Incompressible, Compressible, and Supersonic Flow Fields: Static, Dynamic, and Total Pressure (1)

• In fluid mechanics static pressure is the pressure exerted by a fluid at rest. Examples of static pressure are:

1) Air pressure inside a latex balloon
2) Atmospheric (ambient) pressure (neglecting the effect of wind).
3) Hydrostatic pressure at the bottom of a dam is the static pressure

Strictly Speaking, pressure inside a ventilation duct is ***not*** static pressure, unless the air inside the duct is still.

Strictly Speaking, On an aircraft, the pressure measured on a generic point of the surface of the wing (or fuselage) is not, in general, the static pressure, unless the aircraft is not moving with respect to the air.

Incompressible, Compressible, and Supersonic Flow Fields: Static, Dynamic, and Total Pressure (2)

- For fluids in motion the term static pressure is still applicable (in particular with regard to external flows), and refers strictly to the pressure in the fluid far upstream **(freestream)** of any object immersed into it.

- *Freestream Pressure = Ambient pressure for atmospheric flight*

- When the fluid comes in proximity to a body, its pressure deviates from the *freestream* value and strictly-speaking should no longer be referred to as static pressure …. quantity *should be* called simply "pressure".

Incompressible, Compressible, and Supersonic Flow Fields: Static, Dynamic, and Total Pressure (3)

• However, to distinguish this local surface pressure from local Total AND dynamic pressure, and the freestream pressure …

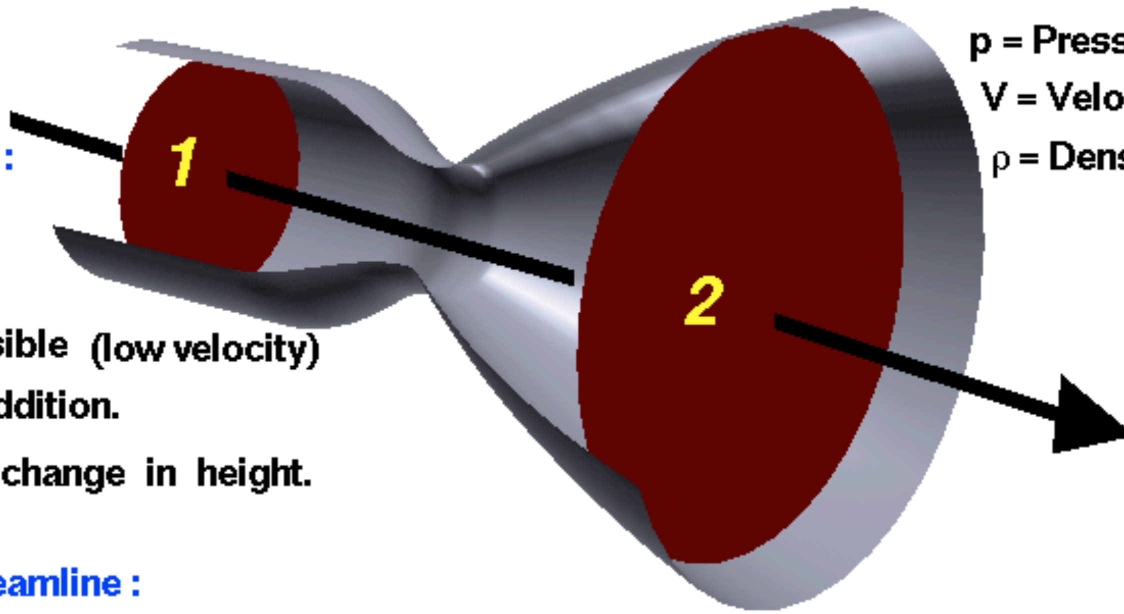
… the local "pressure" by tradition is still called *"static pressure"*

• The confusion between Total pressure, "pressure", and static pressure arises from one of the basic laws of fluid dynamics, *the Bernoulli equation*

• *Bernoulli is Strictly applicable for* incompressible, inviscid flow, With negligible gravitational effects:

Incompressible Bernoulli Equation (1)

- ***Consider Incompressible Flow Case***



p = Pressure
V = Velocity
ρ = Density

Restrictions :

Inviscid
Steady
Incompressible (low velocity)
No heat addition.
Negligible change in height.

Along a streamline :

static pressure + dynamic pressure = total pressure

$$p_s + \frac{\rho V^2}{2} = p_t$$

$$\left(p_s + \frac{\rho V^2}{2}\right)_1 = \left(p_s + \frac{\rho V^2}{2}\right)_2$$

Incompressible Bernoulli Equation (2)

Stagnation (Total) Pressure = Dynamic Pressure + Static Pressure

$$P_{stagnation} = \frac{1}{2}\rho V^2 + P_{static}$$

Incompressible Bernoulli Equation

where: $P_{stagnation}$ is the stagnation (or total) pressure in pascals

ρ is the fluid density in kg/m^3

v is the fluid velocity relative to the stagnation point before it becomes influenced by the object which causes stagnation

P_{static} is the static fluid pressure away from the influence of the moving fluid

$$"qbar" \rightarrow$$

$$\frac{1}{2}\rho V^2 = \frac{1}{2}\frac{\gamma P_{static}}{\gamma R_g T_{static}} V^2 = \frac{\gamma}{2} P_{static} \cdot \frac{V^2}{\gamma R_g T_{static}} = \frac{\gamma}{2} P_{static} M^2 \equiv \bar{q}$$

Fictitious but useful quantity known as *Incompressible Dynamic Pressure*, "qbar"

Incompressible Bernoulli Equation (3)

- More General Definition of Stagnation or Total Pressure

pressure a fluid retains when brought to rest isentropically from mach number *M*.

$$P_{stagnation} = p_{static} \cdot \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

- Applicable to Compressible flow fields

- ***How does this reconcile with the Bernoulli Equation?***

$$p + \frac{1}{2}\rho V^2 = const \qquad = P_{stagnation}$$

Compressible Bernoulli Equation (subsonic flow) (1)

- Adding and subtracting p to the equation

$$P_0 = p + p\left\{\left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{\gamma}{\gamma - 1}} - 1\right\} \equiv p + q_c \Rightarrow$$

Compressible form of Bernoulli Equation

$$q_c \equiv p\left\{\left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{\gamma}{\gamma - 1}} - 1\right\}$$

Compressible Dynamic Pressure

Compressible Bernoulli Equation (subsonic flow) (2)

- Incompressible Flow

$$P_{stagnation} = \bar{q} + P_{static}$$

Fictitious quantity known as Dynamic Pressure, "qbar"

$$\bar{q} = \frac{\gamma}{2} P_{static} M^2$$

- Compressible Flow (Subsonic)

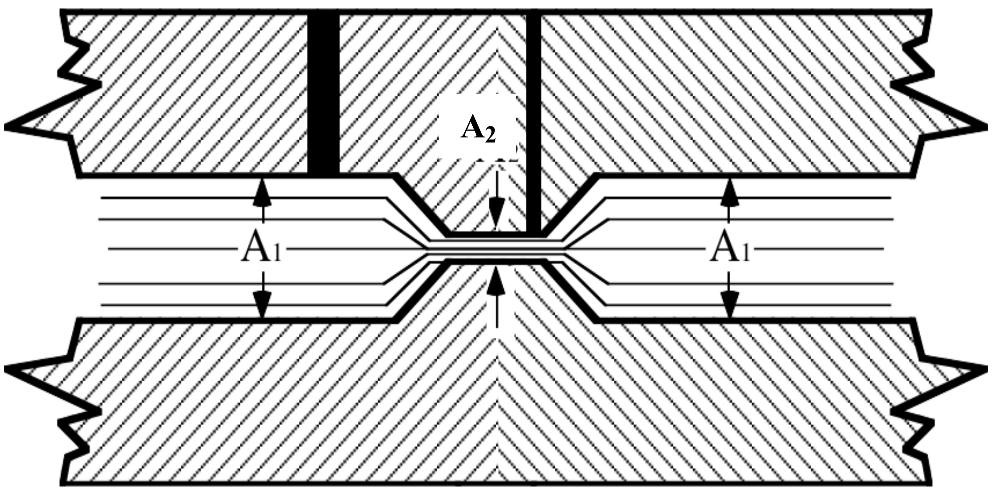
$$P_{stagnation} = q_c + P_{static}$$

$$q_c = \left\{ \left[1 + \frac{\gamma - 1}{2} M^2 \right]^{\frac{\gamma}{\gamma - 1}} - 1 \right\} \cdot P_{static}$$

Compressible Dynamic Pressure or "impact" Pressure

The Compressible Bernoulli Equation

- Consider (M < 1) Compressible Flow Through an orifice



$$Subcritical\ Flow: \left(\frac{p}{P_0}\right) > \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

- Entering Flow is Isentropic

$$\frac{P_0}{p} = \left(1+\left(\frac{\gamma-1}{2}\right)M^2\right)^{\frac{\gamma}{\gamma-1}} \rightarrow \left(\frac{P_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = \left(1+\left(\frac{\gamma-1}{2}\right)M^2\right) = \left(1+\left(\frac{\gamma-1}{2}\right)\frac{V^2}{\gamma R_g T}\right)$$

The Compressible Bernoulli Equation

$$\frac{P_0}{p} = \left(1 + \left(\frac{\gamma - 1}{2}\right) M^2\right)^{\frac{\gamma}{\gamma-1}} \rightarrow \left(\frac{P_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = \left(1 + \left(\frac{\gamma - 1}{2}\right) M^2\right) = \left(1 + \left(\frac{\gamma - 1}{2}\right) \frac{V^2}{\gamma R_g T}\right)$$

Since Entering Flow is Isentropic

$$\left(\frac{P_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = \frac{T_0}{T} \rightarrow substitute \rightarrow \frac{T_0}{T} = \left(1 + \left(\frac{\gamma - 1}{2}\right) \frac{V^2}{\gamma R_g T}\right)$$

$$Solve \rightarrow \frac{1}{2} V^2 = \frac{\gamma}{\gamma - 1} R_g T_0 - \frac{\gamma}{\gamma - 1} R_g T$$

$$Substitute\ Gas\ Law \rightarrow \frac{p}{\rho} = R_g T$$

$$\frac{V^2}{2} + \left(\frac{\gamma}{\gamma - 1}\right) \frac{p}{\rho} = \frac{\gamma}{\gamma - 1} \left(\frac{P_0}{\rho_0}\right) \rightarrow \text{"Compressible Bernoulli Equation"}$$

Compare to Incompressible Bernoulli

"*Compressible Bernoulli Equation*"

$$\frac{V^2}{2}+\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho}=\frac{\gamma}{\gamma-1}\left(\frac{P_0}{\rho_0}\right)$$

"*Compressible Bernoulli Equation*"

$$P_0=p+\frac{1}{2}\cdot\rho\cdot V^2$$

- From Definition of Speed of Sound

$$Sonic\ Velocity\rightarrow c=\sqrt{\gamma\cdot R_g\cdot T}=\sqrt{\left(\frac{\partial p}{\partial\rho}\right)_{\Delta s=0}}$$

$$Incompressible\ Flow\rightarrow\partial\rho=0\rightarrow c_{incr}=\infty$$

$$\rightarrow\lim_{\gamma\rightarrow\infty}\left[\frac{V^2}{2}+\frac{\gamma}{\gamma-1}\frac{p}{\rho}=\frac{\gamma}{\gamma-1}\frac{P_0}{\rho}\right]=\frac{V^2}{2}+\frac{p}{\rho}=\frac{P_0}{\rho}$$

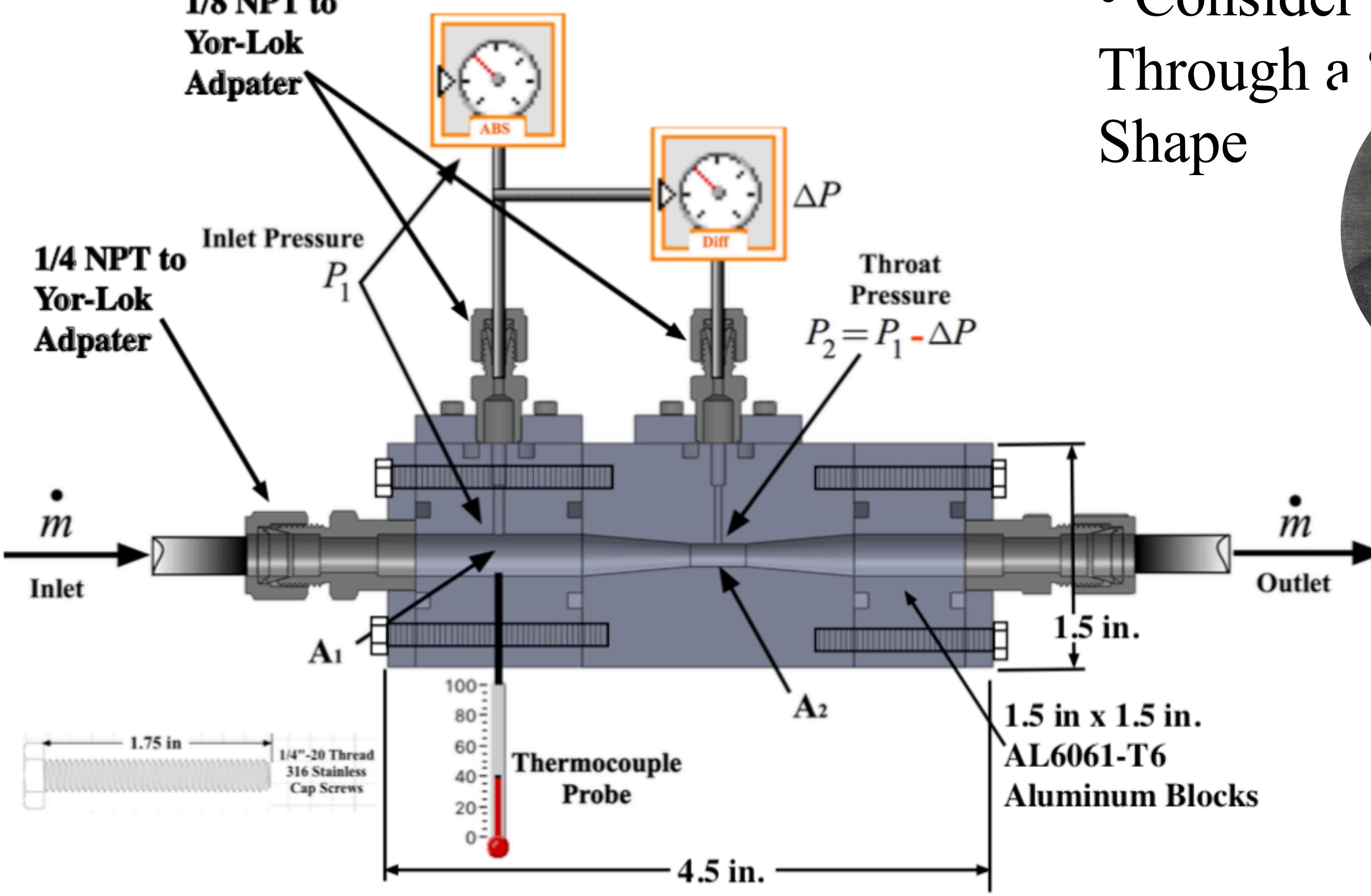
- Solve for P_0

$$\boxed{P_0=p+\frac{1}{2}\rho\cdot V^2}\ !$$

At Mach zero, Compressible Bernoulli Reduces to Incompressible Equation

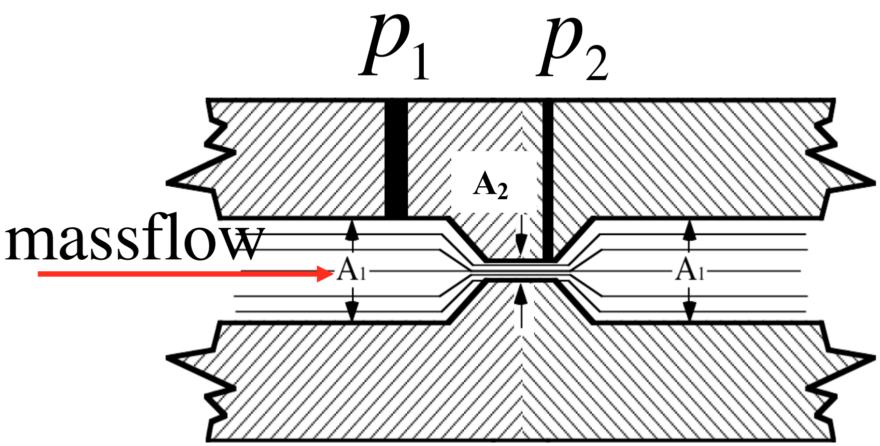
Applications of Bernoulli: The Venturi Massflow Equations

• Consider Flow Through a "Venturi" Shape



Giovanni Battista Venturi (1796)

Incompressible Venturi (1)



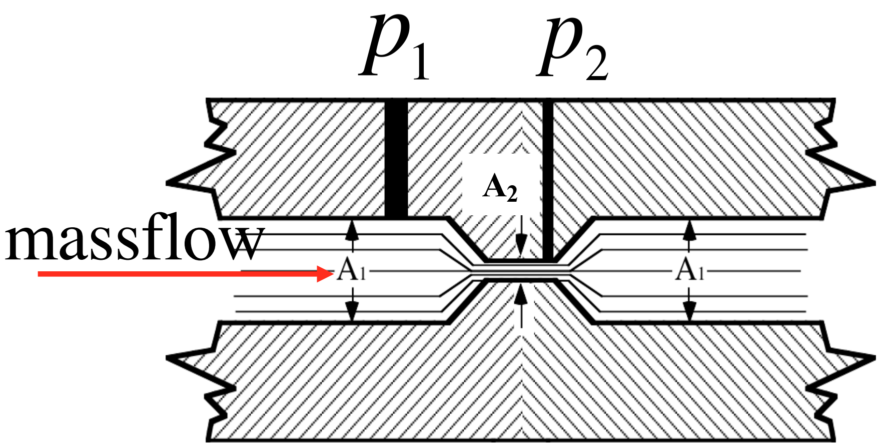
- First look at Incompressible Venturi where density is constant throughout the Flow field.

$$\textit{Bernoulli's Law for Incompressible Flow } (\rho = constant) \rightarrow p_1 + \frac{1}{2}\rho \cdot V_1^2 = const = p_2 + \frac{1}{2}\rho \cdot V_2^2$$

- $\textit{Continuity (Conservation of Mass)} \rightarrow \rho \cdot A_1 \cdot V_1 = \dot{m} = \rho \cdot A_2 \cdot V_2 \rightarrow \boxed{V_1 = \frac{A_2}{A_1} \cdot V_2}$

- $\textit{Solve for pressure differetial in terms of } V_2 \rightarrow p_1 - p_2 = \frac{1}{2}\rho \cdot \left(V_2^2 - V_1^2\right) = \frac{1}{2}\rho \cdot \left(V_2^2 - \left(\frac{A_2}{A_1} \cdot V_2\right)^2\right)$

Incompressible Venturi (2)



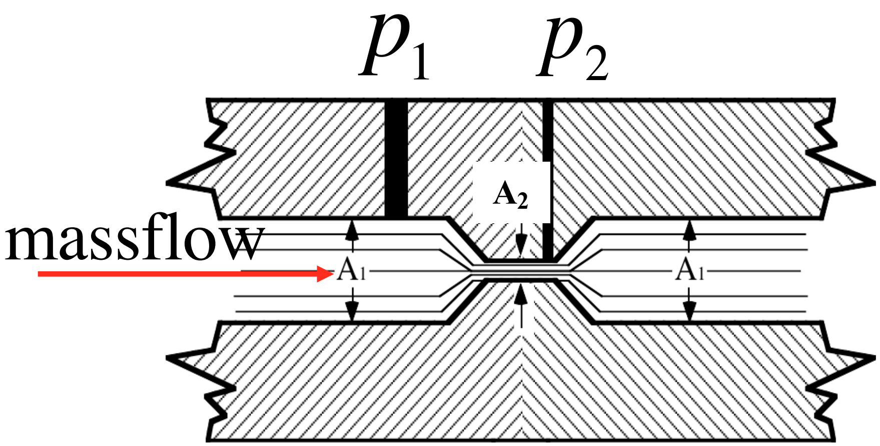
$$\bullet \textit{Solve for } V_2 \rightarrow V_2 = \sqrt{2 \cdot \left(\frac{p_1 - p_2}{\rho \cdot \left(1 - \left(\frac{A_2}{A_1} \right)^2 \right)} \right)}$$

$$\textit{Calculate Massflow} \rightarrow \dot{m} = \rho \cdot A_2 \cdot V_2 = \frac{A_2}{\sqrt{1 - \left(\frac{A_2}{A_1} \right)^2}} \sqrt{2 \cdot \rho \cdot (p_1 - p_2)}$$

Account for Friction Losses with discharge coefficient "fudge factor"

$$\rightarrow \dot{m} = \left(\frac{Cd \cdot A_2}{\sqrt{1 - \left(\frac{A_2}{A_1} \right)^2}} \right) \sqrt{2 \cdot \rho \cdot (p_1 - p_2)}$$

Incompressible Venturi (3)



•*Solve for* $V_2 \rightarrow V_2 = \sqrt{2 \cdot \left(\dfrac{p_1 - p_2}{\rho \cdot \left(1 - \left(\dfrac{A_2}{A_1} \right)^2 \right)} \right)}$

$$\textit{Calculate Massflow} \rightarrow \dot{m} = \rho \cdot A_2 \cdot V_2 = \frac{A_2}{\sqrt{1 - \left(\dfrac{A_2}{A_1} \right)^2}} \sqrt{2 \cdot \rho \cdot (p_1 - p_2)}$$

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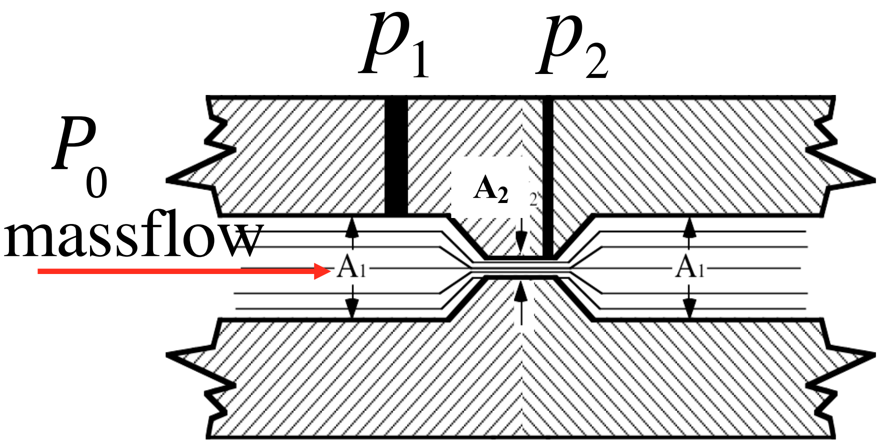
"Flow Coefficient"

$$C_v \equiv \left(\frac{Cd \cdot A_2}{\sqrt{1 - \left(\dfrac{A_2}{A_1} \right)^2}} \right) \rightarrow \boxed{\dot{m} = C_v \cdot \sqrt{2 \cdot \rho \cdot \Delta p}}$$

$$\Delta P = P_1 - P_2$$

Compressible Venturi (1)



- Now look at subcritical Compressible Venturi Flow where density is ***not*** constant throughout the Flow field.

$$\frac{P_0}{p_2} < \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

UnChoked 1-D Compressible Massflow Equation:

$$\dot{m} = A_1 \cdot P_0 \cdot \sqrt{\frac{\gamma}{R_g \cdot T_0}} \cdot \frac{M_1}{\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)}} \rightarrow \textit{Venturi Inlet}$$

$$\dot{m} = A_2 \cdot P_0 \cdot \sqrt{\frac{\gamma}{R_g \cdot T_0}} \cdot \frac{M_2}{\left(1+\frac{\gamma-1}{2}M_2^2\right)^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)}} \rightarrow \textit{Venturi Throat}$$

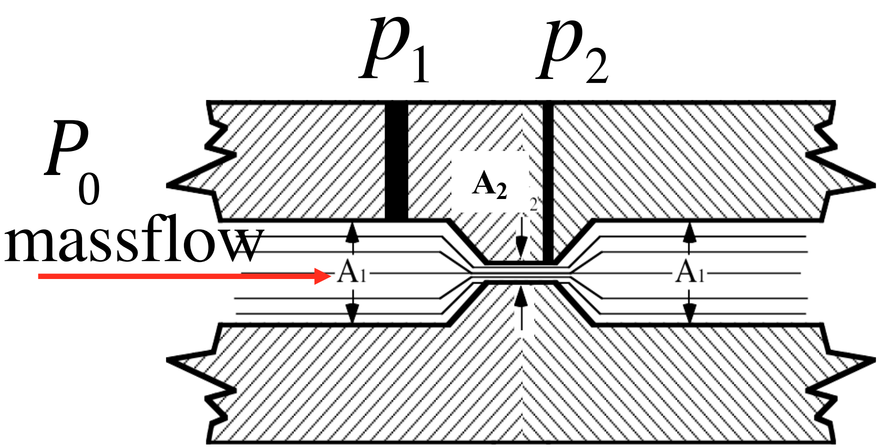
$$\textit{Isentropic Flow} \rightarrow \left(\frac{P_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \left(1+\frac{\gamma-1}{2}M_1^2\right)$$

$$\left(\frac{P_0}{p_2}\right)^{\frac{\gamma-1}{\gamma}} = \left(1+\frac{\gamma-1}{2}M_2^2\right)$$

$$M_1 = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

$$M_2 = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_0}{p_2}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

Compressible Venturi (2)

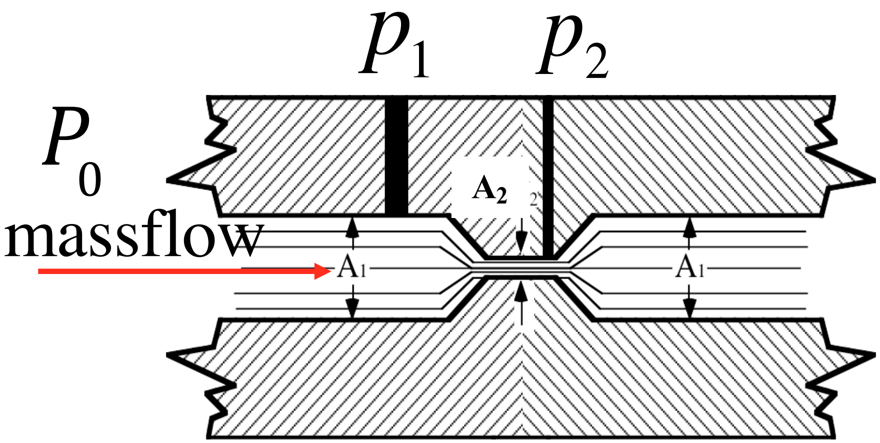


• Substitute and Simplify

$$\rightarrow \dot{m} = A_1 \cdot P_0 \cdot \frac{\sqrt{\left(\frac{\gamma}{R_g \cdot T_0}\right)\frac{2}{\gamma-1}\left[\left(\frac{P_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}}{\left(\frac{P_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}\cdot\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)}} = A_1 \cdot \frac{\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{P_0}{R_g \cdot T_0}\cdot P_0\left[\left(\frac{P_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}}{\left(\frac{P_0}{p_1}\right)^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma}\right)}}$$

$$= A_1 \cdot \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\rho_0 \cdot P_0\left[\left(\frac{P_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]\left(\frac{P_0}{p_1}\right)^{-\left(\frac{\gamma+1}{\gamma}\right)}} = A_1 \cdot \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\rho_0 \cdot P_0\left[\left(\frac{p_1}{P_0}\right)^{\frac{2}{\gamma}} - \left(\frac{p_1}{P_0}\right)^{\left(\frac{\gamma+1}{\gamma}\right)}\right]}$$

Compressible Venturi (3)



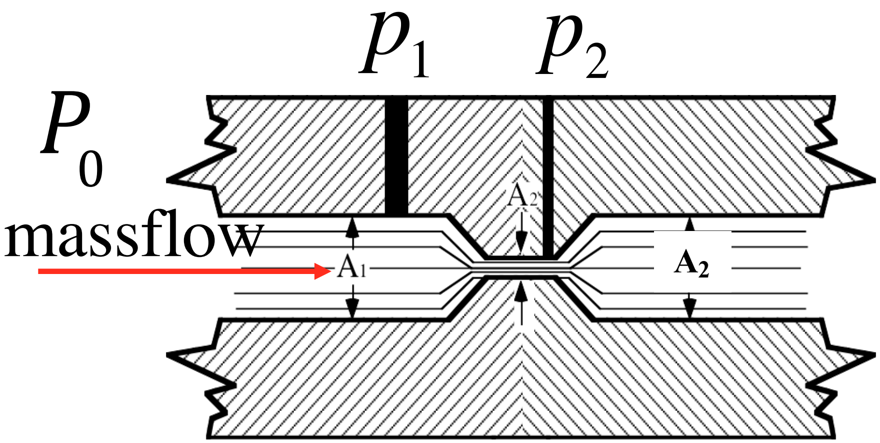
- **But What is P_0?**

$$\dot{m} = A_1 \cdot \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{P_0^{\,2}}{R_g \cdot T_0}\left[\left(\frac{p_1}{P_0}\right)^{\frac{2}{\gamma}} - \left(\frac{p_1}{P_0}\right)^{\left(\frac{\gamma+1}{\gamma}\right)}\right]}$$

and

$$\dot{m} = A_2 \cdot \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{P_0^{\,2}}{R_g \cdot T_0}\left[\left(\frac{p_2}{P_0}\right)^{\frac{2}{\gamma}} - \left(\frac{p_2}{P_0}\right)^{\left(\frac{\gamma+1}{\gamma}\right)}\right]}$$

Compressible Venturi (4)



$$\text{Since} \rightarrow \frac{\dot{m}}{P_0 \cdot \sqrt{\frac{\gamma}{R_g \cdot T_0}}} \text{ is constant}$$

$$A_1 \cdot \frac{M_1}{\left(1+\frac{\gamma-1}{2}M_1^{\,2}\right)^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)}} = A_2 \cdot \frac{M_2}{\left(1+\frac{\gamma-1}{2}M_2^{\,2}\right)^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)}}$$

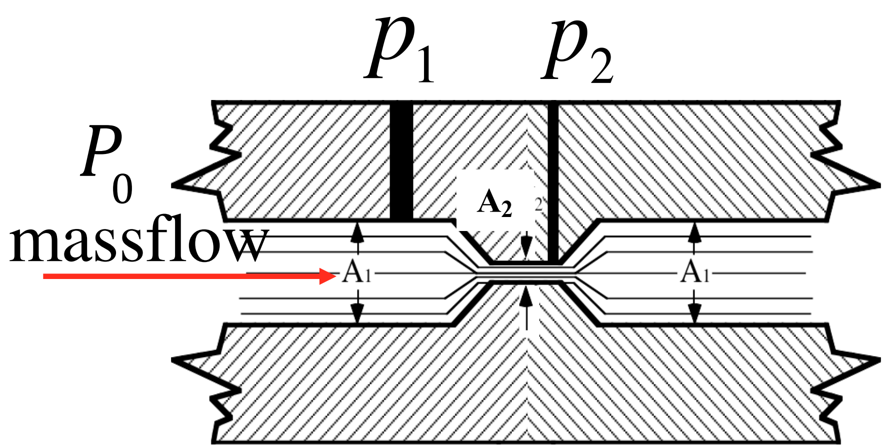
- Substitute

$$M_1 = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

$$M_2 = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_0}{p_2}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

$$\frac{A_1}{A_2} = \frac{M_2}{M_1} \cdot \frac{\left\{\left(\frac{P_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right\}^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)}}{\left\{\left(\frac{P_0}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\right\}^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)}} = \frac{M_2}{M_1} \cdot \left(\frac{p_2}{p_1}\right)^{\frac{\gamma+1}{2\gamma}} = \frac{\sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_0}{p_2}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}}{\sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}} \cdot \left(\frac{p_2}{p_1}\right)^{\frac{\gamma+1}{2\gamma}}$$

Compressible Venturi (5)



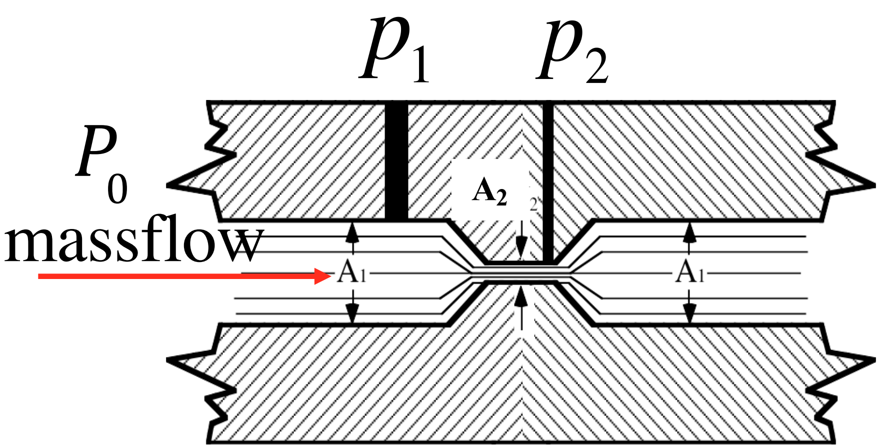
- **SOLVE for P_0**

$$\rightarrow \left(\frac{A_1}{A_2}\right)^2 \cdot (p_1)^{\frac{2}{\gamma}}\left[(P_0)^{\frac{\gamma-1}{\gamma}} - (p_1)^{\frac{\gamma-1}{\gamma}}\right] = (p_2)^{\frac{2}{\gamma}}\left[(P_0)^{\frac{\gamma-1}{\gamma}} - (p_2)^{\frac{\gamma-1}{\gamma}}\right]$$

$$(P_0)^{\frac{\gamma-1}{\gamma}}\cdot\left[\left(\frac{A_1}{A_2}\right)^2\cdot(p_1)^{\frac{2}{\gamma}} - (p_2)^{\frac{2}{\gamma}}\right] = \left[\left(\frac{A_1}{A_2}\right)^2\cdot(p_1)^{\frac{2}{\gamma}}\cdot(p_1)^{\frac{\gamma-1}{\gamma}} - (p_2)^{\frac{2}{\gamma}}\cdot(p_2)^{\frac{\gamma-1}{\gamma}}\right]$$

$$\rightarrow P_0 = \left[\frac{A_1^2\cdot(p_1)^{\frac{\gamma+1}{\gamma}} - A_2^2\cdot(p_2)^{\frac{\gamma+1}{\gamma}}}{A_1^2\cdot(p_1)^{\frac{2}{\gamma}} - A_2^2\cdot(p_2)^{\frac{2}{\gamma}}}\right]^{\frac{\gamma}{\gamma-1}} = \left[\frac{\left(\frac{A_1}{A_2}\right)^2\cdot(p_1)^{\frac{\gamma+1}{\gamma}} - (p_2)^{\frac{\gamma+1}{\gamma}}}{\left(\frac{A_1}{A_2}\right)^2\cdot(p_1)^{\frac{2}{\gamma}} - (p_2)^{\frac{2}{\gamma}}}\right]^{\frac{\gamma}{\gamma-1}}$$

Compressible Venturi (6)



- **Allow for Frictional Losses**

$$\frac{P_0}{p_2} < \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$P_0 = \left[\frac{\left(\frac{A_1}{A_2}\right)^2 \cdot (p_1)^{\frac{\gamma+1}{\gamma}} - (p_2)^{\frac{\gamma+1}{\gamma}}}{\left(\frac{A_1}{A_2}\right)^2 \cdot (p_1)^{\frac{2}{\gamma}} - (p_2)^{\frac{2}{\gamma}}}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\dot{m} = C_d \cdot A_1 \cdot \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\rho_0 \cdot P_0 \left[\left(\frac{p_1}{P_0}\right)^{\frac{2}{\gamma}} - \left(\frac{p_1}{P_0}\right)^{\left(\frac{\gamma+1}{\gamma}\right)}\right]}$$

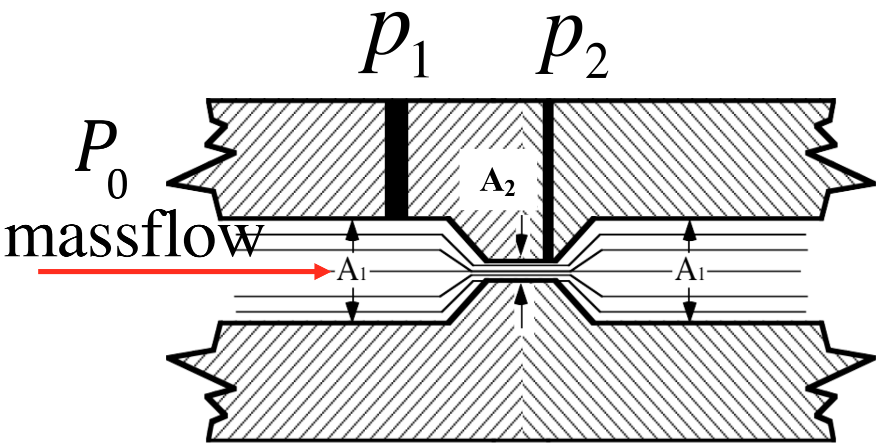
with friction losses
discharge coefficient

$\rightarrow (1) \rightarrow$ *Venturi Inlet*
$(2) \rightarrow$ *Venturi Throat*
$(3) \rightarrow$ *Venturi Outlet*

$$\rightarrow C_d \sim \sqrt{1 - \frac{P_{inlet} - P_{outlet}}{P_{inlet}}} = \sqrt{\frac{P_{inlet} + P_{outlet}}{P_{inlet}}}$$

$p_1 \rightarrow$ *Venturi Inlet Pressure*
$p_1 \rightarrow$ *Venturi Inlet Pressure*
$P_0 \rightarrow$ *Venturi Stagnation Pressure*
$T_0 \rightarrow$ *Venturi Block Temperature*
$Cd \cdot A_1 \rightarrow$ *Venturi InletDischarge Area*

Compressible Venturi (7)



- **Supercritical Flow (Choked Throat)**

$$\frac{P_0}{p_2} \geq \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\dot{m} = C_d \cdot A_2^* \cdot \frac{P_0}{\sqrt{T_0}} \cdot \sqrt{\frac{\gamma}{R_g} \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

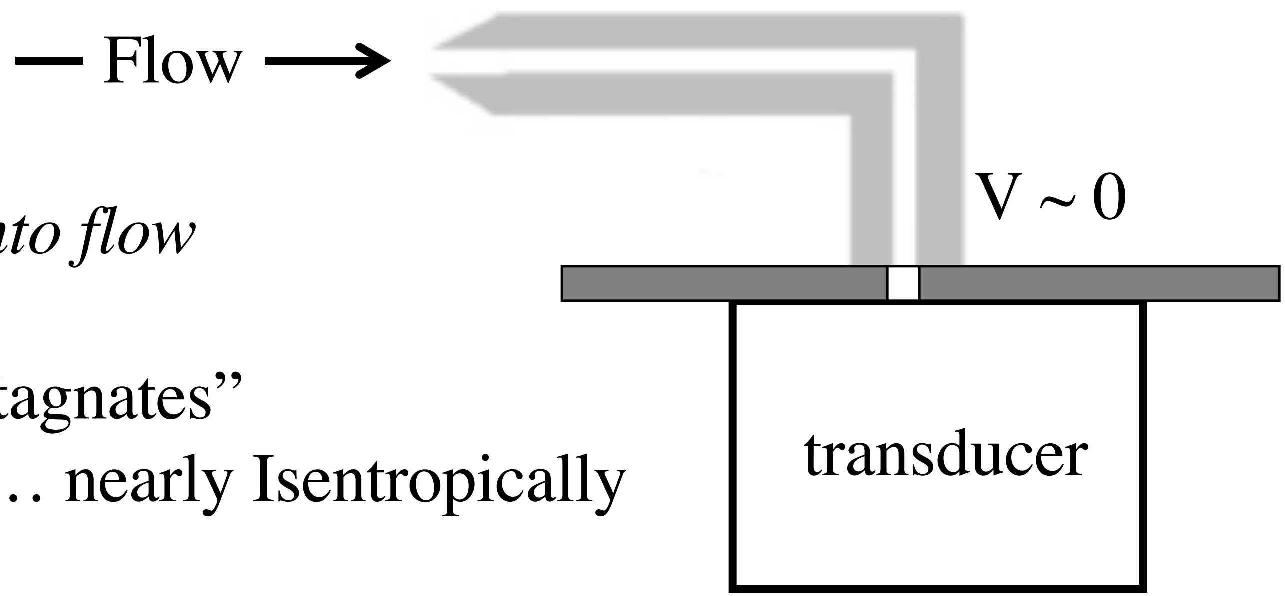
$$P_0 = p_2^* \cdot \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

How do we measure stagnation pressure? (1)

Bernoulli's Equation:

static pressure + dynamic pressure = total pressure

Measure difference in total and static pressure



- Probe *Looks into flow*

- Captures or "stagnates"

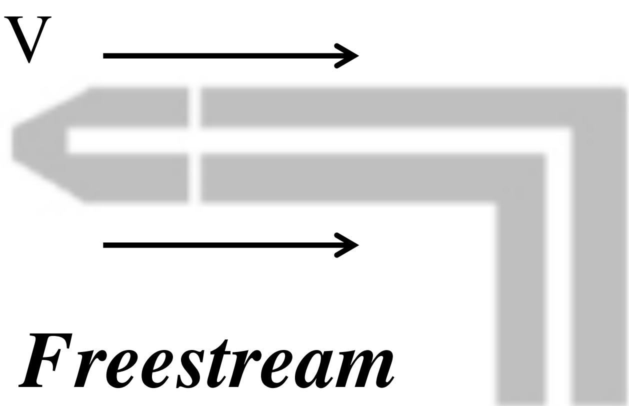
Incoming flow … nearly Isentropically

… transducer senses total pressure
(included effects of flow kinetic energy)

….Pitot Probe

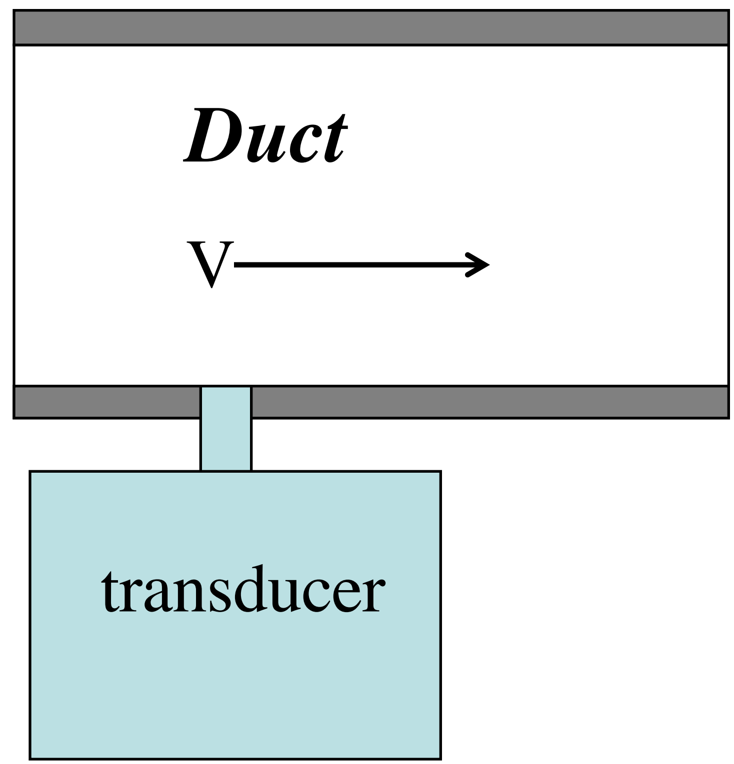
How do we measure *"normal"* or *static* pressure?

….Static Source Port or pressure tap



• Port is *perpendicular* to The flow field ….

Sensed pressure does not capture effect of external flow kinetic energy



How do we Measure Airspeed or Mach number? (1)

- Senses Both Total and Static Pressure from incoming flow field

- Incompressible …. $P_{stagnation} = \frac{1}{2}\rho V^2 + P_{static} \rightarrow \boxed{V = \sqrt{2\frac{\left(P_{stagnation} - P_{static}\right)}{\rho}}}$

- density typically based on sea level number …
  … probe gives "*indicated*" airspeed .. Must be *calibrated* for
  … localized density effects

How do we Measure Airspeed or Mach number? (2)

• Senses Both Total and Static Pressure from incoming flow field

• Compressible, Subsonic ….

$$M = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{P_{stagnation}}{P_{static}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

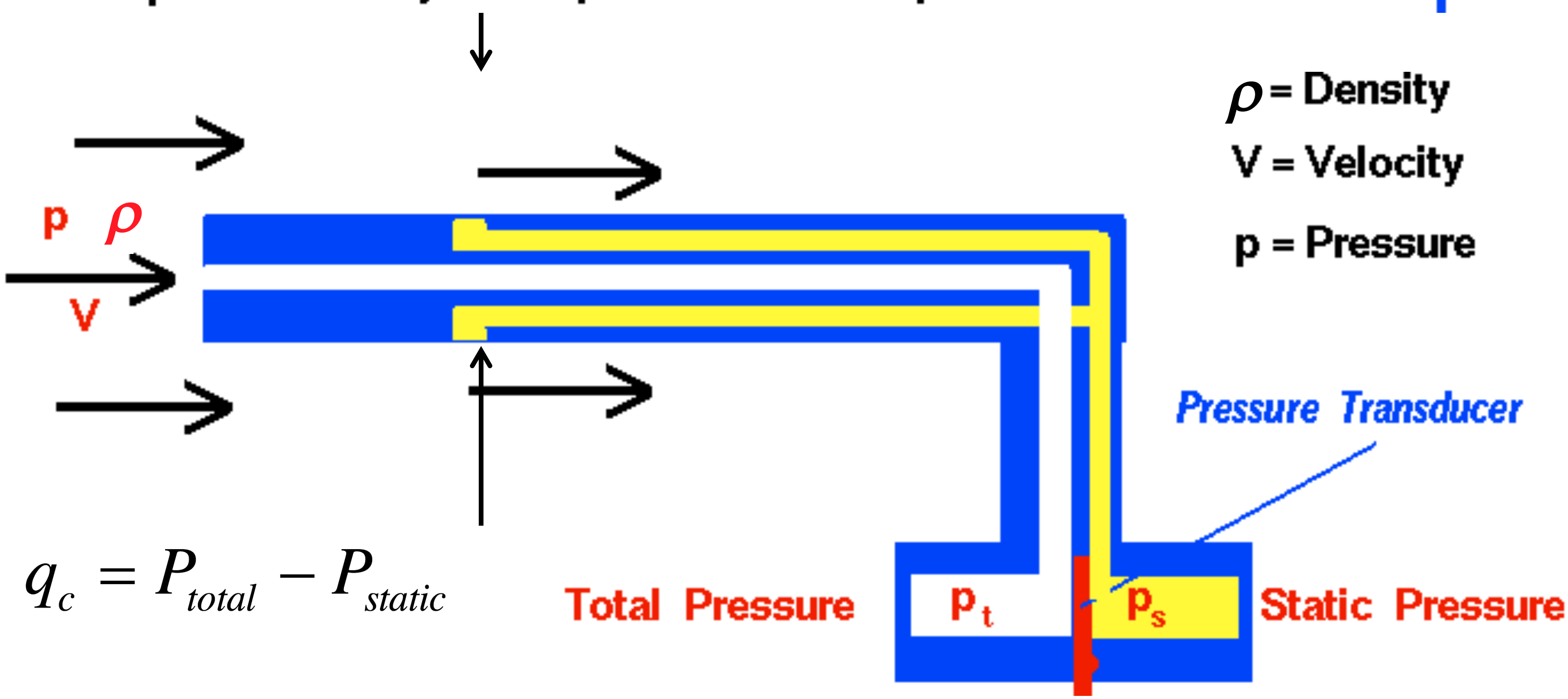
•.. Typically Probe static pressure is influenced by the Aircraft and *indicated Mach* number … must be *calibrated* for … localized vehicle effects

Pitot / Static Probe Details (1)

Bernoulli's Equation:

Measure difference in total and static pressure

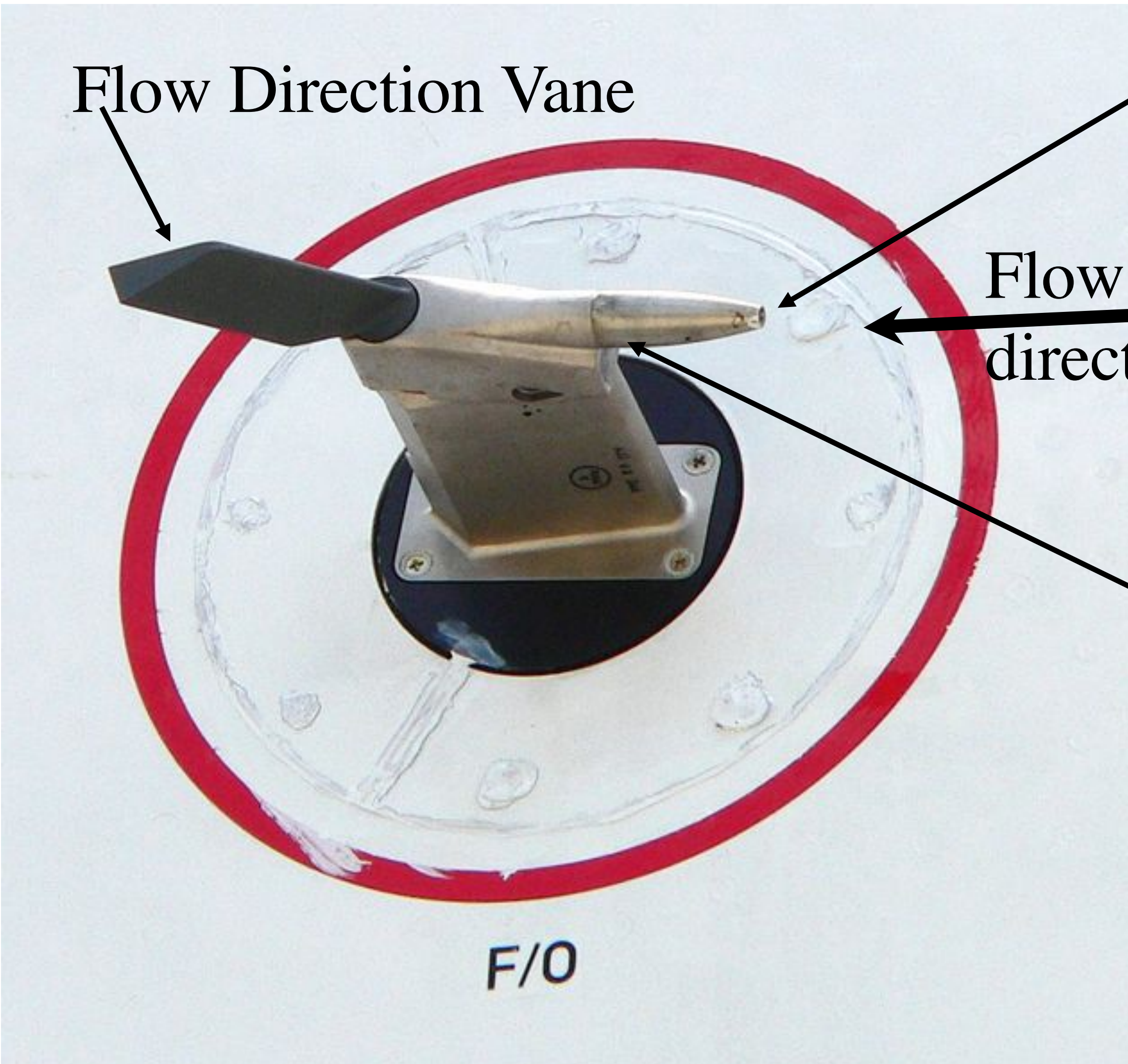
static pressure + dynamic pressure = total pressure



$$q_c = P_{total} - P_{static}$$

- White -- Total or Stagnation Pressure
- Yellow -- “normal” or Static Pressure

Pitot / Static Probe Details (2)

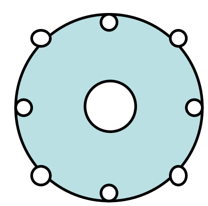


Pitot Probe

Flow direction

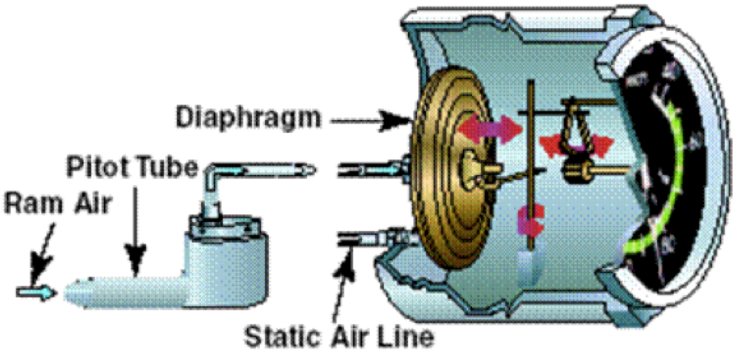
Airliners have pitot/static probes protruding from the cockpit area for measuring airspeed

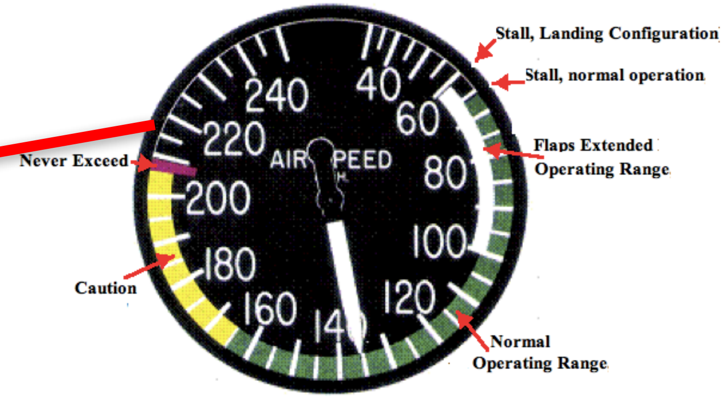
Static Source Holes

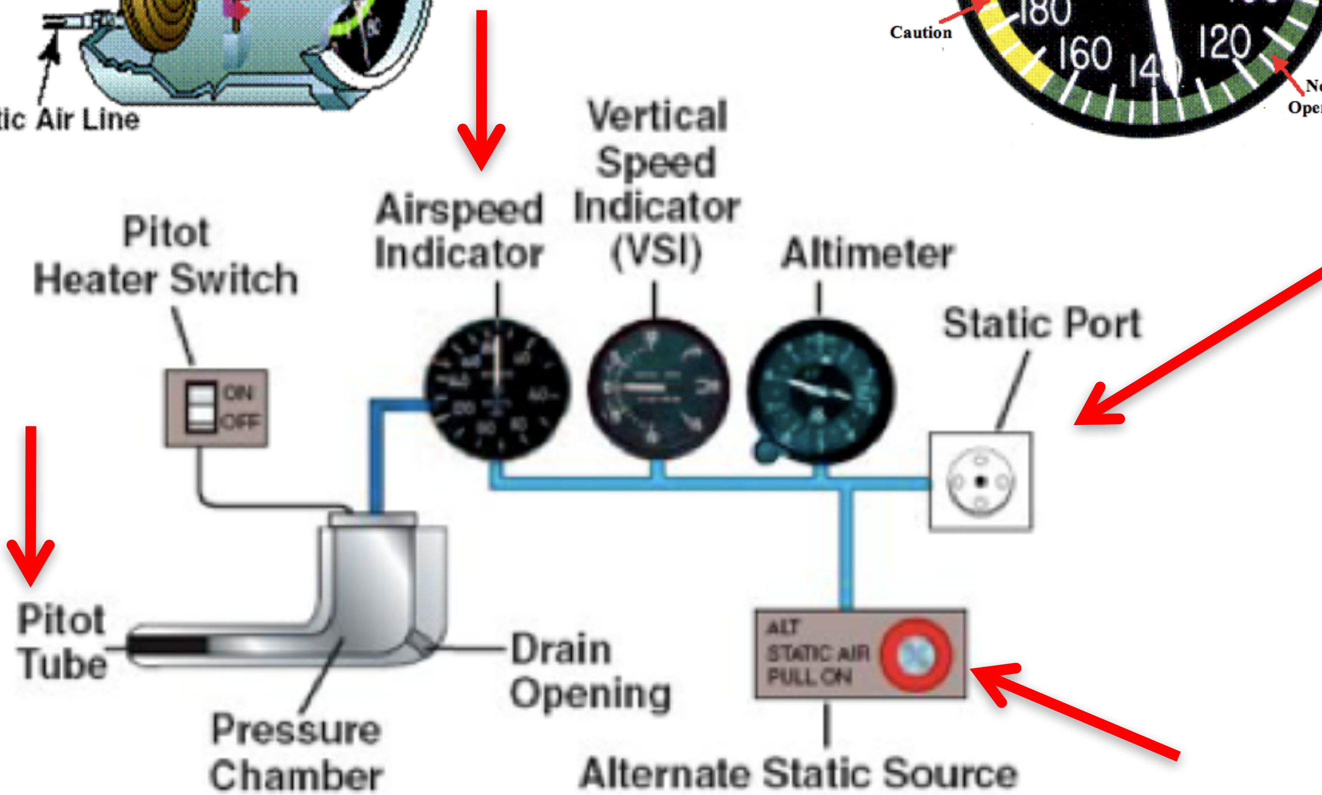


Looking along flow direction

Typical Small Aircraft Airspeed System







US Military Airspeed Definitions

•**Indicated Airspeed (IAS)**—*As shown on the airspeed indicator.*

•**Calibrated Airspeed (CAS)** —*Indicated corrected for instrument and installation error.*

•**Equivalent Airspeed (EAS)** – *CAS corrected for adiabatic compressible flow at a the measurement altitude. EAS and CAS are equal for a standard atmosphere at sea level* **… *typically expressed in units of nautical miles/hour (kts)***

•**True Airspeed**—*CAS corrected for temperature and pressure (actual speed through airstream)*

Indicated Airspeed (IAS)

Indicated airspeed (IAS) means the speed of an aircraft as shown on its pitot static airspeed indicator calibrated to reflect standard atmosphere adiabatic compressible flow at sea level uncorrected for airspeed system errors [1].

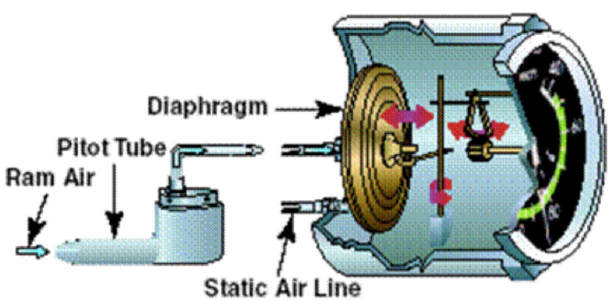
Standard Formula:

$$IAS = \sqrt{7\frac{p_{sl}}{\rho_{sl}} \cdot \left[\left(\frac{P_{total} - p_\infty}{p_{sl}} + 1\right)^{\frac{1}{3.5}} - 1\right]} \rightarrow$$

$$p_{sl} = 101.325_{kPa} = 2116.2_{psf} = 14.696_{psia}$$

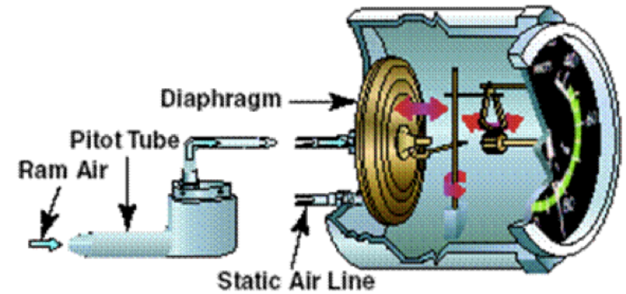
$$\rho_{sl} = 1.225_{kg/m^3} = 7.6314 \times 10^{-2}{}_{lbm/ft^3}$$

$$P_{total} - p_\infty = q_{c_{indicated}}$$



Indicated Airspeed (Derivation)

• *Airspeed as shown on the airspeed indicator – calibrated to reflected standard atmosphere, adiabatic compressible flow with no static sensor position error calibr*



$$\frac{P_{0_{ind}}}{p_{ind}} = \left(1 + \frac{\gamma - 1}{2} \cdot M_{ind}^{\ 2}\right)^{\frac{\gamma}{\gamma-1}}$$

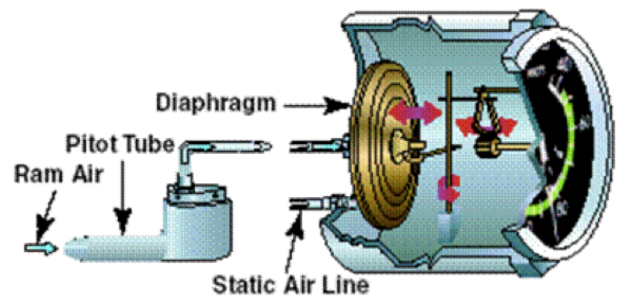
$$\rightarrow M_{ind} = \sqrt{\frac{2}{\gamma - 1} \cdot \left[\left(\frac{P_{0_{ind}}}{p_{ind}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} = \sqrt{\frac{2}{\gamma - 1} \cdot \left[\left(\frac{P_{0_{ind}} - p_{ind}}{p_{ind}} + 1\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} =$$

$$\sqrt{\frac{2}{\gamma - 1} \cdot \left[\left(\frac{q_{c_{ind}}}{p_{ind}} + 1\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} \rightarrow q_{c_{ind}} \text{ "indicated impact (dynamic) pressure"}$$

Indicated Airspeed (Derivation) (2)

$$V_{ind} \equiv M_{ind} \cdot \sqrt{\gamma \cdot R_g \cdot T_{sl}} = \sqrt{\frac{2 \cdot \gamma}{\gamma - 1} \cdot \left(R_g \cdot T_{sl}\right) \left[\left(\frac{q_{c_{ind}}}{p_{ind}} + 1 \right)^{\frac{\gamma-1}{\gamma}} - 1 \right]} =$$

$$\sqrt{\frac{2 \cdot \gamma}{\gamma - 1} \cdot \left(\frac{p_{sl}}{\rho_{sl}} \right) \left[\left(\frac{q_{c_{ind}}}{p_{ind}} + 1 \right)^{\frac{\gamma-1}{\gamma}} - 1 \right]}$$



•Since Typical Airspeed system only measures impact pressure $q_{c_{ind}}$ and not static pressure p_{ind} directly … replace → p_{ind} by p_{sl}

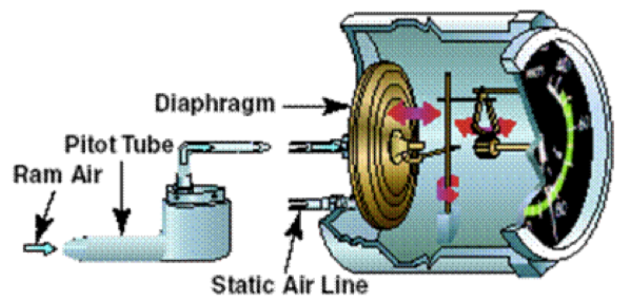
$$IAS = \sqrt{\frac{2 \cdot \gamma}{\gamma - 1} \cdot \left(\frac{p_{sl}}{\rho_{sl}} \right) \left[\left(\frac{q_{c_{ind}}}{p_{sl}} + 1 \right)^{\frac{\gamma-1}{\gamma}} - 1 \right]} = \sqrt{7 \frac{p_{sl}}{\rho_{sl}} \cdot \left[\left(\frac{P_{total} - p_{\infty}}{p_{sl}} + 1 \right)^{\frac{1}{3.5}} - 1 \right]}$$

Indicated Airspeed (Derivation) (3)

- *Evaluate using standard values for parameters …. Express results in Kts (minute of longitude at the equator is equal to 1 nautical mile.)*

$$\left(V_{IAS}\right)_{kts} \equiv 1.9438\sqrt{\frac{2\cdot(1.4)}{(1.4)-1}\cdot\left(\frac{101325_{pa}}{1.225_{\frac{kg}{m^3}}}\right)\left[\left(\frac{q_{c_{ind}}}{101325_{pa}}+1\right)^{\frac{(1.4)-1}{(1.4)}}-1\right]} =$$

$$1479.08\sqrt{\left[\left(\frac{q_{c_{ind}}}{2116.2_{psf}}+1\right)^{0.2857}-1\right]}$$



Indicated Airspeed (Derivation) (4)

- *Alternate version of IAS .. Sometimes still in use*

$$q_{c_{ind}} = \frac{1}{2} \cdot \rho_{sl} \cdot V_{IAS}{}^{2} \rightarrow \boxed{\left(V_{IAS}\right)_{kts} = 1.9438 \sqrt{\frac{2 \cdot \left(q_{c_{ind}}\right)_{psf} \cdot \dfrac{101325_{pa}}{2116.2_{psf}}}{1.225}} \approx 17.186 \cdot \sqrt{\left(q_{c_{ind}}\right)_{psf}}}$$

Example …. $q_{c_{ind}}$ = 250 psf

$$\begin{bmatrix} definition\ 1 \ldots \left(V_{IAS}\right)_{kts} \\ \\ definition\ 2 \ldots \left(V_{IAS}\right)_{kts} \end{bmatrix} = \begin{bmatrix} 1479.08\left(\left(\dfrac{250}{2116.2} + 1\right)^{0.2857} - 1\right)^{0.5} \\ \\ 17.185\ (250^{0.5}) \end{bmatrix} = \begin{bmatrix} 266.31\ \text{kts} \\ \\ 271.72\ \text{kts} \end{bmatrix}$$

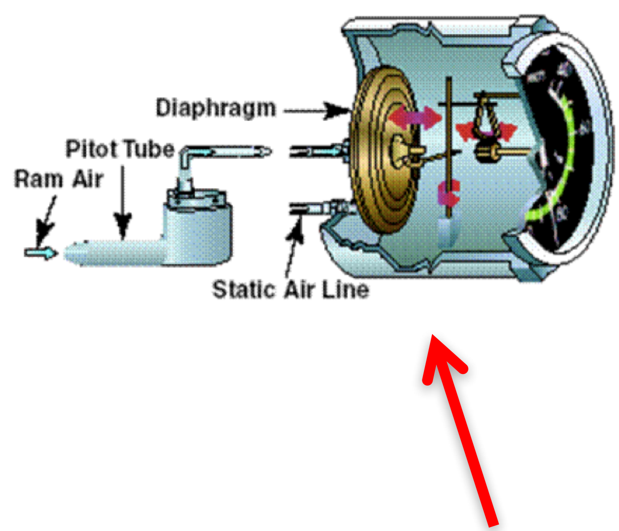
Calibrated Airspeed (CAS)

Calibrated airspeed (CAS) means indicated airspeed of an aircraft, corrected for position and instrument error. Calibrated airspeed is equal to true airspeed in standard atmosphere at sea level [1].

Position and instrument errors are usually small, and if their values are not known, it is reasonable to assume

$$CAS \cong IAS \qquad (2)$$

• *Primary Instrumentation is for the* ***"static source position error"*** *of just* ***"position error"*** *.... corrects the indicated airspeed for known static pressure measurement source errors ...* δp_{static}

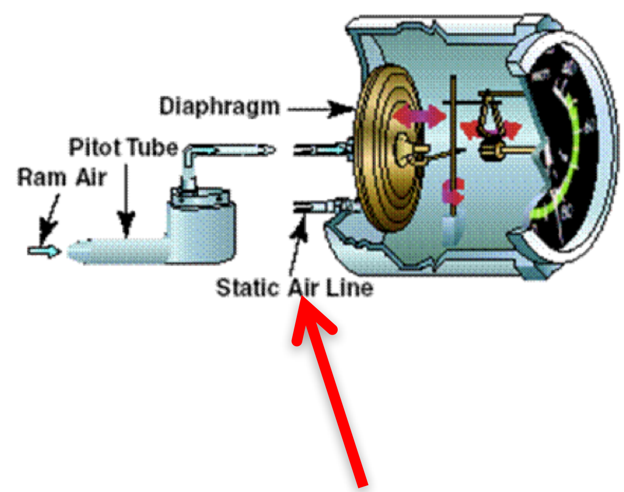


Calibrated Airspeed (CAS) (2)

•*Correct indicated airspeed for known static pressure measurement source errors … δp_{static}*

$$V_{CAS} \equiv \sqrt{\frac{2\cdot\gamma}{\gamma-1}\cdot\left(\frac{p_{sl}}{\rho_{sl}}\right)\left[\left(\frac{P_{0_{ind}}-\left(p_{ind}-\delta p_{static}\right)}{p_{sl}}+1\right)^{\frac{\gamma-1}{\gamma}}-1\right]} =$$

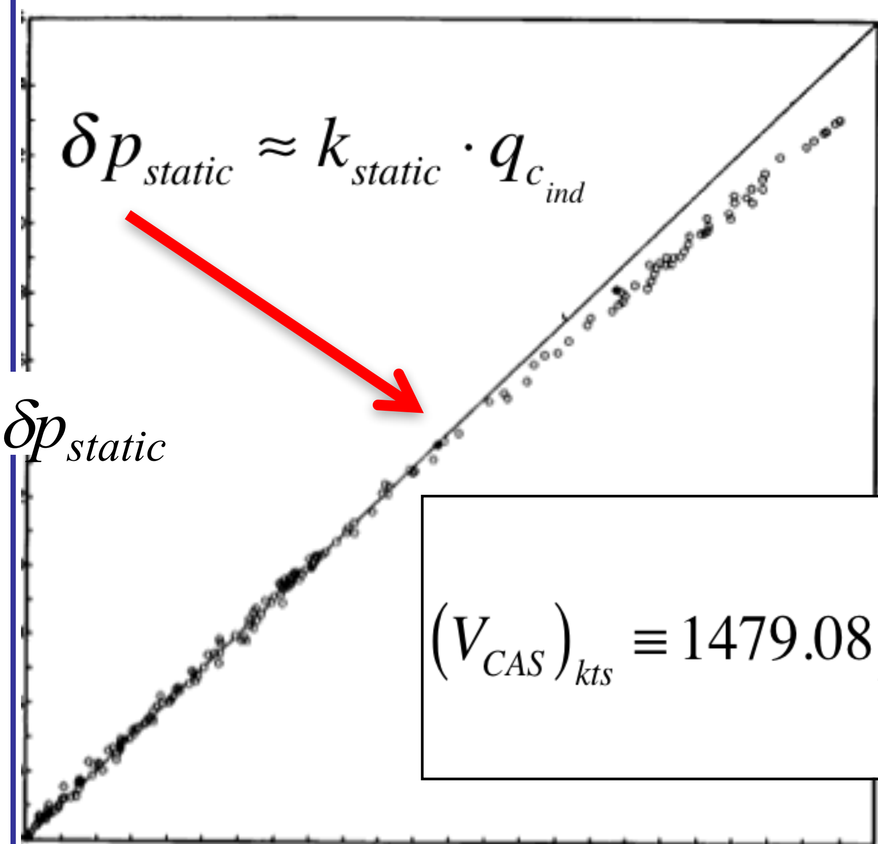
$$\sqrt{\frac{2\cdot\gamma}{\gamma-1}\cdot\left(\frac{p_{sl}}{\rho_{sl}}\right)\left[\left(\frac{q_{c_{ind}}+\delta p_{static}}{p_{sl}}+1\right)^{\frac{\gamma-1}{\gamma}}-1\right]}$$

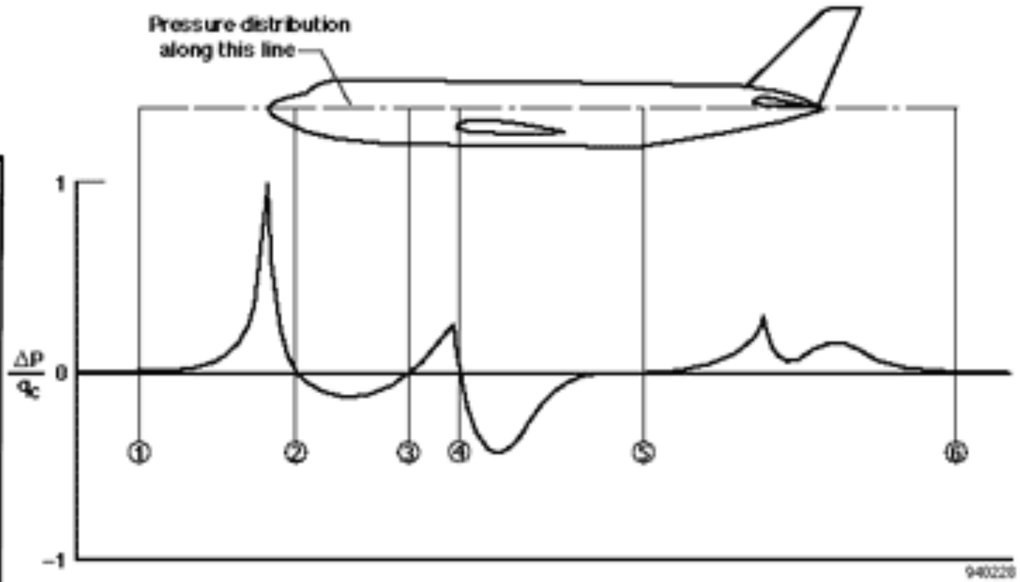


Calibrated Airspeed (CAS) (3)

• *Typically … δp_{static} is a calibrated function of indicated impact pressure*

$$\delta p_{static} \approx k_{static} \cdot q_{c_{ind}}$$





$$\left(V_{CAS}\right)_{kts} \equiv 1479.08\sqrt{\left[\left(\left(\frac{q_{c_{ind}}}{2116.2_{psf}}\right)\left(1+k_{static}\right)+1\right)^{0.2857}-1\right]}$$

Equivalent Airspeed (EAS)

Equivalent airspeed (EAS) means the calibrated airspeed of an aircraft corrected for adiabatic compressible flow for the particular altitude. Equivalent airspeed is equal to calibrated airspeed in standard atmosphere at sea level [1]. By this definition, assuming that position and instrument errors are small $p_{ind} \approx p_\infty$

Standard Formula:

$$EAS = \sqrt{7\frac{p_\infty}{\rho_{sl}} \cdot \left[\left(\frac{P_{total} - p_\infty}{p_\infty} + 1\right)^{\frac{1}{3.5}} - 1\right]}$$

Equivalent Airspeed (Derivation) (2)

• *CAS corrected for adiabatic compressible flow at the measurement altitude.*

$$V_{EAS} = \sqrt{\frac{2\cdot\gamma}{\gamma-1}\cdot\left(\frac{p_{ind}}{\rho_{sl}}\right)\left[\left(\frac{q_{c_{ind}}}{p_{ind}}\left(1+k_{static}\right)+1\right)^{\frac{\gamma-1}{\gamma}}-1\right]} \rightarrow$$

$$KEAS = \left(V_{EAS}\right)_{kts} = 1.9438\sqrt{\frac{2\cdot(1.4)}{(1.4)-1}\cdot\left(\frac{p_{ind}}{1.225\,\frac{kg}{m^3}}\right)\left[\left(\frac{q_{c_{ind}}}{p_{ind}}\left(1+k_{static}\right)+1\right)^{\frac{(1.4)-1}{(1.4)}}-1\right]} =$$

$$32.1523\cdot\sqrt{\left(p_{ind}\right)_{psf}\left[\left(\left(\frac{q_{c_{ind}}}{p_{ind}}\right)_{psf}\left(1+k_{static}\right)+1\right)^{0.2857}-1\right]}$$

True Airspeed (TAS)

•*CAS corrected for temperature and pressure (actual speed through airstream)*

True airspeed (TAS) means the airspeed of an aircraft relative to undisturbed air. True airspeed is equal to equivalent airspeed multiplied by

$$\sigma = \frac{\rho_\infty}{\rho_{\substack{sea\\level}}}$$

Air density of free-stream flow divided by standard sea level air density

$$\left(V_{TAS}\right)_{kts} = \frac{KEAS}{\sqrt{\sigma}} \rightarrow \sigma = local\ density\ ratio \rightarrow \frac{\rho_\infty}{\rho_{\substack{sea\\level}}}$$

True Airspeed (2)

•*CAS corrected for temperature and pressure (actual speed through airstream)*

$$V_{TAS} = \sqrt{\frac{2\cdot\gamma}{\gamma-1}\cdot\left(\frac{p_{ind}}{\rho_{true}}\right)\left[\left(\frac{q_{c_{ind}}}{p_{ind}}\left(1+k_{static}\right)+1\right)^{\frac{\gamma-1}{\gamma}}-1\right]} \rightarrow V_{TAS} = V_{EAS}\cdot\sqrt{\frac{\rho_{sl}}{\rho_{true}}}$$

$$M_{true} = \frac{TAS}{c_\infty}$$

Mach number means the ratio of true air speed to the speed of sound [1] at the flight altitude.

where c_∞ is the speed of sound at the flight altitude.

Airspeed Definition Comparisons

For a standard day, assuming that there are no position and instrument errors.

h [ft]	IAS [kts]	CAS [kts]	EAS [kts]	TAS [kts]	M
0	280	280	280	280	0.42
10000	280	280	277	323	0.51
20000	280	280	273	375	0.61
25000	280	280	271	405	0.67
30000	280	280	268	437	0.74

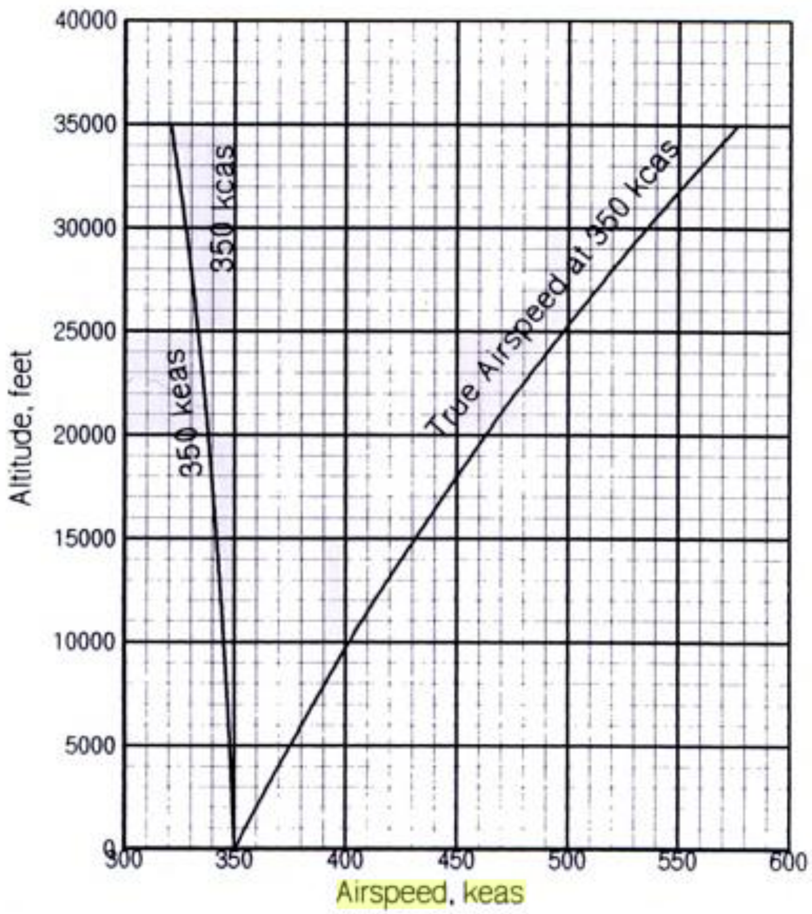
This example shows that *TAS* and Mach number increases by altitude although *IAS* is constant for all altitudes. This situation implies that the air traffic controller must be careful when two aircraft are descending towards same fix from two different altitudes but reporting same indicated airspeeds.

Airspeed Definition Comparisons (2)

	Airspeed[a]			
Altitude, ft	kcas	keas	ktas	Mach no.
Sea level	250	250.0	250.0	0.378
10,000	250	248.1	288.7	0.452
20,000	250	245.2	335.9	0.547
25,000	250	243.3	363.4	0.604
30,000	250	240.8	393.7	0.668
35,000	250	237.8	427.2	0.741
40,000	250	234.2	472.0	0.823
Sea level	350	350.0	350.0	0.529
10,000	350	345.1	401.5	0.629
20,000	350	337.9	462.9	0.754
25,000	350	333.2	497.7	0.827
30,000	350	327.6	535.5	0.909
35,000	350	320.8	576.4	1.0

[a]Where kcas = knots calibrated airspeed, keas = knots equivalent airspeed, and ktas = knots true airspeed (actual speed over the ground in still air).

Airspeed Definition Comparisons (3)



Calibrated and true airspeeds vs equivalent airspeeds.

Altitude, ft	Airspeed[a] kcas	keas	ktas	Mach no.
Sea level	250	250.0	250.0	0.378
10,000	250	248.1	288.7	0.452
20,000	250	245.2	335.9	0.547
25,000	250	243.3	363.4	0.604
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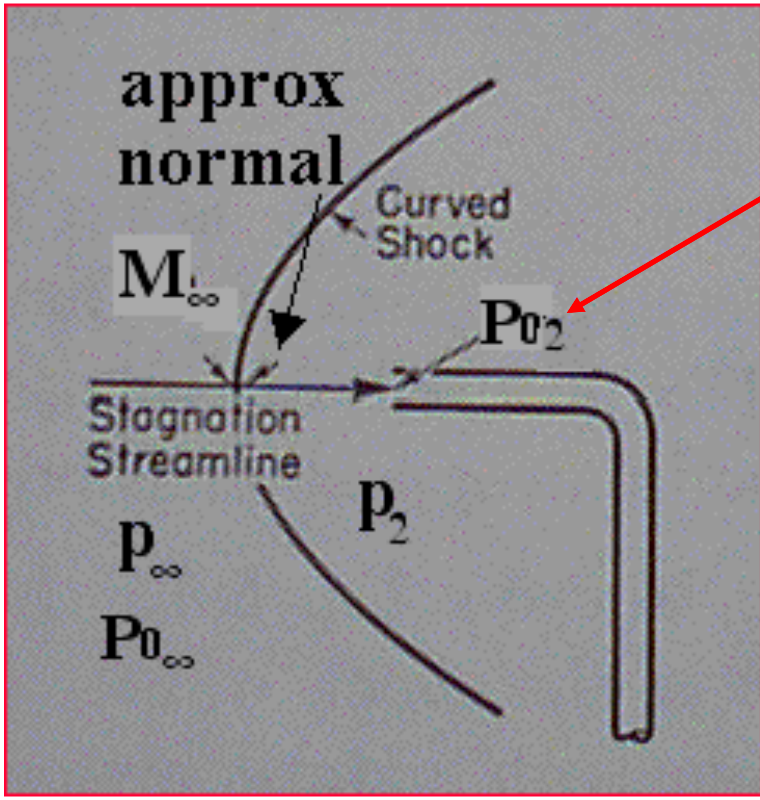
[a]Where kcas = knots calibrated airspeed, keas = knots equivalent airspeed, and ktas = knots true airspeed (actual speed over the ground in still air).

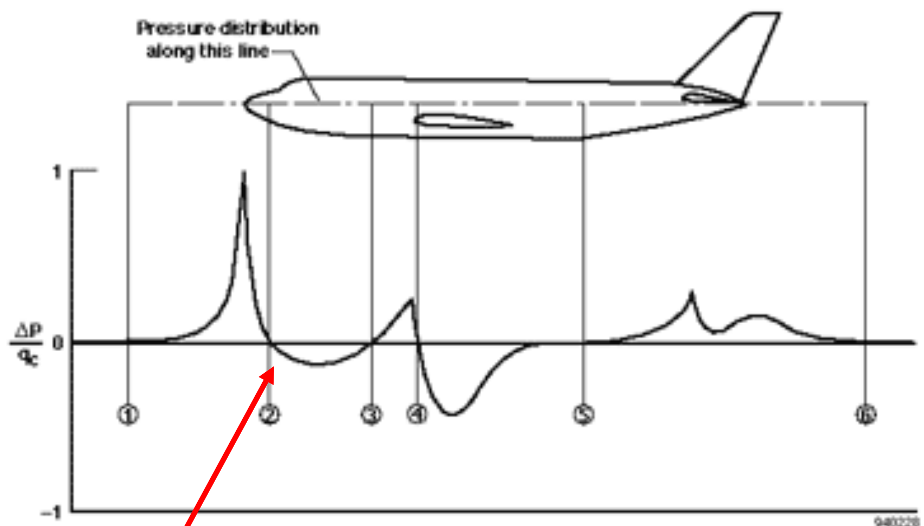
What About Supersonic Flow? (1)

- Cannot Directly Measure Supersonic Total Pressure
- Normal Shock wave forms in front of probe …



- Probe senses P_{02} ***(behind shock wave)***

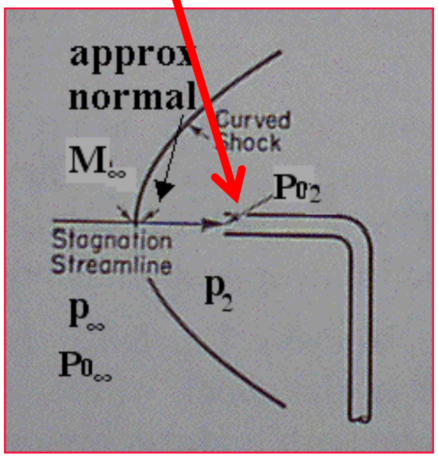
Not! .. P_{01}



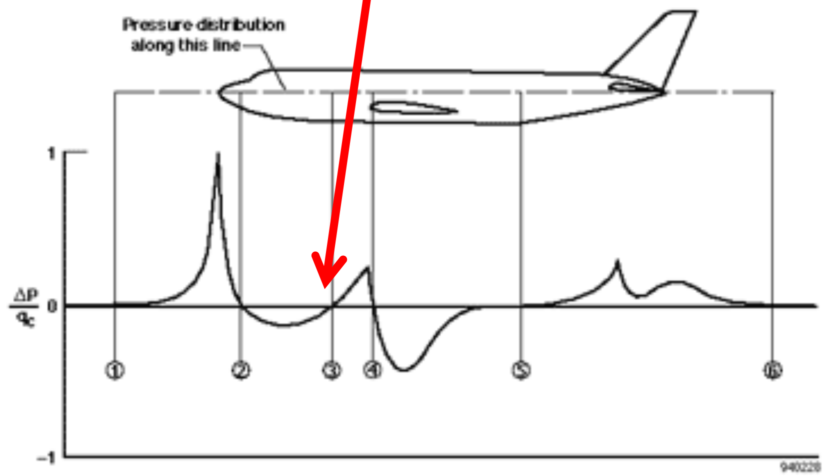
- *Static orifices far* enough aft to not be strongly influenced by normal shockwave (*position error must be calibrated for*)

Supersonic Pitot Probe Calculation (2)

P_{0_2}



- Assume direct $P_{\infty}(indicated)$ measurement



- Calculate Indicated Mach Number from **Rayleigh-Pitot Equation**

Lets derive this ….

$$\frac{P_{0_2}}{P_{\infty(indicated)}} = \frac{\left[\frac{\gamma+1}{2} M_{ind}^2\right]^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1} M_{ind}^2 - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}}$$

Rayleigh-Pitot Equation Derivation

$$\frac{P_{0_2}}{p_\infty} = \frac{P_{0_2}}{P_{0_\infty}} \cdot \frac{P_{0_\infty}}{p_\infty}$$

- ***From Isentropic Flow***

$$\frac{P_{0_\infty}}{p_\infty} = \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}$$

- ***From Normal Shockwave Eqs.***

$$\frac{P_{0_2}}{P_{0_\infty}} = \left\{ \frac{2}{(\gamma+1)\cdot\left(\gamma M_\infty^2 - \frac{\gamma-1}{2}\right)^{\frac{1}{\gamma-1}}} \left[\frac{\left(\frac{\gamma+1}{2} M_\infty\right)^2}{1 + \frac{\gamma-1}{2} M_\infty^2} \right]^{\frac{\gamma}{\gamma-1}} \right\}$$

Rayleigh-Pitot Equation Derivation

$$\frac{P_{0_2}}{p_\infty} = \frac{P_{0_2}}{P_{0_1}} \times \frac{P_{0_1}}{p_\infty} = \left\{ \frac{2}{(\gamma+1)\left(\gamma M_\infty^2 - \frac{\gamma-1}{2}\right)^{\frac{1}{\gamma-1}}} \left[\frac{\left[\left(\frac{\gamma+1}{2}\right) M_\infty\right]^2}{\left(1 + \left(\frac{\gamma-1}{2}\right) M_\infty^2\right)} \right]^{\frac{\gamma}{\gamma-1}} \right\} \times \left(1 + \left(\frac{\gamma-1}{2}\right) M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$= \frac{2}{(\gamma+1)\left(\gamma M_\infty^2 - \frac{\gamma-1}{2}\right)^{\frac{1}{\gamma-1}}} \left[\left(\frac{\gamma+1}{2}\right)^2 M_\infty^2\right]^{\frac{\gamma}{\gamma-1}} = \frac{1}{\left(\gamma M_\infty^2 - \frac{\gamma-1}{2}\right)^{\frac{1}{\gamma-1}}} \left[\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma-1}{\gamma}} \left(\frac{\gamma+1}{2}\right)\right]^{\frac{\gamma}{\gamma-1}} \left[\left(\frac{\gamma+1}{2}\right) M_\infty^2\right]^{\frac{\gamma}{\gamma-1}} =$$

$$\frac{\left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}}}{\left(\gamma M_\infty^2 - \frac{\gamma-1}{2}\right)^{\frac{1}{\gamma-1}}} \left[\left(\frac{\gamma+1}{2}\right) M_\infty^2\right]^{\frac{\gamma}{\gamma-1}} = \frac{\left[\left(\frac{\gamma+1}{2}\right) M_\infty^2\right]^{\frac{\gamma}{\gamma-1}}}{\left(\left(\frac{2}{\gamma+1}\right)\left(\gamma M_\infty^2 - \frac{\gamma-1}{2}\right)\right)^{\frac{1}{\gamma-1}}} \rightarrow \boxed{\frac{P_{0_2}}{p_\infty} = \frac{\left[\frac{\gamma+1}{2} M_\infty^2\right]^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1} M_\infty^2 - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}}}$$

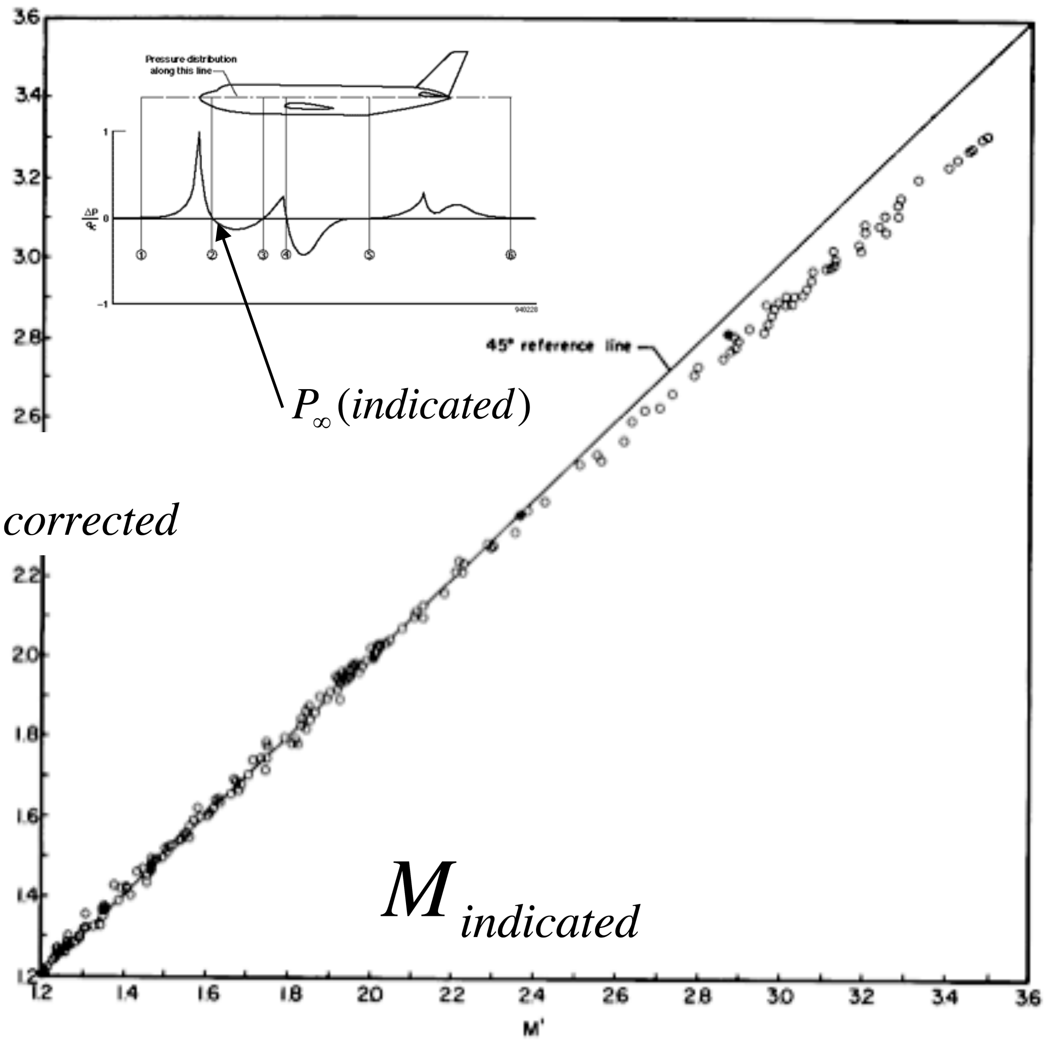
Rayleigh-Pitot Equation

Supersonic Mach Number Calibration

- Apply Calibration Correction for errors In p_∞ (indicated)

Rayleigh-Pitot Equation

$$\frac{P_{0_2}}{P_{\infty(indicated)}} = \frac{\left[\frac{\gamma+1}{2} M_{ind}^{2}\right]^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1} M_{ind}^{2} - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}}$$



Supersonic position-error calibration for the nose-boom system.

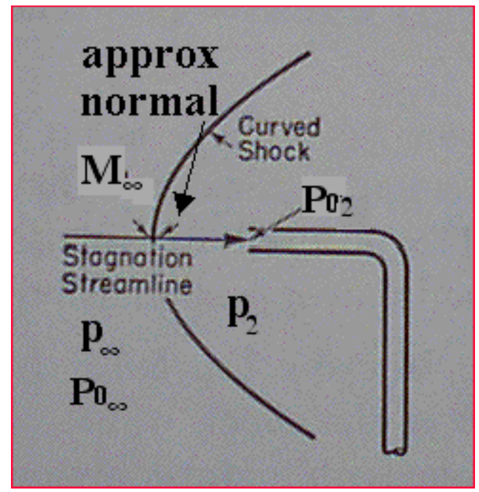
Alternate Form For Rayleigh Pitot Equation

- Use …. p_2 (indicated)

$$\frac{P_{0_2}}{p_2} = \frac{P_{0_2}}{P_{0_\infty}} \cdot \frac{P_{0_\infty}}{p_\infty} \cdot \frac{p_\infty}{p_2} = \left\{ \frac{2}{(\gamma+1)\cdot\left(\gamma M_\infty^2 - \frac{\gamma-1}{2}\right)^{\frac{1}{\gamma-1}}} \left[\frac{\left(\frac{\gamma+1}{2} M_\infty\right)^2}{1+\frac{\gamma-1}{2}M_\infty^2} \right]^{\frac{\gamma}{\gamma-1}} \right\} \times \left\{ \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{\gamma}{\gamma-1}} \right\} \times \left\{ \frac{1}{1+\frac{2\gamma}{\gamma+1}\left(M_\infty^2-1\right)} \right\}$$

- Simplify

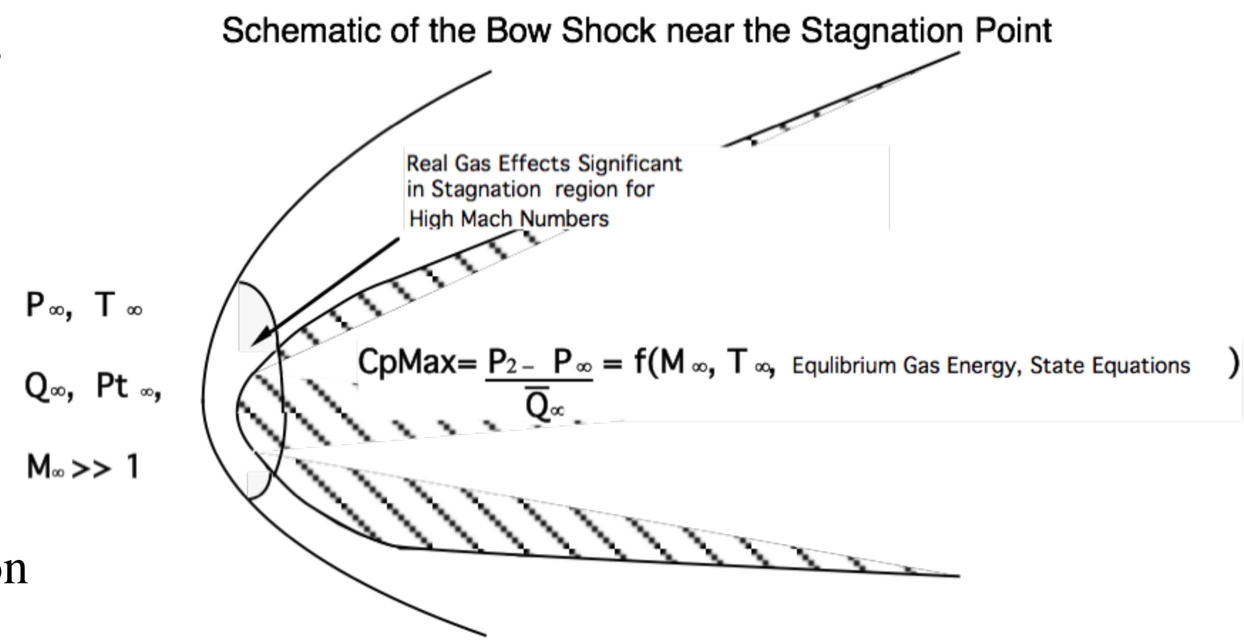
$$\frac{P_{0_2}}{p_2} = \frac{P_{0_2}}{p_\infty} \cdot \frac{p_\infty}{p_2} = \frac{\left[\frac{\gamma+1}{2} M_\infty{}^2\right]^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1} M_\infty{}^2 - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}} \times \left\{ \frac{1}{1+\frac{2\gamma}{\gamma+1}\left(M_\infty^2-1\right)} \right\}$$



Maximum Pressure Coefficient

- Across a Normal Shock wave (bow shock at nose of aircraft) Maximum pressure coefficient ($C_{p_{max}}$) on a flying body is

$$C_{p_{\max}} = \frac{P0_2 - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^{\ 2}}$$



∞--> free stream condition

Maximum Pressure Coefficient (cont'd)

- Rewriting in terms of free stream mach number

$$C_{p_{max}} = \frac{P_{0_2} - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^{\ 2}} = \frac{P_{0_2} - p_\infty}{\frac{1}{2}\frac{\gamma\, p_\infty}{\gamma R_g T_\infty} V_\infty^{\ 2}} = \frac{P_{0_2} - p_\infty}{\frac{\gamma}{2} p_\infty M_\infty^{\ 2}} = \frac{\frac{P_{0_2}}{p_\infty} - 1}{\frac{\gamma}{2} M_\infty^{\ 2}}$$

- For supersonic Conditions:

$$C_{p_{max}} = \left[\frac{\left(P_{0_2} - p_\infty\right)}{\frac{1}{2}\rho_\infty V_\infty^{\ 2}}\right] = \frac{\left(P_{0_2} - p_\infty\right)}{\frac{\gamma}{2} p_\infty M_\infty^{\ 2}} = \frac{1}{\frac{\gamma}{2} M_\infty^{\ 2}} \left\{ \frac{\left(\frac{\gamma+1}{2} M_\infty^{\ 2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{(\gamma+1)} M_\infty^{\ 2} - \frac{(\gamma-1)}{(\gamma+1)}\right)^{\left(\frac{1}{\gamma-1}\right)}} - 1 \right\}$$

- Proof left as exercise

Compressible Bernoulli Equation (Supersonic Flow) (1)

- Adding and subtracting p to the Rayleigh Pitot equation

$$P_{02} = p_\infty + p_\infty \left\{ \frac{\left(\frac{\gamma+1}{2} M_\infty^{\ 2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{(\gamma+1)} M_\infty^{\ 2} - \frac{(\gamma-1)}{(\gamma+1)}\right)^{\left(\frac{1}{\gamma-1}\right)}} - 1 \right\} = p_\infty + q_{c_2}$$

Supersonic form of Bernoulli Equation

$$q_{c_2} = p_\infty \left\{ \frac{\left(\frac{\gamma+1}{2} M_\infty^{\ 2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{(\gamma+1)} M_\infty^{\ 2} - \frac{(\gamma-1)}{(\gamma+1)}\right)^{\left(\frac{1}{\gamma-1}\right)}} - 1 \right\}$$

$$q_{c_2} \equiv impact\ pressure$$

$$q_{c_2} = C_{p_{\max}} \cdot \frac{\gamma}{2} \cdot p_\infty \cdot M_\infty^2$$

1 hr 19 min ago

The rocket just experienced "Max Q." Here's what that means.

The New Shepard rocket and capsule just experienced what's called "Max Q," an aerospace term that refers to the point during flight at which a vehicle experiences its maximum dynamic pressure.

Put simply: It's when the rocket is drumming up high speeds at a time when the atmosphere is still pretty thick, putting a lot of pressure on the vehicle.

Here's NASA's explanation:

$$q_{c_2} = p_\infty \left\{ \frac{\left(\frac{\gamma+1}{2} M_\infty^2 \right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{(\gamma+1)} M_\infty^2 - \frac{(\gamma-1)}{(\gamma+1)} \right)^{\left(\frac{1}{\gamma-1}\right)}} - 1 \right\}$$

> *The density of the air decreases with altitude in a complex manner. The velocity of a rocket during launch is constantly increasing with altitude. Therefore, the dynamic pressure on a rocket during launch is initially zero because the velocity is zero. The dynamic pressure increases because of the increasing velocity to some maximum value, called the maximum dynamic pressure, or Max Q. Then the dynamic pressure decreases because of the decreasing density."*

(or static pressure)

Compressible Bernoulli Equation (Supersonic Flow)(continued)

- *From earlier*

$$\frac{P_0 - p}{\bar{q}} = Cp_{max}$$

- *Comparing Equations*

$$P_0 = p + Cp_{max}\,\bar{q}$$

For Supersonic Conditions

$$Cp_{max_{supersonic}} = \frac{1}{\frac{\gamma}{2}M^2}\left\{\frac{\left(\frac{\gamma+1}{2}M^2\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{(\gamma+1)}M^2 - \frac{(\gamma-1)}{(\gamma+1)}\right)^{\left(\frac{1}{\gamma-1}\right)}} - 1\right\}$$

For Subsonic Conditions

$$\frac{p_{02}}{p_\infty} = \frac{p_{0\infty}}{p_\infty} = \left[1 + \frac{(\gamma-1)}{2}M_\infty^{\;2}\right]^{\frac{\gamma}{\gamma-1}}$$

.. *thus*

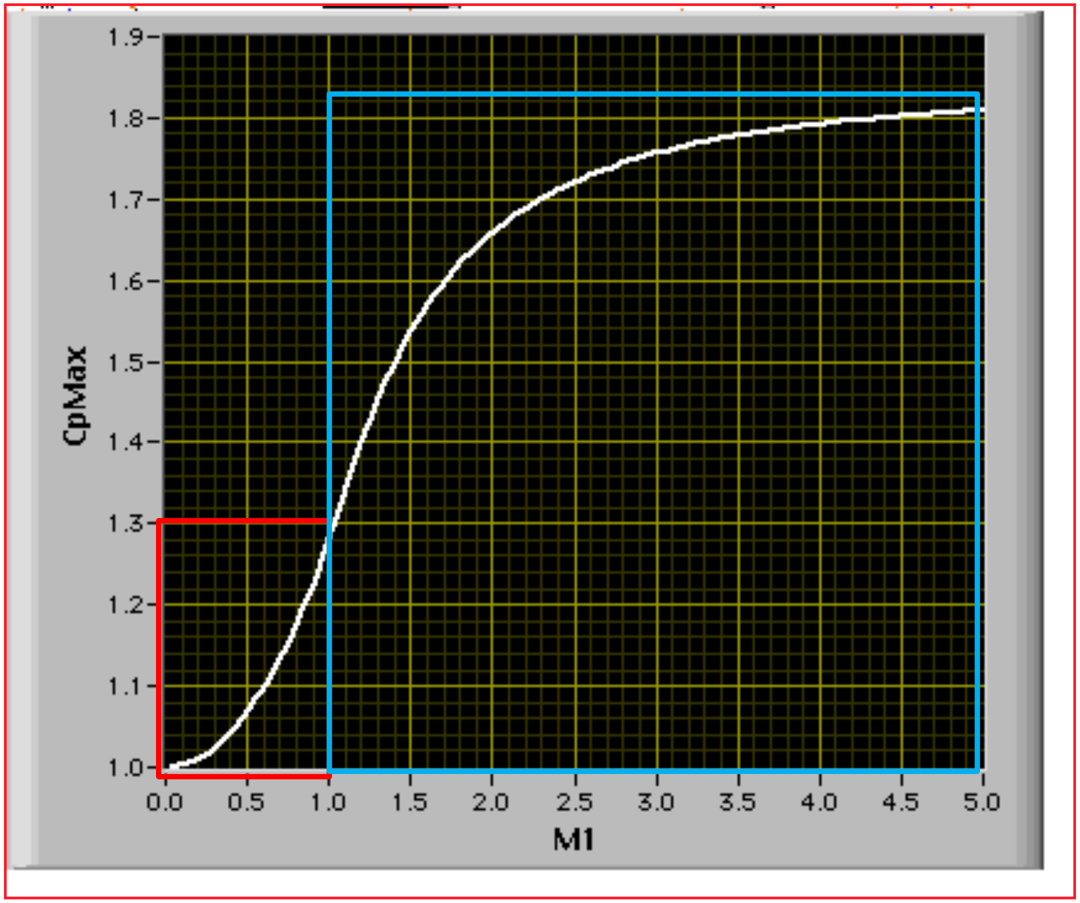
$$Cp_{max_{subsonic}} = \frac{\left\{\left[1 + \frac{(\gamma-1)}{2}M_\infty^{\;2}\right]^{\frac{\gamma}{\gamma-1}} - 1\right\}}{\frac{\gamma}{2}M_\infty^{\;2}}$$

Compressible Form of Bernoulli Equation .. Valid for both Subsonic and Supersonic Flows

Maximum Pressure Coefficient (cont'd)

$$Cp_{\max} = \frac{1}{\frac{\gamma}{2}M^2}\left\{\frac{\left(\frac{\gamma+1}{2}M^2\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{(\gamma+1)}M^2 - \frac{(\gamma-1)}{(\gamma+1)}\right)^{\left(\frac{1}{\gamma-1}\right)}} - 1\right\}$$

$$Cp_{\max} = \frac{\left\{\left[1 + \frac{(\gamma-1)}{2}M_\infty^{\ 2}\right]^{\frac{\gamma}{\gamma-1}} - 1\right\}}{\frac{\gamma}{2}M_\infty^{\ 2}}$$



- M_∞ --> 0 (incompressible flow)
  Cp_{max} -->1

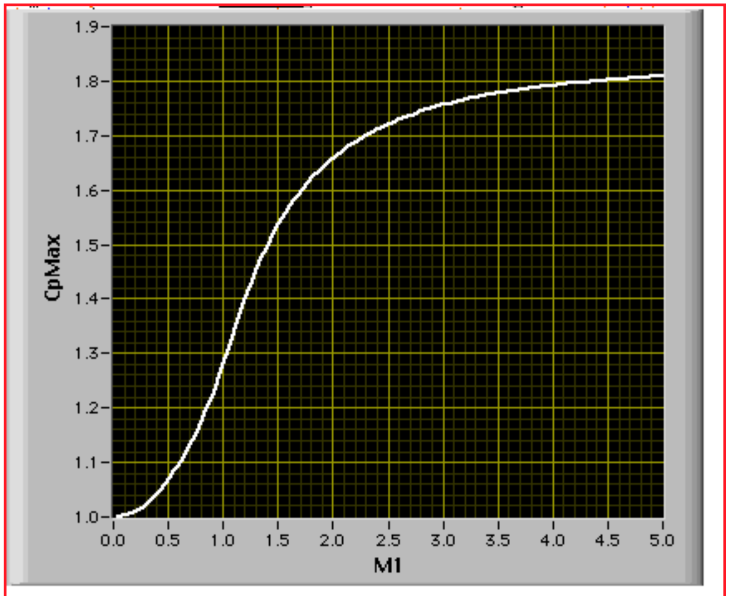
$$\left(P_0 = p_\infty + \bar{q}\right)_{|M=0}$$

Compressible Bernoulli Equation (revisited)

$$P_0 = p_\infty + C_{p\,\max} \cdot \bar{q} \rightarrow \boxed{\bar{q} = \frac{1}{2}\rho_\infty V_\infty^{\ 2} = \frac{\gamma}{2} P_\infty M_\infty^{\ 2}} \rightarrow M_\infty = 0_{(incompressible\ flow)} \rightarrow \boxed{C_{p\,\max} = 1}$$

$$0 < M_\infty < 1 \rightarrow \boxed{C_{p\,\max} = \frac{\left(1 + \frac{\gamma - 1}{2} M_\infty^{\ 2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)} - 1}{\frac{\gamma}{2} M_\infty^{\ 2}}}$$

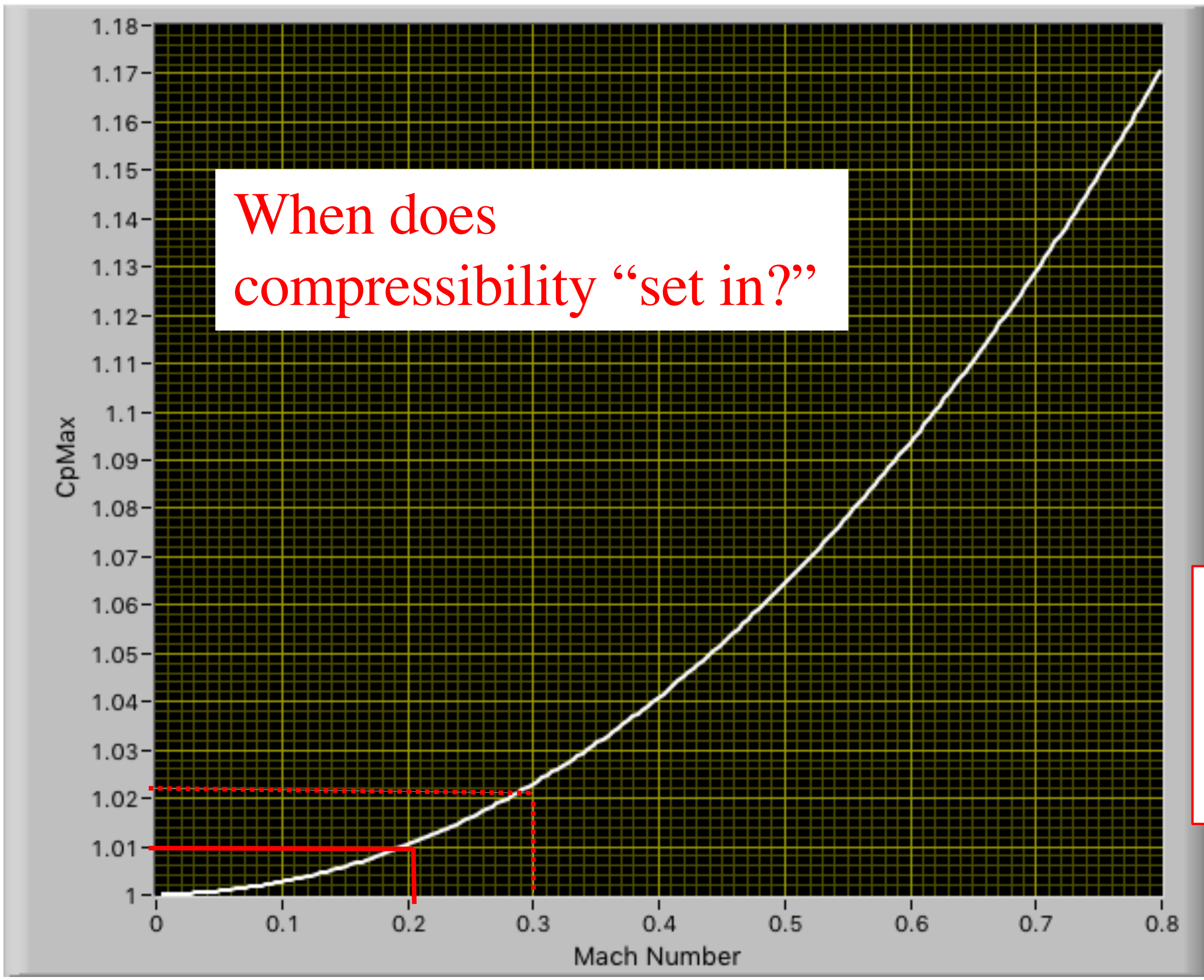
$$M_\infty = 1 \rightarrow \boxed{C_{p\,\max} = \frac{\left(1 + \frac{\gamma - 1}{2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)} - 1}{\frac{\gamma}{2}} = \frac{2}{\gamma}\left(\left(\frac{\gamma+1}{2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)} - 1\right)}$$



$$1 < M_\infty < \infty \rightarrow \boxed{C_{p\,\max} = \frac{1}{\frac{\gamma}{2} M_\infty^{\ 2}} \left(\frac{\left(\frac{\gamma+1}{2} M_\infty^{\ 2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\left(\frac{2\gamma}{\gamma+1}\right) M_\infty^{\ 2} - \left(\frac{\gamma-1}{\gamma+1}\right)\right)^{\left(\frac{1}{\gamma-1}\right)}} - 1 \right)}$$

Compressible Bernoulli Equation (revisited)

Cp Max vs Mach Number



~ between Mach 0.2 and 0.3 or between… 70 m/s (156 mph) and 105 m/s (123.4 mph) … On a Warm day at Indy .. (86 ^{0}F)

- IndyCars hit some of the highest top speeds in motorsport, getting **up to 380 km/h (Mach = 0.3) at the end of some straights.**

Compressible Bernoulli Equation (revisited)

What happens at "infinite" mach number?

$$\frac{C_{p\max}}{\lim\limits_{M_\infty\to\infty}} = \frac{1}{\frac{\gamma}{2}\left(\frac{M_\infty}{\lim\limits_{M_\infty\to\infty}}\right)^2}\left(\frac{\left(\frac{\gamma+1}{2}\left(\frac{M_\infty}{\lim\limits_{M_\infty\to\infty}}\right)^2\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\left(\frac{2\gamma}{\gamma+1}\right)\left(\frac{M_\infty}{\lim\limits_{M_\infty\to\infty}}\right)^2-\left(\frac{\gamma-1}{\gamma+1}\right)\right)^{\left(\frac{1}{\gamma-1}\right)}}-1\right) =$$

$$\frac{1}{\frac{\gamma}{2}\left(\frac{M_\infty}{\lim\limits_{M_\infty\to\infty}}\right)^2}\left(\frac{\left(\frac{\gamma+1}{2}\left(\frac{M_\infty}{\lim\limits_{M_\infty\to\infty}}\right)^2\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\left(\frac{2\gamma}{\gamma+1}\right)\left(\frac{M_\infty}{\lim\limits_{M_\infty\to\infty}}\right)^2\right)^{\left(\frac{1}{\gamma-1}\right)}}\right) = \frac{\frac{2}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{\gamma+1}\right)^{\left(\frac{1}{\gamma-1}\right)}}\left(\frac{\left(\frac{M_\infty}{\lim\limits_{M_\infty\to\infty}}\right)^{\left(\frac{2\gamma}{\gamma-1}\right)}}{\left(\left(\frac{M_\infty}{\lim\limits_{M_\infty\to\infty}}\right)^{2(\gamma-1)}\left(\frac{M_\infty}{\lim\limits_{M_\infty\to\infty}}\right)^2\right)^{\left(\frac{1}{\gamma-1}\right)}}\right)$$

Compressible Bernoulli Equation (Hypersonic Limit (2))

Collecting exponents on $\left(\lim_{M_\infty \to \infty} M_\infty\right)$

$$\lim_{M_\infty\to\infty} C_{p\,\max} = \frac{\frac{2}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{\gamma+1}\right)^{\left(\frac{1}{\gamma-1}\right)}}\left(\left(\lim_{M_\infty\to\infty} M_\infty\right)^{\left(\frac{2\gamma}{\gamma-1}\right)-\left(2(\gamma-1)+2\right)\left(\frac{1}{\gamma-1}\right)}\right) = \frac{\frac{2}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{\gamma+1}\right)^{\left(\frac{1}{\gamma-1}\right)}}\left(\left(\lim_{M_\infty\to\infty} M_\infty\right)^{0}\right) = \frac{\frac{2}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{\gamma+1}\right)^{\left(\frac{1}{\gamma-1}\right)}}$$

$$= \frac{\frac{2}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{\gamma+1}\right)^{\left(\frac{1}{\gamma-1}\right)}} = \frac{1}{\gamma}\cdot\frac{2}{(2)^{\left(\frac{\gamma}{\gamma-1}\right)}\cdot(2)^{\left(\frac{1}{\gamma-1}\right)}}\frac{(\gamma+1)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{\gamma}{\gamma+1}\right)^{\left(\frac{1}{\gamma-1}\right)}} = \frac{1}{\gamma}\cdot\frac{1}{(2)^{\left(\frac{2}{\gamma-1}\right)}}\frac{(\gamma+1)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{\gamma}{\gamma+1}\right)^{\left(\frac{1}{\gamma-1}\right)}} =$$

$$\frac{1}{\gamma^{1+\frac{1}{\gamma-1}}}\cdot\frac{1}{(2)^{\left(\frac{2}{\gamma-1}\right)}}(\gamma+1)^{\left(\frac{\gamma}{\gamma-1}+\gamma-1\right)} = \frac{\gamma^{\left(\frac{\gamma}{1-\gamma}\right)}}{(2)^{\left(\frac{2}{\gamma-1}\right)}}(\gamma+1)^{\left(\frac{\gamma+1}{\gamma-1}\right)}$$

Compressible Bernoulli Equation (Hypersonic Limit (3))

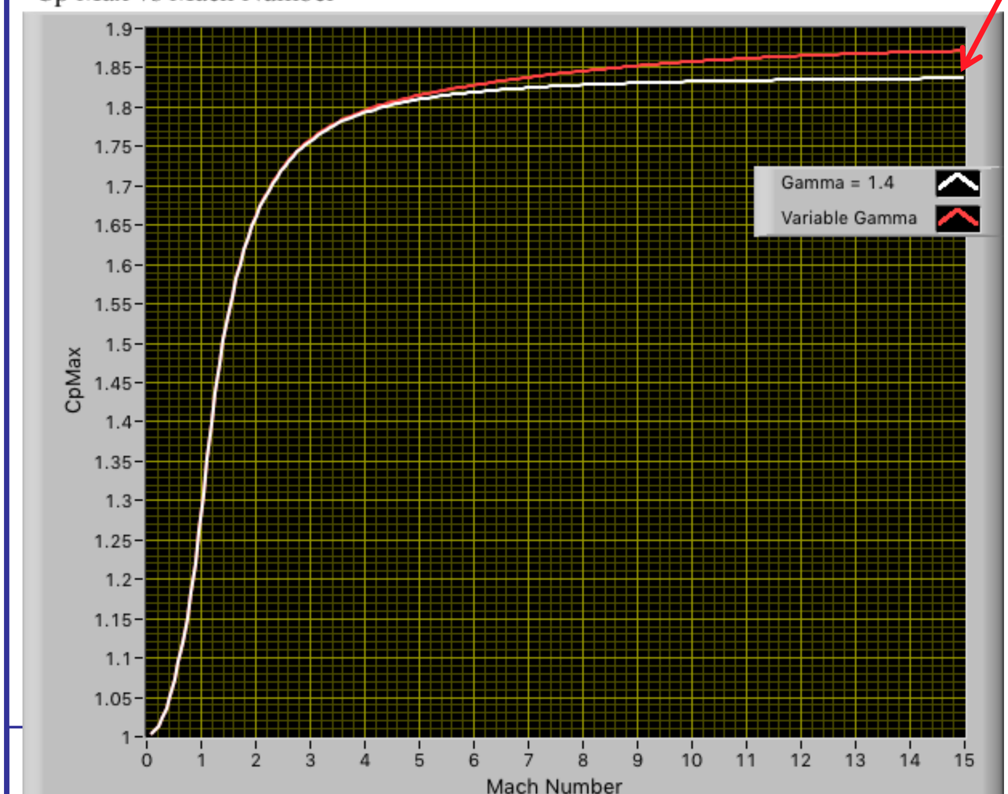
Check .. Substitute $\gamma = 1.40$

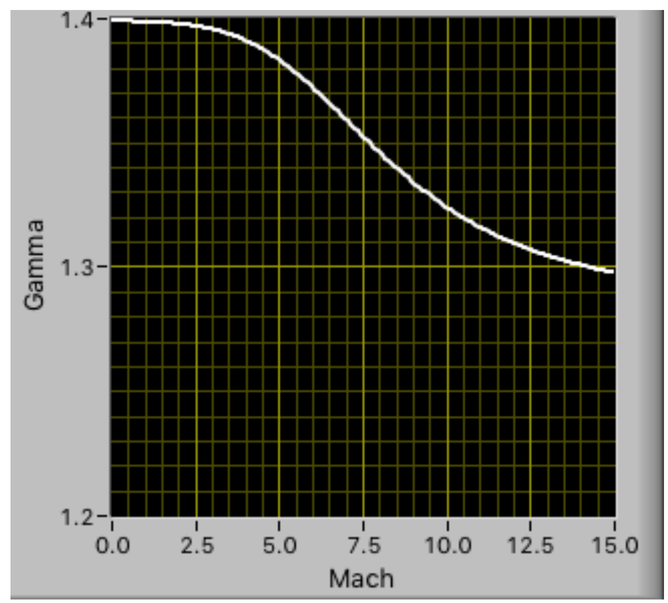
$$\frac{C_{p_{max}}}{\lim_{M_\infty \to \infty}} = \frac{\gamma^{\left(\frac{\gamma}{1-\gamma}\right)}}{(2)^{\left(\frac{2}{\gamma-1}\right)}}(\gamma+1)^{\left(\frac{\gamma+1}{\gamma-1}\right)} = \frac{1.4^{\left(\frac{1.4}{1-1.4}\right)}}{2^{\left(\frac{2}{1.4-1}\right)}}\left((1.4+1)^{\left(\frac{1.4+1}{1.4-1}\right)}\right) = 1.8394$$

- Fundamental property of Hypersonic flow … Mechanical Properties on surface (i.e. pressure) become independent of free stream Mach number

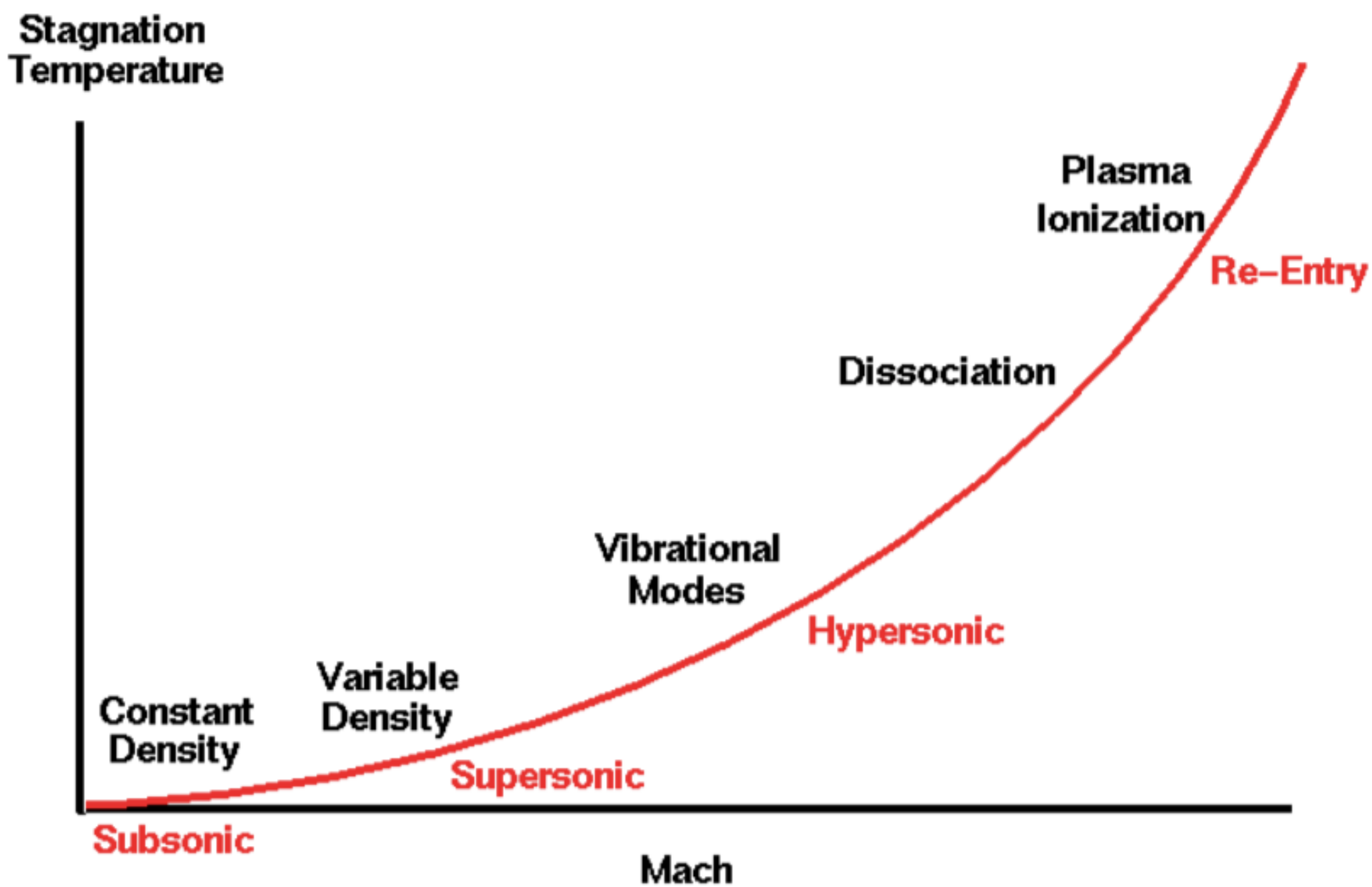
- Real-gas effects start to Kick in about Mach 3.5 (or SR-71 speeds)

Cp Max vs Mach Number





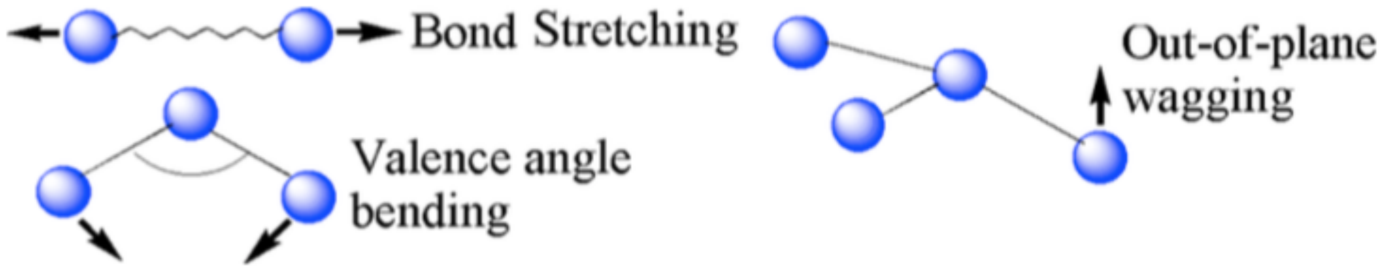
Hypersonic Flow of a Real Gas (cont'd)



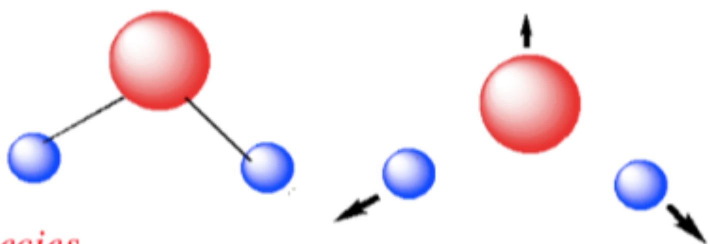
Hypersonic Flow of a Real Gas (cont'd)

Vibration

Species remain intact

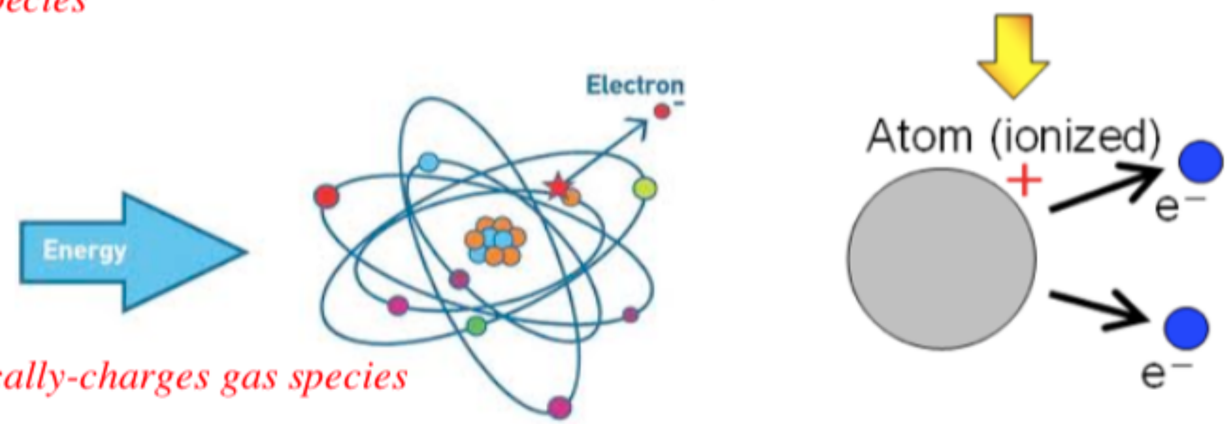


Dissociation



Creates different gas species

Ionization

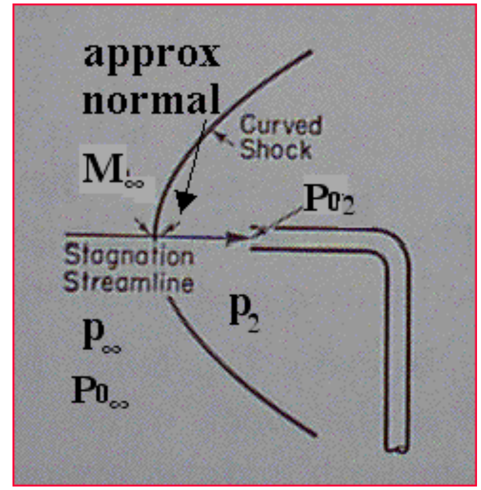


Creates different, electrically-charges gas species

Supersonic Pitot Probe Summary

- Max Pressure Coefficient

$$C_{p_{\max}} = \frac{1}{\frac{\gamma}{2} M_\infty^2} \left\{ \frac{\left(\frac{\gamma+1}{2} M_\infty^2\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{(\gamma+1)} M_\infty^2 - \frac{(\gamma-1)}{(\gamma+1)}\right)^{\left(\frac{1}{\gamma-1}\right)}} - 1 \right\}$$



- Rayleigh Pitot Equation

$$\frac{P_{0_2}}{p_\infty} = 1 + \frac{\gamma}{2} \cdot M_\infty^2 \cdot C_{p_{\max}} = \frac{\left(\frac{\gamma+1}{2} \cdot M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1} \cdot M_\infty^2 - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}}$$

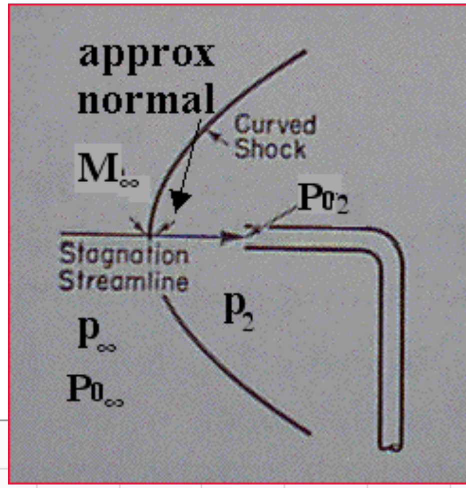
- Stagnation Pressure Ratio Behind Shock

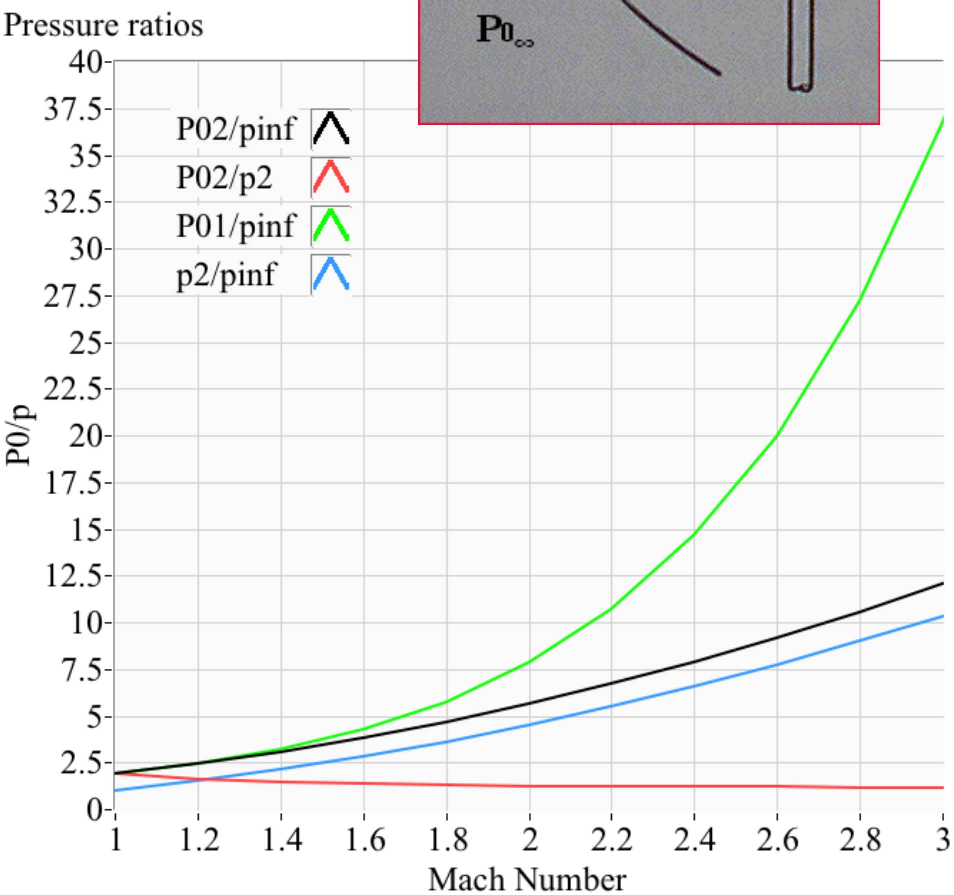
$$\frac{P_{0_2}}{p_2} = \frac{P_{0_2}}{p_\infty} \Big/ \frac{p_2}{p_\infty} = \frac{1 + \frac{\gamma}{2} \cdot M_\infty^2 \cdot C_{p_{\max}}}{1 + \left(\frac{2 \cdot \gamma}{\gamma+1}\right) \cdot \left(M_\infty^2 - 1\right)}$$

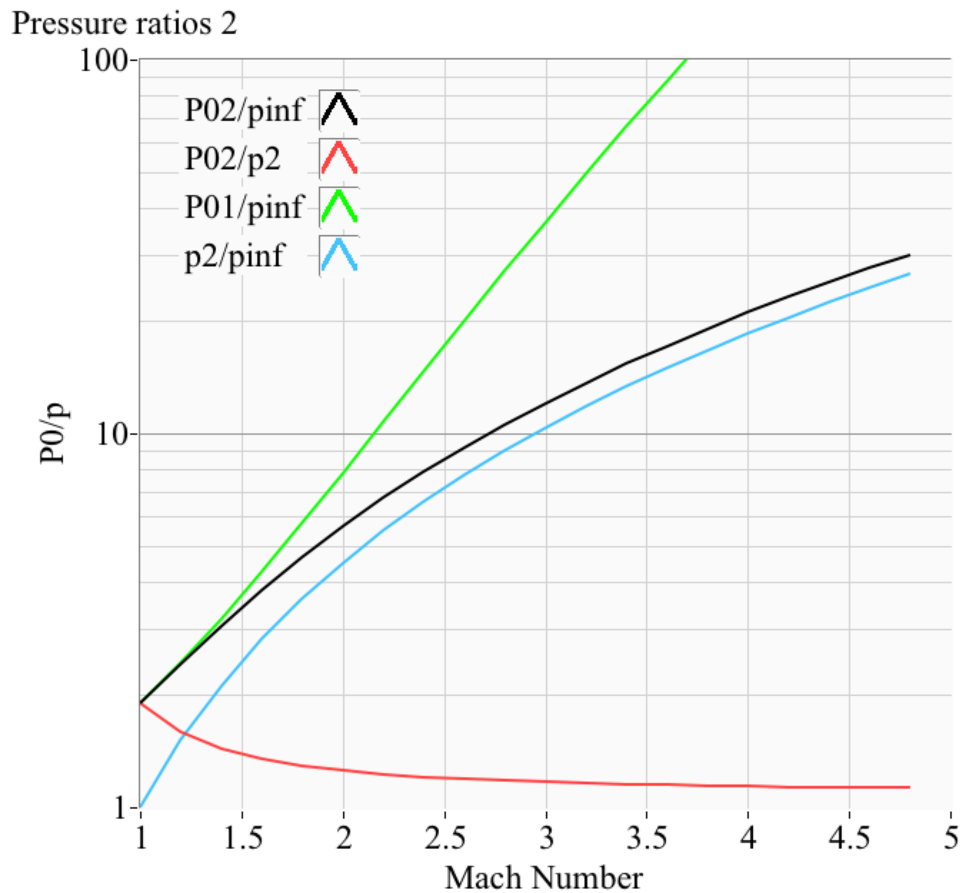
- Static Pressure Ratio Across Shock

$$\frac{p_2}{p_\infty} = 1 + \left(\frac{2 \cdot \gamma}{\gamma+1}\right) \cdot \left(M_\infty^2 - 1\right)$$

Supersonic Pitot Probe Summary (2)

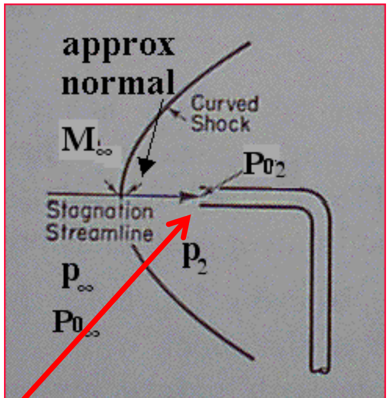






Example Problem

- Supersonic Pitot Tube

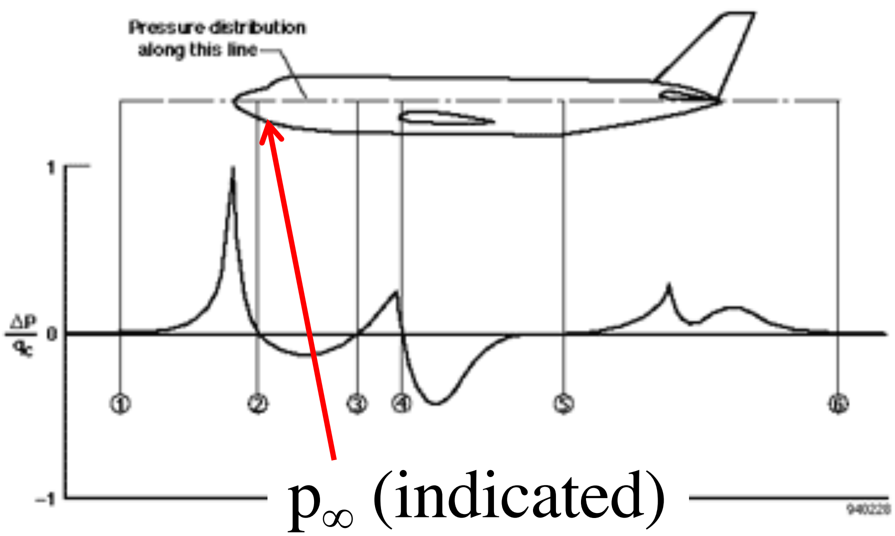


P_{0_2}

- p_∞ (indicated) ~ 250 kPa
  P_{0_2} ~ 1200 kPa

- Estimate Indicated M_∞

Assume $\gamma = 1.4$



$$\frac{P_{0_2}}{P_{\infty(indicated)}} = \frac{\left[\frac{\gamma+1}{2}M_{ind}^{\;2}\right]^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1}M_{ind}^{\;2}-\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}}$$

Example Problem

- Supersonic Pitot Tube .. in terms of P_{02}/P_∞

$$\frac{P_{02}}{p_\infty} = \frac{1200}{250} = 4.8$$

- trial and error solution

$$\frac{P_{0_2}}{P_{\infty(indicated)}} = \frac{\left[\frac{\gamma+1}{2} M_{ind}^{\;2}\right]^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1} M_{ind}^{\;2} - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}} = \frac{\left(\frac{1.4+1}{2} 1.8282^2\right)^{\frac{1.4}{(1.4-1)}}}{\left(\frac{2 \cdot 1.4}{(1.4+1)} 1.8282^2 - \frac{(1.4-1)}{1.4+1}\right)^{\frac{1}{(1.4-1)}}}$$

$=4.800011 \text{-->} M_\infty \sim 1.8282$

Homework 6, part 1

Consider Venturi Flow with a Gaseous Oxygen O_2 with Measured Static Pressures p_1, p_2

$$\begin{bmatrix} p_1 \\ D_1 \\ T_0 \end{bmatrix} = \begin{bmatrix} 80\ kPa \\ 1\ cm \\ 300\ {}^oK \end{bmatrix} \rightarrow \begin{bmatrix} p_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} 60 kPa \\ 0.5\ cm \end{bmatrix}$$

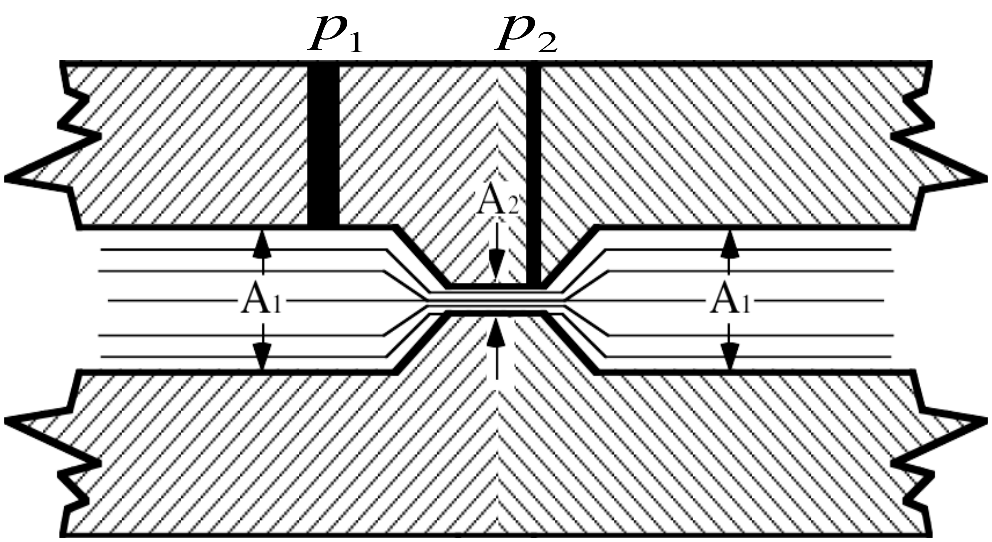
$$A = \frac{\pi}{4} D^2$$

- ***Part 2: Assuming Isentropic Flow .. Calculate the Massflow and Flow Velocity Through Venturi at stations (1) and (2)***

$\gamma = 1.4$, $Mw = 32.0$ kg/kg-mol

Homework 6, part 1

Compare results of compressible Venturi calculations to values calculated using incompressible Venturi Equation Massflow, V_1, V_2



- *Assume $T_1 = T_0 =$ 300 K Base incompressible Density on P_o from previous calculation*
- *Use Cd=1 for calculations*

$$V_2 = \sqrt{\frac{2\cdot(p_1 - p_2)}{\rho\cdot\left(1-\left(\frac{A_2}{A_1}\right)^2\right)}} \rightarrow \dot{m} = A_2 \cdot \sqrt{\frac{2\cdot\rho\cdot(p_1 - p_2)}{\left(1-\left(\frac{A_2}{A_1}\right)^2\right)}}$$

Incompressible Venturi Equations

Homework 6, part 2

- Show That for supersonic free stream conditions

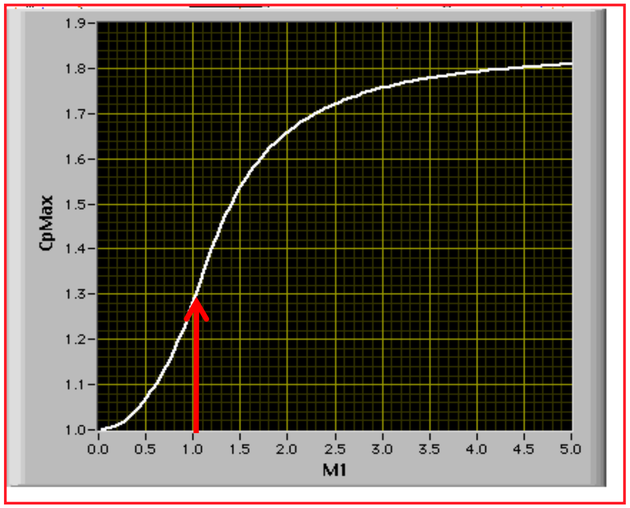
$$C_{P_{\max}} = \frac{P_{0_2} - p_\infty}{\left(\frac{1}{2}\cdot\rho_\infty\cdot V_\infty^{\ 2}\right)} = \frac{1}{\frac{\gamma}{2}M_\infty^{\ 2}}\left\{\frac{\left(\frac{\gamma+1}{2}M_\infty^{\ 2}\right)^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1}M_\infty^{\ 2} - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}} - 1\right\}$$

- Prove that when …. $M \rightarrow 0$ …. $Cp_{Max} \rightarrow 1.0$

Careful here … subsonic flow when M=0!

Homework 6, part 2

- Verify that Cp_{Max} is a continuous curve at *Mach 1*.



i.e. .. Show that When … $M_\infty \rightarrow 1$

$$\lim_{M_\infty \rightarrow 1} \left(C_{P_{\max}}\right)_{supersonic} = \frac{1}{\frac{\gamma}{2} M_\infty^2} \left\{ \frac{\left(\frac{\gamma+1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1} M_\infty^2 - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}} - 1 \right\} =$$

$$\lim_{M_\infty \rightarrow 1} \left(C_{P_{\max}}\right)_{subsonic} = \left[\frac{\left(1 + \frac{\gamma-1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1}{\frac{\gamma}{2} M_\infty^2} \right] = \frac{2}{\gamma}\left[\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

Homework 6: Part 3

- Write iterative Solver for "Rayleigh Pitot Equation" .. Given pressure ratio …. Calculate Mach number

$$\frac{P_{0_2}}{p_\infty} = \left\{ \frac{\left(\frac{\gamma+1}{2} M_\infty{}^2\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{(\gamma+1)} M_\infty{}^2 - \frac{(\gamma-1)}{(\gamma+1)}\right)^{\left(\frac{1}{\gamma-1}\right)}} \right\}$$

Supersonic Flow

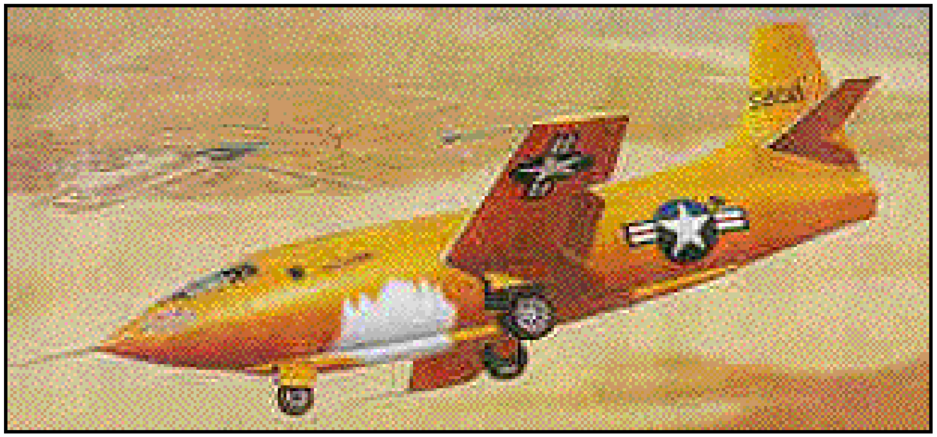
$$\frac{P_0}{p_\infty} = \left(1 + \frac{\gamma-1}{2} M_\infty{}^2\right)^{\frac{\gamma}{\gamma-1}}$$

Supersonic Flow

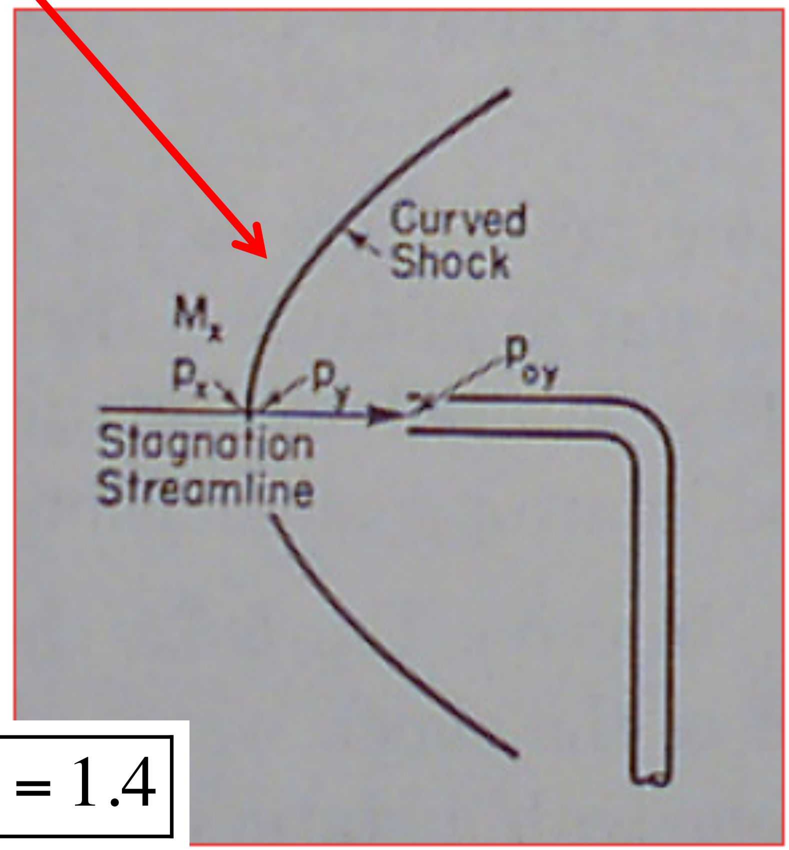
Carefully comment your code .. Hand in code with assignment …. Make program "smart enough" to solve for both supersonic and subsonic free stream conditions

Homework 6: Part 3

- Consider a Pitot / Static Probe on X-1 experimental rocket plane



- Probe measures *free stream static pressure* and *local impact pressure* .. *May or MAY NOT! Be shockwave in front of probe*



- Use Rayleigh-Pitot Code to compute free stream Mach number for following conditions

$$a)P_{0_y} = 1.22\times10^5\ Nt/m^2 \rightarrow p_x = 1.01\times10^5\ Nt/m^2$$

$$b)P_{0_y} = 7222\ lbf/ft^2 \rightarrow p_x = 2116\ lbf/ft^2$$

$$c)P_{0_y} = 13107\ lbf/ft^2 \rightarrow p_x = 1020\ lbf/ft^2$$

$$assume...\gamma = 1.4$$

Homework 6: Part 3

- Key Piece of Knowledge

$$\frac{\partial}{\partial M}\left[\frac{\left(\frac{\gamma+1}{2}M^2\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}{\left(\frac{2\gamma}{(\gamma+1)}M^2-\frac{(\gamma-1)}{(\gamma+1)}\right)^{\left(\frac{1}{\gamma-1}\right)}}\right]=\frac{\gamma M\left(2M^2-1\right)\left[M^2\frac{(\gamma+1)}{2}\right]^{\left(\frac{1}{\gamma-1}\right)}}{\left(\frac{2\gamma}{(\gamma+1)}M^2-\frac{\gamma-1}{(\gamma+1)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}}$$