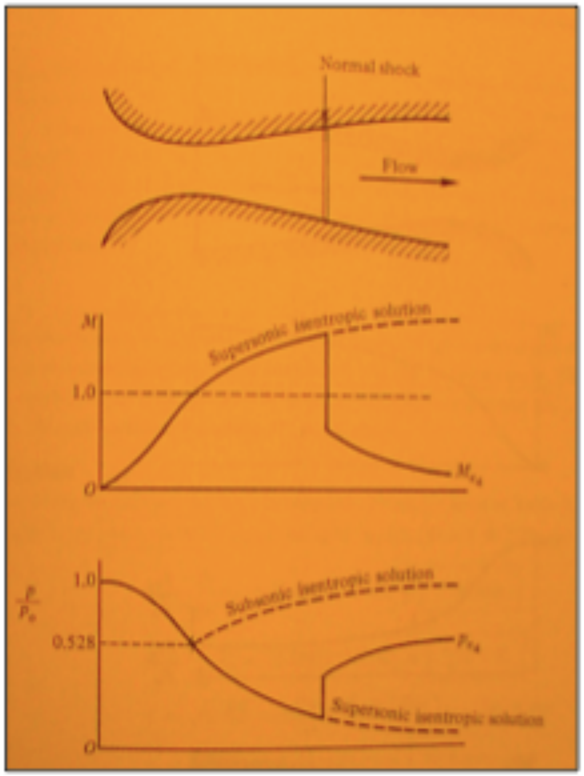
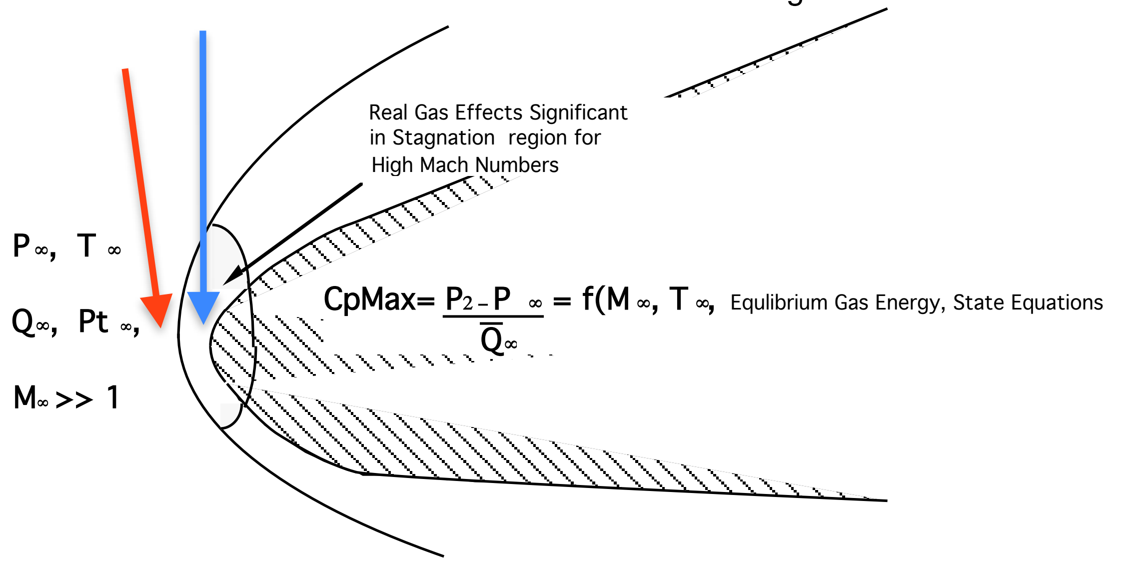
Section 5: Lecture 1
Normal Shockwaves



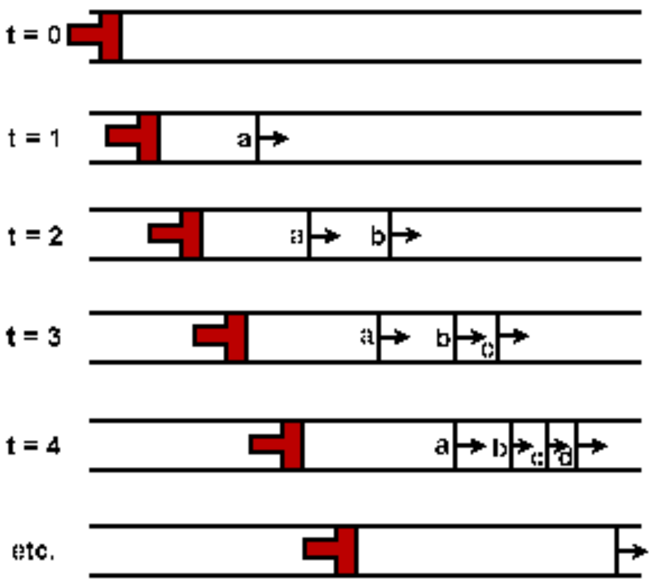
Schematic of the Bow Shock near the Stagnation Point



Anderson: Chapter 3 pp. 88-125

What is a Normal Shockwave?

- Flow discontinuity caused by object traveling faster than sounds waves can transmit pressure disturbances



- Sound waves eventually "pile-up" causing very thin flow discontinuity

- Normal Shockwaves are perpendicular To the direction of the flow

- Normal Shock waves at moderate Mach numbers are *adiabatic*, but not *Isentropic*

- *i.e.* T_0 *is constant,* P_0 *is not*

Mach Number Relationship

- As derived in section 3, for 1-D steady adiabatic flow

Continuity $$\rho_1 V_1 = \rho_2 V_2$$ **Delta S != 0**

Momentum $$p_1 + \rho_1 V_1^{\,2} = p_2 + \rho_2 V_2^{\,2}$$

Energy
$$c_{p1} T_1 + \frac{V_1^{\,2}}{2} = c_{p2} T_2 + \frac{V_2^{\,2}}{2}$$

Mach Number Relationship(cont'd)

- Divide continuity equation into momentum

$$\frac{p_1 + \rho_1 V_1^{\,2}}{\rho_1 V_1} = \frac{p_2 + \rho_2 V_2^{\,2}}{\rho_2 V_2}$$

$$\left(\frac{p_1}{\rho_1 V_1} + V_1\right) - \left(\frac{p_2}{\rho_2 V_2} + V_2\right) = 0$$

$$\left(\frac{R_g T_1}{V_1} + V_1\right) - \left(\frac{R_g T_2}{V_2} + V_2\right) = 0 \rightarrow$$

$$\left(\frac{c_1^{\,2}}{\gamma V_1} + V_1\right) - \left(\frac{c_2^{\,2}}{\gamma V_2} + V_2\right) = 0$$

Mach Number Relationship (cont'd)

- Write Alternate form of Energy Equation

$$c_p T + \frac{V^2}{2} = c_p T^* + \frac{c^{*2}}{2} \rightarrow$$

$$\frac{\gamma}{\gamma - 1} R_g T + \frac{V^2}{2} = \frac{\gamma}{\gamma - 1} R_g T^* + \frac{c^{*2}}{2} \rightarrow$$

$$\frac{c^2}{\gamma - 1} + \frac{V^2}{2} = \frac{c^{*2}}{\gamma - 1} + \frac{c^{*2}}{2} = \frac{\gamma + 1}{2(\gamma - 1)} c^{*2}$$

Mach Number Relationship (cont'd)

- Re-write and substitute into previously derived momentum/continuity equation

$$\frac{c^2}{\gamma V} = \frac{\frac{\gamma+1}{2}c^{*2}}{\gamma V} - \left(\frac{\gamma-1}{2}\right)\frac{V^2}{\gamma V}$$

$$\frac{c^2}{\gamma V} = \frac{\gamma+1}{2\gamma}\frac{c^{*2}}{V} - \left(\frac{\gamma-1}{2\gamma}\right)V$$

$$\left(\frac{\gamma+1}{2\gamma}\frac{c^{*2}}{V_1} - \left(\frac{\gamma-1}{2\gamma}\right)V_1 + V_1\right) - \left(\frac{\gamma+1}{2\gamma}\frac{c^{*2}}{V_2} - \left(\frac{\gamma-1}{2\gamma}\right)V_2 + V_2\right) = 0$$

Mach Number Relationship (cont'd)

- Collect V_2-V_1

$$\left(\frac{\gamma+1}{2\gamma}\frac{c^{*2}}{V_1}-\left(\frac{\gamma-1}{2\gamma}\right)V_1+V_1\right)-\left(\frac{\gamma+1}{2\gamma}\frac{c^{*2}}{V_2}-\left(\frac{\gamma-1}{2\gamma}\right)V_2+V_2\right)=0$$

$$\left(\frac{\gamma+1}{2\gamma}\frac{c^{*2}}{V_1}-\frac{\gamma+1}{2\gamma}\frac{c^{*2}}{V_2}\right)+\left[\left(\frac{\gamma-1}{2\gamma}\right)V_2-V_2-\left(\frac{\gamma-1}{2\gamma}\right)V_1-V_1\right]=0\rightarrow$$

$$\left(\frac{\gamma+1}{2\gamma}\frac{c^{*2}}{V_2V_1}\left(V_2-V_1\right)\right)+\left[\left(\frac{\gamma-1}{2\gamma}\right)-1\right]\left(V_2-V_1\right)=0\rightarrow$$

Mach Number Relationship (cont'd)

- Divide by V_2-V_1

$$\left(\frac{\gamma+1}{2\gamma}\frac{c^{*2}}{V_2V_1}\right)+\left[\left(\frac{\gamma-1}{2\gamma}\right)-1\right]=0$$

- But

$$\left[\left(\frac{\gamma-1}{2\gamma}\right)-1\right]=\frac{\gamma-1-2\gamma}{2\gamma}=-\left(\frac{\gamma-1}{2\gamma}\right)$$

- Solving for c^{*2}

$$c^{*2}=V_1V_2 \rightarrow M_1^{\ *}=\frac{1}{M_2^{\ *}}$$

Mach Number Relationship (cont'd)

- From earlier

$$\frac{\gamma+1}{2(\gamma-1)}c^{*2}=\frac{c^2}{\gamma-1}+\frac{V^2}{2}\rightarrow\frac{\gamma+1}{(\gamma-1)}=2\frac{c^2/c^{*2}}{\gamma-1}+\frac{V^2}{c^{*2}}\rightarrow$$

$$\frac{\gamma+1}{(\gamma-1)}=2\frac{c^2/c^{*2}}{\gamma-1}+M^{*2}=2\frac{\left(c^2/V^2\right)\left(V^2/c^{*2}\right)}{\gamma-1}+M^{*2}\rightarrow$$

$$\frac{\gamma+1}{(\gamma-1)}M^2=\frac{2}{\gamma-1}M^{*2}+M^2M^{*2}\rightarrow$$

$$\frac{\gamma+1}{(\gamma-1)}M^2=\left(\frac{2}{\gamma-1}+M^2\right)M^{*2}\rightarrow$$

$$M^{*2}=\frac{\gamma+1}{(\gamma-1)}M^2\frac{1}{\left(\frac{2}{\gamma-1}+M^2\right)}=\frac{(\gamma+1)M^2}{\left(2+(\gamma-1)M^2\right)}$$

Mach Number Relationship (cont'd)

- Since

$$M_2^{*2} = \frac{1}{M_1^{*2}}$$

- and

$$M^{*2} = \frac{(\gamma + 1)M^2}{\left(2 + (\gamma - 1)M^2\right)}$$

- Then

$$\frac{(\gamma + 1)M_2^2}{\left(2 + (\gamma - 1)M_2^2\right)} = \frac{\left(2 + (\gamma - 1)M_1^2\right)}{(\gamma + 1)M_1^2}$$

Mach Number Relationship (concluded)

- Solving for M_2

$$\left(\gamma+1\right)M_2^{\,2}=\frac{\left(2+\left(\gamma-1\right)M_1^{\,2}\right)}{\left(\gamma+1\right)M_1^{\,2}}\left(2+\left(\gamma-1\right)M_2^{\,2}\right)$$

$$\left(\gamma+1\right)M_2^{\,2}-\left[\frac{\left(2+\left(\gamma-1\right)M_1^{\,2}\right)}{\left(\gamma+1\right)M_1^{\,2}}\right]\left(\gamma-1\right)M_2^{\,2}=2\left[\frac{\left(2+\left(\gamma-1\right)M_1^{\,2}\right)}{\left(\gamma+1\right)M_1^{\,2}}\right]\rightarrow$$

$$\left[\left(\gamma+1\right)-\left[\frac{\left(2+\left(\gamma-1\right)M_1^{\,2}\right)}{\left(\gamma+1\right)M_1^{\,2}}\right]\left(\gamma-1\right)\right]M_2^{\,2}=2\left[\frac{\left(2+\left(\gamma-1\right)M_1^{\,2}\right)}{\left(\gamma+1\right)M_1^{\,2}}\right]\rightarrow$$

$$\left[\left(\gamma+1\right)\left(\gamma+1\right)M_1^{\,2}-\left[\left(2+\left(\gamma-1\right)M_1^{\,2}\right)\right]\left(\gamma-1\right)\right]M_2^{\,2}=2\left[\left(2+\left(\gamma-1\right)M_1^{\,2}\right)\right]\rightarrow$$

$$\left[\left(\gamma+1\right)^2M_1^{\,2}-\left(\gamma-1\right)^2M_1^{\,2}-2\left(\gamma-1\right)\right]M_2^{\,2}=2\left[\left(2+\left(\gamma-1\right)M_1^{\,2}\right)\right]\rightarrow$$

Mach Number Relationship (concluded)

- Expanding the squares

$$(\gamma+1)^2 M_1^{\,2} - (\gamma-1)^2 M_1^{\,2} = \left(\gamma^2 + 2\gamma + 1\right) M_1^{\,2} - \left((\gamma^2 - 2\gamma + 1)\right) M_1^{\,2} = 4\gamma M_1^{\,2}$$

$$\left[4\gamma M_1^{\,2} - 2\left(\gamma-1\right)\right] M_2^{\,2} = 2\left[\left(2 + \left(\gamma-1\right) M_1^{\,2}\right)\right] \rightarrow$$

$$\left[\gamma M_1^{\,2} - \frac{\left(\gamma-1\right)}{2}\right] M_2^{\,2} = \left[\left(1 + \frac{\left(\gamma-1\right)}{2} M_1^{\,2}\right)\right] \rightarrow$$

$$\boxed{M_2 = \sqrt{\frac{\left(1 + \frac{\left(\gamma-1\right)}{2} M_1^{\,2}\right)}{\left(\gamma M_1^{\,2} - \frac{\left(\gamma-1\right)}{2}\right)}}}$$

Whew!

Density Ratio Across Normal Shock

- From Continuity

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{V_1/c^*}{V_2/c^*} = \frac{M_1^{\ *}}{M_2^{\ *}} \rightarrow M_1^{\ *} = \frac{1}{M_2^{\ *}}$$

$$\frac{\rho_2}{\rho_1} = M_1^{\ *2} = \frac{(\gamma+1)M_1^{\ 2}}{\left(2+(\gamma-1)M_1^{\ 2}\right)}$$

- Only a *FUNCTION OF UPSTREAM MACH*

Pressure Ratio Across Normal Shock

- From Earlier derived momentum/continuity equation

$$\rho_1 V_1 = \rho_2 V_2$$

$$p1 + \rho_1 V_1^{\,2} = p_2 + \rho_2 V_2^{\,2} \rightarrow$$

$$p_2 - p1 = \rho_1 V_1^{\,2} - \rho_2 V_2^{\,2} = \rho_1 V_1 \left(V_1 - V_2\right) = \rho_1 V_1^{\,2}\left(1 - \frac{V_2}{V_1}\right)$$

- Divide by p_1

$$\frac{p_2 - p_1}{p_1} = \frac{\rho_1}{p1} V_1^{\,2}\left(1 - \frac{V_2}{V_1}\right) = \gamma \frac{1}{\gamma R_g T} V_1^{\,2}\left(1 - \frac{V_2}{V_1}\right) = \gamma M_1^{\,2}\left(1 - \frac{V_2}{V_1}\right) =$$

$$\gamma M_1^{\,2}\left(1 - \frac{V_2 / c^*}{V_1 / c^*}\right) = \gamma M_1^{\,2}\left(1 - \frac{M_2^{\,*}}{M_1^{\,*}}\right) = \gamma M_1^{\,2}\left(1 - M_1^{\,*2}\right) \rightarrow$$

Pressure Ratio Across Normal Shock (cont'd)

- Collect terms and simplify

$$\frac{p_2 - p_1}{p_1} = \gamma M_1^{\ 2}\left(1 - \frac{\left(2 + (\gamma - 1)M_1^{\ 2}\right)}{(\gamma + 1)M_1^{\ 2}}\right) \rightarrow$$

$$\frac{p_2}{p_1} = 1 + \gamma M_1^{\ 2}\left(\frac{(\gamma + 1)M_1^{\ 2} - \left(2 + (\gamma - 1)M_1^{\ 2}\right)}{(\gamma + 1)M_1^{\ 2}}\right) = 1 + \frac{2\gamma}{(\gamma + 1)}\left(M_1^{\ 2} - 1\right)$$

- Again ….
- Only a *FUNCTION OF UPSTREAM MACH*

Temperature Ratio Across Normal Shock (cont'd)

- Finally

$$\frac{T_2}{T_1} = \frac{\frac{p_2}{p_1}}{\frac{\rho_2}{\rho_1}} = \frac{1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\,2} - 1\right)}{\frac{(\gamma+1)M_1^{\,2}}{\left(2 + (\gamma-1)M_1^{\,2}\right)}} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\,2} - 1\right)\right]\left[\frac{\left(2 + (\gamma-1)M_1^{\,2}\right)}{(\gamma+1)M_1^{\,2}}\right]$$

What Happens to Stagnation Conditions Across Shock Wave?

- Flow is no longer Isentropic and Total pressure loss will occur

$$\frac{P_{02}}{P_{01}} = \frac{P_{02}}{P_2} \times \frac{p_2}{p_1} \times \frac{p_1}{P_{01}} = \frac{\left(1 + \frac{\gamma - 1}{2} M_2^{\,2}\right)^{\left(\frac{\gamma}{\gamma - 1}\right)} \left(1 + \frac{2\gamma}{(\gamma + 1)}\left(M_1^{\,2} - 1\right)\right)}{\left(1 + \frac{\gamma - 1}{2} M_1^{\,2}\right)^{\left(\frac{\gamma}{\gamma - 1}\right)}} =$$

$$\left(\frac{1 + \frac{\gamma - 1}{2} M_2^{\,2}}{1 + \frac{\gamma - 1}{2} M_1^{\,2}}\right)^{\left(\frac{\gamma}{\gamma - 1}\right)} \left(1 + \frac{2\gamma}{(\gamma + 1)}\left(M_1^{\,2} - 1\right)\right)$$

P01 P02

- But across the Normal Shock wave

$$M_2^{\,2} = \frac{\left(1 + \frac{(\gamma - 1)}{2} M_1^{\,2}\right)}{\left(\gamma M_1^{\,2} - \frac{(\gamma - 1)}{2}\right)}$$

- Substitution gives:-->

What Happens to Stagnation Conditions Across Shock Wave? (cont'd)

$$\frac{P_{02}}{P_{01}} = \left(\left. \left(1 + \frac{\gamma - 1}{2} \frac{\left(1 + \frac{(\gamma - 1)}{2} M_1^{\ 2}\right)}{\left(\gamma M_1^{\ 2} - \frac{(\gamma - 1)}{2}\right)} \right) \middle/ 1 + \frac{\gamma - 1}{2} M_1^{\ 2} \right. \right)^{\left(\frac{\gamma}{\gamma - 1}\right)} \left(1 + \frac{2\gamma}{(\gamma + 1)} \left(M_1^{\ 2} - 1 \right) \right)$$

- Simplifying…

What Happens to Stagnation Conditions Across Shock Wave? (concluded)

$$\left(\frac{1}{1+\frac{\gamma-1}{2}M_1^2}+\frac{\gamma-1}{2}\frac{1}{\left(\gamma M_1^2-\frac{(\gamma-1)}{2}\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}\left(1+\frac{2\gamma}{(\gamma+1)}\left(M_1^2-1\right)\right)=$$

$$\left(\frac{\left(\gamma M_1^2-\frac{(\gamma-1)}{2}\right)+\frac{\gamma-1}{2}\left(1+\frac{\gamma-1}{2}M_1^2\right)}{\left(1+\frac{\gamma-1}{2}M_1^2\right)\left(\gamma M_1^2-\frac{(\gamma-1)}{2}\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}\left(1+\frac{2\gamma}{(\gamma+1)}\left(M_1^2-1\right)\right)=$$

$$\frac{P_{02}}{P_{01}}=\left(\frac{\gamma M_1^2+\frac{\gamma-1}{2}\left(\frac{\gamma-1}{2}M_1^2\right)}{\left(1+\frac{\gamma-1}{2}M_1^2\right)\left(\gamma M_1^2-\frac{(\gamma-1)}{2}\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}\left(1+\frac{2\gamma}{(\gamma+1)}\left(M_1^2-1\right)\right)$$

What Happens to Stagnation Conditions Across Shock Wave? (concluded)

- Simplifying

$$\frac{P_{02}}{P_{01}} = \left(\frac{\gamma M_1^2 + \frac{\gamma - 1}{2}\left(\frac{\gamma - 1}{2} M_1^2\right)}{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)\left(\gamma M_1^2 - \frac{(\gamma - 1)}{2}\right)} \right)^{\left(\frac{\gamma}{\gamma - 1}\right)} \left(1 + \frac{2\gamma}{(\gamma + 1)}\left(M_1^2 - 1\right)\right) = \left(\frac{\left[\frac{(\gamma + 1)}{2} M_1\right]^2}{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)\left(\gamma M_1^2 - \frac{(\gamma - 1)}{2}\right)} \right)^{\left(\frac{\gamma}{\gamma - 1}\right)} \left(\frac{2\gamma M_1^2}{(\gamma + 1)} - \frac{(\gamma - 1)}{(\gamma + 1)}\right) =$$

$$\left(\frac{\left[\frac{(\gamma + 1)}{2} M_1\right]^2}{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)\left(\gamma M_1^2 - \frac{(\gamma - 1)}{2}\right)} \right)^{\left(\frac{\gamma}{\gamma - 1}\right)} \left(\frac{2}{(\gamma + 1)}\left(\gamma M_1^2 - \frac{(\gamma - 1)}{2}\right)\right) = \frac{2}{(\gamma + 1)\left(\gamma M_1^2 - \frac{(\gamma - 1)}{2}\right)^{\frac{1}{\gamma - 1}}} \left(\frac{\left[\frac{(\gamma + 1)}{2} M_1\right]^2}{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)} \right)^{\left(\frac{\gamma}{\gamma - 1}\right)}$$

$$\boxed{\frac{P_{02}}{P_{01}} = \frac{2}{(\gamma + 1)\left(\gamma M_1^2 - \frac{(\gamma - 1)}{2}\right)^{\frac{1}{\gamma - 1}}} \left(\frac{\left[\frac{(\gamma + 1)}{2} M_1\right]^2}{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)} \right)^{\left(\frac{\gamma}{\gamma - 1}\right)}}$$

Normal Shock Summary

$$M_2 = \sqrt{\frac{\left(1 + \frac{(\gamma - 1)}{2} M_1^2\right)}{\left(\gamma M_1^2 - \frac{(\gamma - 1)}{2}\right)}}$$

$$\frac{\rho_2}{\rho_1} = M_1^{*2} = \frac{(\gamma + 1) M_1^2}{\left(2 + (\gamma - 1) M_1^2\right)}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma + 1)}\left(M_1^2 - 1\right)$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma + 1)}\left(M_1^2 - 1\right)\right]\left[\frac{\left(2 + (\gamma - 1) M_1^2\right)}{(\gamma + 1) M_1^2}\right]$$

$$T_{0_2} = T_{0_1}$$

$$\frac{P_{02}}{P_{01}} = \frac{2}{(\gamma + 1)\left(\gamma M_1^2 - \frac{(\gamma - 1)}{2}\right)^{\frac{1}{\gamma - 1}}} \left(\frac{\left[\frac{(\gamma + 1)}{2} M_1\right]^2}{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)}\right)^{\left(\frac{\gamma}{\gamma - 1}\right)}$$

Entropy Change across normal shock wave

- The change in stagnation pressure indicates that the process is no longer isentropic (reversible, adiabatic)

$$\frac{P_{02}}{P_{01}} = \frac{2}{(\gamma+1)\left(\gamma M_1^2 - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}}\left(\frac{\left[\frac{(\gamma+1)}{2}M_1\right]^2}{\left(1+\frac{\gamma-1}{2}M_1^2\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}$$

- Prove It!

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Entropy Change across normal shock wave

(cont'd)

- From section 1: Second law for adiabatic process

$$s_2 - s_1 = c_p \ln\left[\frac{T_2}{T_1}\right] - R_g \ln\left[\frac{p_2}{p_1}\right] = c_p\left(\ln\left[\frac{T_2}{T_1}\right] - \frac{R_g}{c_p}\ln\left[\frac{p_2}{p_1}\right]\right) =$$

$$c_p\left(\ln\left[\frac{T_2}{T_1}\right] - \frac{c_p - c_v}{c_p}\ln\left[\frac{p_2}{p_1}\right]\right) = c_p\left(\ln\left[\frac{T_2}{T_1}\right] - \frac{\gamma - 1}{\gamma}\ln\left[\frac{p_2}{p_1}\right]\right) =$$

$$c_p\left(\ln\left[\frac{T_2}{T_1}\right] - \ln\left[\frac{p_2}{p_1}^{\frac{\gamma-1}{\gamma}}\right]\right) = c_p \ln\left[\frac{\left(\frac{T_2}{T_1}\right)}{\frac{p_2}{p_1}^{\frac{\gamma-1}{\gamma}}}\right]$$

Entropy Change across normal shock wave

(cont'd)

- but from normal shock relationships

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\;2} - 1\right)$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\;2} - 1\right)\right]\left[\frac{\left(2 + (\gamma-1)M_1^{\;2}\right)}{(\gamma+1)M_1^{\;2}}\right]$$

- Subbing into the entropy relationship

Entropy Change across normal shock wave

(cont'd)

$$s_2 - s_1 = c_p \ln \left[\frac{\left[1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^2 - 1\right)\right]\left[\frac{\left(2 + (\gamma-1)M_1^2\right)}{(\gamma+1)M_1^2}\right]}{\left(1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^2 - 1\right)\right)^{\frac{\gamma-1}{\gamma}}} \right] =$$

$$c_p \ln \left[\left[1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^2 - 1\right)\right]^{\frac{1}{\gamma}} \left[\frac{\left(2 + (\gamma-1)M_1^2\right)}{(\gamma+1)M_1^2}\right] \right]$$

Entropy Change across normal shock wave

(cont'd)

- Factoring out $\dfrac{2}{\gamma + 1}$

$$s_2 - s_1 = c_p \ln\left[\left[\frac{2}{(\gamma+1)}\left[\frac{(\gamma+1)}{2} + \gamma\left(M_1^{\,2} - 1\right)\right]\right]^{\frac{1}{\gamma}}\left[\frac{\left(1 + \frac{(\gamma-1)}{2} M_1^{\,2}\right)}{\frac{(\gamma+1)}{2} M_1^{\,2}}\right]\right] =$$

$$c_p \ln\left[\left[\gamma M_1^{\,2} - \frac{(\gamma-1)}{2}\right]^{\frac{1}{\gamma}}\left[\frac{\left(1 + \frac{(\gamma-1)}{2} M_1^{\,2}\right)}{\left[\frac{(\gamma+1)}{2}\right]^2 M_1^{\,2}}\right]\left(\frac{2}{(\gamma+1)}\right)^{\frac{1}{\gamma}}\left(\frac{2}{(\gamma+1)}\right)^{-1}\right] =$$

$$c_p \ln\left[\left[\gamma M_1^{\,2} - \frac{(\gamma-1)}{2}\right]^{\frac{1}{\gamma}}\left[\frac{\left(1 + \frac{(\gamma-1)}{2} M_1^{\,2}\right)}{\left[\frac{(\gamma+1)}{2}\right]^2 M_1^{\,2}}\right]\frac{1}{\left(\frac{2}{(\gamma+1)}\right)^{\frac{(\gamma-1)}{\gamma}}}\right]$$

Entropy Change across normal shock wave

(cont'd)

- Since $\ln[x] = -\ln[\frac{1}{x}]$

$$s_2 - s_1 = -c_p \ln\left[\left(\frac{2}{(\gamma+1)}\right)^{\frac{(\gamma-1)}{\gamma}} \frac{1}{\left[\gamma M_1^{\,2} - \frac{(\gamma-1)}{2}\right]^{\frac{1}{\gamma}}} \frac{\left[\frac{(\gamma+1)}{2}\right]^2 M_1^{\,2}}{\left(1 + \frac{(\gamma-1)}{2} M_1^{\,2}\right)}\right]$$

Entropy Change across normal shock wave

(cont'd)

- But from previous derivation for stagnation pressure ratio

$$\frac{P_{02}}{P_{01}} = \frac{2}{(\gamma+1)\left(\gamma M_1^2 - \frac{(\gamma-1)}{2}\right)}^{\frac{1}{\gamma-1}} \left(\frac{\left[\frac{(\gamma+1)}{2} M_1\right]^2}{\left(1 + \frac{\gamma-1}{2} M_1^2\right)} \right)^{\left(\frac{\gamma}{\gamma-1}\right)}$$

- Taking

$$\left(\frac{P_{02}}{P_{01}}\right)^{\frac{\gamma-1}{\gamma}}$$

Entropy Change across normal shock wave

(cont'd)

- Then

$$\left(\frac{P_{02}}{P_{01}}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{2}{(\gamma+1)\left(\gamma M_1^2 - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}}\left(\frac{\left[\frac{(\gamma+1)}{2}M_1\right]^2}{\left(1+\frac{\gamma-1}{2}M_1^2\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}\right)^{\frac{\gamma-1}{\gamma}} =$$

$$\left(\frac{2}{(\gamma+1)}\right)^{\frac{\gamma-1}{\gamma}} \frac{1}{\left(\gamma M_1^2 - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma}}} \frac{\left[\frac{(\gamma+1)}{2}M_1\right]^2}{\left(1+\frac{\gamma-1}{2}M_1^2\right)}$$

Entropy Change across normal shock wave

(cont'd)

- Comparing

$$s_2 - s_1 = -c_p \ln\left[\left(\frac{2}{(\gamma+1)}\right)^{\frac{(\gamma-1)}{\gamma}} \frac{1}{\left[\gamma M_1^{\,2} - \frac{(\gamma-1)}{2}\right]^{\frac{1}{\gamma}}} \frac{\left[\frac{(\gamma+1)}{2}\right]^2 M_1^{\,2}}{\left(1+\frac{(\gamma-1)}{2}M_1^{\,2}\right)}\right]$$

$$\left(\frac{P_{02}}{P_{01}}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{2}{(\gamma+1)}\right)^{\frac{\gamma-1}{\gamma}} \frac{1}{\left(\gamma M_1^{\,2} - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma}}} \frac{\left[\frac{(\gamma+1)}{2}M_1\right]^2}{\left(1+\frac{\gamma-1}{2}M_1^{\,2}\right)}$$

Entropy Change across normal shock wave

(cont'd)

- substituting

$$s_2 - s_1 = -c_p \ln\left[\left(\frac{P_{02}}{P_{01}}\right)^{\frac{\gamma-1}{\gamma}}\right] = -c_p \frac{\gamma - 1}{\gamma}\ln\left[\frac{P_{02}}{P_{01}}\right] = -c_p \frac{\frac{c_p}{c_v} - 1}{\frac{c_p}{c_v}}\ln\left[\frac{P_{02}}{P_{01}}\right] =$$

$$-\left(c_p - c_v\right)\ln\left[\frac{P_{02}}{P_{01}}\right] = -R_g \ln\left[\frac{P_{02}}{P_{01}}\right]$$

$$s_2 - s_1 = -R_g \ln\left[\frac{P_{02}}{P_{01}}\right] \Rightarrow \boxed{\frac{P_{02}}{P_{01}} = e^{-\left(\frac{s_2 - s_1}{R_g}\right)}}$$

Entropy Change across normal shock wave

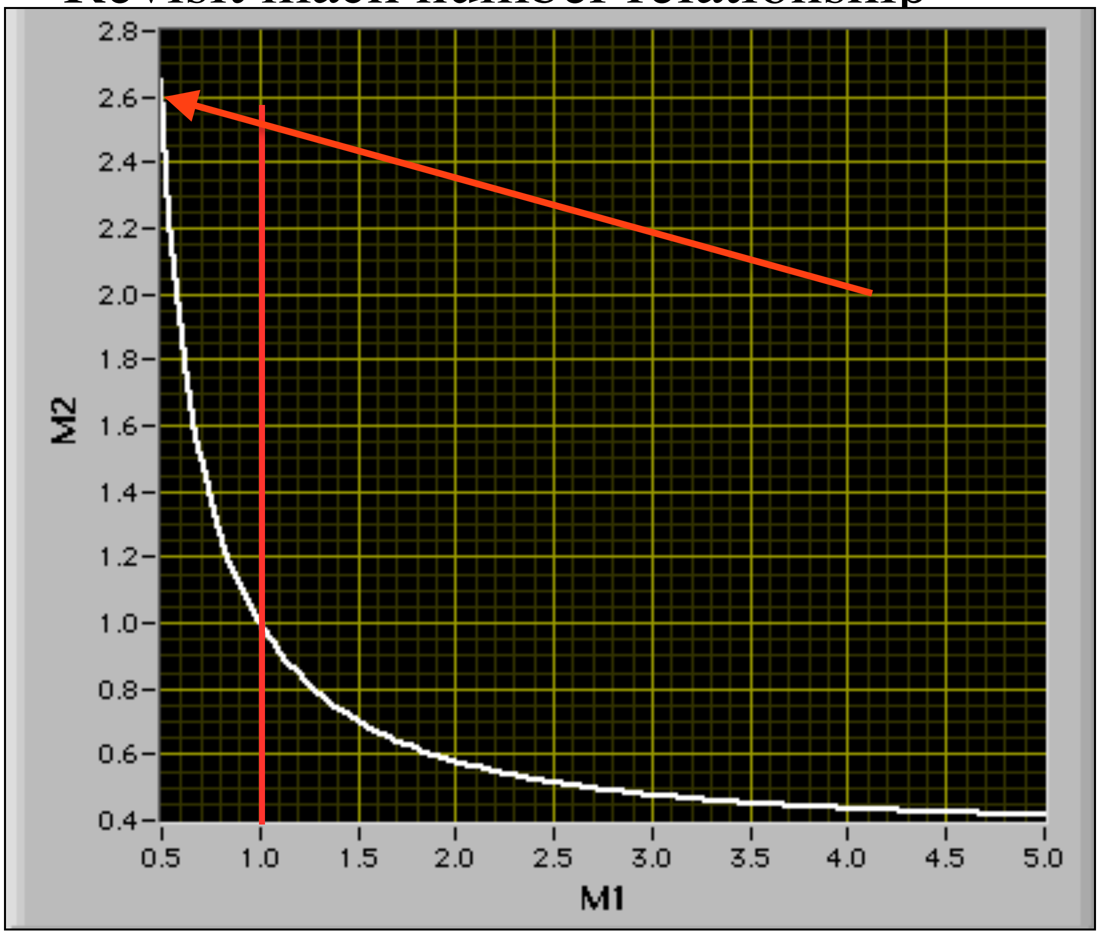
(cont'd)

- Entropy always increases. … stagnation pressure always drops across shock wave … stagnation pressure drop is a measure of the irreversibility of the process

$$\boxed{\frac{P_{02}}{P_{01}} = e^{-\left(\frac{s_2 - s_1}{R_g}\right)}}$$

Can Mach ever increase across a normal shock wave?

- Revisit mach number relationship



Plot …

$$M_2 = \sqrt{\frac{\left(1 + \frac{(\gamma - 1)}{2} M_1^{\,2}\right)}{\left(\gamma M_1^{\,2} - \frac{(\gamma - 1)}{2}\right)}}$$

$M_2 < M_1$ for $M_1 > 1$

$M_2 > M_1$ for $M_1 < 1$

- But is this physically Possible?

Can Mach ever increase across a normal shock wave? (cont'd)

- From General 1-D momentum equation

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 = p_1 + \frac{\gamma\, p_1}{\gamma R_g T_1} V_1^2 = p_2 + \frac{\gamma\, p_2}{\gamma R_g T_2} V_2^2$$

$$p_2 - p_1 = \gamma\, p_1 M_1^{\ 2} - \gamma\, p_2 M_2^{\ 2} \Rightarrow \boxed{\frac{p_2}{p_1} = \frac{1+\gamma M_1^{\ 2}}{1+\gamma M_2^{\ 2}}}$$

- And since flow is *isentropic* on *either* side of the shock

$$\frac{P_{0_2}}{P_{0_1}} = \frac{p_2}{p_1}\left[\frac{1+\frac{\gamma-1}{2}M_2^{\ 2}}{1+\frac{\gamma-1}{2}M_1^{\ 2}}\right]^{\frac{\gamma}{\gamma-1}} = \left[\frac{1+\frac{\gamma-1}{2}M_2^{\ 2}}{1+\frac{\gamma-1}{2}M_1^{\ 2}}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{1+\gamma M_1^{\ 2}}{1+\gamma M_2^{\ 2}}\right]$$

Can Mach ever increase across a normal shock wave? (cont'd)

- Second law:

$$\boxed{\frac{P_{02}}{P_{01}} = e^{-\left(\frac{s_2 - s_1}{R_g}\right)}}$$

- Momentum:

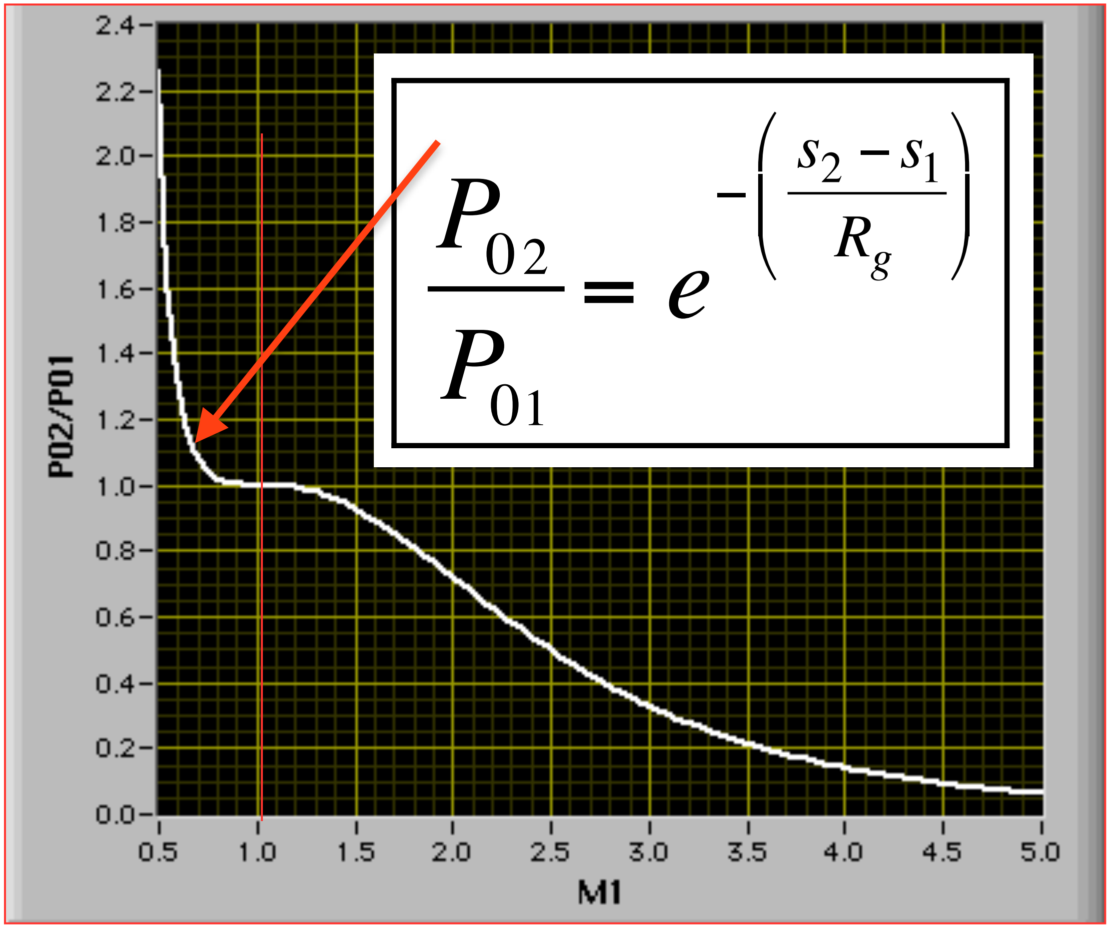
$$\frac{P_{0_2}}{P_{0_1}} = \left[\frac{1 + \frac{\gamma - 1}{2} M_2{}^2}{1 + \frac{\gamma - 1}{2} M_1{}^2}\right]^{\frac{\gamma}{\gamma - 1}} \left[\frac{1 + \gamma M_1{}^2}{1 + \gamma M_2{}^2}\right]$$

Can Mach ever increase across a normal shock wave? (cont'd)

- Plot Stagnation pressure ratio versus M_1:

$P_{0_2} < P_{0_1}$ for $M_1 > 1$

$P_{0_2} > P_{0_1}$ for $M_1 < 1$



$$\frac{P_{02}}{P_{01}} = e^{-\left(\frac{s_2 - s_1}{R_g}\right)}$$

Can Mach ever increase across a normal shock wave? (cont'd)

- Plot Stagnation pressure ratio versus M_1:

$P_{0_2} < P_{0_1}$ for $M_1 > 1$

$P_{0_2} > P_{0_1}$ for $M_1 < 1$

$$\boxed{\frac{P_{02}}{P_{01}} = e^{-\left(\frac{s_2 - s_1}{R_g}\right)}}$$

- But when

$$\frac{P_{0_2}}{P_{0_1}} > 0$$

- Implies

$$s_2 < s_1$$

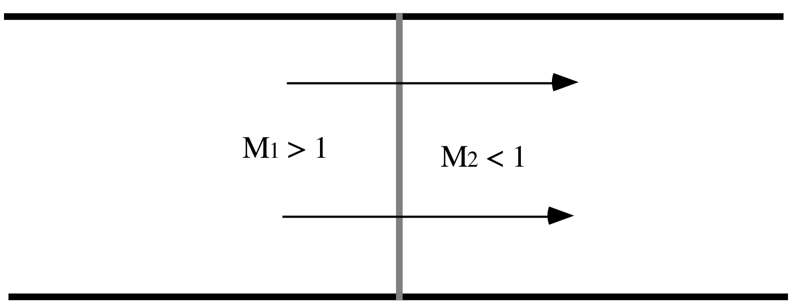
- Entropy always increases. … not physically possible!

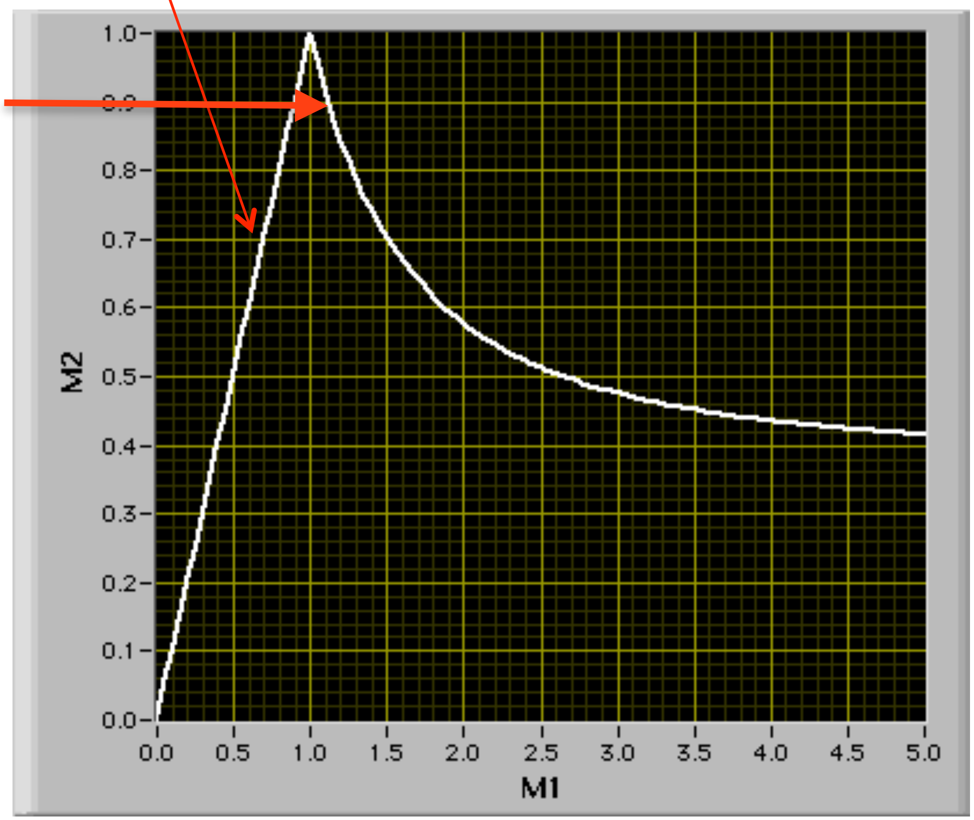
What happens when $M_1<1$?

Normal Shock Plots: Mach Number

- Nothing! .. Trivial flow solution… $M_1=M_2$

$$M_2 = \sqrt{\frac{\left(1 + \frac{(\gamma - 1)}{2} M_1^{\,2}\right)}{\left(\gamma M_1^{\,2} - \frac{(\gamma - 1)}{2}\right)}}$$



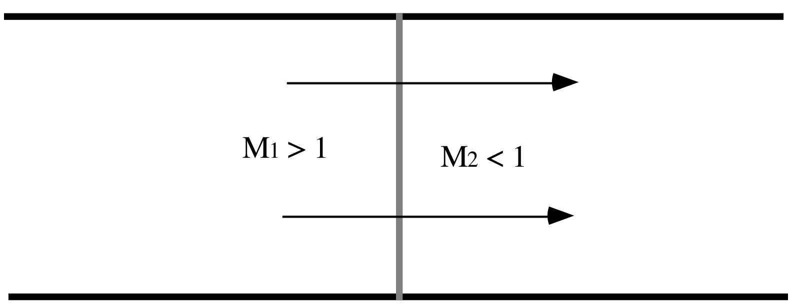


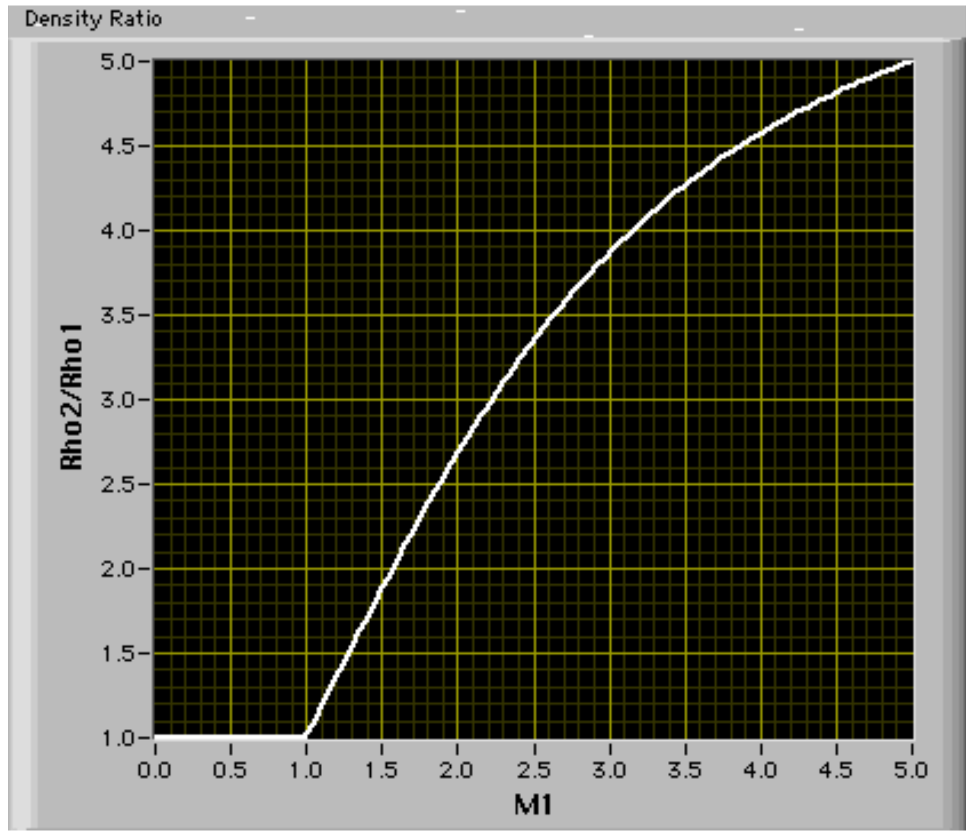
- Mach Number across an (adiabatic) Normal shock wave

Normal Shock Plots: Density

- Density across an (adiabatic) Normal shock wave

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_1^{\ 2}}{\left(2 + (\gamma - 1) M_1^{\ 2}\right)}$$

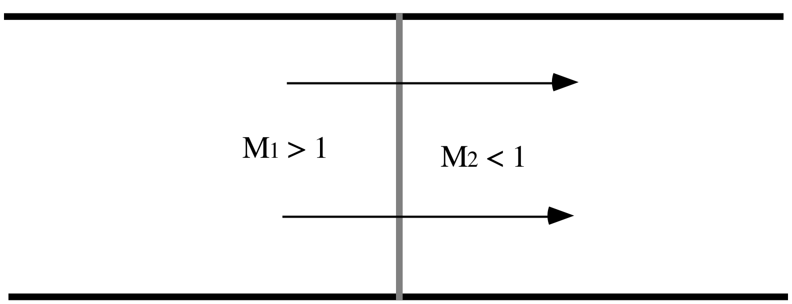


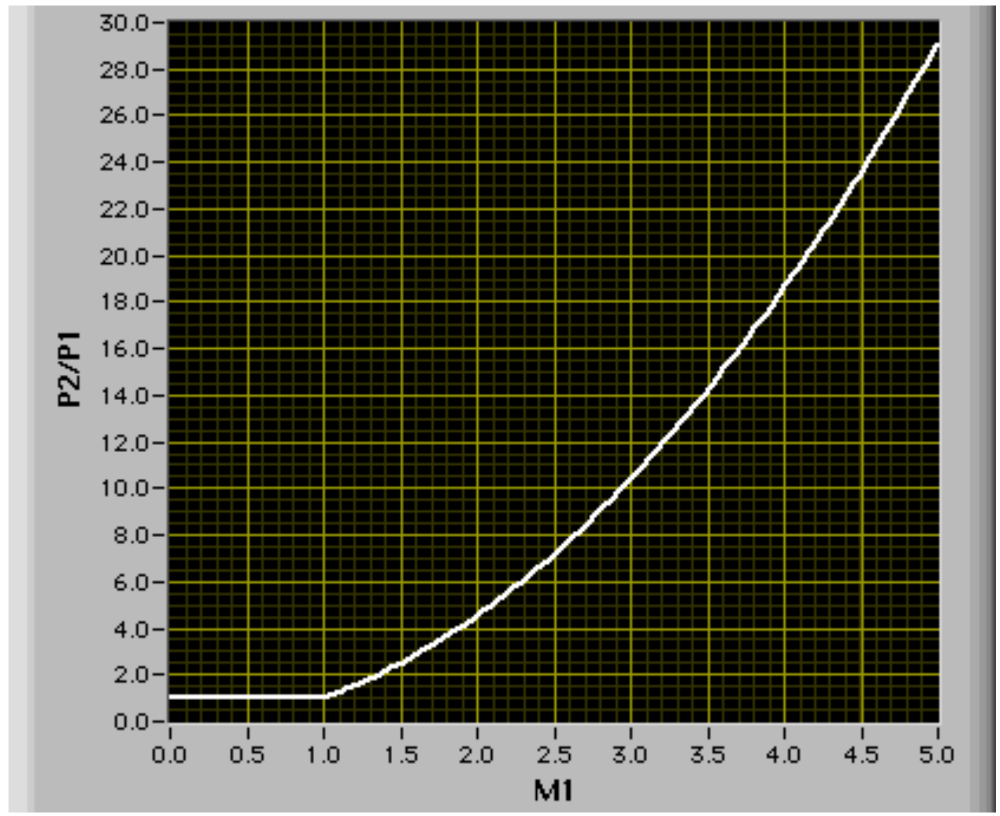


Normal Shock Plots: Pressure

- Pressure across an (adiabatic) Normal shock wave

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma + 1)}\left(M_1^{\,2} - 1\right)$$

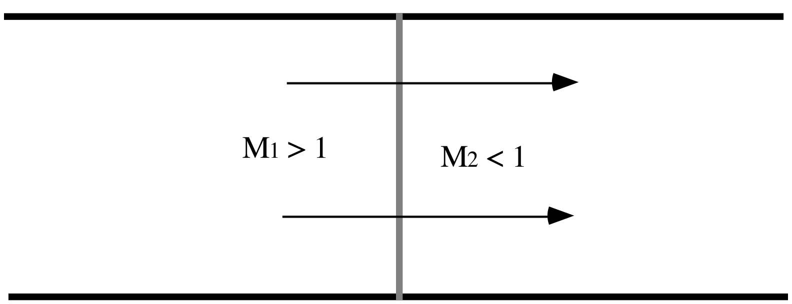


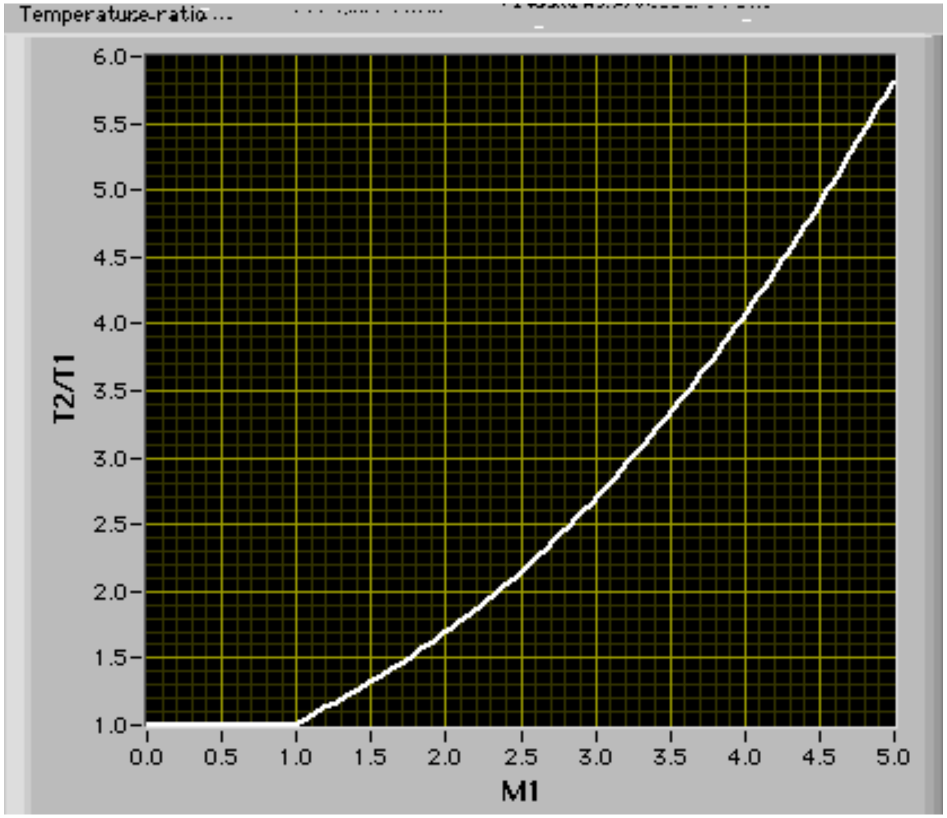


Normal Shock Plots: Temperature

- Temperature across an (adiabatic) Normal shock wave

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left(M_1^{\,2} - 1\right)\right]\left[\frac{\left(2 + (\gamma - 1)M_1^{\,2}\right)}{(\gamma+1)M_1^{\,2}}\right]$$





Normal Shock Plots: Stagnation Pressure

- Stagnation pressure across an (adiabatic) Normal shock wave

$$\frac{P_{02}}{P_{01}} = \frac{2}{(\gamma+1)\left(\gamma M_1^{\,2} - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}} \left(\frac{\left[\frac{(\gamma+1)}{2} M_1\right]^2}{\left(1 + \frac{\gamma-1}{2} M_1^{\,2}\right)} \right)^{\left(\frac{\gamma}{\gamma-1}\right)}$$