}{P_0}\right)^{\frac{2}{\gamma}}-\left(\frac{p_\infty}{P_0}\right)^{\frac{(\gamma+1)}{\gamma}}\right]\left(\frac{A_{exit}}{A^*}\right)^2-\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}=0$$

$$\left[\left(\frac{p_\infty}{P_0}\right)^{\frac{2}{\gamma}}-\left(\frac{p_\infty}{P_0}\right)^{\frac{(\gamma+1)}{\gamma}}\right]-\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)\left(\frac{A_{exit}}{A^*}\right)^2}=0$$

Solve for Design Altitude of Given Nozzle

(cont'd)

- Newton Again?

- No … there is an easier way

- User Iterative Solve to Compute Exit Mach Number

$$\frac{A_{exit}}{A^*} = 36.87 = \left[\frac{1}{M_{exit}} \left[\left(\frac{2}{\gamma+1} \right) \left(1 + \frac{(\gamma-1)}{2} M_{exit}{}^2 \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}} \right] \Rightarrow$$

$$M_{exit} = 4.2954$$

Solve for Design Altitude of Given Nozzle

(cont'd)

- Compute Exit Pressure

$$p_{exit} = \frac{P_{0_{exit}}}{\left(1+\frac{\gamma-1}{2}M_{exit}^{2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}} = \frac{25.7\cdot 1000}{\left(1+\frac{1.220155-1}{2}4.2624^{2}\right)^{\left(\frac{1.220155}{1.220155-1}\right)}} = 55.06 \text{ kPa}$$

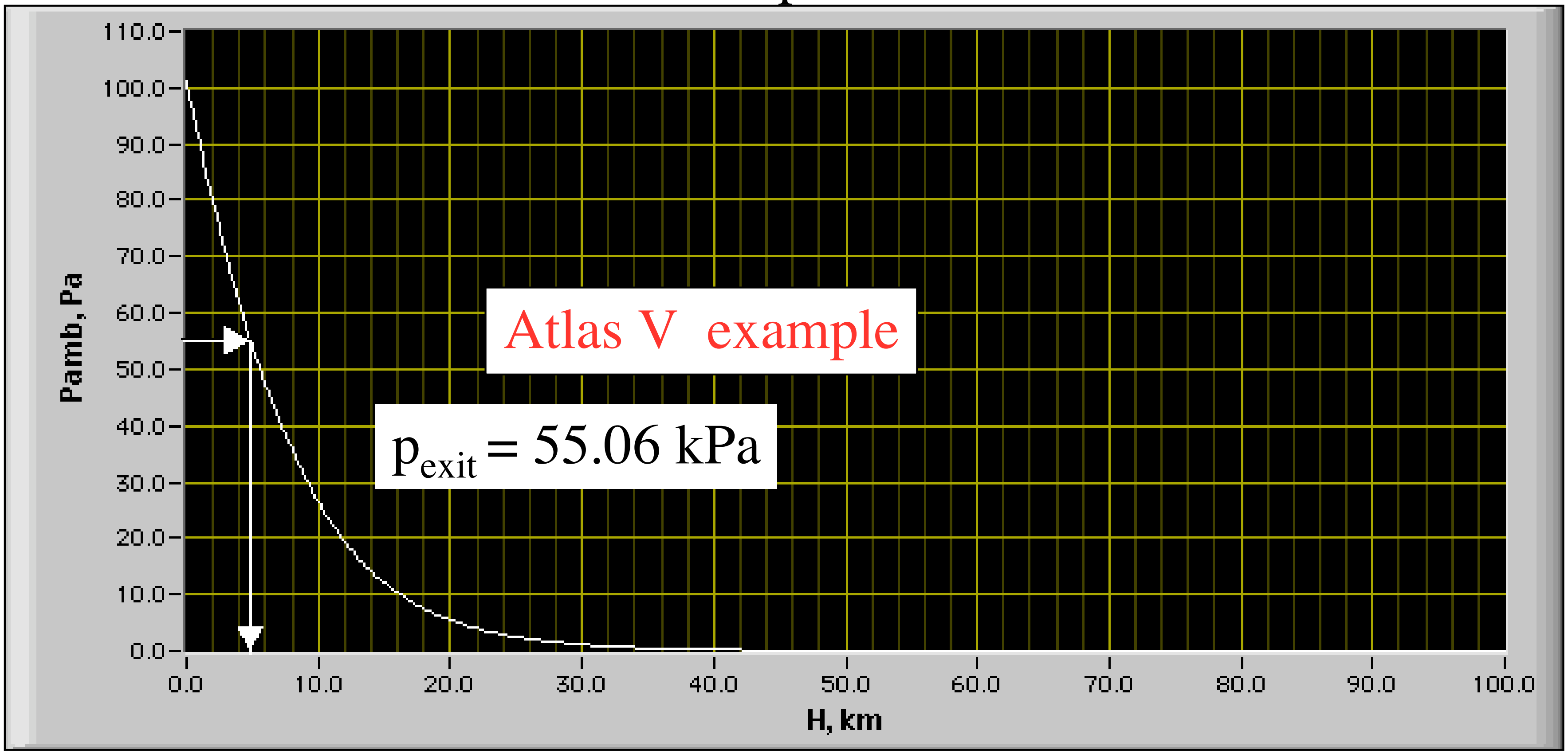
- Set

$$\boxed{p_{exit} = p_{\infty}}_{opt}$$

Solve for Design Altitude of Given Nozzle

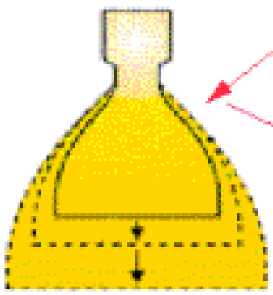
(cont'd)

- Table look up of US 1976 Standard Atmosphere or World GRAM 99 Atmosphere



Space Shuttle Optimum Nozzle?

$$\frac{A_{exit}}{A_t} \Rightarrow p_{exit} = p_a$$



Unfeasible because of the large weight penalty and complexity of deployment mechanisms, also requires that nozzle expand to very large area ratios

What is A/A* Optimal for SSME at 80,000 ft altitude (24.384 km)?

At 80kft

$$p_\infty = 2.76144 \text{ kPa}$$

$$p_\infty = 2.76144 \text{ kPa} \rightarrow \frac{P_0}{p_\infty} = \frac{18.9 \times 10}{2.76144} = 6844.3$$

$$\rightarrow M_{exit} = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{P_0}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} = \left(\frac{2}{1.196 - 1}\left(\left(\frac{18900}{2.76144}\right)^{\frac{1.196-1}{1.196}} - 1\right)\right)^{0.5} =$$

$$5.7592$$

Space Shuttle Optimum Nozzle? (cont'd)

$$p_\infty = 2.76144\ \text{kPa} \rightarrow \frac{P_0}{p_\infty} = \frac{18.9 \times 10}{2.76144} = 6844.3$$

$$\rightarrow M_{exit} = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{P_0}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} = 5.7592$$

What is A/A* Optimal for SSME at 80,000 ft altitude (24.384 km)?

At 80kft

$$p_\infty = 2.76144\ \text{kPa}$$

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1 + \frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} =$$

$$\frac{\left(\left(\frac{2}{1.196+1}\right)\left(1 + \frac{1.196-1}{2}(5.7592^2)\right)\right)^{\frac{1.196+1}{2(1.196-1)}}}{5.7592} = 340.98$$

(originally 77.52)

Space Shuttle Optimum Nozzle? (cont'd)

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$= 340.98$$

What is A/A* Optimal for SSME at 80,000 ft altitude (24.384 km)?

At 80kft

$$p_\infty = 2.76144 \text{ kPa}$$

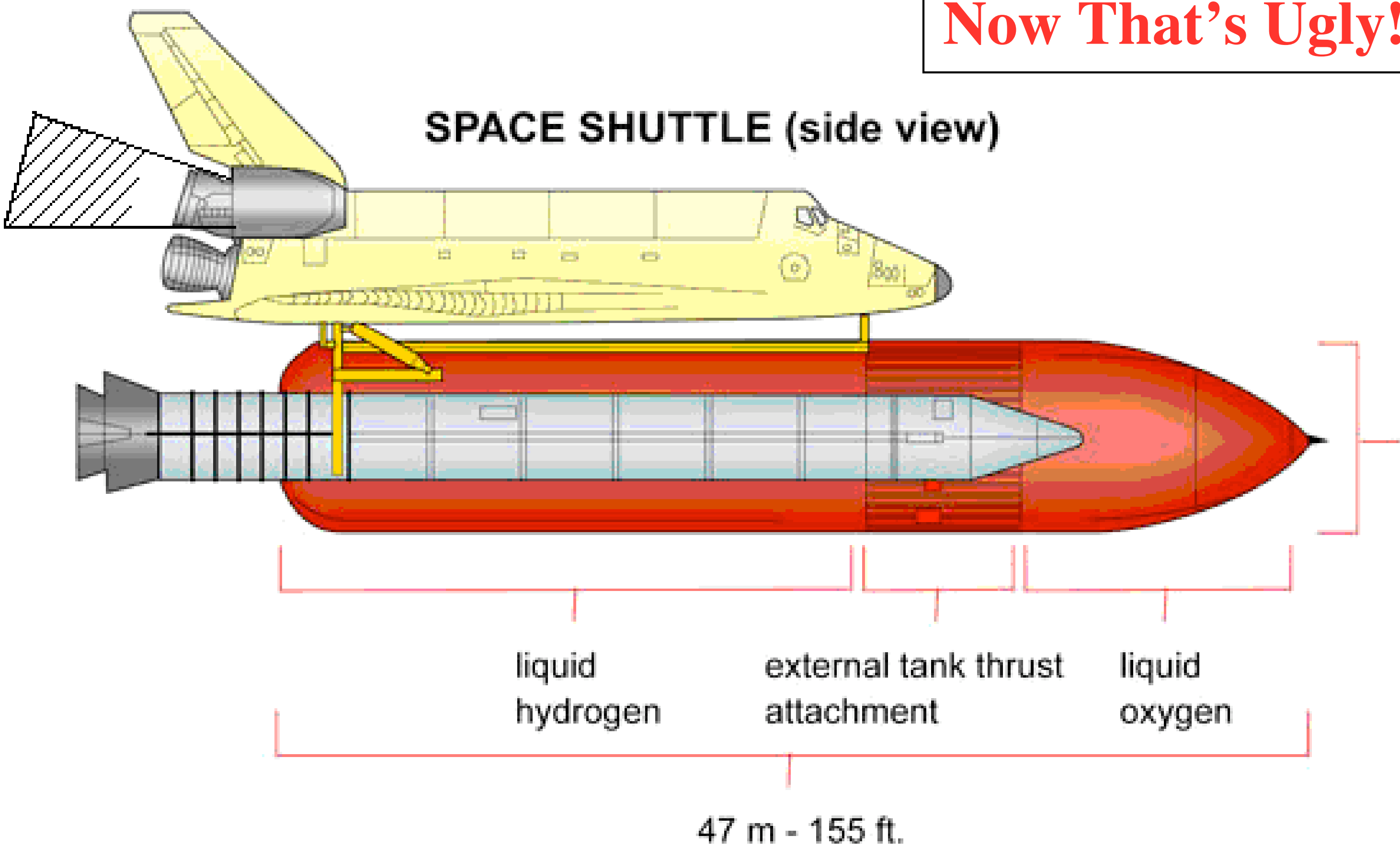
- Compute Throat Area

$$\left(\frac{26}{100}\right)^2 \frac{\pi}{4} \quad = 0.05297 \text{ m}^2$$

→ Aexit=18.062 m^2---> 4.8 (15.7 ft) meters in diameter
As opposed to 2.286 meters for original shuttle

Space Shuttle Optimum Nozzle? (cont'd)

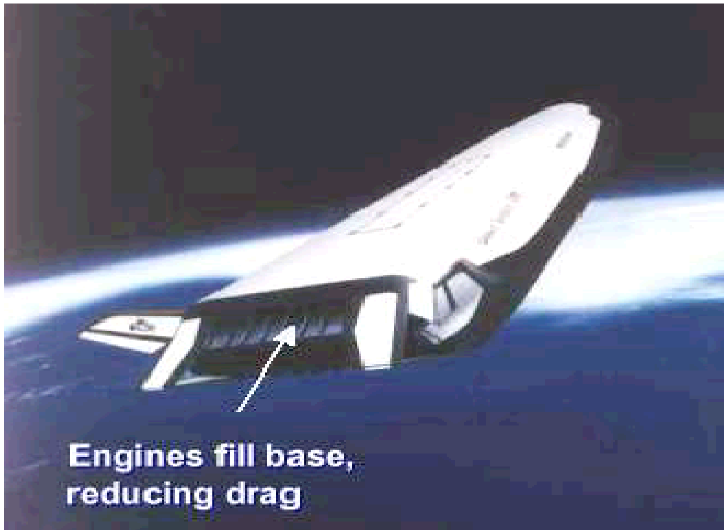
Now That's Ugly!

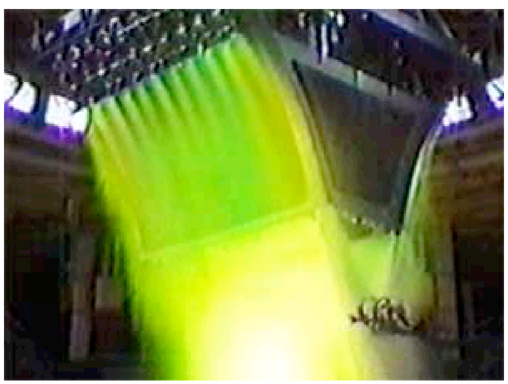


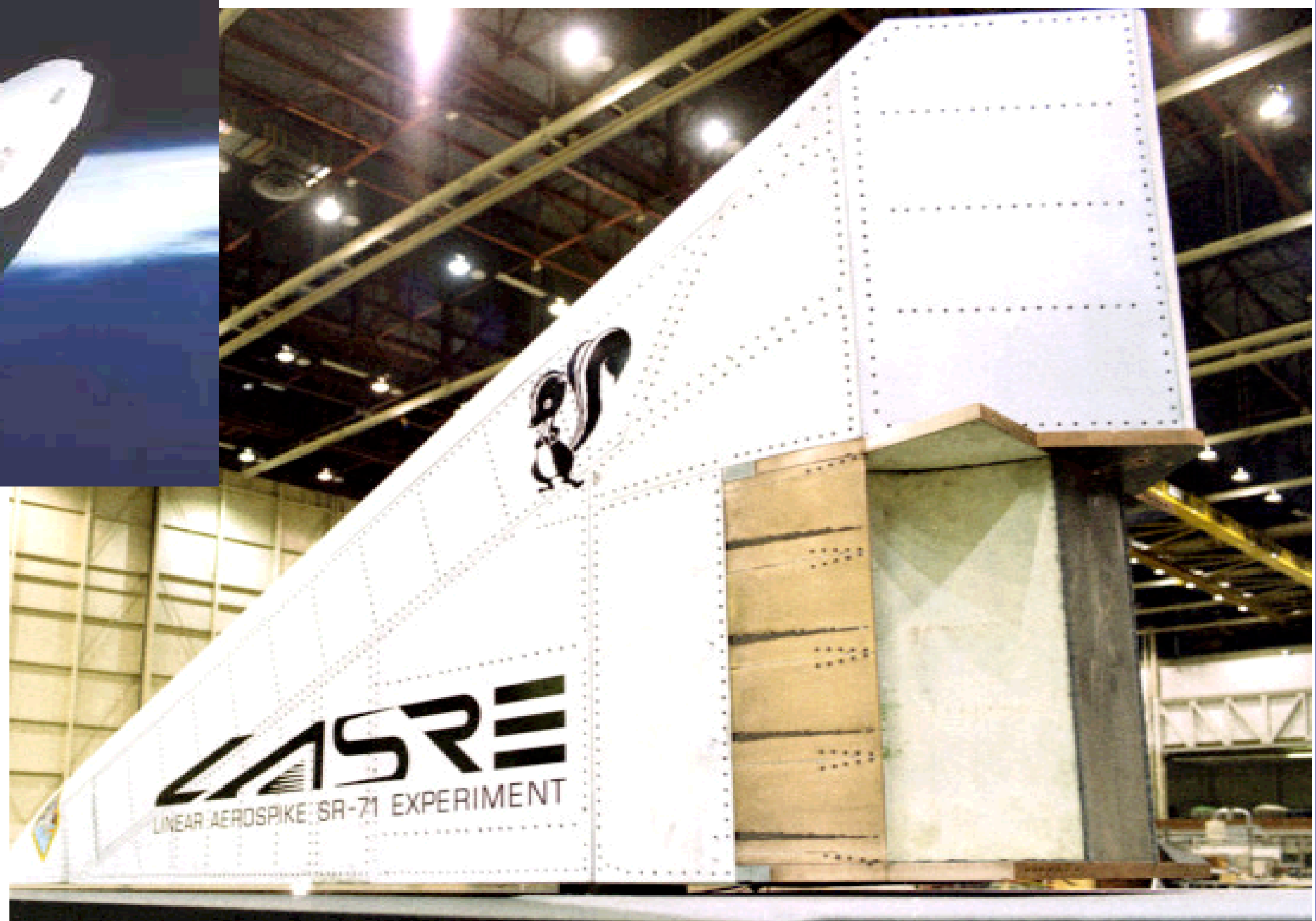
- So What are the Alternatives?

"The Linear Aerospike Rocket Engine"

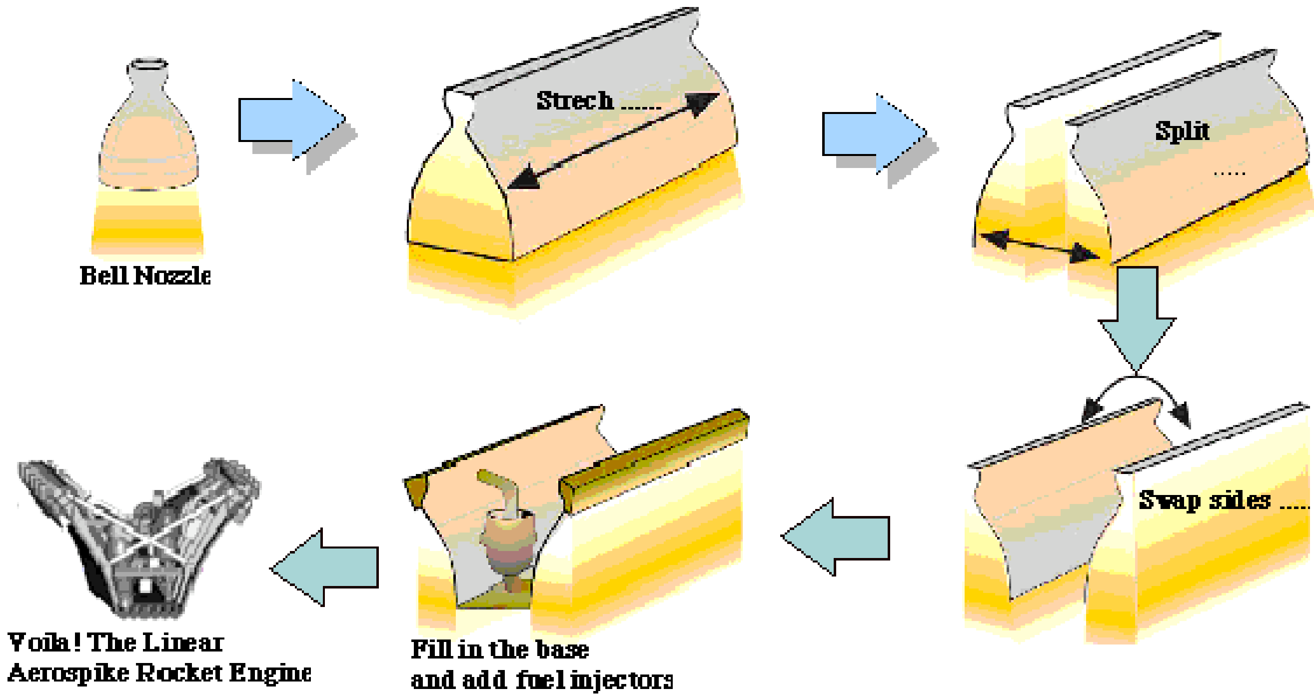
… Which leads us to the … real alternative

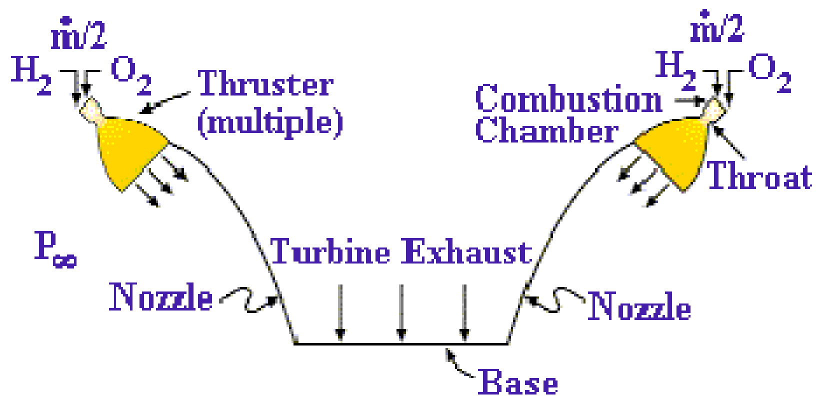






A New Nozzle Shape





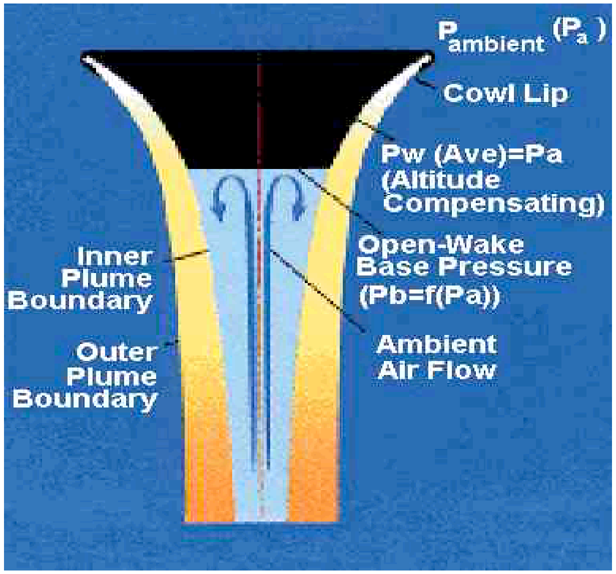
UtahState UNIVERSITY

Mechanical & Aerospace Engineering

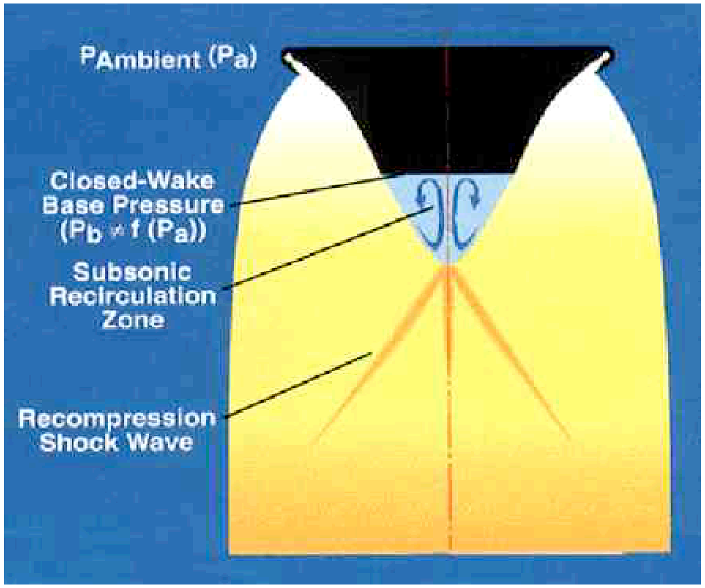
Linear Aerospike Rocket Engine

Nozzle has same effect as telescope nozzle

Lift off



Vacuum (Space)



$$F = F_{Thruster} + F_{Ramp} + F_{Base}$$

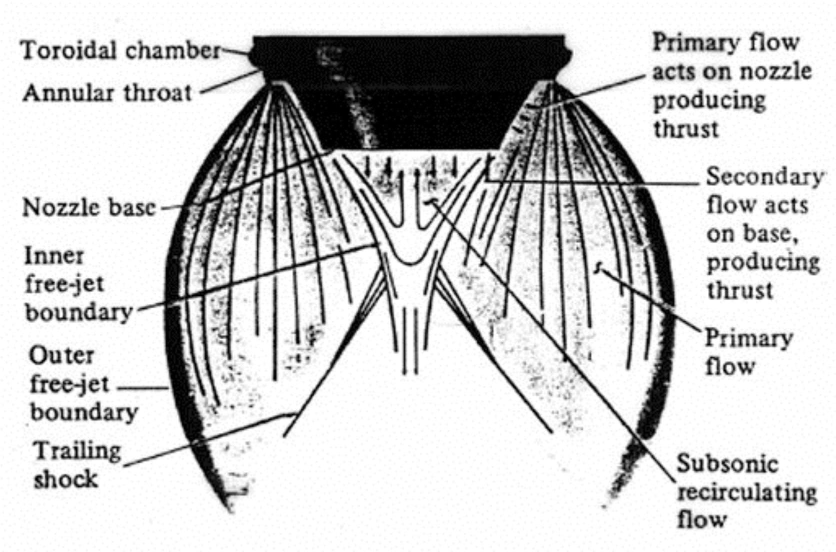
$$F_{Thruster} = \cos\theta\,(\dot{m}Vexit + Aexit\,(Pexit - P_\infty))$$

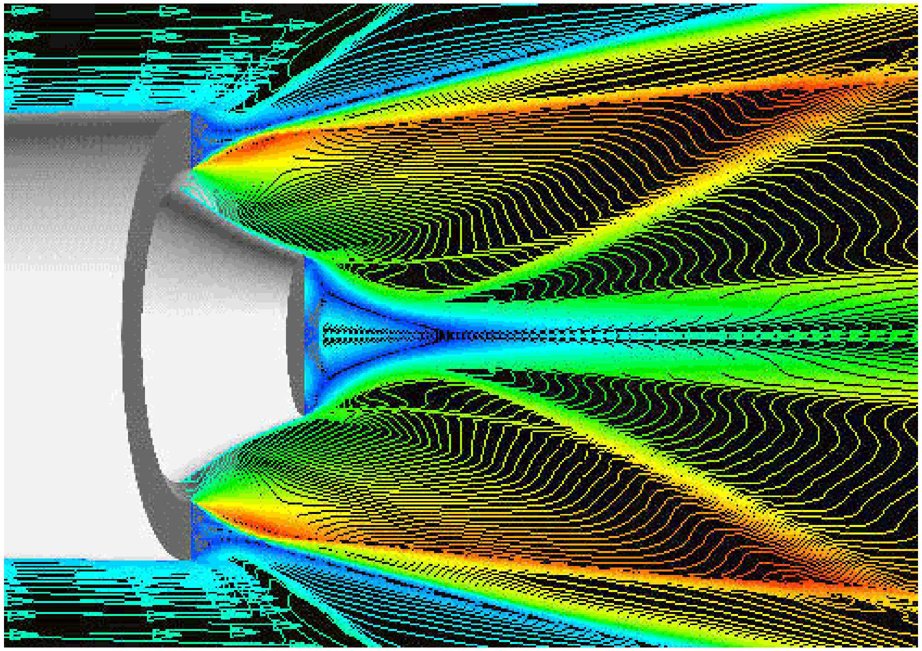
$$F_{Ramp} = \int^{A_{Ramp}} (P_{Ramp} - P_\infty)\, dA$$

$$F_{Base} = A_{Base}\,(P_{Base} - P_\infty)$$

- Aerospike's flow unconstrained, allows best performance

More Aerospike





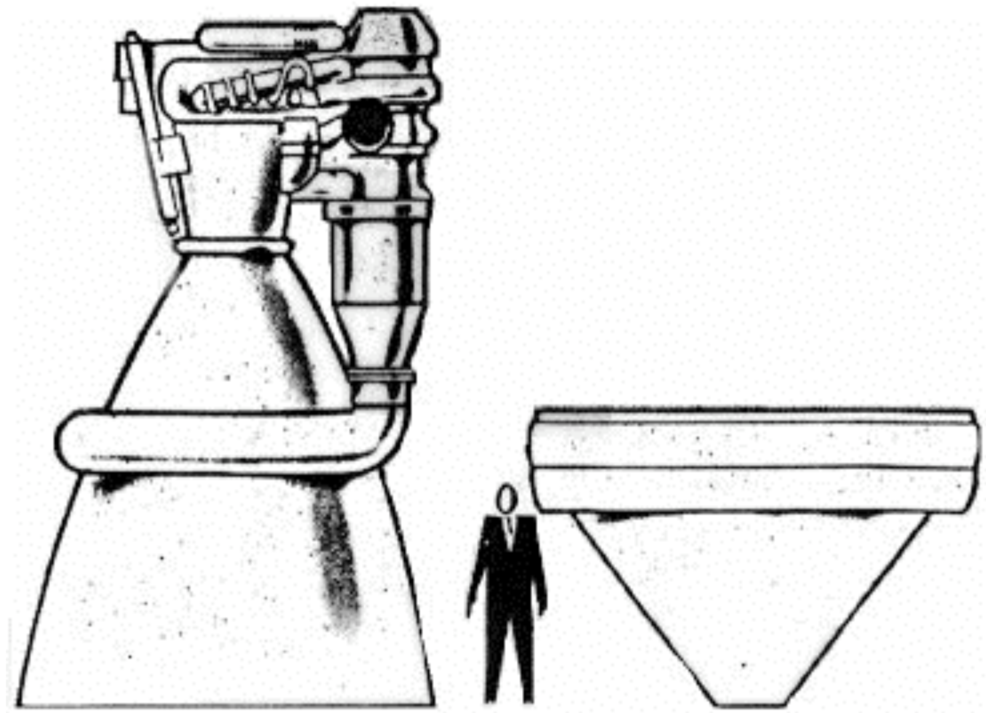
Credit: Aerospace web

Advantages of Aerospike High Expansion Ratio Experimental Nozzle

- Truncated aerospike nozzles can be as short as 25% the length of a conventional bell nozzle.
  - Provide savings in packing volume and weight for space vehicles.
- Aerospike nozzles allow higher expansion ratio than conventional nozzle for a given space vehicle base area.
  - Increase vacuum thrust and specific impulse.
- For missions to the Moon and Mars, advanced nozzles can increase the thrust and specific impulse by 5-6%, resulting in a 8-9% decrease in propellant mass.
- Lower total vehicle mass and provide extra margin for the mass inclusion of other critical vehicle systems.
- New nozzle technology also applicable to RCS, space tugs, etc…

Spike Nozzle … Other advantages (cont'd)

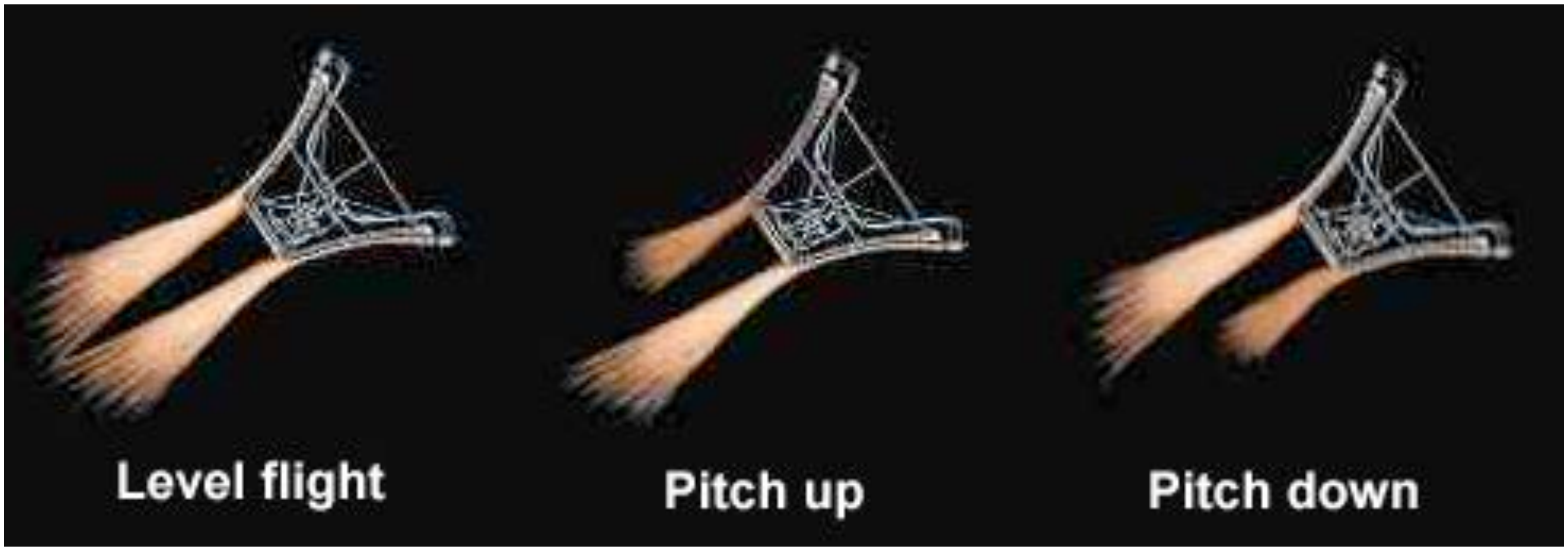
- Higher expansion ratio for smaller size II



Credit: Aerospace web

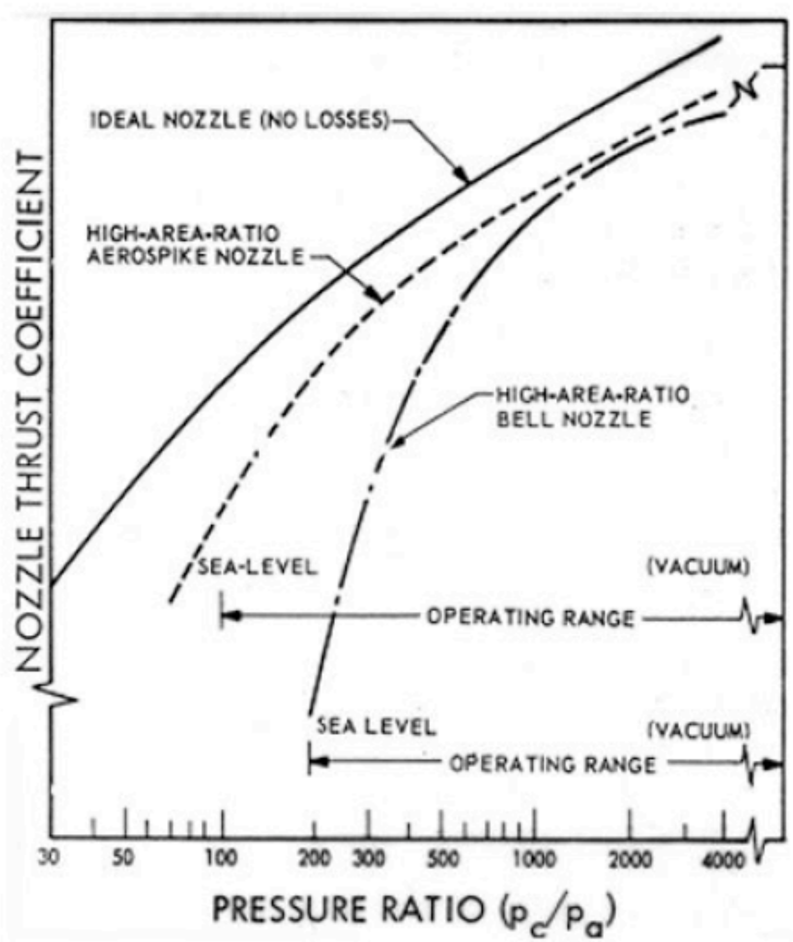
Spike Nozzle … Other advantages (cont'd)

- Thrust vectoring without Gimbals



Credit: Aerospace web

Performance Comparison

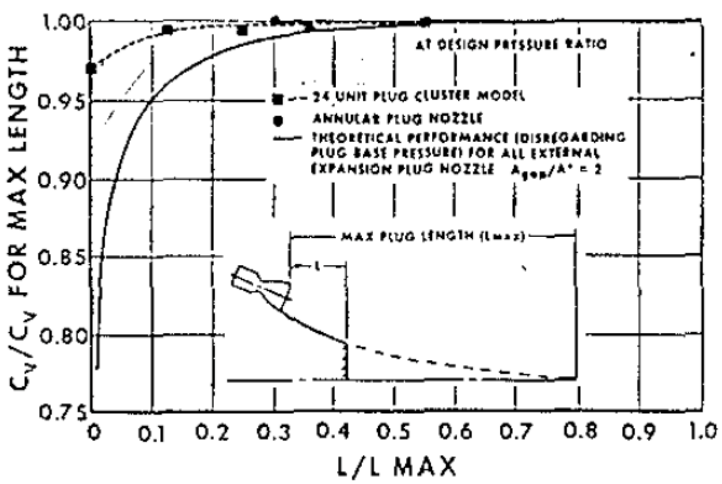


[from Huzel and Huang, 1967]

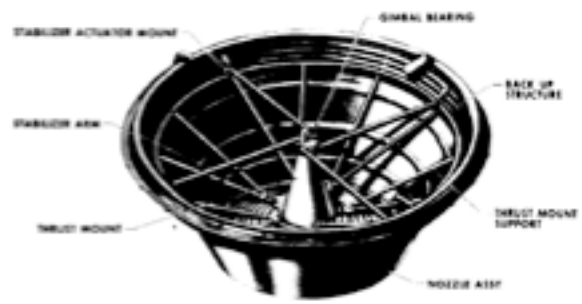
- Although less than Ideal The significant Isp recovery of Spike Nozzles offer significant advantage

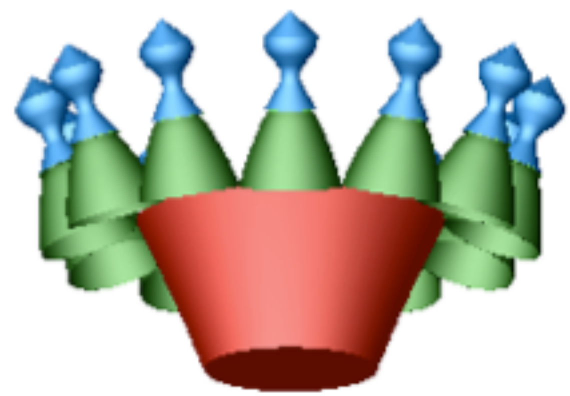
Aerospike on the Moon?

G. Mungas, M. Johnson, D. Fisher, C. Mungas, B. Rishikoff, (2008) "***NOFB Monopropulsion System for Lunar Ascent Vehicle Utilizing Plug Nozzle with Clustered Engines for Ascent Main Engine***", LPS-II-33, 2008 JANNAF Conference, 6th Modeling and Simulation / 4th Liquid Propulsion / 3rd Spacecraft Propulsion Joint Subcommittee Meeting



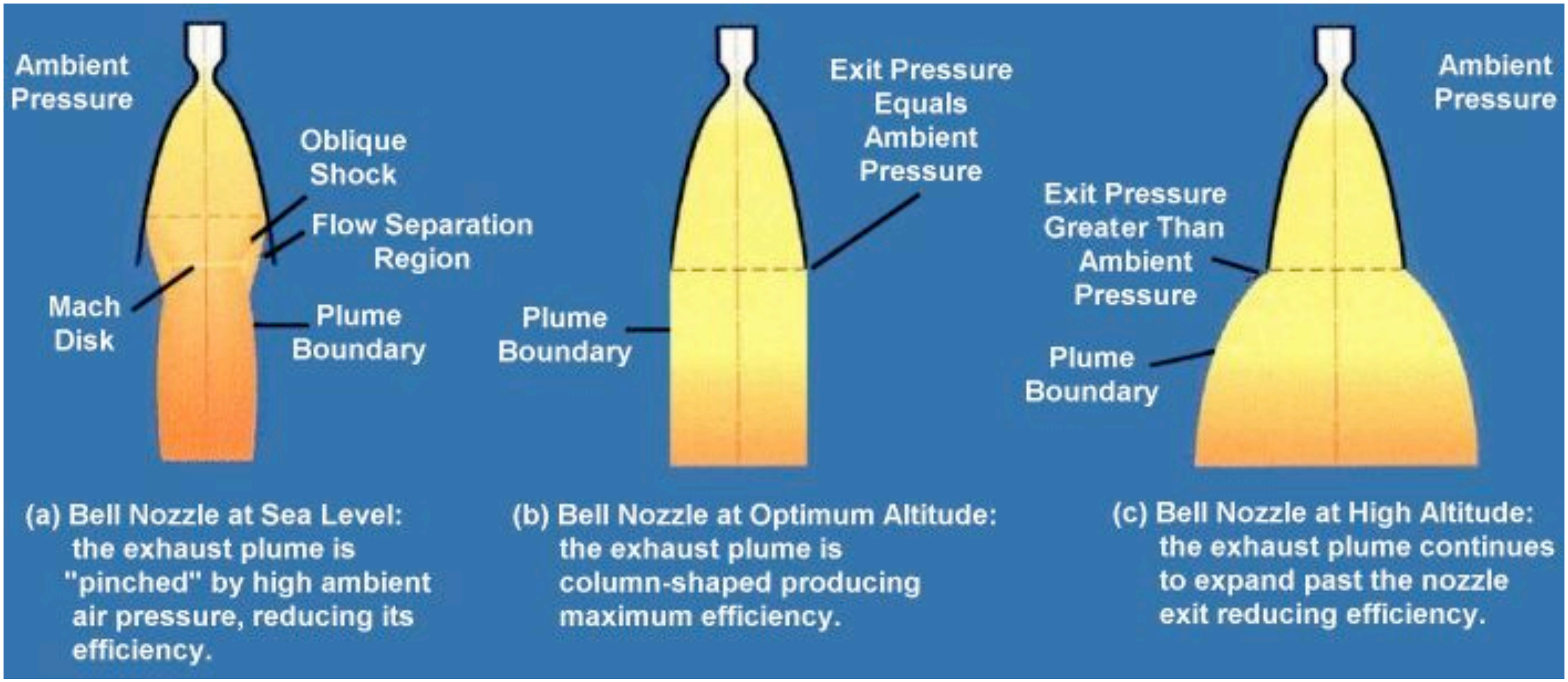
Nozzle Coefficient vs. Aerospike Nozzle Length [2]





NOFB37 12 x 500 lbf (6000lbf) Monopropellant Engine

Optimal Nozzle Summary



Credit: Aerospace web

Optimal Nozzle Summary (cont'd)

- Thrust equation $Thrust = \dot{m} V_{exit} + A_{exit}(p_{exit} - p_\infty)$

can be re-written as

$$Thrust = \gamma P_0 A^* \sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \left[1 - \left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + A_{exit}(p_{exit} - p_\infty)$$

and $$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}} \sqrt{\frac{\left(\frac{P_0}{p_{exit}}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]}}$$

Optimal Nozzle Summary (cont'd)

- Eliminating A_{exit} from the expression

$$Thrust = \gamma P_0 A^* \sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \left\{ \left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + \frac{\gamma-1}{2\gamma}\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\left[\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right] \right\}$$

P_0, γ, drive by combustion process, only p_e is effected by nozzle

- Optimal Nozzle given by

$$\frac{\partial\left(\frac{Thrust}{P_0 A^*}\right)}{\partial p_{exit}} = 0 \Rightarrow \boxed{p_{exit} = p_\infty}_{opt}$$

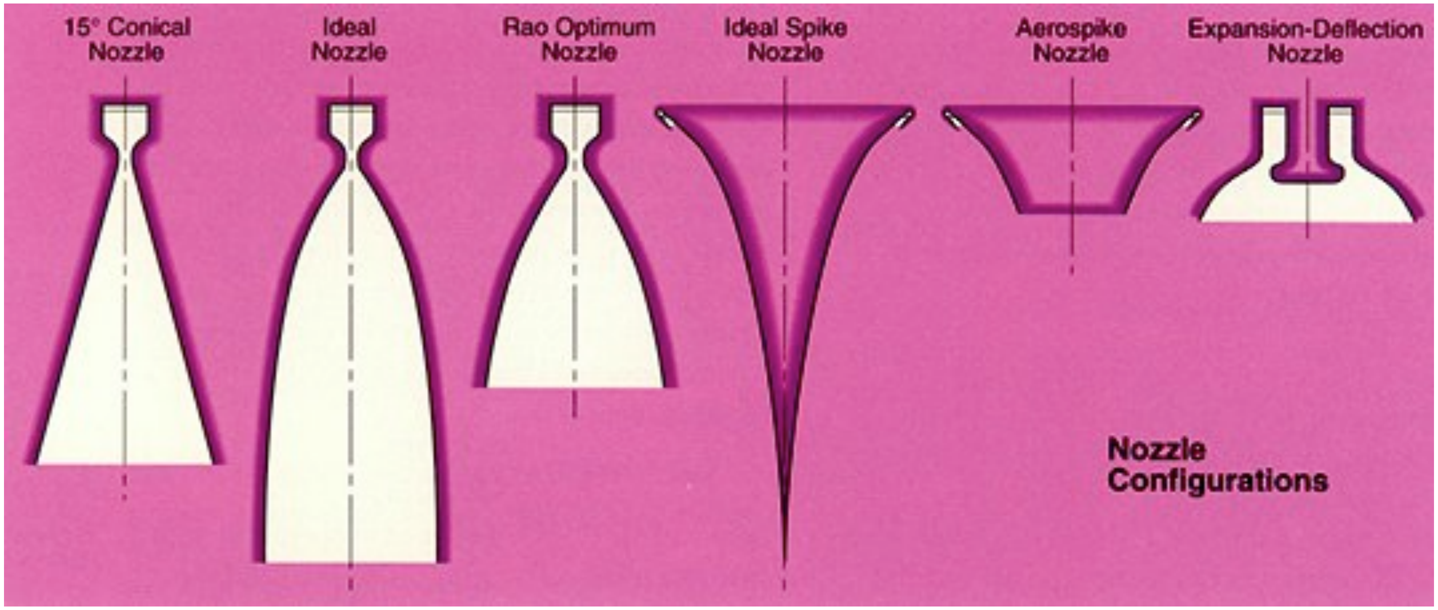
Optimal Nozzle Summary (cont'd)

- Optimal Thrust (or thrust at design condition)

$$Thrust_{opt} = \gamma P_0 A^* \sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left[1-\left(\frac{p_\infty}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

$$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}}\sqrt{\frac{\left(\frac{P_0}{p_\infty}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_\infty}\right)^{\frac{(\gamma-1)}{\gamma}}-1\right]}} \Rightarrow forces ... p_{exit} = p_\infty$$

Optimal Nozzle Summary (concluded)



- *Optimum nozzle configuration for a particular mission depends upon system trades involving performance, thermal issues, weight, fabrication, vehicle integration and cost.*

http://www.k-makris.gr/RocketTechnology/Nozzle_Design/nozzle_design.htm