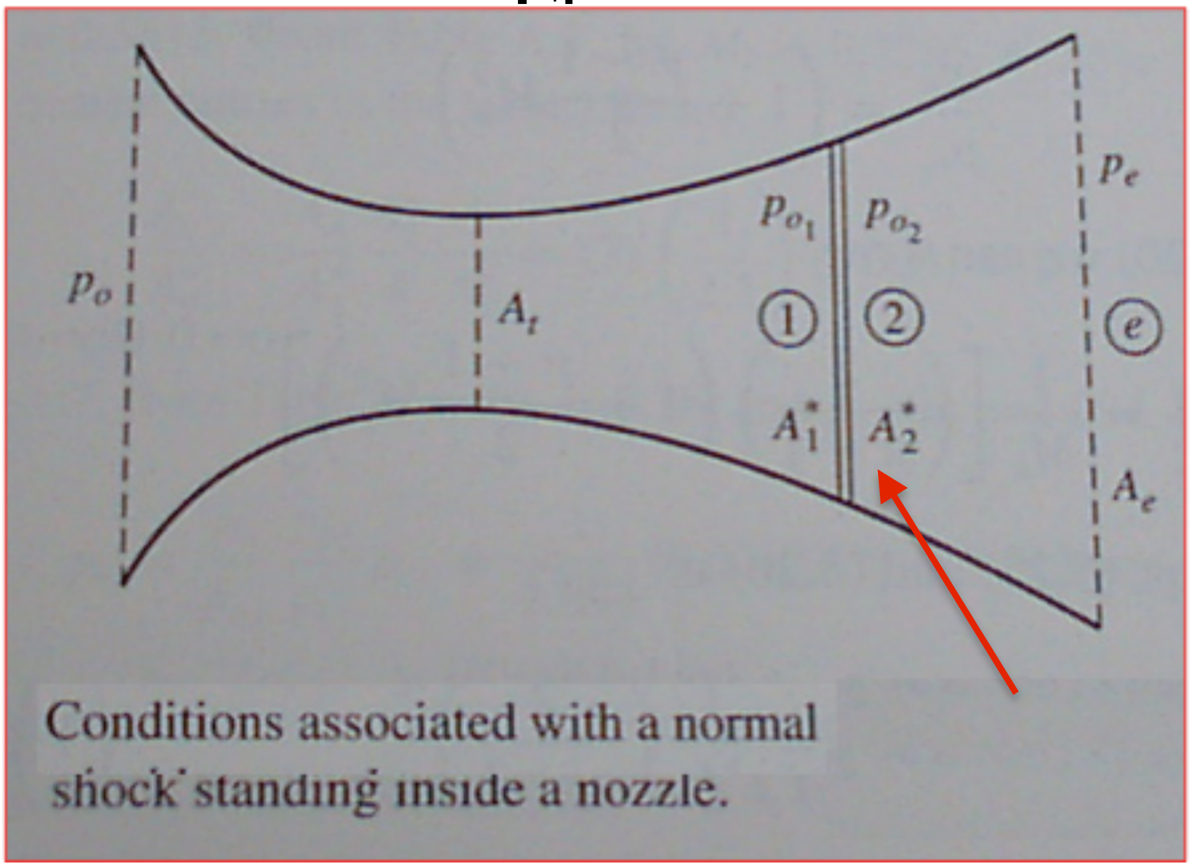
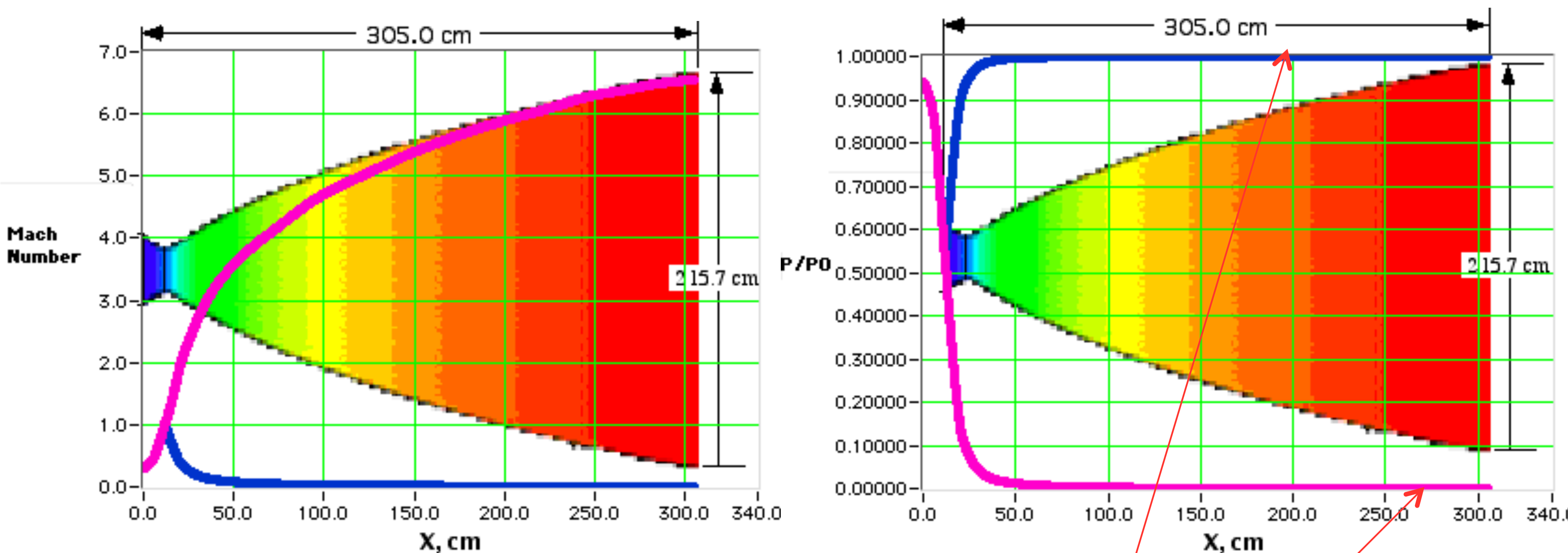
Section 5 Lecture 2: Non-Isentropic Nozzle Fl



Conditions associated with a normal shock standing inside a nozzle.

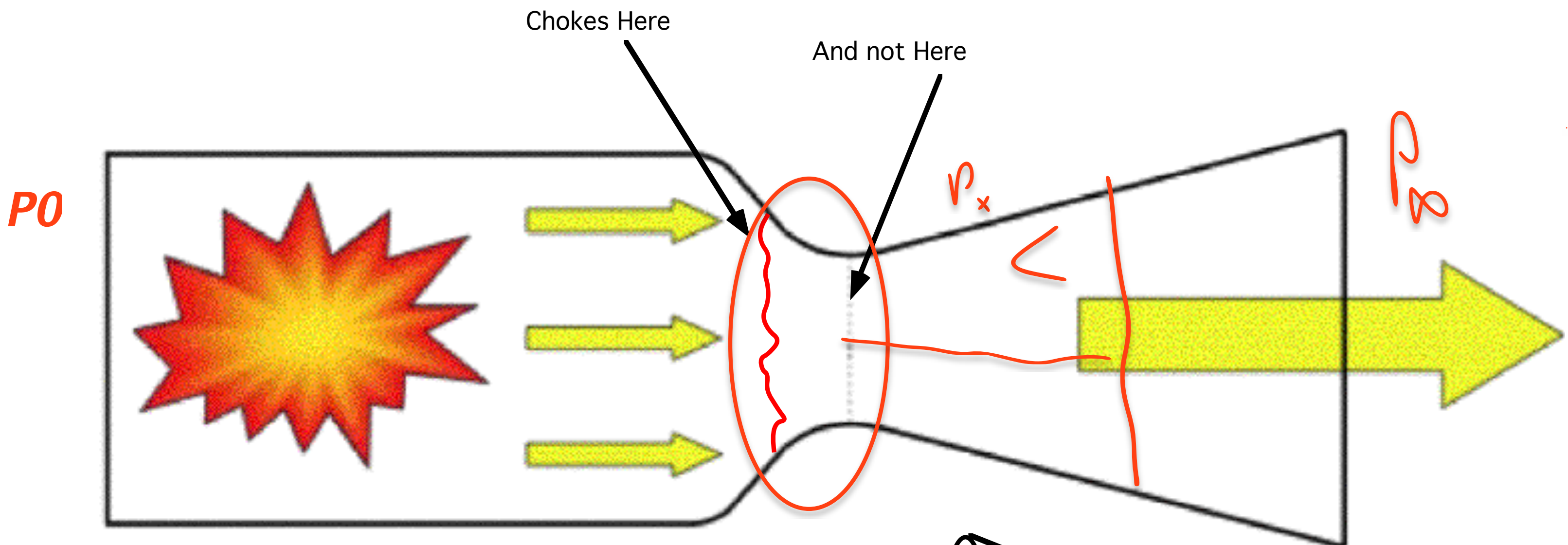
Anderson: Chapter 5 pp. 214-226

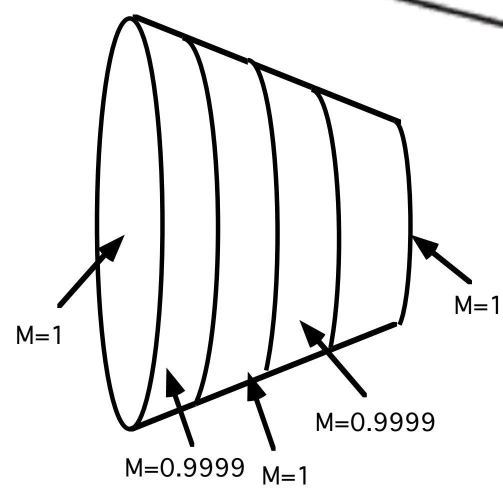
Two Distinct Solutions for Isentropic Flow in Duct with Choked Throat



• What happens when pressure ratio across nozzle is between these two options?

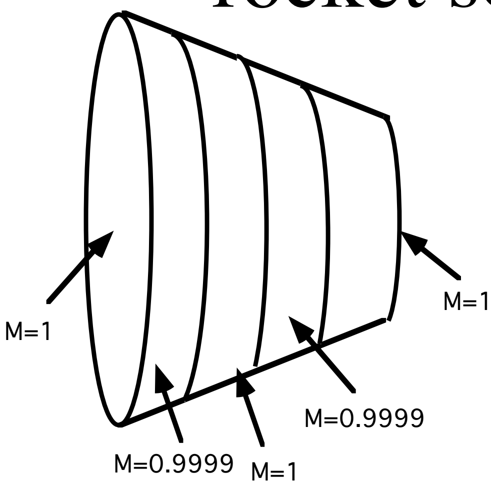
Review: What happens if we “hard start” a rocket so that we choke ahead of throat? (1)





- Flow cannot remain isentropic forms train of mini shockwaves

Review: What happens if we "hard start" a rocket so that we choke ahead of throat? (2)



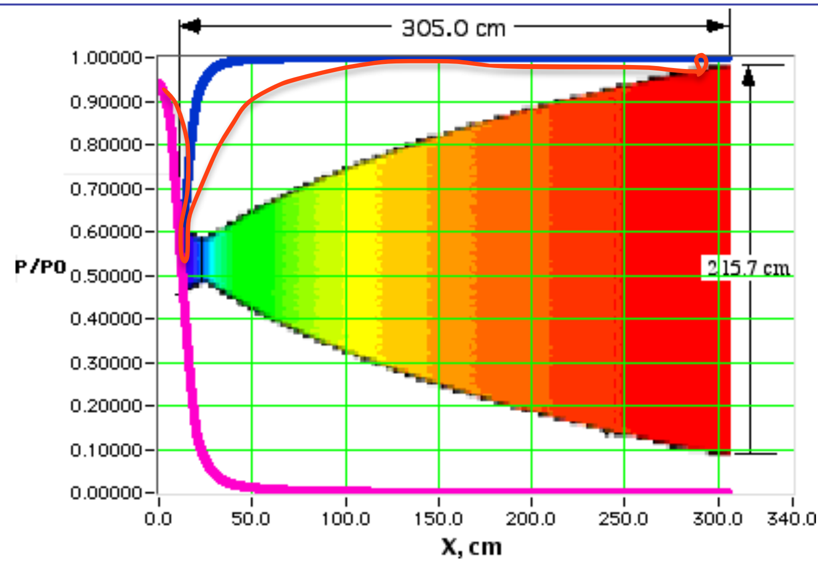
$$\dot{m} = \frac{P_0}{\sqrt{T_0}} A^* \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \rightarrow \boxed{A^* = \frac{1}{\dot{m}} \frac{\sqrt{T_0}}{P_0\sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}}$$

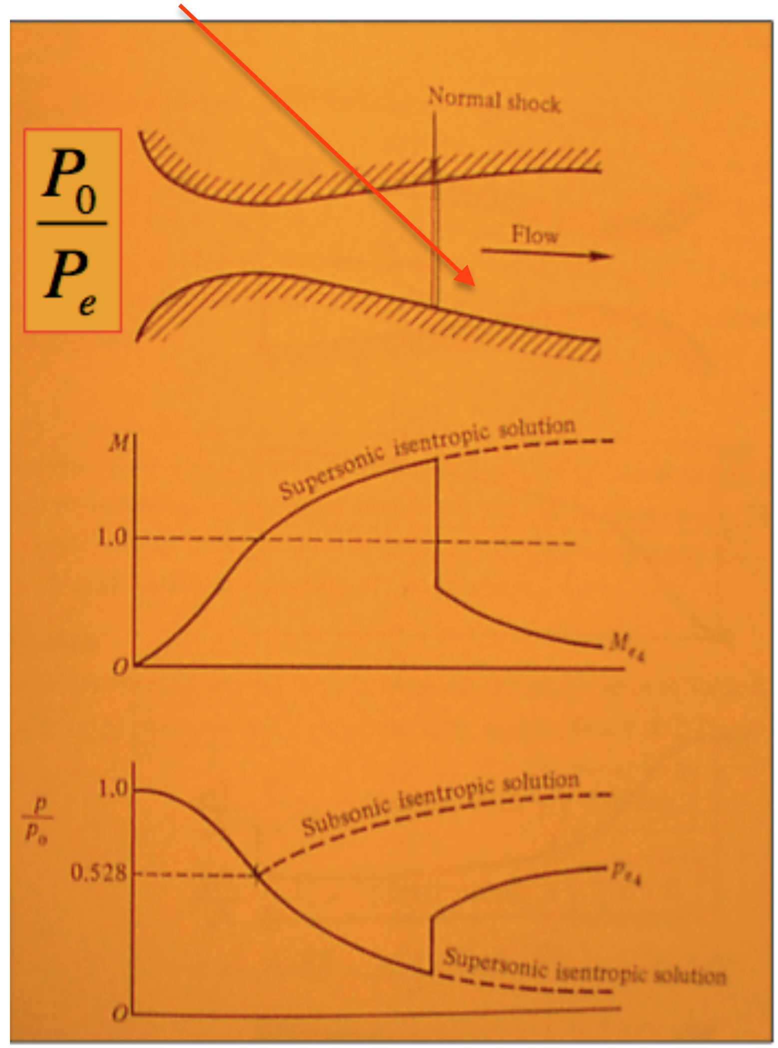
- Combustion Produces High temperature gaseous By-products
- Gases Escape Through Nozzle
- Nozzle Chokes at area A* (maximum mass flow for P_0, T_0, γ)
- Choke acts as blockage … *chamber Pressure builds up*
- As chamber Pressure Builds .. A* gets smaller
- Sonic point moves down stream
- Eventually Steady State Condition is reached with choking at smallest point in nozzle

Review: Two Distinct Solutions for Isentropic Flow with Choked Throat

- What happens when pressure ratio across nozzle is between these two options? $\left\{P_0, A^*\right\} \neq const$

- Flow in Nozzle is no longer isentropic?
- Normal shockwave appears in divergent section of Nozzle



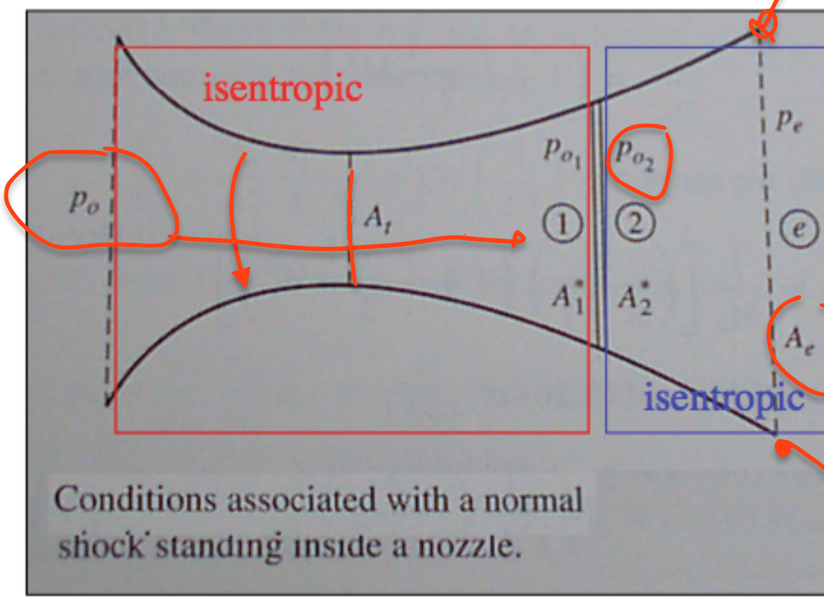


Non-Isentropic Nozzle
Solve for Exit Mach Number

- Massflow at Choked Throat (Homework 3)

$$\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\frac{P_0}{\sqrt{T_0}} \rightarrow$$

$$\frac{\dot{m}}{\sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}} = A^*_1\frac{P_{0_1}}{\sqrt{T_0}} = A^*_2\frac{P_{0_2}}{\sqrt{T_0}}$$



- **Adiabatic Shock Wave …** $T_{0_1} = T_{0_2}$ --->

$$\boxed{P_{0_1}A^*_1 = P_{0_2}A^*_2 = P_{0_e}A^*_e}$$

Non-Isentropic Nozzle
Solve for Exit Mach Number (cont'd)

- Since $\boxed{P_{0_1} A^*_1 = P_{0_2} A^*_2 = P_{0_e} A^*_e}$ .... then

$$\frac{P_{0_e}}{p_e} = \frac{P_{0_1}}{p_e} \cdot \frac{P_{0_e}}{P_{0_1}} \rightarrow \frac{P_{0_e}}{P_{0_1}} = \frac{A_e^*}{A_1^*} \rightarrow \boxed{\frac{P_{0_e}}{p_e} = \frac{P_{0_1}}{p_e} \cdot \frac{A_1^*}{A_e^*}}$$

$$\rightarrow \frac{P_{0_e}}{p_e} = \frac{P_{0_1}}{p_e} \cdot \left(\frac{A_1^*}{A_e} \cdot \frac{A_e}{A_e^*} \right) \rightarrow \boxed{\frac{p_e}{P_{0_e}} \cdot \frac{A_e}{A_e^*} = \frac{p_e}{P_{0_1}} \cdot \frac{A_e}{A_1^*}}$$

$$but, \ldots \frac{p_e}{P_{0_1}} \cdot \frac{A_e}{A_1^*} = \frac{p_e \cdot A_e}{P_{0_1} A_1^*} = \frac{p_e \cdot A_e}{P_{0e} A_e^*} = \frac{p_e}{P_{0e}} \cdot \frac{A_e}{A_e^*}$$

Non-Isentropic Nozzle
Solve for Exit Mach Number (cont'd)

- Since $\frac{p_e}{P_{0_1}} \cdot \frac{A_e}{A_1^*} = \frac{p_e}{P_{0e}} \cdot \frac{A_e}{A_e^*}$ .... then

from isentropic flow relationships $\rightarrow$

$$\frac{p_e}{P_{0e}} \rightarrow \frac{1}{\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{\gamma}{\gamma - 1}}}$$

$$\frac{A_e}{A_e^*} = \frac{1}{M_e}\left[\left(\frac{2}{\gamma + 1}\right)\left(1 + \frac{\gamma - 1}{2} M_e^2\right)\right]^{\frac{\gamma + 1}{2(\gamma - 1)}}$$

$$\frac{p_e}{P_{0_1}} \cdot \frac{A_e}{A_1^*} = \frac{\frac{1}{M_e}\left[\left(\frac{2}{\gamma + 1}\right)\left(1 + \frac{\gamma - 1}{2} M_e^2\right)\right]^{\frac{\gamma + 1}{2(\gamma - 1)}}}{\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{\gamma}{\gamma - 1}}}$$

Non-Isentropic Nozzle
Solve for Exit Mach Number (cont'd)

- Simplify expression

$$\frac{\frac{1}{M_e}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_e^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}{\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma}{\gamma-1}}} = \frac{1}{M_e}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma+1}{2(\gamma-1)}-1} =$$

$$\frac{1}{M_e}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma+1-2\gamma}{2(\gamma-1)}} = \frac{1}{M_e}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}\left(1+\frac{\gamma-1}{2}M_e^2\right)^{-\frac{1}{2}}$$

$$\rightarrow \boxed{\frac{p_e}{P_{0_1}}\cdot\frac{A_e}{A_1^*} = \frac{\frac{1}{M_e}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}}{\sqrt{\left(1+\frac{\gamma-1}{2}M_e^2\right)}}}$$

Non-Isentropic Nozzle
Solve for Exit Mach Number (cont'd)

- Square both sides and regroup in terms of mach number

$$\boxed{\frac{p_e}{P_{0_1}} \cdot \frac{A_e}{A_1^*} = \frac{\frac{1}{M_e}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}}{\sqrt{\left(1+\frac{\gamma-1}{2}M_e^2\right)}}} \rightarrow M_e^2\left(1+\frac{\gamma-1}{2}M_e^2\right) = \cdot\frac{\left[\frac{2}{\gamma+1}\right]^{\left(\frac{\gamma+1}{\gamma-1}\right)}}{\left(\frac{p_e}{P_{0_1}} \cdot \frac{A_e}{A_1^*}\right)^2}$$

- Collect like powers in M_e

$$M_e^4 + \left(\frac{2}{\gamma-1}\right) \cdot M_e^2 - \left(\frac{2}{\gamma-1}\right) \cdot \left[\frac{2}{\gamma+1}\right]^{\left(\frac{\gamma+1}{\gamma-1}\right)} \left(\frac{P_{0_1} \cdot A_1^*}{p_e \cdot A_e}\right)^2 = 0$$

Quadratic Expression in M_e^2

Non-Isentropic Nozzle
Solve for Exit Mach Number (cont'd)

- Quadratic Formula

$$M_e^{\,4} + \left(\frac{2}{\gamma - 1}\right) \cdot M_e^{\,2} - \left(\frac{2}{\gamma - 1}\right) \cdot \left[\frac{2}{\gamma + 1}\right]^{\left(\frac{\gamma+1}{\gamma-1}\right)} \left(\frac{P_{0_1} \cdot A_1^{\,*}}{p_e \cdot A_e}\right)^2 = 0$$

$$M_e^{\,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\dfrac{2}{(\gamma-1)} \pm \sqrt{\left(\dfrac{2}{(\gamma-1)}\right)^2 + 4\dfrac{2}{(\gamma-1)}\left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left(\dfrac{P_0}{p_e}\dfrac{A^*}{A_e}\right)^2}}{2} =$$

$$-\frac{1}{(\gamma-1)} \pm \sqrt{\left(\frac{1}{(\gamma-1)}\right)^2 + \frac{2}{(\gamma-1)}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left(\frac{P_0}{p_e}\frac{A^*}{A_e}\right)^2}$$

Non-Isentropic Nozzle
Solve for Exit Mach Number (concluded)

- Complex Root excluded

$$M_e = \sqrt{-\frac{1}{(\gamma-1)} + \sqrt{\left(\frac{1}{(\gamma-1)}\right)^2 + \frac{2}{(\gamma-1)}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left(\frac{P_{0_1}\cdot A_1^*}{p_e\cdot A_e}\right)^2}}$$

$$\boxed{\left\{P_{0_1}\cdot A_1^*, p_e, A_e\right\} \to \text{"}known\ quantities\text{"}}$$

- Explicit Solution for ***exit Mach number***

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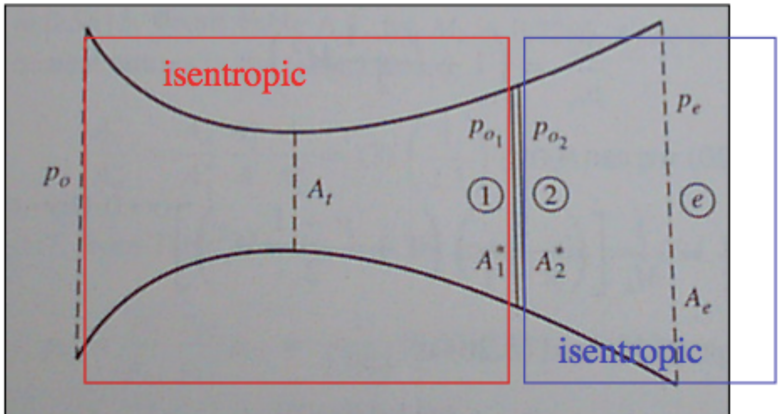
Non-Isentropic Nozzle Solve for Total Pressure Ratio Across Shock Wave



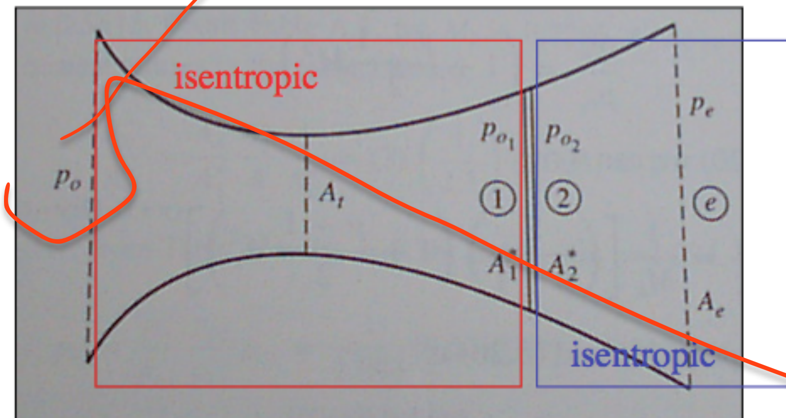
Conditions associated with a normal shock standing inside a nozzle.

- Downstream of shock wave flow is isentropic

$$P_{0_2} = P_{0_e} = p_e\left[1 + \frac{(\gamma - 1)}{2} M_{e^2}\right]^{\frac{\gamma}{\gamma-1}} \rightarrow$$

$$\frac{P_{0_2}}{P_{0_1}} = \frac{p_e}{P_{0_1}}\left[1 + \frac{(\gamma - 1)}{2} M_{e^2}\right]^{\frac{\gamma}{\gamma-1}}$$

Non-Isentropic Nozzle Solve for Mach Number Ahead of Shock Wave



Conditions associated with a normal shock standing inside a nozzle.

- REQUIRES ITERATIVE NUMERICAL SOLVER

- USE Newton's method

$$\frac{P_{02}}{P_{01}} = \left[\frac{2}{(\gamma+1)\left(\gamma M_1^{\,2} - \frac{(\gamma-1)}{2}\right)}\right]^{\frac{1}{\gamma-1}} \left[\left(\frac{\left[\frac{(\gamma+1)}{2} M_1\right]^2}{\left(1+\frac{\gamma-1}{2} M_1^{\,2}\right)}\right)\right]^{\left(\frac{\gamma}{\gamma-1}\right)}$$

Non-Isentropic Nozzle Solve for Mach Number Ahead of Shock Wave (cont'd)

- Solve for roots of $G(M_1) = 0$

$$G(M_1) = \frac{2}{(\gamma+1)\left(\gamma M_1^2 - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}} \left(\frac{\left[\frac{(\gamma+1)}{2} M_1\right]^2}{\left(1 + \frac{\gamma-1}{2} M_1^2\right)} \right)^{\left(\frac{\gamma}{\gamma-1}\right)} - \left(\frac{P_{0_2}}{P_{0_1}}\right)$$

- Expand in Taylor's series

$$G(M_1) = G(M_{1(j)}) + \left(\frac{\partial G}{\partial M_1}\right)_{(j)} \left(M_1 - M_{1(j)}\right) + O(M_1^2) + ...$$

Non-Isentropic Nozzle Solve for Mach Number Ahead of Shock Wave (cont'd)

$$\boxed{G(M_1) = 0} \rightarrow G(M_{1(j)}) + \left(\frac{\partial G}{\partial M_1}\right)_{(j)} \left(M_1 - M_{1(j)}\right) + O(M_1^{\,2}) + \ldots.$$

$$M_1 = M_{1(j)} - \frac{G(M_{1(j)}) + O(M_1^{\,2})}{\left(\frac{\partial G}{\partial M_1}\right)_{(j)}}$$

- Truncate after first order …

$$M_{1(j+1)} = M_{1(j)} - \frac{G(M_{1(j)})}{\left(\frac{\partial G}{\partial M_1}\right)_{(j)}}$$

Non-Isentropic Nozzle Solve for Mach Number Ahead of Shock Wave (cont'd)

- Derivative term

$$\frac{\partial G(M)}{\partial M} = \frac{\partial}{\partial M}\left[\frac{2}{(\gamma+1)\left(\gamma M^2 - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}}\left(\frac{\left[\frac{(\gamma+1)}{2}M\right]^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}\right] =$$

$$-\left\{\frac{2^{\left(3-\frac{2\gamma}{\gamma-1}\right)}\gamma\left(M^2-1\right)^2\left(\frac{\left[(\gamma+1)M\right]^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}\left[\frac{1}{2}+\gamma\left(M^2-\frac{1}{2}\right)\right]^{\left(\frac{-1}{\gamma-1}\right)}}{(\gamma+1)M\left(2+M^2(\gamma-1)\right)\left[1+\gamma\left(2M^2-1\right)\right]}\right\}$$

Non-Isentropic Nozzle Solve for Mach Number Ahead of Shock Wave (concluded)

$$M_{1(j+1)} = M_{1(j)} - \frac{G(M_{1(j)})}{\left(\frac{\partial G}{\partial M_1}\right)_{(j)}} \rightarrow$$

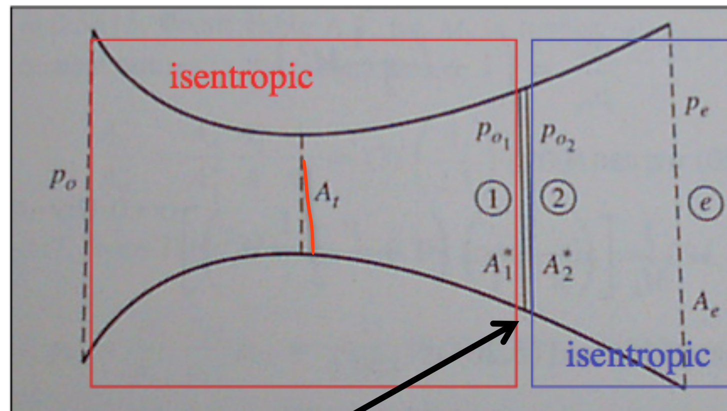
$$G(M_{1(j)}) = \frac{2}{(\gamma+1)\left(\gamma M_{1(j)}^{2} - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}}\left(\frac{\left[\frac{(\gamma+1)}{2}M_{1(j)}\right]^{2}}{\left(1+\frac{\gamma-1}{2}M_{1(j)}^{2}\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)} - \left(\frac{P_{0_2}}{P_{0_1}}\right)$$

$$\left(\frac{\partial G}{\partial M_1}\right)_{(j)} = -\frac{2^{\left(3-\frac{2\gamma}{\gamma-1}\right)}\gamma\left(M_{1(j)}^{2}-1\right)^{2}\left(\frac{\left[(\gamma+1)M_{1(j)}\right]^{2}}{\left(1+\frac{\gamma-1}{2}M^{2}\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}\left[\frac{1}{2}+\gamma\left(M_{1(j)}^{2}-\frac{1}{2}\right)\right]^{\left(\frac{-1}{\gamma-1}\right)}}{(\gamma+1)M_{1(j)}\left(2+M_{1(j)}^{2}(\gamma-1)\right)\left[1+\gamma\left(2M_{1(j)}^{2}-1\right)\right]}$$

- Similar to - M(A/A*) - algorithm … given starting mach number Iterate to convergence

Non-Isentropic Nozzle Solve for A/A* ahead of shock wave

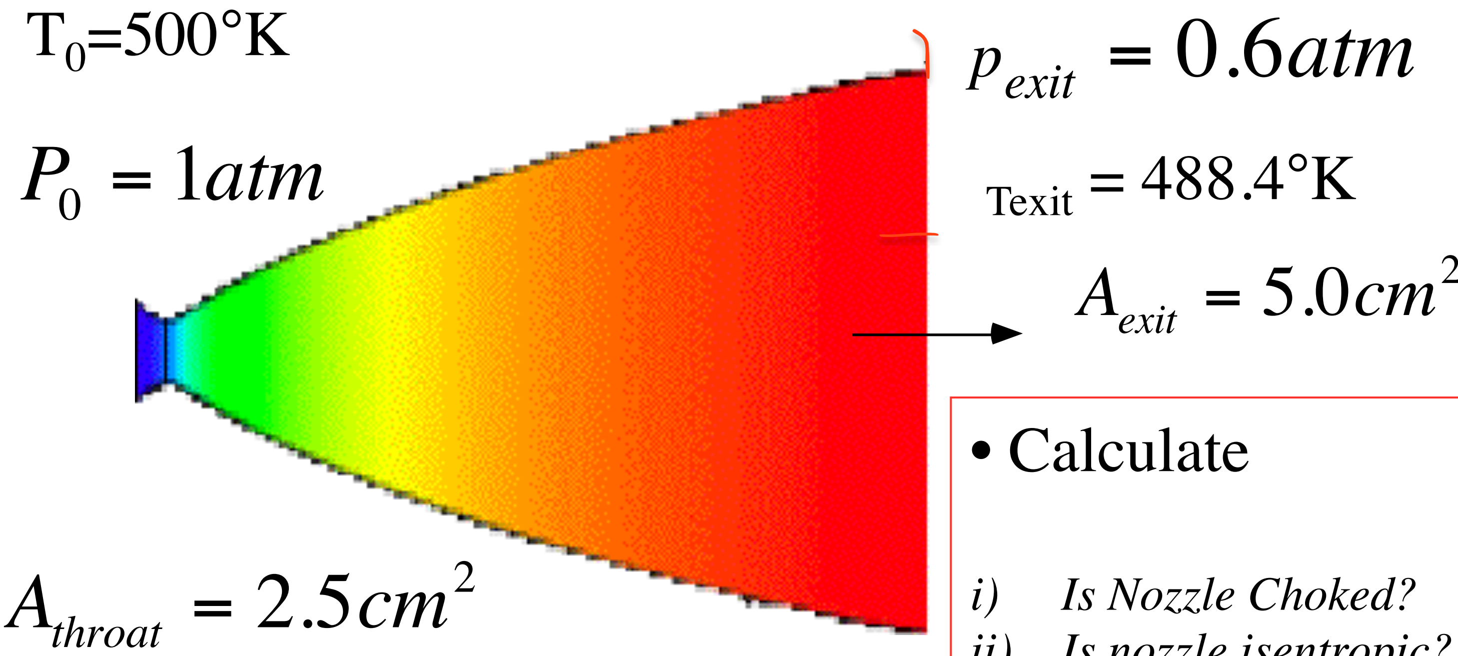
$$\frac{A_1}{A^*} = \frac{1}{M_1}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M_1^{\,2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$



Conditions associated with a normal shock standing inside a nozzle.

- Since throat is choked … represents
Point in nozzle where shock wave stands

Example 1



- Calculate

i) Is Nozzle Choked?
ii) Is nozzle isentropic?
iii) Mach number at exit plane, M_e
iv) Total Pressure at Exit, $P0_e$
v) Cross sectional area of Nozzle at Normal shock location

- Assume γ =1.2
(i.e. tables are not valid)

$$R_g = 287.056_{J/kg-K}$$

Example 1 (cont'd)

Is Nozzle Choked?

- Compute Exit Mach Number

$$M_e = \sqrt{\frac{2}{(\gamma - 1)}\left(\frac{T_0}{T_e} - 1\right)} = \left(\frac{2}{1.2 - 1}\left(\frac{500}{488.4} - 1\right)\right)^{0.5} = 0.48759$$

- Compute Exit Velocity

$$V_e = M_e\sqrt{\gamma R_g T_e} = 0.48759\sqrt{1.2(287.056)(488.4)} = 200.0 \frac{m}{\sec}$$

Example 1 (cont'd)

Is Nozzle Choked?

- Compute Exit Density

$$\rho_e = \frac{p_e}{R_g T_e} = \frac{0.6 \times 101325}{287.056 \times 488.4} = 0.4336 \, \frac{kg}{m^3}$$

- Compute mass flow

$$\dot{m} = \rho_e \cdot A_e \cdot V_e = \quad 0.4336\left(\frac{5}{100^2}\right)200$$

$$= 0.04336_{kg/\sec}$$

Example 1 (cont'd)

Is Nozzle Choked?

- Compute Mass flow required to choke Nozzle

$$\dot{m} = \frac{P_0}{\sqrt{T_0}} A^* \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} =$$

$$\frac{1 \cdot 101325}{500^{0.5}}\left(\frac{2.5}{100^2}\right)\left(\frac{1.2}{287.056}\left(\frac{2}{1.2+1}\right)^{\frac{1.2+1}{1.2-1}}\right)^{0.5} = 0.04336_{kg/\sec}$$

Nozzle is Choked!

Example 1 (cont'd)

Is Nozzle Isentropic?

- p_e=0.6 atm, M_e=0.48759

- Compute P_{0_e}

$$P0_2 = p_e\left[1+\frac{(\gamma-1)}{2}(M_e)^2\right]^{\frac{\gamma}{\gamma-1}} =$$

$$0.6\left(1+\frac{1.2-1}{2}(0.487592^2)\right)^{\left(\frac{1.2}{(1.2-1)}\right)}$$ =0.691 atm < 1 atm

Nozzle is Not Isentropic!

Example 1 (cont'd)

- Find Mach Number ahead of Normal Shock
- Compute $P_{0_1}/P_{0_e} = 1/0.691 = 1.4475$
- from Iterative solver (or trial and error)

$$\boxed{\frac{P_{02}}{P_{01}} = \frac{2}{(\gamma+1)\left(\gamma M_1^2 - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}}\left(\frac{\left[\frac{(\gamma+1)}{2}M_1\right]^2}{\left(1+\frac{\gamma-1}{2}M_1^2\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}$$

$M_1 = 1.9746$

Example 1 (cont'd)

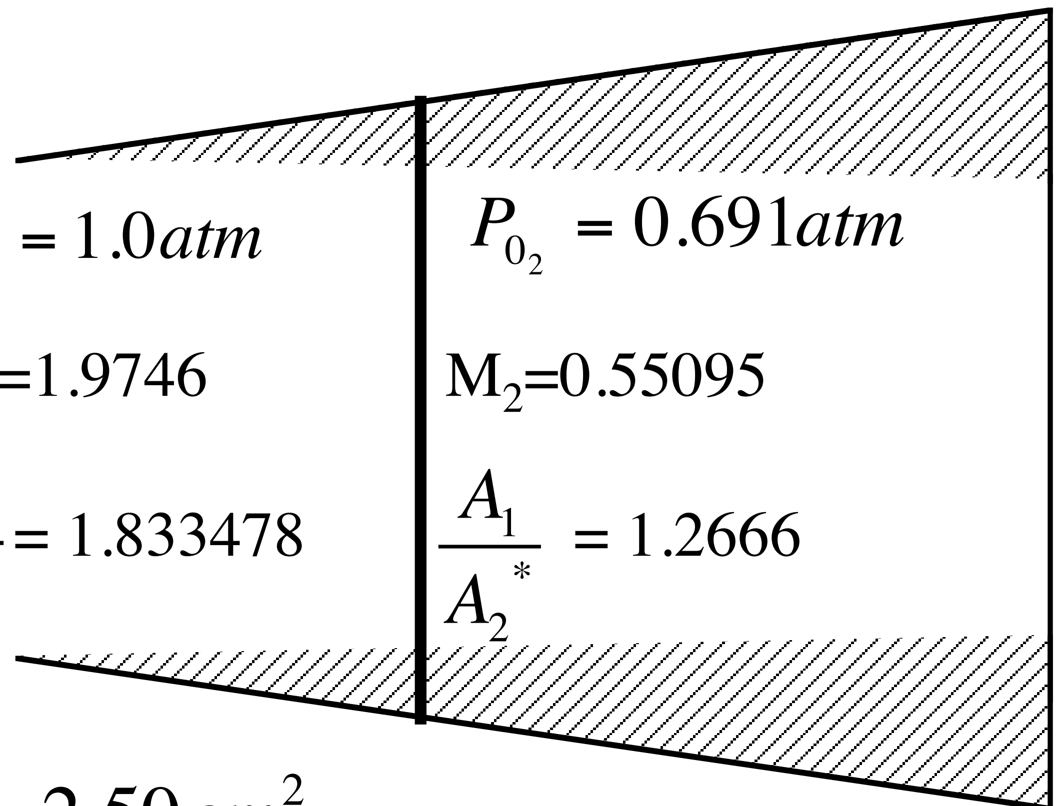
- Compute A/A*

$$\frac{A_1}{A^*} = \left[\frac{1}{M_1} \left[\left(\frac{2}{\gamma + 1} \right) \left(1 + \frac{(\gamma - 1)}{2} M_1^{\ 2} \right) \right]^{\frac{\gamma + 1}{2(\gamma - 1)}} \right] =$$

$$\frac{1}{1.974594} \left(\left(\frac{2}{1.2 + 1} \right) \left(1 + \frac{1.2 - 1}{2} 1.974594^2 \right) \right)^{\frac{1.2 + 1}{2(1.2 - 1)}} = 1.833478$$

$$A_1 = 1.833478 \times 2.5 = 4.5837 \text{ cm}^2$$

Example 1 (cont'd)



$$P_{0_1} = 1.0 atm$$

$$M_1 = 1.9746$$

$$\frac{A_1}{A_1^*} = 1.833478$$

$$A_1^* = 2.50 cm^2$$

$$P_{0_2} = 0.691 atm$$

$$M_2 = 0.55095$$

$$\frac{A_1}{A_2^*} = 1.2666$$

$$A_2^* = 3.619 cm^2$$

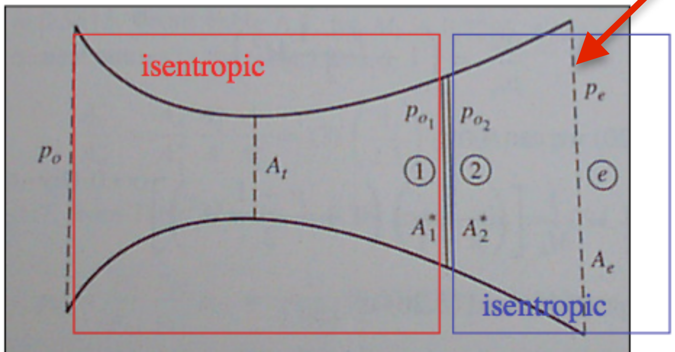
Example 2: Nozzle Flow With Normal Shockwave in Divergent Section

- P_e=204 KPa, P_0=2.4 Mpa, T_0=3500°K ,

γ =1.2, A_t=0.04714 m^2, A_e=3.6542 m^2, MW=16

i) *Is Throat Choked?*
ii) *Is Nozzle Isentropic?*
iii) *Where is Normal Shockwave? (if any)*

Ae/A*=77.5



Conditions associated with a normal shock standing inside a nozzle.

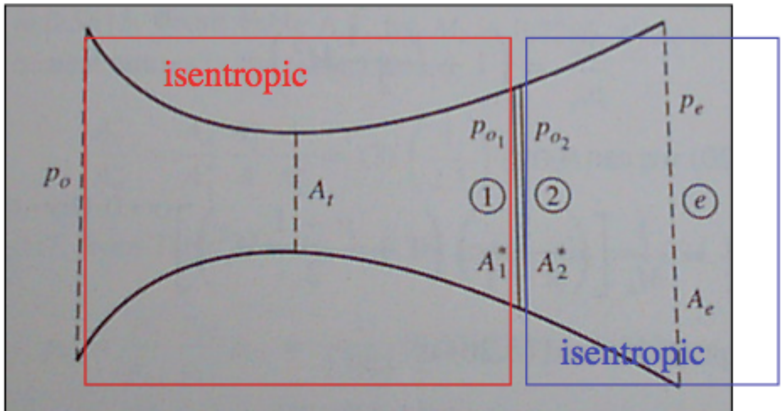
Example 2: Nozzle Flow With Normal Shockwave in Divergent Section (cont'd)

- Check to see if throat is choked

--> If throat is not choked then $P0_e = P0$

--> Compute Mach from P_o/p_e

$$M = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{P_0}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} =$$



Conditions associated with a normal shock standing inside a nozzle.

$$\left(\left(\frac{2}{1.2-1}\right)\left(\left(\frac{2.4\cdot 10^3}{204}\right)^{\frac{1.2-1}{1.2}} - 1\right)\right)^{0.5} = 2.2541$$

- Supersonic Flow

… Nozzle is Choked

Example 2: Nozzle Flow With Normal Shockwave in Divergent Section (cont'd)

- Compute Pressure Ratio required for Isentropic Flow:

$$A_t = A^* \rightarrow \frac{A_e}{A^*} = \frac{3.6542}{0.04714} = 77.52 = \left[\frac{1}{M_e}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M_e^{\,2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right] \rightarrow$$

$$(M_e)_{isentropic} = 4.7068$$

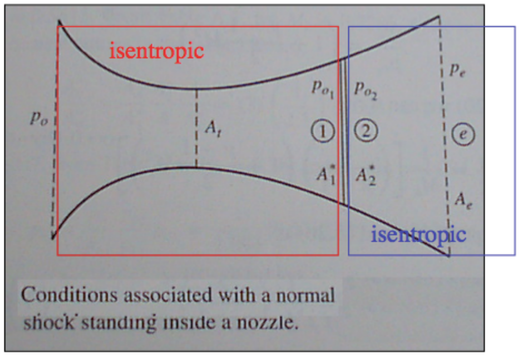
- $$\left(\frac{P_0}{p_e}\right)_{isentropic} = \left[1+\frac{(\gamma-1)}{2}(M_e)_{isentropic}^{\;\;2}\right]^{\frac{\gamma}{\gamma-1}} = \left(\left(1+\frac{1.2-1}{2}(4.7068^2)\right)^{\left(\frac{1.2}{1.2-1}\right)}\right)$$

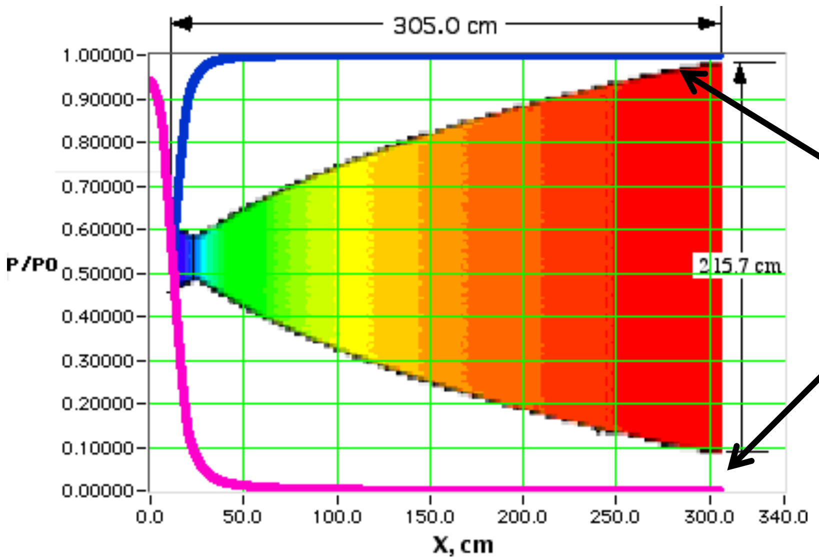
$$= 1105.1$$

Example 2: Nozzle Flow With Normal Shockwave in Divergent Section (cont'd)

- Compare Isentropic pressure ratio to Actual Pressure ratio

$$\left(\frac{P_0}{p_e}\right)_{isentropic} = 1105.1 \rightarrow \left(\frac{P_0}{p_e}\right)_{actual} = \frac{2.4 \times 1000}{204} = 11.76$$





Isentropic Solutions

- Nozzle is *not* isentropic

$$\left\{P_0, A^*\right\} \neq const$$

Example 2: Nozzle Flow With Normal Shockwave in Divergent Section (cont'd)

- Compute Exit mach Number

$$M_e = \sqrt{-\frac{1}{(\gamma - 1)} + \sqrt{\left(\frac{1}{(\gamma - 1)}\right)^2 + \frac{2}{(\gamma - 1)}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left(\frac{P_0}{p_e}\frac{A^*}{A_e}\right)^2}}$$

$$\left(\left(\left(\frac{1}{(1.2-1)}\right)^2 + \frac{2}{(1.\ -1)}\left(\left(\frac{2}{1.\ +1}\right)^{\frac{(1.2+1)}{1.2-1}}\right)\left(\frac{2.4\,\frac{1000}{204}}{77.52}\right)^2\right)^{0.5} - \frac{1}{1.2-1}\right)^{0.5}$$

$= 0.089814$

Example 2: Nozzle Flow With Normal Shockwave in Divergent Section(cont'd)

• Compute Stagnation pressure ratio across shock

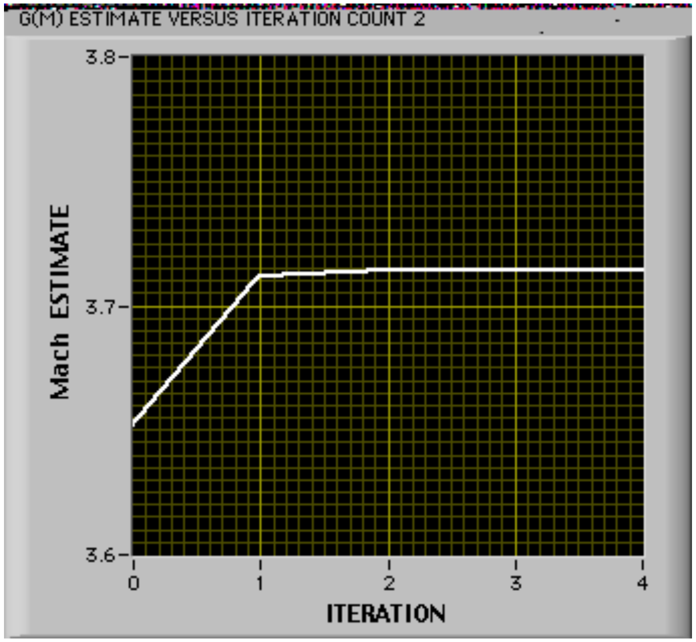
$$\frac{P_{0_2}}{P_{0_1}} = \frac{p_e}{P_{0_1}}\left[1+\frac{(\gamma-1)}{2}M_e^{\,2}\right]^{\frac{\gamma}{\gamma-1}} =$$

$$\frac{1}{11.76}\left(1+\frac{1.2-1}{2}(0.08981)^2\right)^{\left(\frac{1.2}{1.2-1}\right)} = 0.08545$$

Example 2: Nozzle Flow With Normal Shockwave in Divergent Section (cont'd)

- Find Mach Number Upstream of Shockwave (iterative solver)

$$\frac{P_{02}}{P_{01}} = \frac{2}{(\gamma+1)\left(\gamma M_1^2 - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}} \left(\frac{\left[\frac{(\gamma+1)}{2} M_1\right]^2}{\left(1+\frac{\gamma-1}{2} M_1^2\right)} \right)^{\left(\frac{\gamma}{\gamma-1}\right)} \rightarrow \boxed{M_1 = 3.7514}$$



Example 2: Nozzle Flow With Normal Shockwave in Divergent Section(cont'd)

- Find A/A* Corresponding to Upstream Mach Number

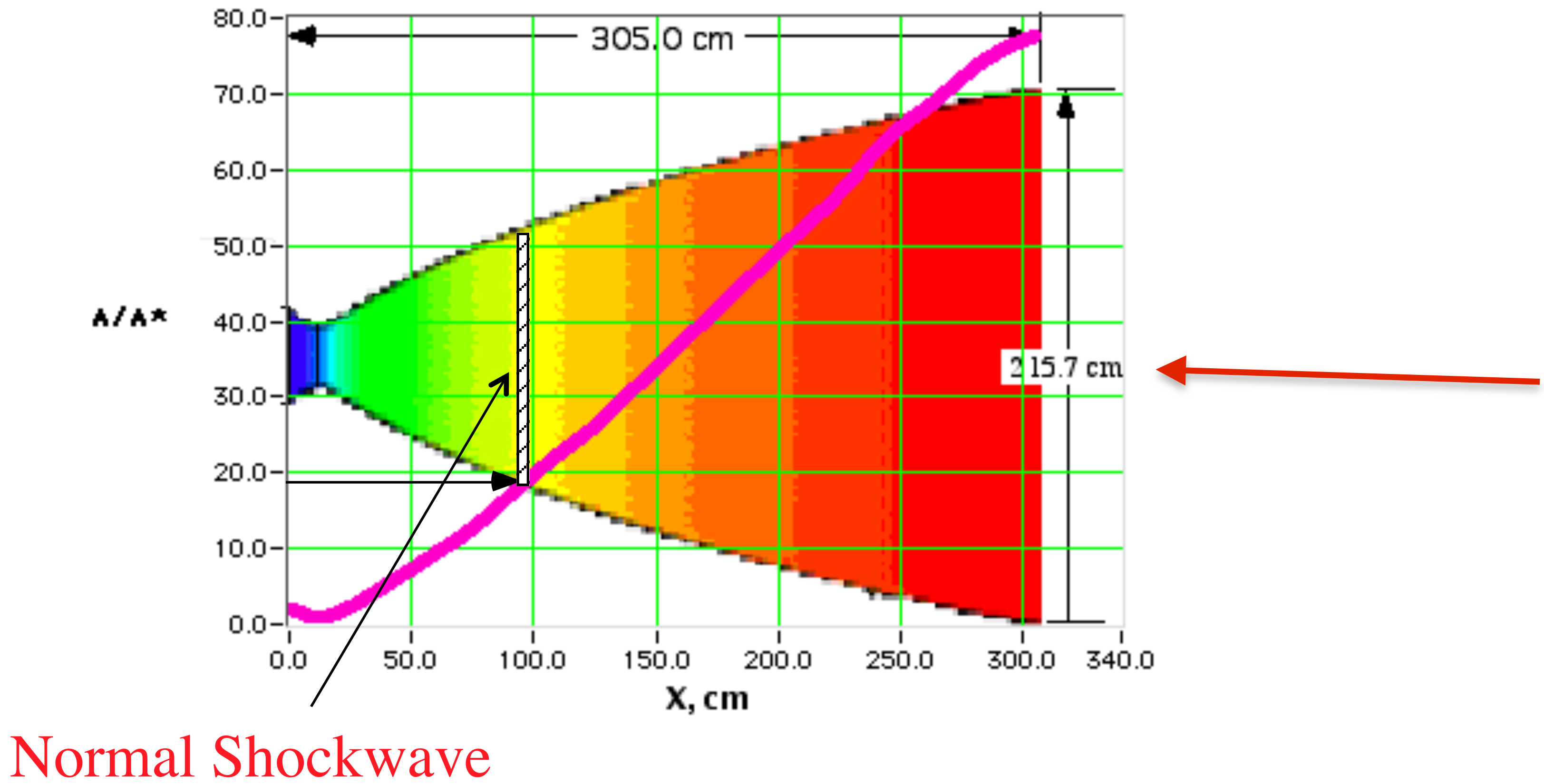
$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} =$$

$$\frac{1}{3.7514}\left(\left(\frac{2}{1.2+1}\right)\left(1+\left(\frac{1.2-1}{2}\right)(3.7514^2)\right)\right)^{\frac{1.2+1}{2(1.2-1)}}$$

$$=19.795$$

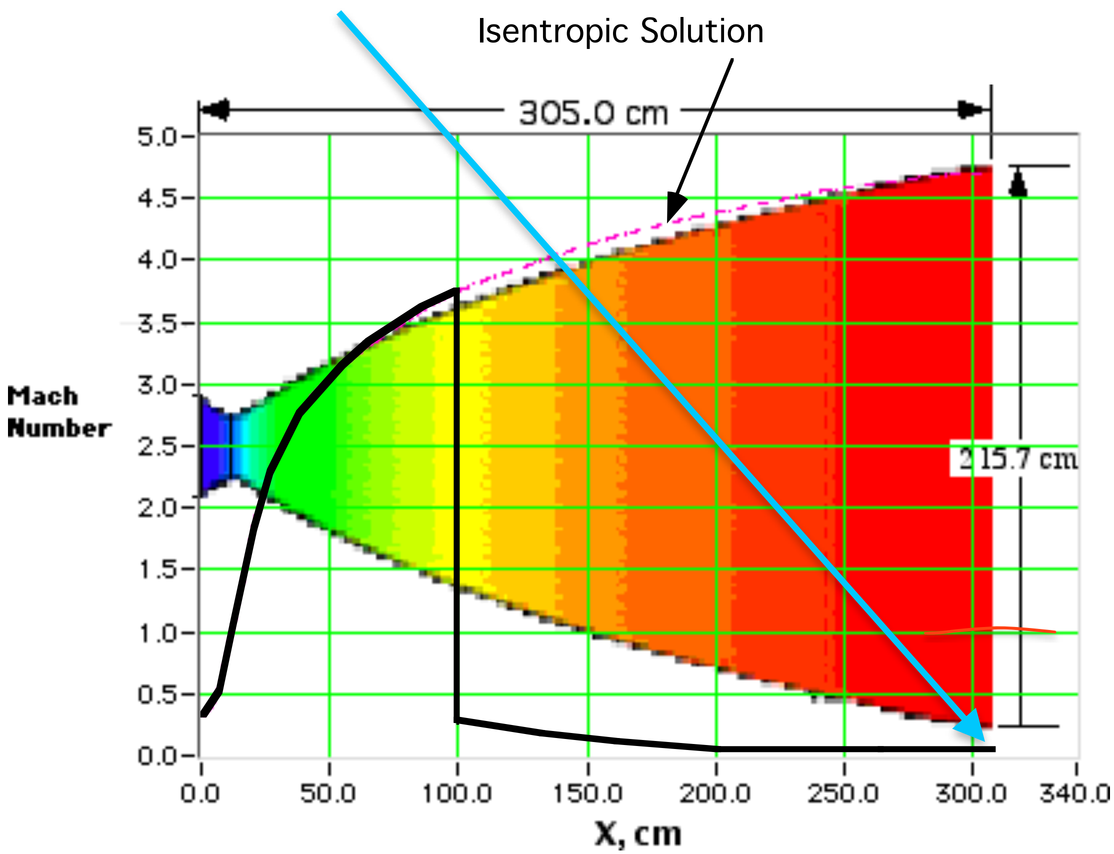
Example 2: Nozzle Flow With Normal Shockwave in Divergent Section(cont'd)

- Location of Normal Shock in Nozzle



Example 2: Nozzle Flow With Normal Shockwave in Divergent Section (concluded)

- Mach Number Distribution in Nozzle

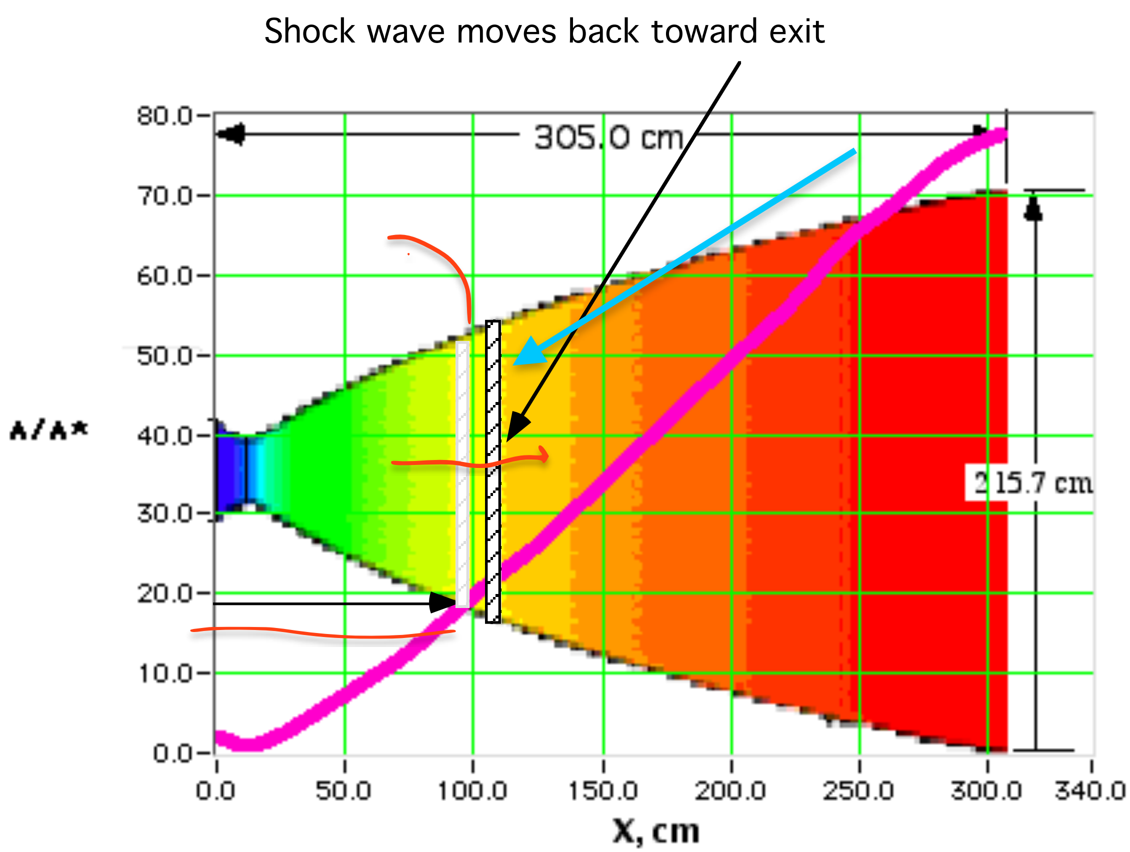


Example 3: Lower Exit Pressure

- P_e=194 KPa, P_0=2.4 Mpa, T_0=3500°K ,

γ =1.2, A_t=0.04714 m^2, A_e=3.6542 m^2, MW=16

- P_0/p_e=12.3711
- M_e = 0.094437
- P_{0_2}/P_{0_1} =0.081267
- M_1= 3.78815
- A/A^*_1=20.877



Example 4: Shock at Exit plane

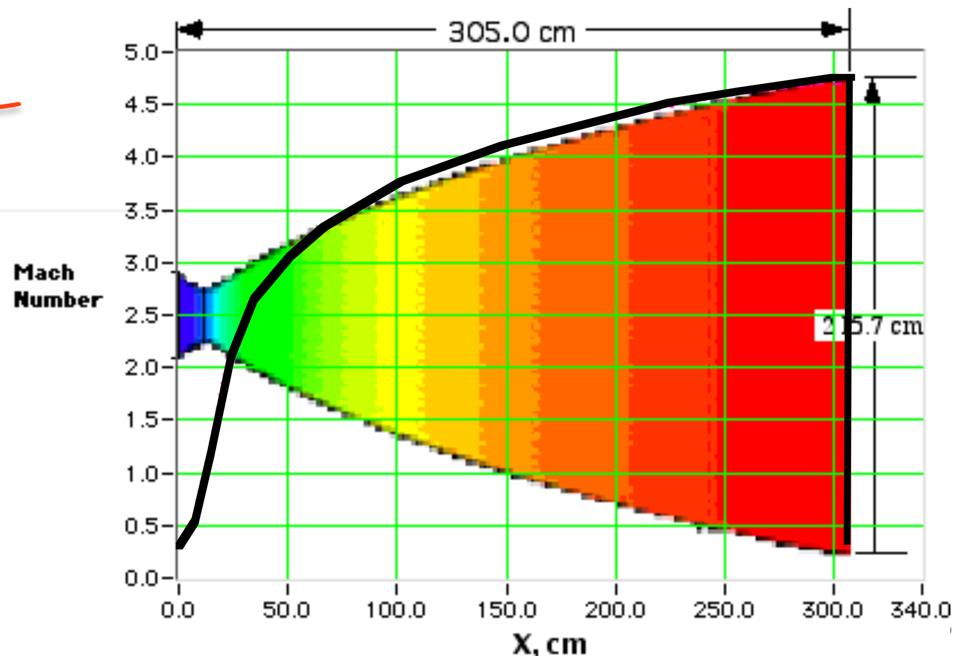
- P_e=52.29 KPa, P_0=2.4 Mpa, T_0=3500°K ,

  γ =1.2, A_t=0.04714 m^2, A_e=3.6542 m^2, MW=16

- Recall from earlier

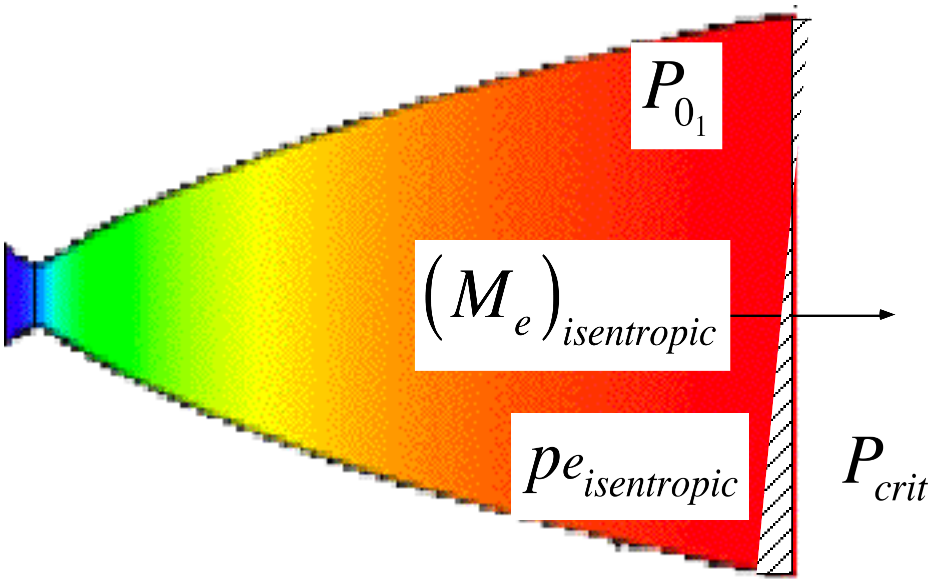
  $$\left(M_e\right)_{isentropic} = 4.7068$$

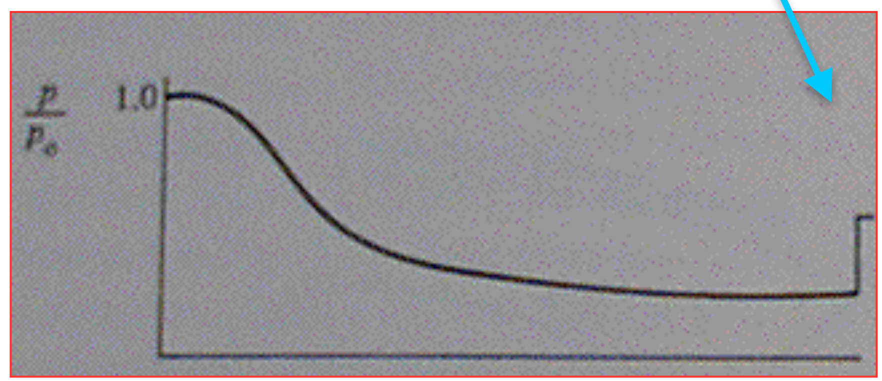
- P_0/p_e= 45.900
- M_e = 0.348417
- P_{0_2}/P_{0_1} = 0.023423
- M_1= 4.706786
- A/A^*_1= 77.52



Example 4: Shock at Exit plane (cont'd)

- $\eta_{crit} = P_{0_1}/P_{crit}$…Pressure ratio required for normal shock at exit plane … Pressure ratio across Shock wave with strength $(M_e)_{isentropic}$





Example 4: Shock at Exit plane (concluded)

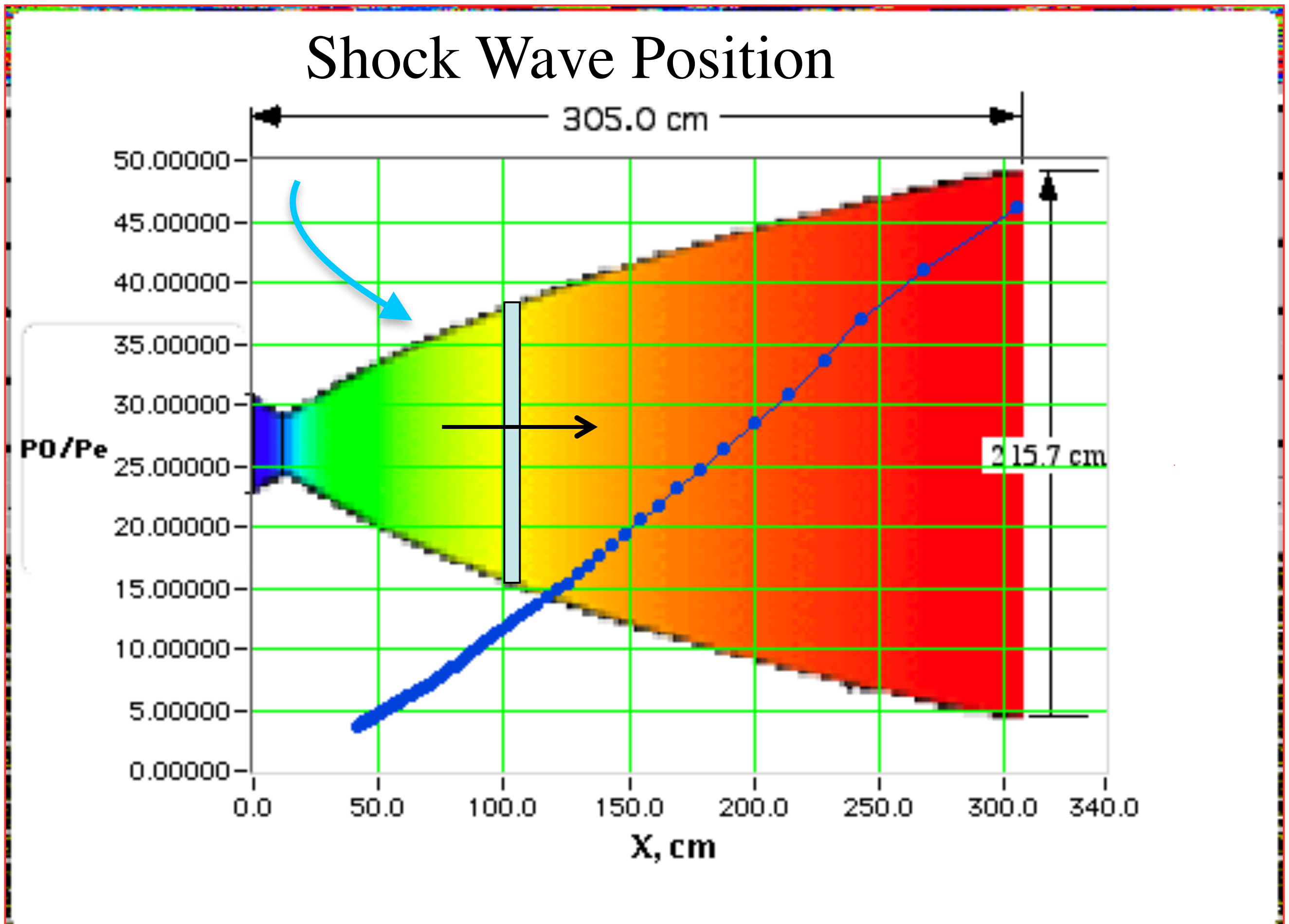
$$\left(M_e\right)_{isentropic} = 4.7068 = F\left(\frac{A_e}{A^*}\right)_{isentropic} \rightarrow pe_{isentropic} = \frac{P_{0_1}}{\left[1+\frac{(\gamma-1)}{2}\left(M_e\right)^2_{isentropic}\right]^{\frac{\gamma}{\gamma-1}}}$$

$$\eta_{crit} = \frac{P_{0_1}}{P_{crit}} = \frac{pe_{isentropic}}{P_{crit}} \times \frac{P_{0_1}}{pe_{isentropic}} = \frac{\left[1+\frac{(\gamma-1)}{2}\left(M_e\right)^2_{isentropic}\right]^{\frac{\gamma}{\gamma-1}}}{\left(1+\frac{2\gamma}{(\gamma+1)}\left(\left(M_e\right)^2_{isentropic}-1\right)\right)}$$

$$\left(\frac{\left(1+\frac{1.2-1}{2}\left(4.7068\right)^2\right)^{\frac{1.2}{(1.2-1)}}}{\left(1+\frac{2\cdot 1.2}{1.2+1}\left(\left(4.7068\right)^2-1\right)\right)}\right) = 45.90$$

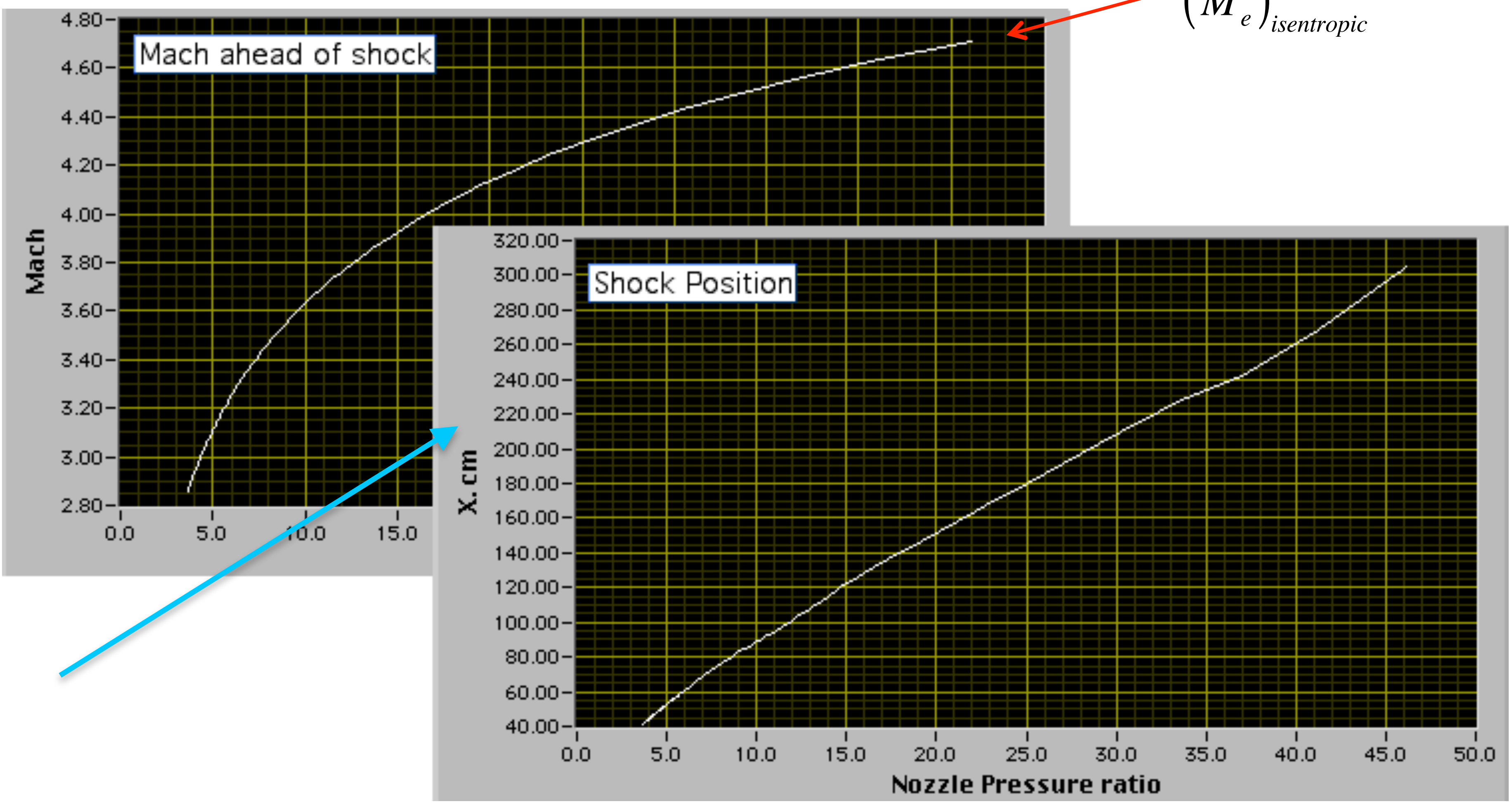
- Just what we found By Trial and error two Slides earlier

General Behavior of Shock Position versus Nozzle pressure ratio



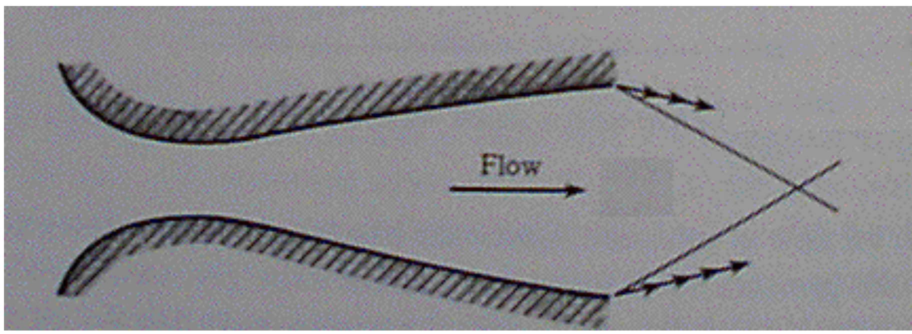
General Behavior of Shock Position versus Nozzle pressure ratio (cont'd)

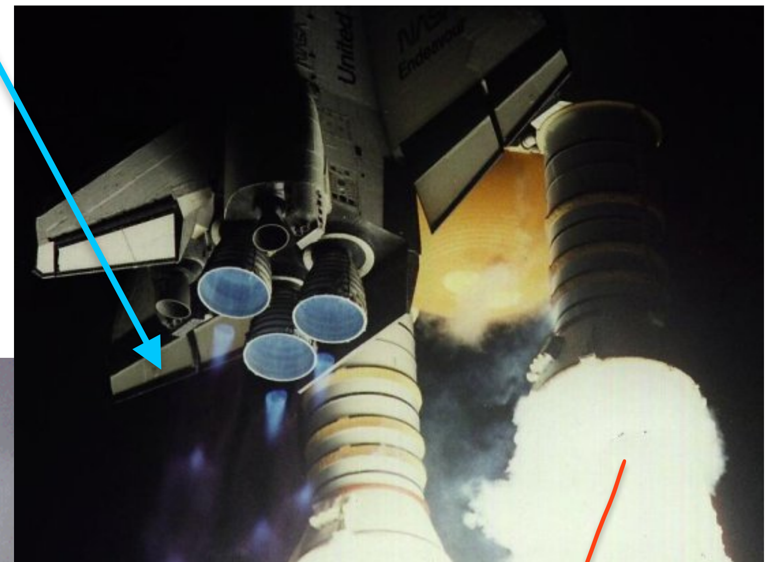
$\left(M_e\right)_{isentropic}$



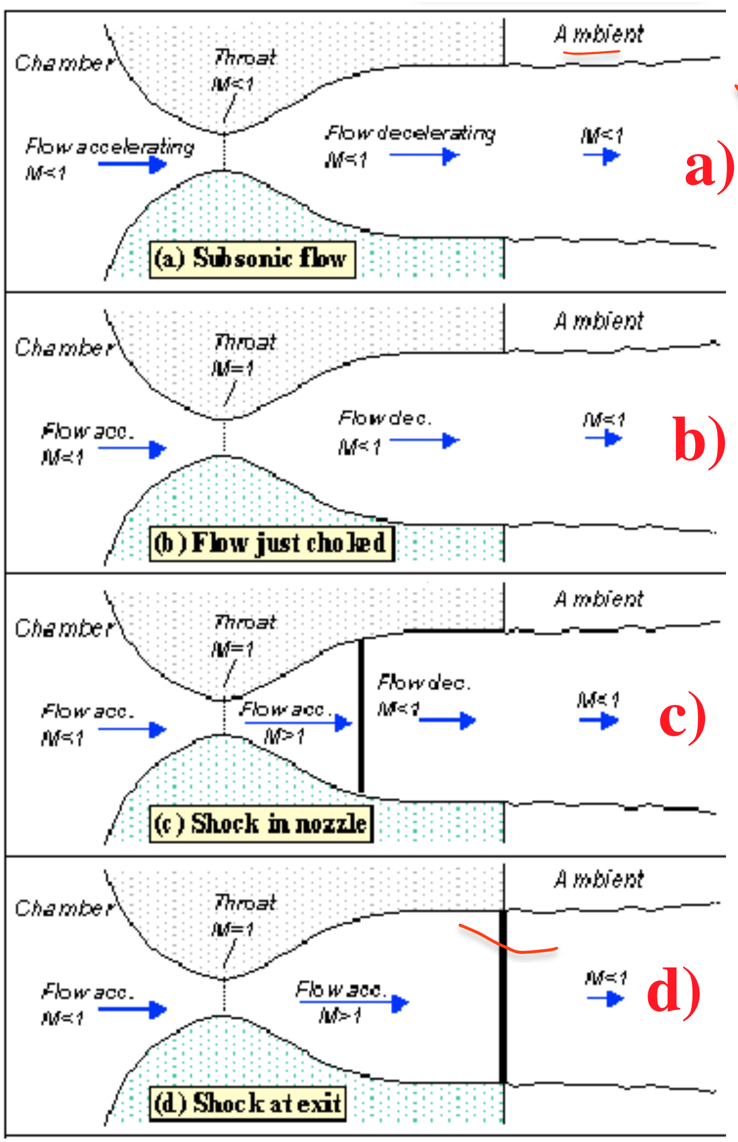
What happens when Nozzle pressure ratio Exceeds η_{crit}

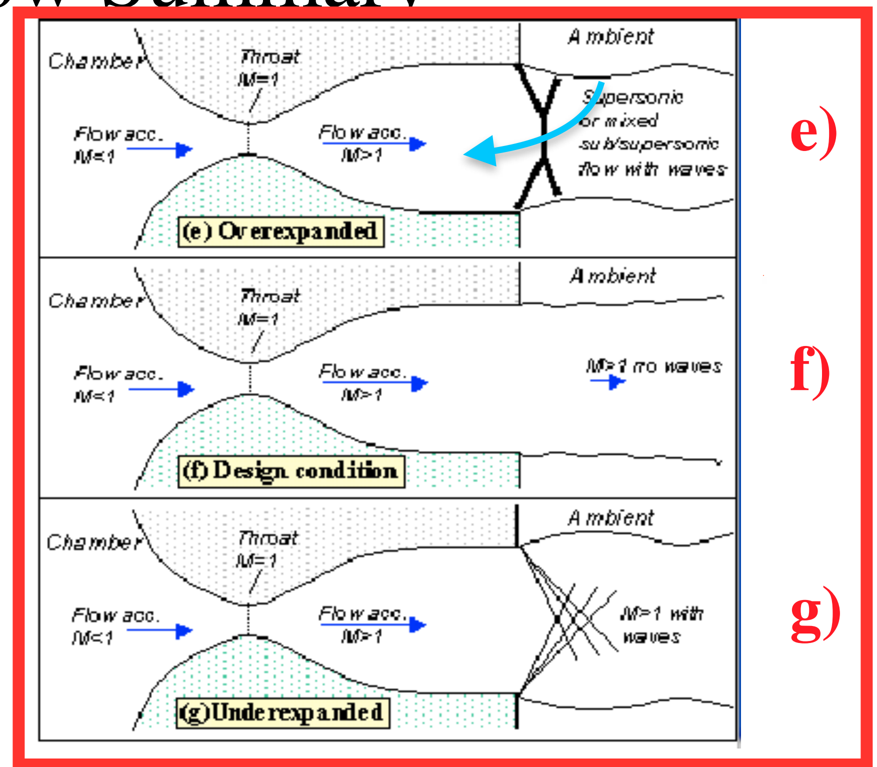
- Shock wave is pushed out of nozzle and pressure is relieved by *Oblique shock waves or expansion fan (section 6)*

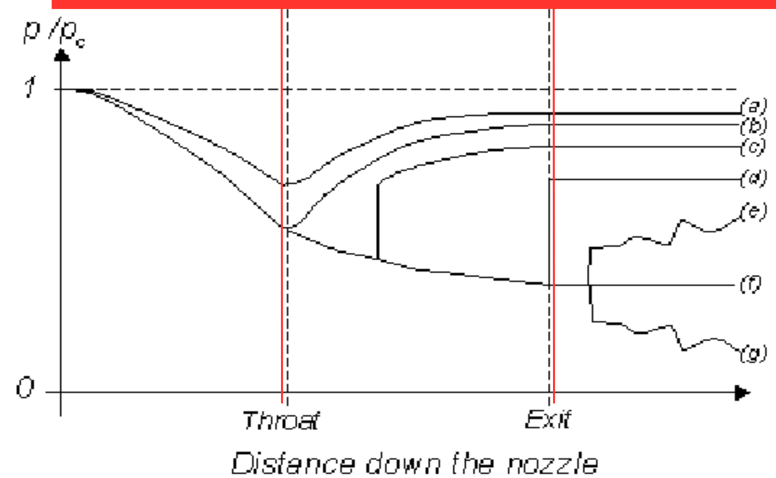


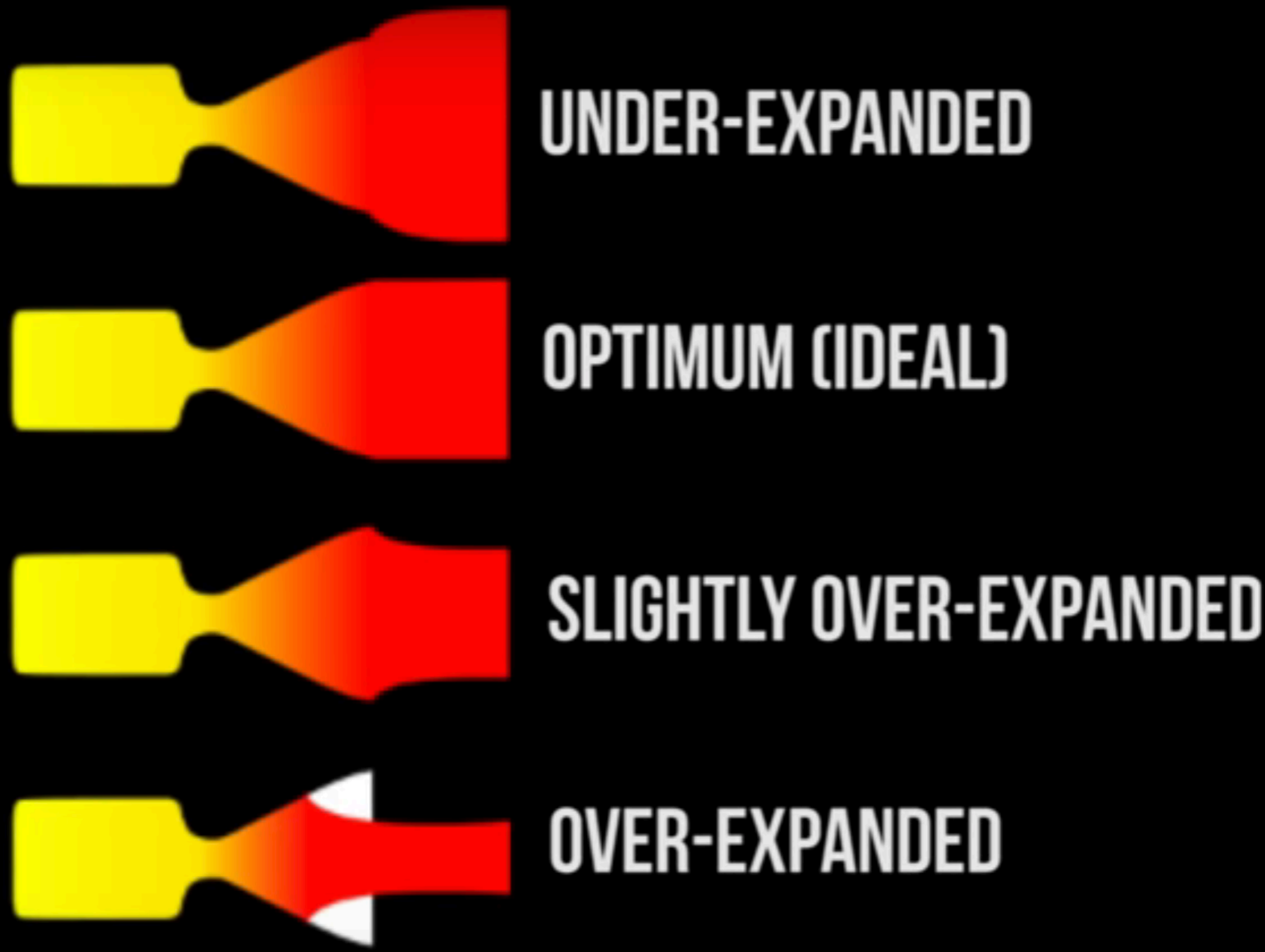


Nozzle Flow Summary



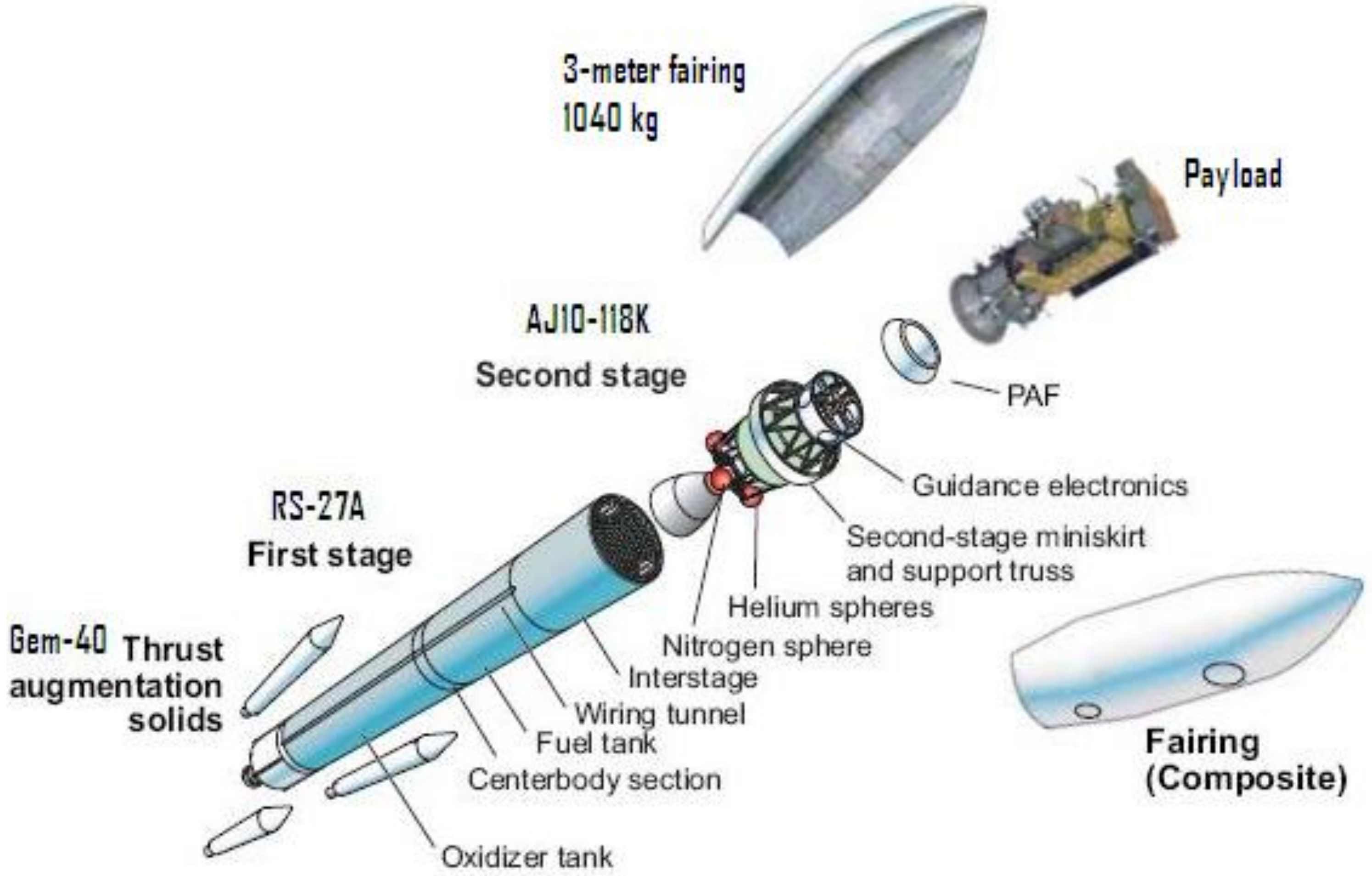





UNDER-EXPANDED
OPTIMUM (IDEAL)
SLIGHTLY OVER-EXPANDED
OVER-EXPANDED

Example Problem

Delta II 7320 Launch Vehicle

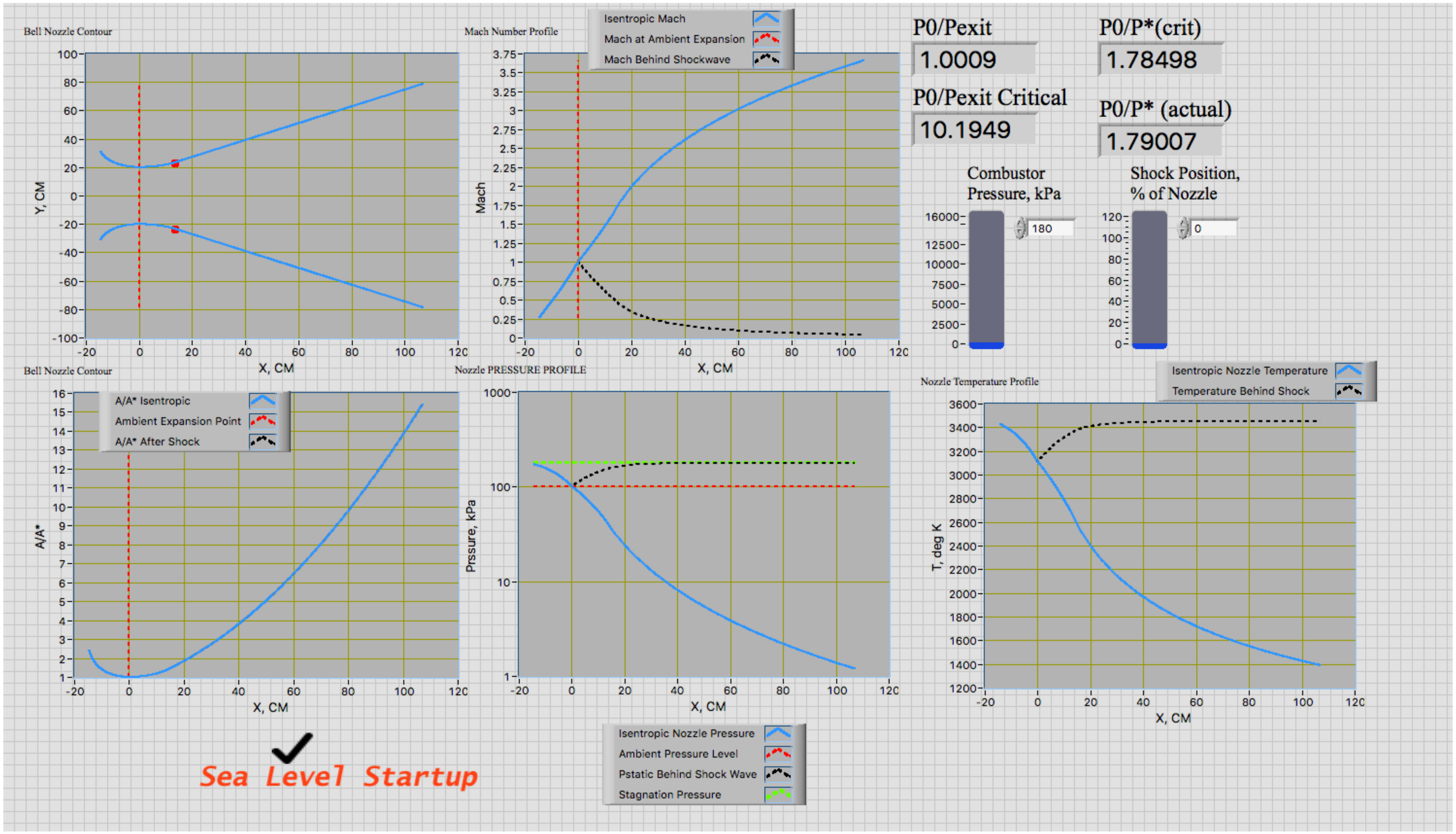


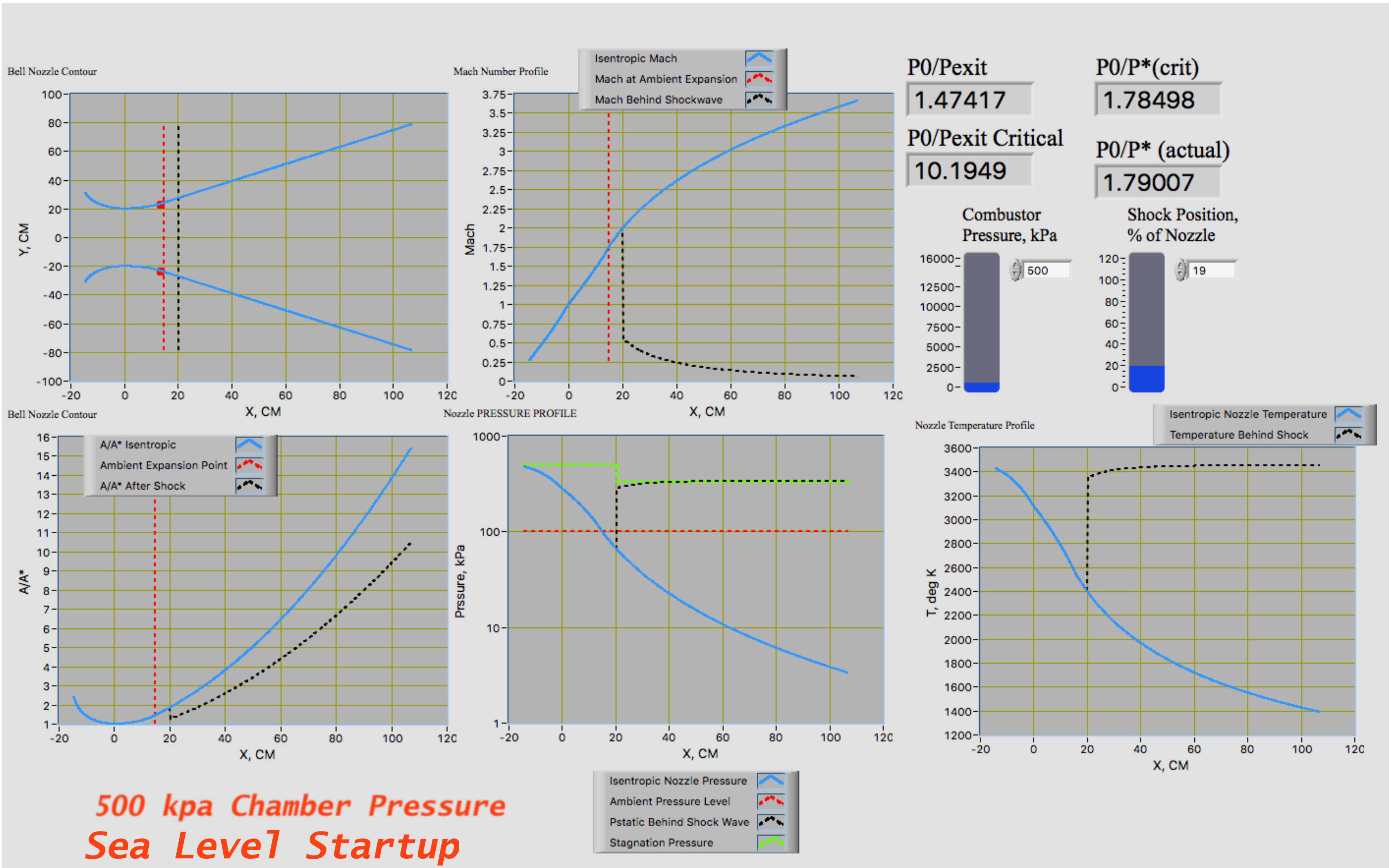
Stage 1 Properties

- Boeing Delta II Rocket…Stage 1
  - Sea Level Thrust: 890kN
  - Vacuum Thrust: 1085.8 kN
  - **Nozzle Expansion Ratio: 15.25:1**
  - **Conical Nozzle, 30.5 deg exit angle**

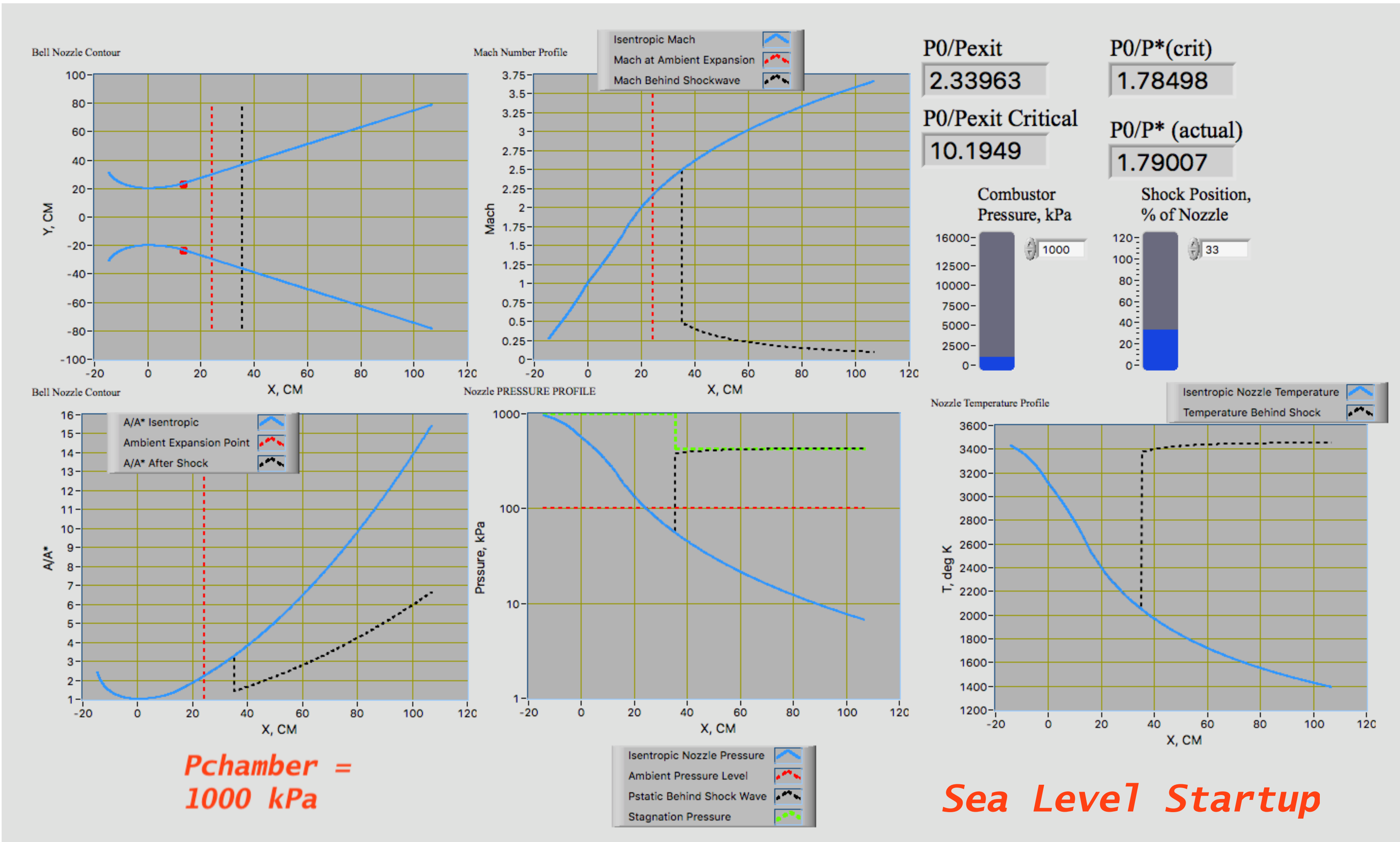
- Combustion Properties: (RS-27A Rocketdyne Engine)
  - Lox/Kerosene, Mixture Ratio: 2.24:1
  - **Chamber Pressure (P_0): 5161.463 kPa**
  - Combustion temperature (T_0): 3455 K
  - γ = 1.2220
  - M_W = 21.28 *kg/kg-mol*

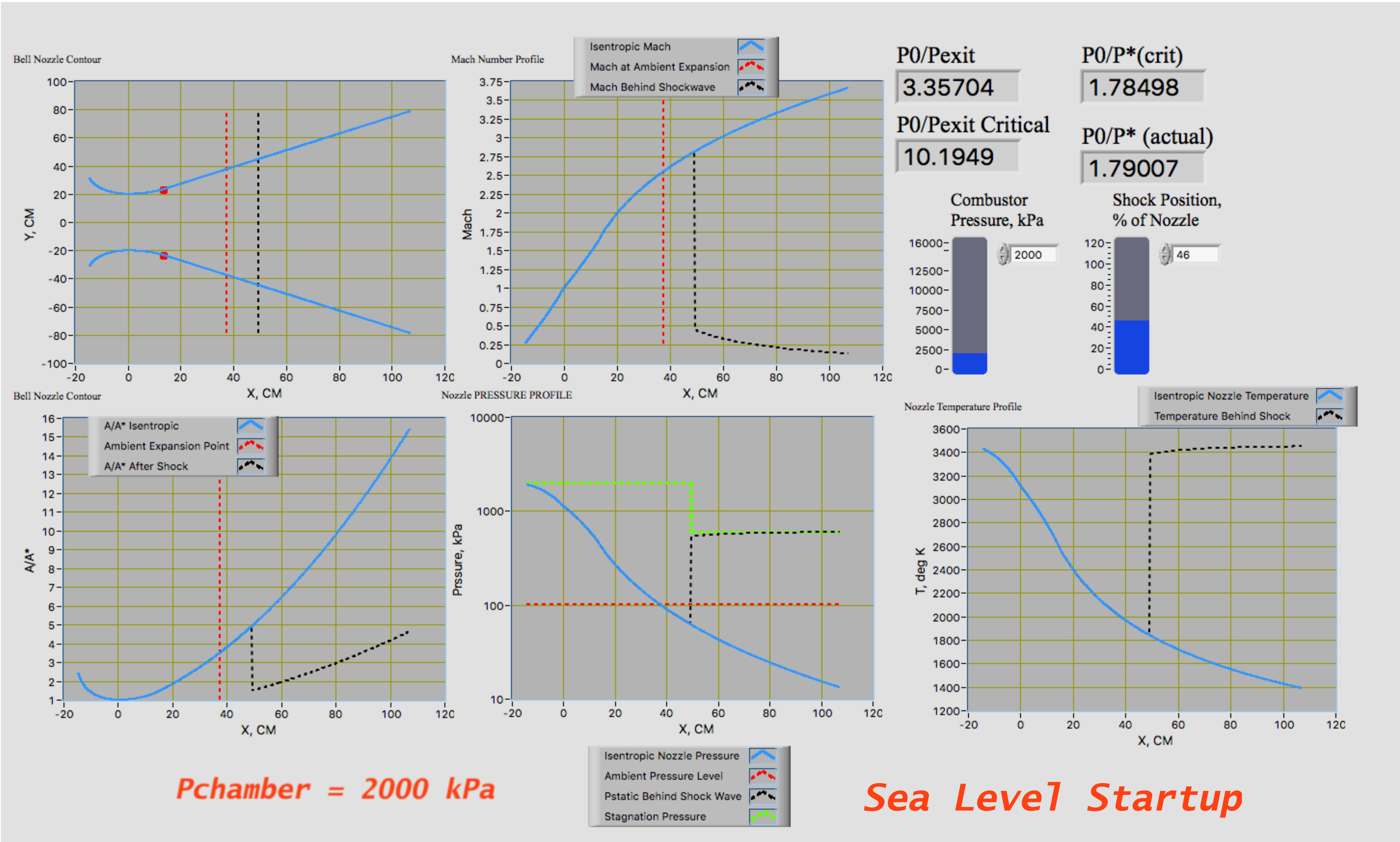
- Propellant Mass: 97.08 Metric Tons
- Stage 1 Launch Mass: 101.8 Metric Tons

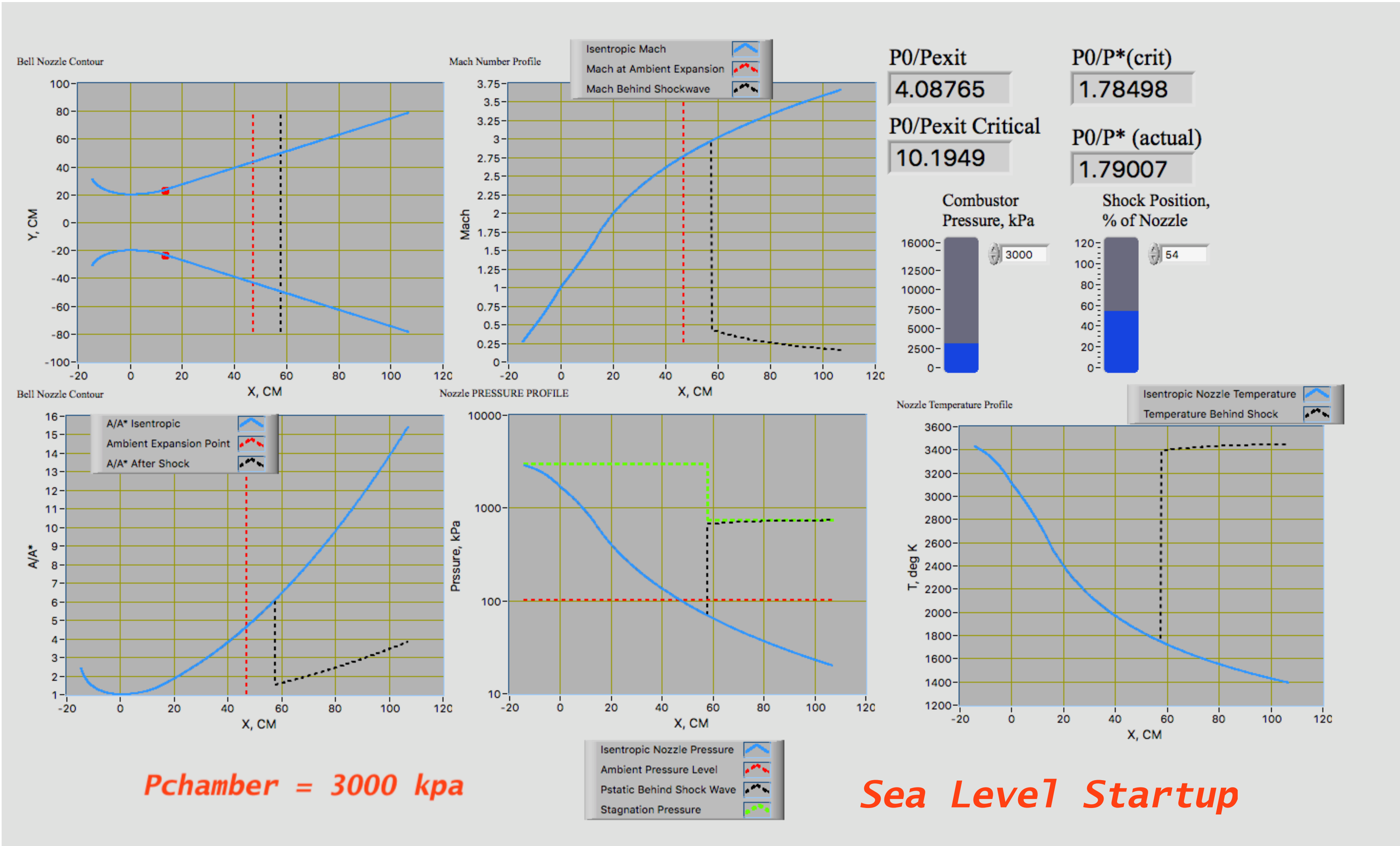

Bell Nozzle Contour
Y, CM
X, CM
Mach Number Profile
Isentropic Mach
Mach at Ambient Expansion
Mach Behind Shockwave
Mach
X, CM
P0/Pexit
1.0009
P0/Pexit Critical
10.1949
P0/P*(crit)
1.78498
P0/P* (actual)
1.79007
Combustor Pressure, kPa
180
Shock Position, % of Nozzle
0
Bell Nozzle Contour
A/A* Isentropic
Ambient Expansion Point
A/A* After Shock
A/A*
X, CM
Nozzle PRESSURE PROFILE
Prssure, kPa
X, CM
Isentropic Nozzle Pressure
Ambient Pressure Level
Pstatic Behind Shock Wave
Stagnation Pressure
Nozzle Temperature Profile
Isentropic Nozzle Temperature
Temperature Behind Shock
T, deg K
X, CM
Sea Level Startup

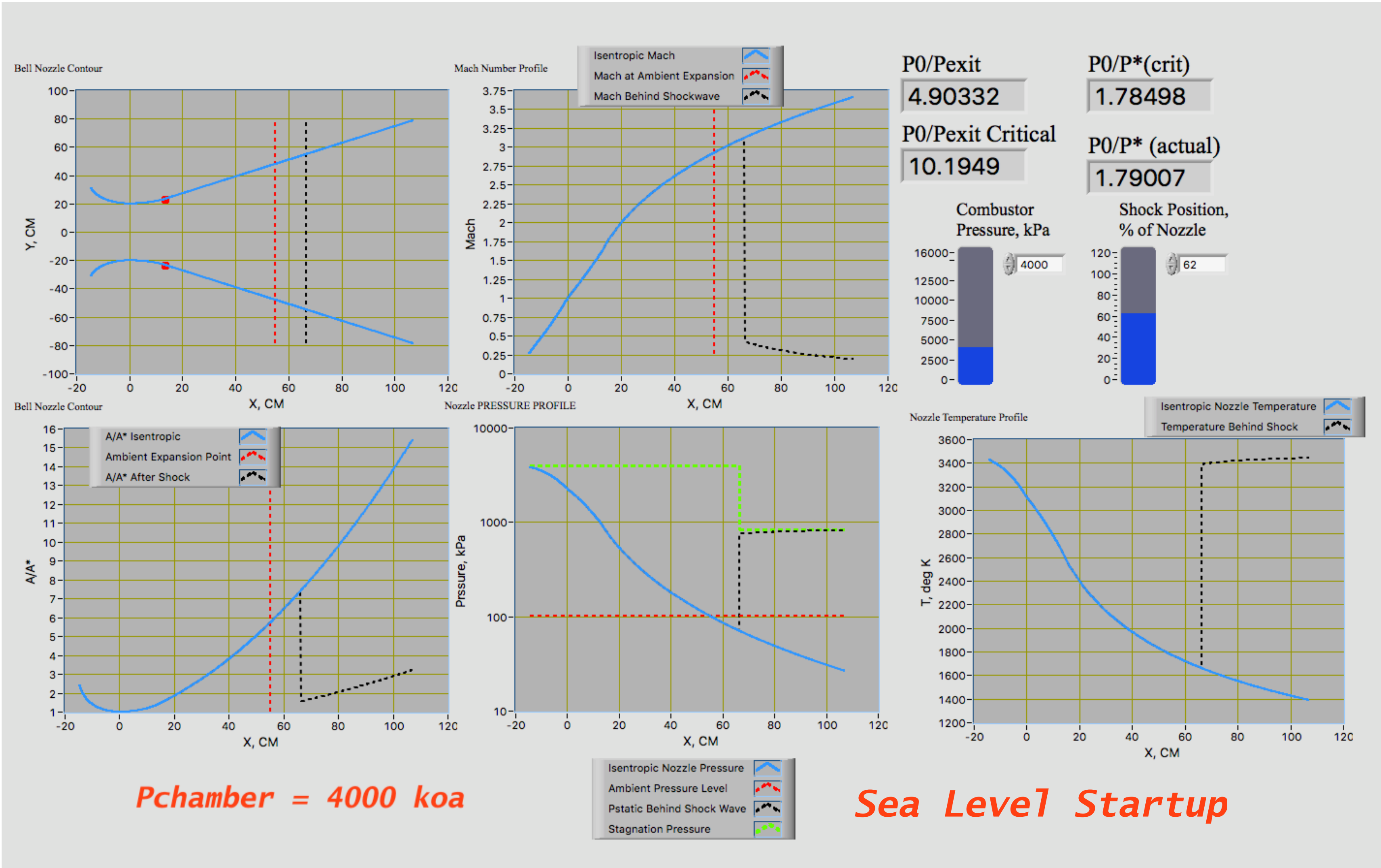


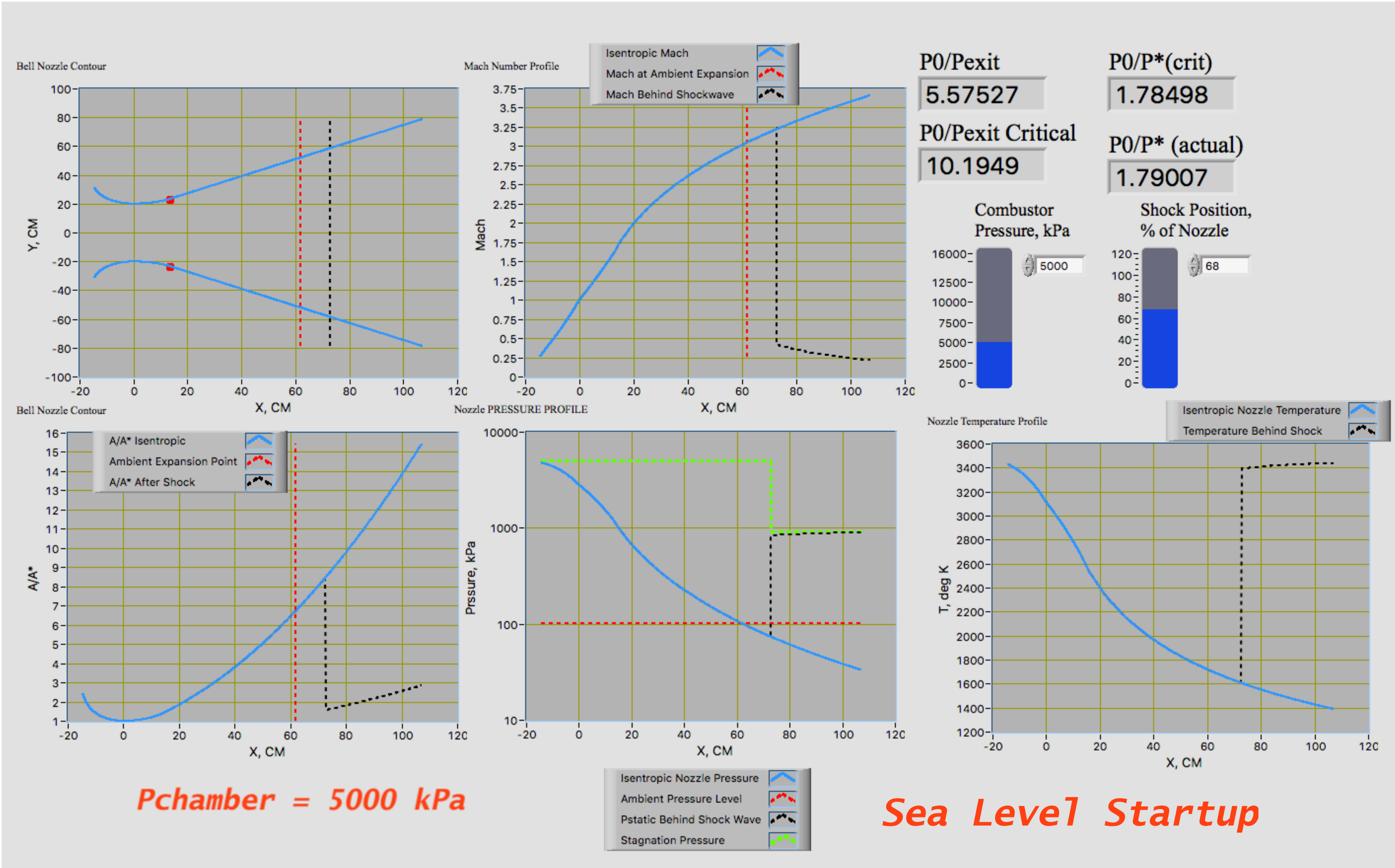
Bell Nozzle Contour
Y, CM
X, CM
Mach Number Profile
Isentropic Mach
Mach at Ambient Expansion
Mach Behind Shockwave
Mach
X, CM
P0/Pexit
1.47417
P0/Pexit Critical
10.1949
P0/P*(crit)
1.78498
P0/P* (actual)
1.79007
Combustor Pressure, kPa
500
Shock Position, % of Nozzle
19
Bell Nozzle Contour
A/A* Isentropic
Ambient Expansion Point
A/A* After Shock
A/A*
X, CM
Nozzle PRESSURE PROFILE
Prssure, kPa
X, CM
Isentropic Nozzle Pressure
Ambient Pressure Level
Pstatic Behind Shock Wave
Stagnation Pressure
Nozzle Temperature Profile
Isentropic Nozzle Temperature
Temperature Behind Shock
T, deg K
X, CM
500 kpa Chamber Pressure
Sea Level Startup

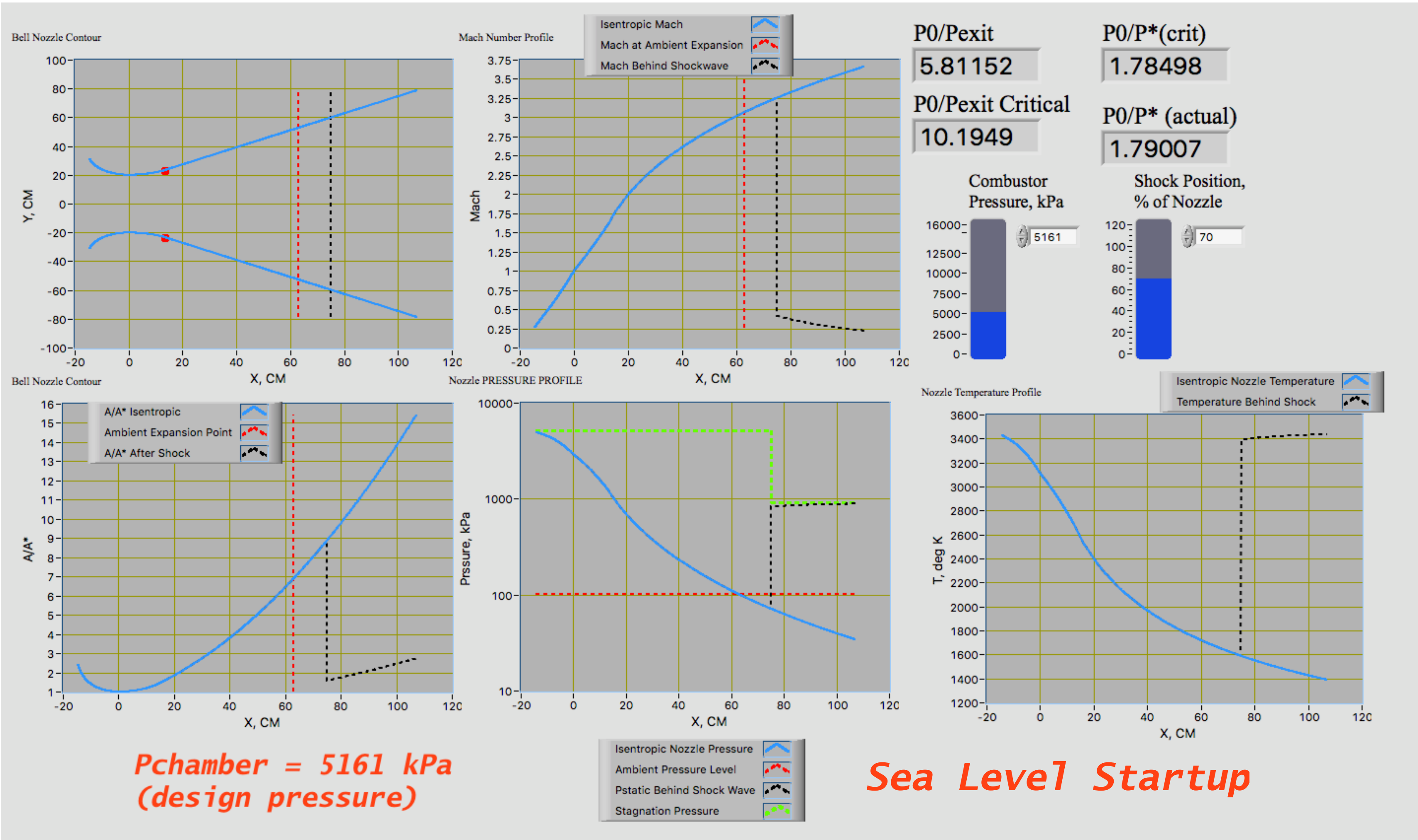

Bell Nozzle Contour
Y, CM
X, CM
Mach Number Profile
Isentropic Mach
Mach at Ambient Expansion
Mach Behind Shockwave
Mach
X, CM
P0/Pexit
2.33963
P0/Pexit Critical
10.1949
P0/P*(crit)
1.78498
P0/P* (actual)
1.79007
Combustor Pressure, kPa
1000
Shock Position, % of Nozzle
33
Bell Nozzle Contour
A/A* Isentropic
Ambient Expansion Point
A/A* After Shock
A/A*
X, CM
Nozzle PRESSURE PROFILE
Prssure, kPa
X, CM
Isentropic Nozzle Pressure
Ambient Pressure Level
Pstatic Behind Shock Wave
Stagnation Pressure
Nozzle Temperature Profile
Isentropic Nozzle Temperature
Temperature Behind Shock
T, deg K
X, CM
Pchamber = 1000 kPa
Sea Level Startup


Bell Nozzle Contour
Y, CM
X, CM
Mach Number Profile
Isentropic Mach
Mach at Ambient Expansion
Mach Behind Shockwave
Mach
P0/Pexit
3.35704
P0/P*(crit)
1.78498
P0/Pexit Critical
10.1949
P0/P* (actual)
1.79007
Combustor Pressure, kPa
2000
Shock Position, % of Nozzle
46
Bell Nozzle Contour
A/A* Isentropic
Ambient Expansion Point
A/A* After Shock
A/A*
Nozzle PRESSURE PROFILE
Prssure, kPa
Isentropic Nozzle Pressure
Ambient Pressure Level
Pstatic Behind Shock Wave
Stagnation Pressure
Nozzle Temperature Profile
Isentropic Nozzle Temperature
Temperature Behind Shock
T, deg K
Pchamber = 2000 kPa
Sea Level Startup

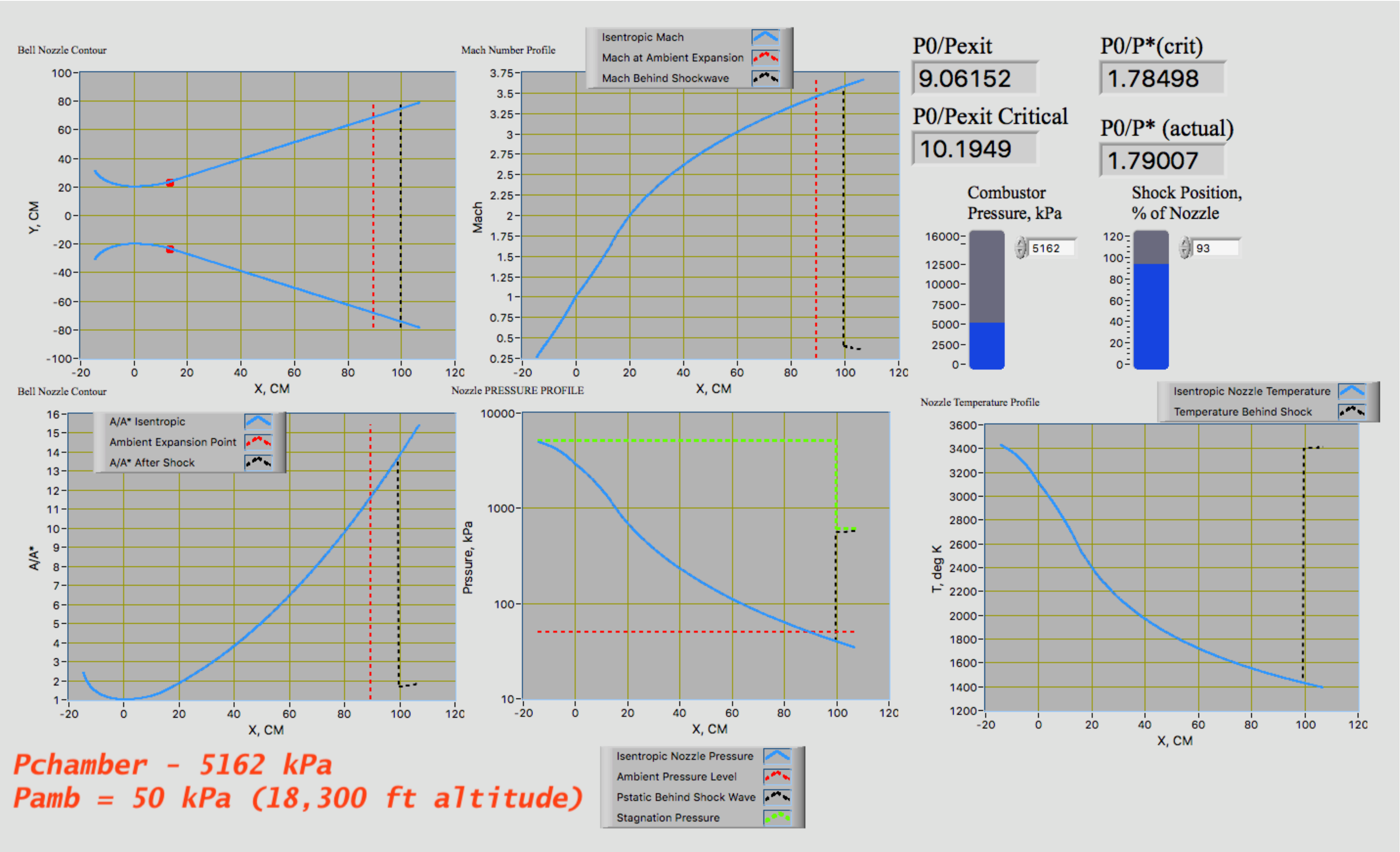

Bell Nozzle Contour
Y, CM
X, CM
Mach Number Profile
Isentropic Mach
Mach at Ambient Expansion
Mach Behind Shockwave
Mach
P0/Pexit
4.08765
P0/Pexit Critical
10.1949
P0/P*(crit)
1.78498
P0/P* (actual)
1.79007
Combustor Pressure, kPa
3000
Shock Position, % of Nozzle
54
Bell Nozzle Contour
A/A* Isentropic
Ambient Expansion Point
A/A* After Shock
A/A*
Nozzle PRESSURE PROFILE
Prssure, kPa
Isentropic Nozzle Pressure
Ambient Pressure Level
Pstatic Behind Shock Wave
Stagnation Pressure
Nozzle Temperature Profile
Isentropic Nozzle Temperature
Temperature Behind Shock
T, deg K
Pchamber = 3000 kpa
Sea Level Startup


Bell Nozzle Contour
Y, CM
X, CM
Mach Number Profile
Isentropic Mach
Mach at Ambient Expansion
Mach Behind Shockwave
Mach
X, CM
P0/Pexit
4.90332
P0/Pexit Critical
10.1949
P0/P*(crit)
1.78498
P0/P* (actual)
1.79007
Combustor Pressure, kPa
4000
Shock Position, % of Nozzle
62
Bell Nozzle Contour
A/A* Isentropic
Ambient Expansion Point
A/A* After Shock
A/A*
X, CM
Nozzle PRESSURE PROFILE
Prssure, kPa
X, CM
Isentropic Nozzle Pressure
Ambient Pressure Level
Pstatic Behind Shock Wave
Stagnation Pressure
Nozzle Temperature Profile
Isentropic Nozzle Temperature
Temperature Behind Shock
T, deg K
X, CM
Pchamber = 4000 koa
Sea Level Startup

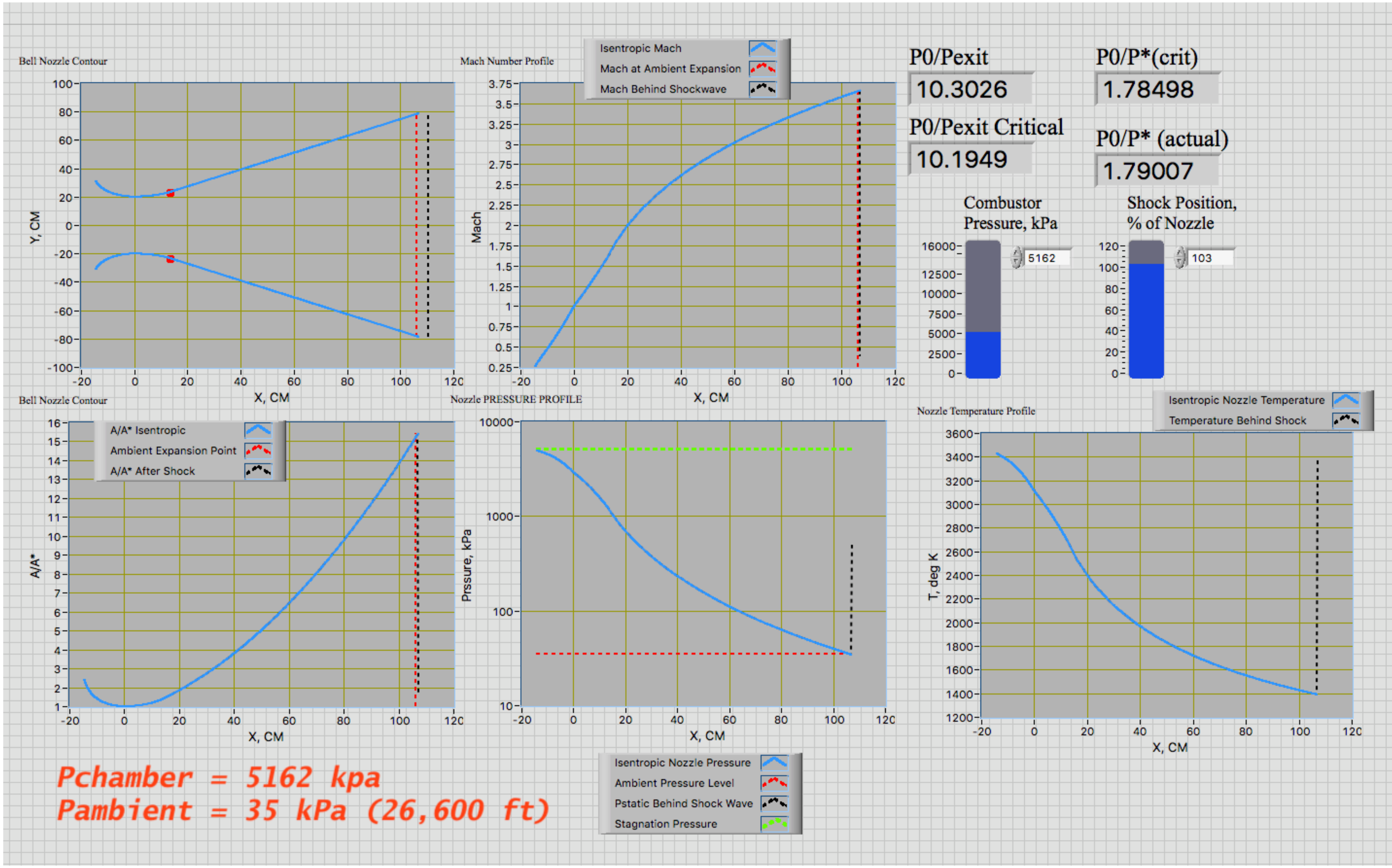

Bell Nozzle Contour
Y, CM
X, CM
Mach Number Profile
Isentropic Mach
Mach at Ambient Expansion
Mach Behind Shockwave
Mach
X, CM
P0/Pexit
5.57527
P0/Pexit Critical
10.1949
P0/P*(crit)
1.78498
P0/P* (actual)
1.79007
Combustor Pressure, kPa
5000
Shock Position, % of Nozzle
68
Bell Nozzle Contour
A/A* Isentropic
Ambient Expansion Point
A/A* After Shock
A/A*
X, CM
Nozzle PRESSURE PROFILE
Prssure, kPa
X, CM
Isentropic Nozzle Pressure
Ambient Pressure Level
Pstatic Behind Shock Wave
Stagnation Pressure
Nozzle Temperature Profile
Isentropic Nozzle Temperature
Temperature Behind Shock
T, deg K
X, CM
Pchamber = 5000 kPa
Sea Level Startup

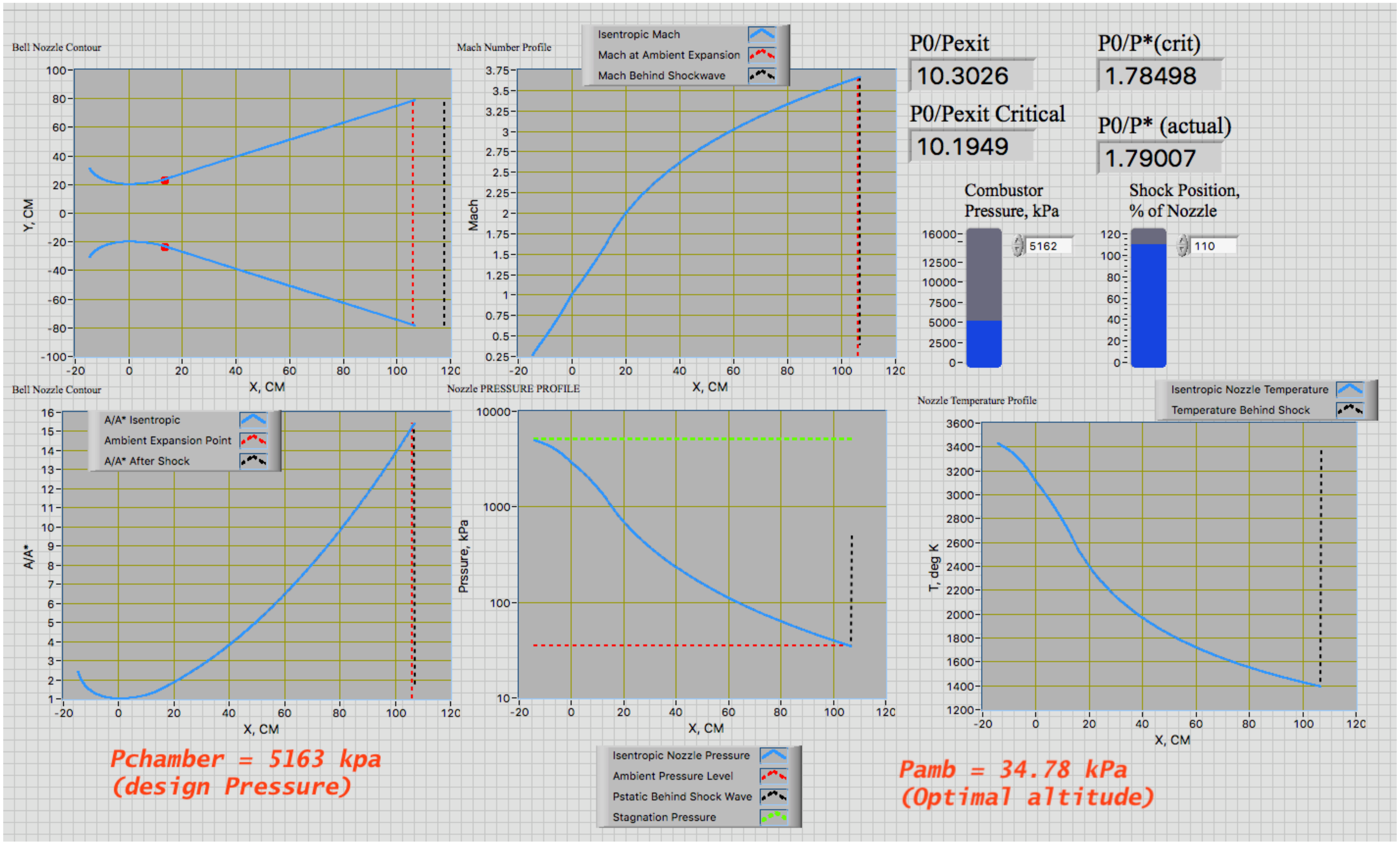


Bell Nozzle Contour
Y, CM
X, CM
Mach Number Profile
Isentropic Mach
Mach at Ambient Expansion
Mach Behind Shockwave
Mach
X, CM
P0/Pexit
5.81152
P0/Pexit Critical
10.1949
P0/P*(crit)
1.78498
P0/P* (actual)
1.79007
Combustor Pressure, kPa
5161
Shock Position, % of Nozzle
70
Bell Nozzle Contour
A/A* Isentropic
Ambient Expansion Point
A/A* After Shock
A/A*
X, CM
Nozzle PRESSURE PROFILE
Prssure, kPa
X, CM
Isentropic Nozzle Pressure
Ambient Pressure Level
Pstatic Behind Shock Wave
Stagnation Pressure
Nozzle Temperature Profile
Isentropic Nozzle Temperature
Temperature Behind Shock
T, deg K
X, CM
Pchamber = 5161 kPa
(design pressure)
Sea Level Startup



Pchamber - 5162 kPa
Pamb = 50 kPa (18,300 ft altitude)


Bell Nozzle Contour
Y, CM
X, CM
Mach Number Profile
Isentropic Mach
Mach at Ambient Expansion
Mach Behind Shockwave
Mach
P0/Pexit
10.3026
P0/Pexit Critical
10.1949
P0/P*(crit)
1.78498
P0/P* (actual)
1.79007
Combustor Pressure, kPa
5162
Shock Position, % of Nozzle
103
Bell Nozzle Contour
A/A* Isentropic
Ambient Expansion Point
A/A* After Shock
A/A*
Nozzle PRESSURE PROFILE
Prssure, kPa
Isentropic Nozzle Pressure
Ambient Pressure Level
Pstatic Behind Shock Wave
Stagnation Pressure
Nozzle Temperature Profile
Isentropic Nozzle Temperature
Temperature Behind Shock
T, deg K
Pchamber = 5162 kpa
Pambient = 35 kPa (26,600 ft)

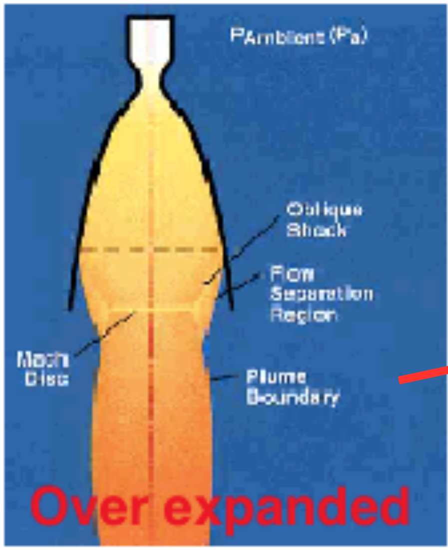


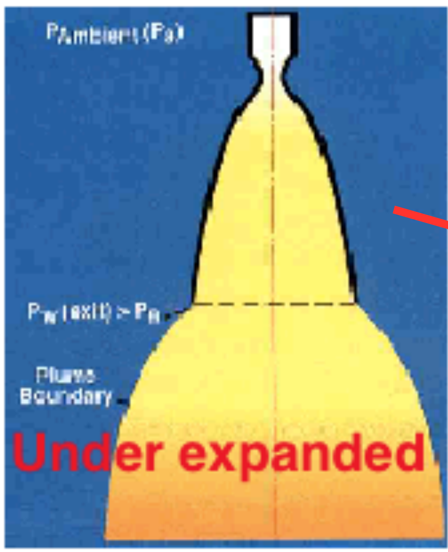
Bell Nozzle Contour
Y, CM
X, CM
Mach Number Profile
Isentropic Mach
Mach at Ambient Expansion
Mach Behind Shockwave
Mach
P0/Pexit
10.3026
P0/P*(crit)
1.78498
P0/Pexit Critical
10.1949
P0/P* (actual)
1.79007
Combustor Pressure, kPa
5162
Shock Position, % of Nozzle
110
Bell Nozzle Contour
A/A* Isentropic
Ambient Expansion Point
A/A* After Shock
A/A*
Nozzle PRESSURE PROFILE
Prssure, kPa
Nozzle Temperature Profile
Isentropic Nozzle Temperature
Temperature Behind Shock
T, deg K
Isentropic Nozzle Pressure
Ambient Pressure Level
Pstatic Behind Shock Wave
Stagnation Pressure
Pchamber = 5163 kpa
(design Pressure)
Pamb = 34.78 kPa
(Optimal altitude)

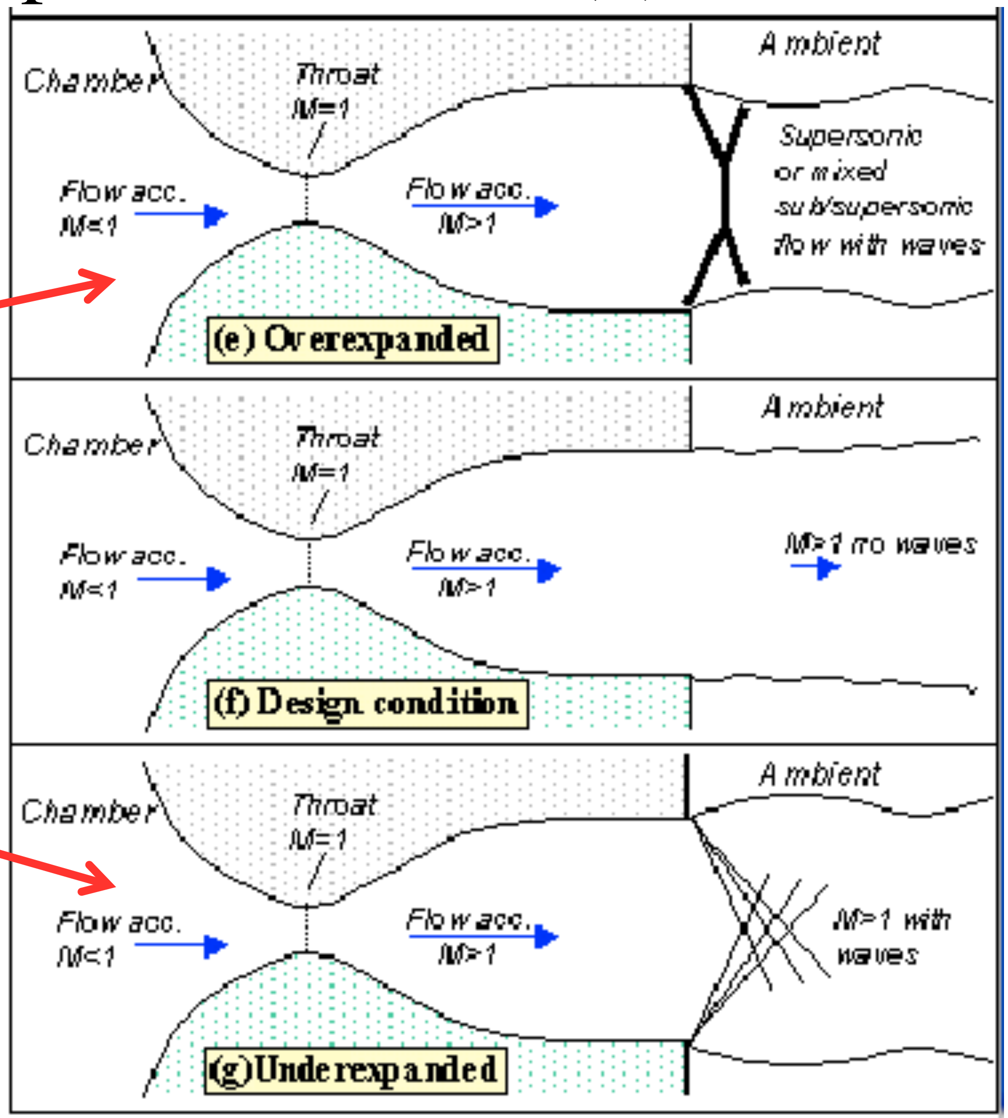
EXIT PRESSURE IS <u>LESS</u>
THAN ATMOSPHERIC PRESSURE

EXIT PRESSURE GREATER THAN ATMOSPHERIC PRESSURE

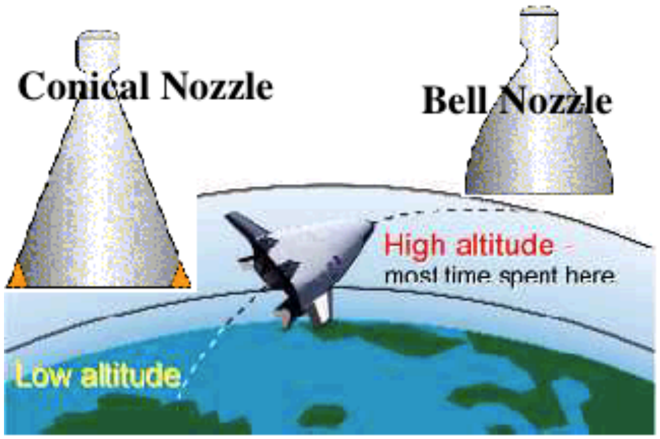
Next: The Optimum Nozzle (1)



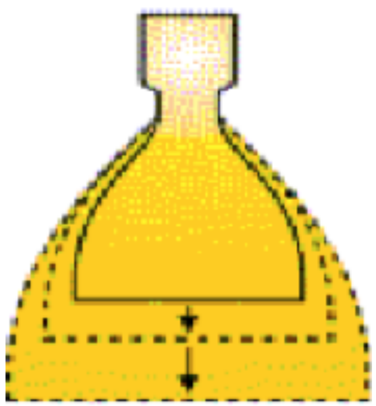




Next: The Optimum Nozzle (2)



$$Thrust = \dot{m} V_{exit} + A_{exit}(p_{exit} - p_\infty)$$



$$for\ given\ \dot{m} \rightarrow \boxed{\begin{array}{l} V_{exit} \propto \dfrac{A_{exit}}{A^*} \\ \dfrac{1}{P_{exit}} \propto \dfrac{A_{exit}}{A^*} \end{array}}$$

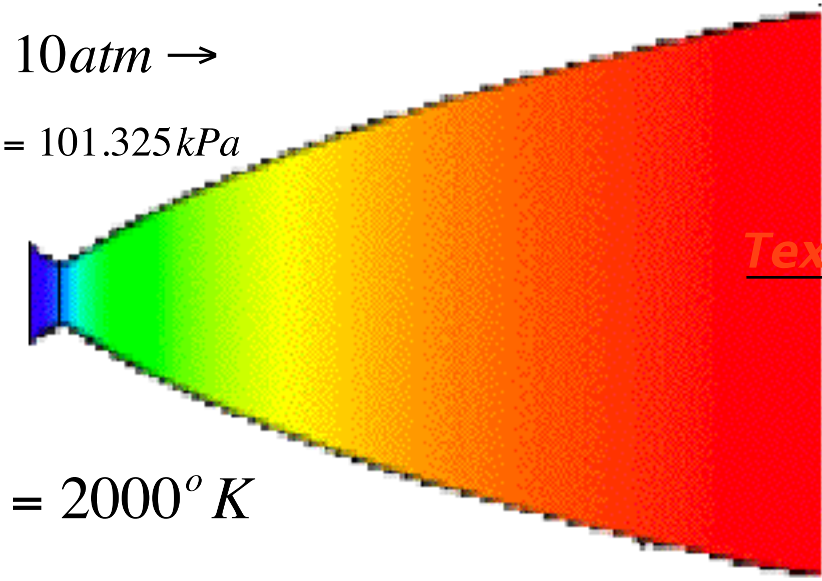
$$\rightarrow both\ \{V_{exit}, P_{exit}\}\ contribute\ to\ thrust$$

$$\rightarrow what\ \frac{A_{exit}}{A^*}\ is\ "optimal"?$$

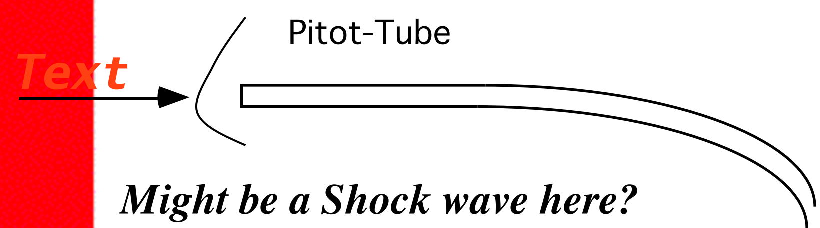
Project 1 (counts as double homework)
Due Friday October 10, 2021

$P_0 = 10 atm \rightarrow$

$1 atm = 101.325 kPa$



$P0_2 = 0.5 atm$



Might be a Shock wave here?

$T_0 = 2000^o K$

$$A_{throat} = 0.30 m^2$$

$$A_{exit} = 23.256 m^2$$

- Assume γ =1.2, MW = 22

Is nozzle isentropic?

- Calculate

i) *Exit mach,* M_e
ii) *Exit pressure* ,p_e
iii) *Exit Temperature,* T_e
iv) *Mass Flow Through Nozzle, dm/dt*
v) *Thrust of Nozzle in vacuum*
vi) *Compare Thrust to Thrust of isentropic nozzle*

Hint:

- Compute exit mach number for isentropic nozzle

- Employ normal shock wave equations to determine if there is a shock wave standing in front of Pitot tube

- If nozzle is non isentropic …
  You'll have to write a solver for (or use trial and error)

$$\frac{P_{02}}{P_{01}} = \frac{2}{(\gamma+1)\left(\gamma M_1^2 - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}} \left(\frac{\left[\frac{(\gamma+1)}{2} M_1\right]^2}{\left(1 + \frac{\gamma-1}{2} M_1^2\right)} \right)^{\left(\frac{\gamma}{\gamma-1}\right)}$$

… and

$$P_0 \neq constant, A^* \neq constant \rightarrow \left\{P_0 \cdot A^*\right\} = constant$$