$$\int \frac{\sqrt{e^{mx}-1}}{1+be^{mx}}dx = \int \frac{e^{mx}-1}{(1+be^{mx})\sqrt{e^{mx}-1}}dx = \int \frac{(e^{mx}+be^{mx})-(1+be^{mx})}{(1+be^{mx})\sqrt{e^{mx}-1}}dx =$$

$$\int \left(\frac{e^{mx}+be^{mx}}{(1+be^{mx})\sqrt{e^{mx}-1}} - \frac{1+be^{mx}}{(1+be^{mx})\sqrt{e^{mx}-1}}\right)dx = \int \left(\underbrace{\frac{e^{mx}+be^{mx}}{(1+be^{mx})\sqrt{e^{mx}-1}}}_{\text{Part 1}} - \underbrace{\frac{1}{\sqrt{e^{mx}-1}}}_{\text{Part 2}}\right)dx$$

Part 1:

$$\int \frac{e^{mx}+be^{mx}}{(1+be^{mx})\sqrt{e^{mx}-1}}dx = \int \frac{e^{mx}(1+b)}{(1+be^{mx})\sqrt{e^{mx}-1}}dx$$

$$u = e^{mx}$$

$$du = me^{mx}dx$$

$$= \int \frac{\frac{1}{m}(1+b)}{(1+bu)\sqrt{u-1}}du$$

$$v = \sqrt{u-1}$$

$$dv = \frac{1}{2\sqrt{u-1}}du = \frac{1}{2v}du$$

$$= \frac{1}{m}\int \frac{2v(1+b)}{\left(1+b(v^2+1)\right)v}dv = \frac{2}{m}\int \frac{(1+b)}{1+b(v^2+1)}dv = \frac{2}{m}\int \frac{(1+b)}{1+b+bv^2}dv$$

$$= \frac{2}{m} \int \frac{1}{1 + \frac{b}{1+b} v^2} dv \qquad w = \sqrt{\frac{b}{1+b}} v \qquad dw = \sqrt{\frac{b}{1+b}} dv$$

$$= \frac{2}{m} \sqrt{\frac{b+1}{b}} \int \frac{1}{1+w^2} dw = \frac{2}{m} \sqrt{\frac{b+1}{b}} \tan^{-1}(w) = \frac{2}{m} \sqrt{\frac{b+1}{b}} \tan^{-1}\left(\sqrt{\frac{b}{1+b}} v \right)$$

$$= \frac{2}{m} \sqrt{\frac{b+1}{b}} \tan^{-1}\left(\sqrt{\frac{b}{1+b}} \sqrt{u-1} \right) = \boxed{\frac{2}{m} \sqrt{\frac{b+1}{b}} \tan^{-1}\left(\sqrt{\frac{b}{1+b}} \sqrt{e^{mx}-1} \right)}$$

Part 2:

$$\int \frac{1}{\sqrt{e^{mx}-1}} dx \qquad u = e^{mx} \qquad du = me^{mx} dx$$

$$= \int \frac{1}{mu\sqrt{u-1}} du \qquad v = \sqrt{u-1} \qquad dv = \frac{1}{2\sqrt{u-1}} du = \frac{1}{2v} du$$

$$= \int \frac{2v}{m(v^2+1)v} dv = \frac{2}{m} \int \frac{1}{v^2+1} dv = \frac{2}{m} \tan^{-1}(v) = \frac{2}{m} \tan^{-1}\left(\sqrt{u-1}\right) = \boxed{\frac{2}{m} \tan^{-1}\left(\sqrt{e^{mx}-1}\right)}$$

Combine parts 1 and 2:

$$\int \frac{\sqrt{e^{mx}-1}}{1+be^{mx}}\,dx = \frac{2}{m}\sqrt{\frac{b+1}{b}}\tan^{-1}\left(\sqrt{\frac{b}{1+b}}\sqrt{e^{mx}-1}\right) - \frac{2}{m}\tan^{-1}\left(\sqrt{e^{mx}-1}\right)$$

$$= \boxed{\frac{\frac{2(b+1)}{m}\tan^{-1}\left(\sqrt{\frac{b}{1+b}}\sqrt{e^{mx}-1}\right)}{\sqrt{b+1}\sqrt{b}} - \frac{2}{m}\tan^{-1}\left(\sqrt{e^{mx}-1}\right)}$$