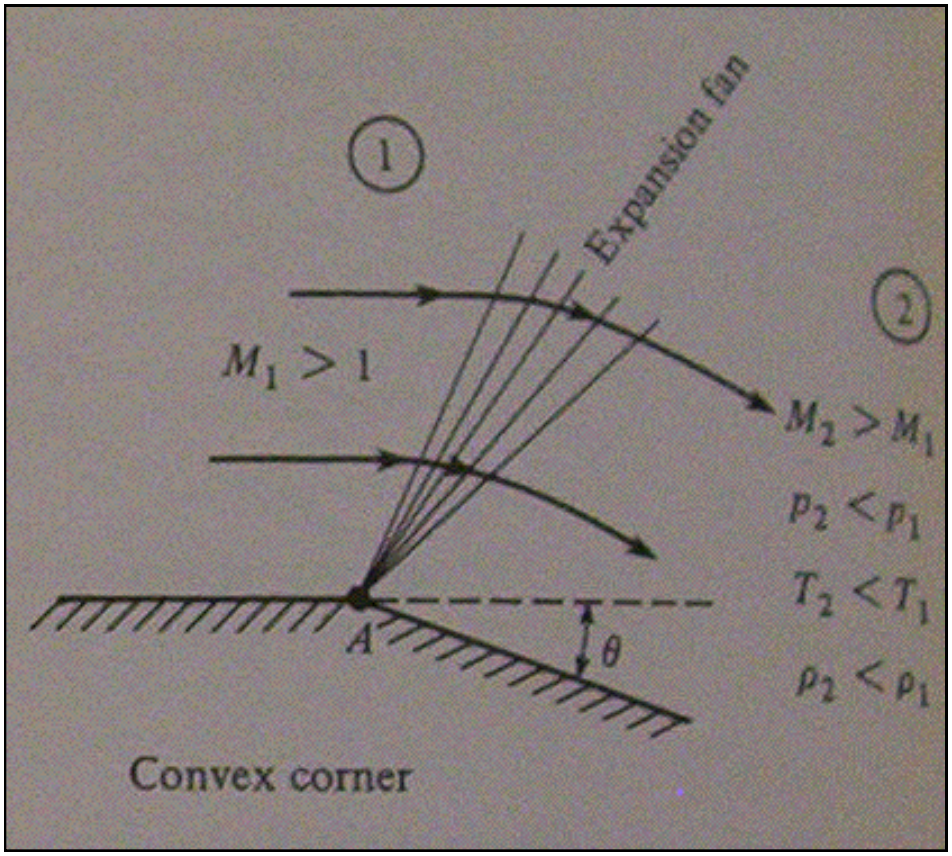
Section 6 Lecture 2:Prandtl-Meyer Expansion Waves



- Anderson,
  Chapter 4 pp.167-190

What Happens when

- $M_1 = 3.0$, p_1=1atm, γ =1.4, T1=288°K θ =0.00001°
- Explicit Solver for β

$$\lambda = \sqrt{\left(M_1^2 - 1\right)^2 - 3\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\left[1 + \frac{\gamma + 1}{2} M_1^2\right]\tan^2(\theta)} \quad = 8.0$$

$$\chi = \frac{\left(M_1^2 - 1\right)^3 - 9\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\left[1 + \frac{\gamma - 1}{2} M_1^2 + \frac{\gamma + 1}{4} M_1^4\right]\tan^2(\theta)}{\lambda^3} \quad = 1.0$$

What Happens when (cont'd)

- $M_1 = 3.0$, $p_1 = 1 atm$, $\gamma = 1.4$, T1=288°K $\theta = 0.00001°$

$$\tan(\beta) = \frac{\left(M_1^2 - 1\right) + 2\lambda \cos\left(\frac{4\pi\delta + \cos^{-1}(\chi)}{3}\right)}{3\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\tan(\theta)} \longrightarrow$$

$$\beta = 19.47° \longrightarrow \mu = \frac{180}{\pi}\sin^{-1}\left[\frac{1}{M_1}\right] = 19.47^o$$

- "mach line"

What Happens when (cont'd)

- $M_1 = 3.0$, $p_1 = 1\,atm$, $\gamma = 1.4$, $T1 = 288°K$ $\theta = 0.00001°$
- Normal Component of Free stream mach Number

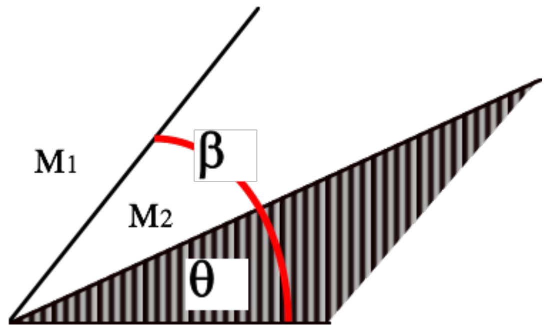
$$Mn_1 = M_1 \sin\beta = \quad 1.0000$$

- $$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma + 1)}\left(Mn_1{}^2 - 1\right) \quad = 1.0$$ **(NO COMPRESSION!)**

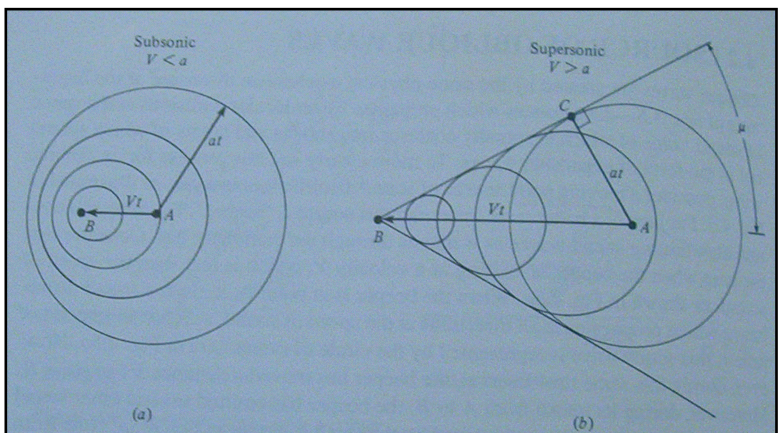
Expansion Waves

- So if

θ >0 .. Compression around corner



θ =0 … no compression across shock

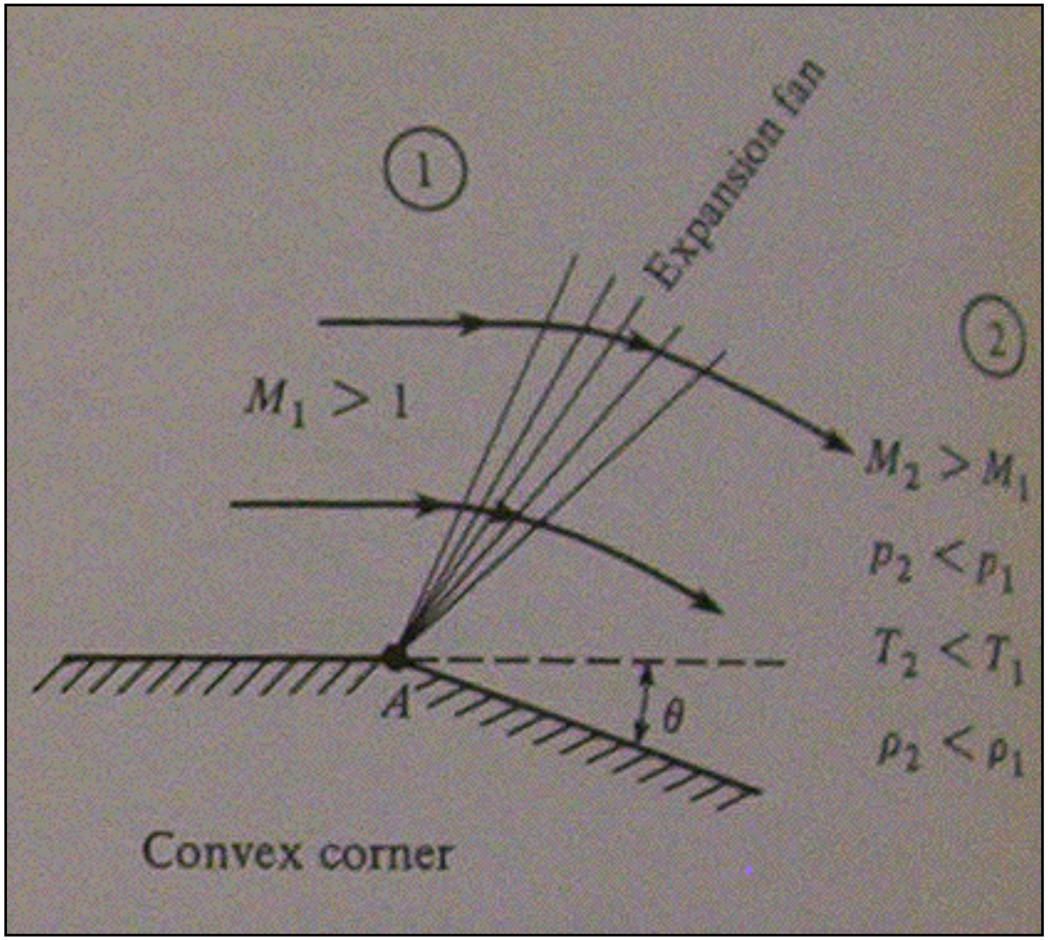


The propagation of disturbances in (a) subsonic and (b) supersonic flow.

Expansion Waves (concluded)

- Then it follows that

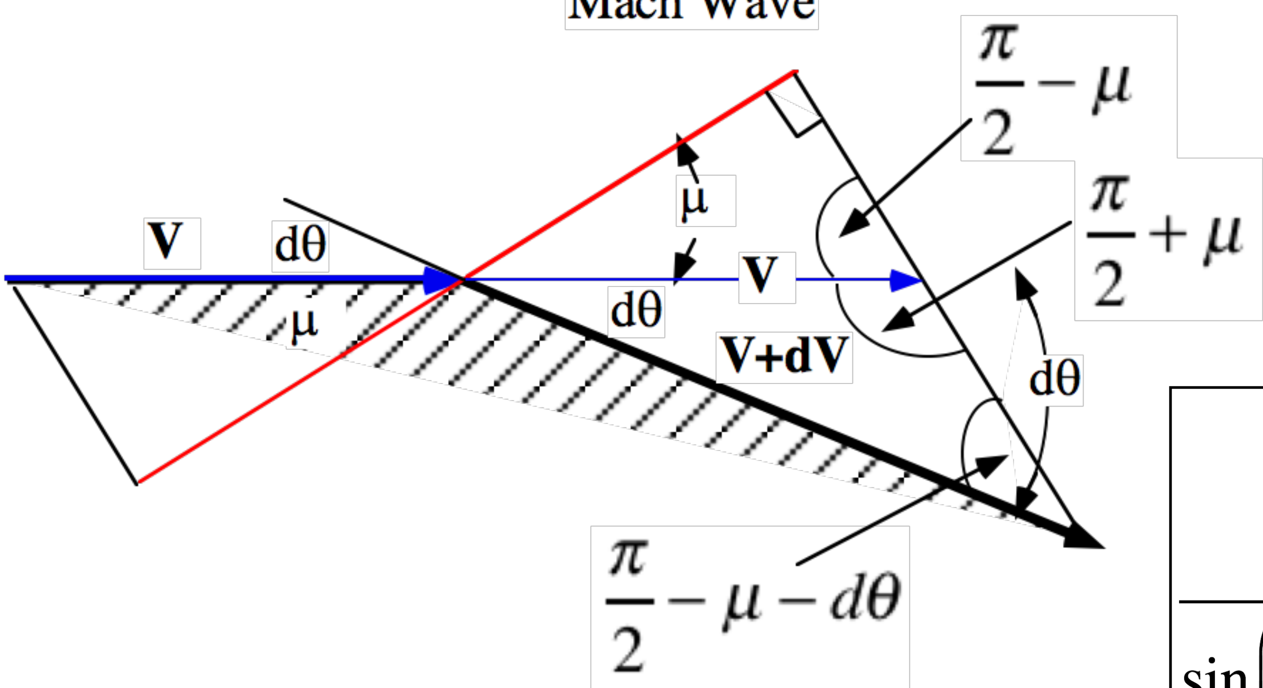
θ <0 .. We get an *expansion wave (Prandtl-Meyer)*



- Flow accelerates around corner
- Continuous flow region …
  sometimes called
  "expansion fan"
- Each mach wave is infinitesimally weak
  isentropic flow region
- Flow stream lines are curved and smooth through fan

Prandtl-Meyer Expansion Fan: Mathematical Analysis

- Consider flow expansion around an infinitesimal corner



Infinitesimal Expansion Fan Flow Geometry

- From Law of Sines

$$\frac{V}{\sin\left(\frac{\pi}{2}-\mu-d\theta\right)}=\frac{V+dV}{\sin\left(\frac{\pi}{2}+\mu\right)}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Using the trigonometric identities

$$\sin\left(\frac{\pi}{2}+\mu\right)=\sin\left(\frac{\pi}{2}\right)\cos(\mu)+\sin(\mu)\cos\left(\frac{\pi}{2}\right)=\cos(\mu)$$

$$\sin\left(\frac{\pi}{2}-\mu\right)=\sin\left(\frac{\pi}{2}\right)\cos(\mu)-\sin(\mu)\cos\left(\frac{\pi}{2}\right)=\cos(\mu)$$

- And …

$$\sin\left(\frac{\pi}{2}-\mu-d\theta\right)=\sin\left(\frac{\pi}{2}-\mu\right)\cos(d\theta)-\cos\left(\frac{\pi}{2}-\mu\right)\sin(d\theta)=$$

$$\cos(\mu)\cos(d\theta)-\left[\cos\left(\frac{\pi}{2}\right)\cos(\mu)+\sin(\mu)\sin\left(\frac{\pi}{2}\right)\right]\sin(d\theta)=$$

$$\cos(\mu)\cos(d\theta)-\sin(\mu)\sin(d\theta)$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Substitution gives

$$\frac{V}{\cos(\mu)\cos(d\theta)-\sin(\mu)\sin(d\theta)}=\frac{V+dV}{\cos(\mu)}\rightarrow$$

$$1+\frac{dV}{V}=\frac{\cos(\mu)}{\cos(\mu)\cos(d\theta)-\sin(\mu)\sin(d\theta)}$$

- Since dθ is considered to be infinitesimal

$$\cos(d\theta)=1$$

$$\sin(d\theta)=d\theta$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- and the equation reduces to

$$1+\frac{dV}{V}=\frac{\cos(\mu)}{\cos(\mu)-\sin(\mu)(d\theta)}=\frac{1}{1-\tan(\mu)(d\theta)}$$

- Exploiting the form of the power series (expanded about x=0)

$$\frac{1}{1-x}=(1-x)_{|x=0}-\left[\frac{1}{(1-x)^2_{\;|x=0}}(-1)\right](x-0)+\ldots O(x^2)\rightarrow$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Then

$$\frac{1}{1-\frac{dV}{V}} = 1 + \frac{1}{\left(1-\frac{dV}{V}\right)^2}\Bigg|_{\frac{dV}{V}=0}\left(\frac{dV}{V}-0\right) + O\left(\frac{dV}{V}^2\right)$$

- Since dV is infinitesimal … truncate after first order term

$$\frac{1}{1-\frac{dV}{V}} \approx 1 + \frac{dV}{V} \rightarrow \frac{1}{1-\frac{dV}{V}} \approx \frac{1}{1-\tan(\mu)(d\theta)}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Solve for $d\theta$ in terms of dV/V

$$\frac{1}{1-\frac{dV}{V}} \approx 1+\frac{dV}{V} \rightarrow \frac{1}{1-\frac{dV}{V}} \approx \frac{1}{1-\tan(\mu)(d\theta)}$$

$$1-\tan(\mu)(d\theta) = 1-\frac{dV}{V} \rightarrow \boxed{d\theta = \frac{1}{\tan(\mu)}\frac{dV}{V}}$$

- Since disturbance is infinitesimal (mach wave)

$$\sin(\mu) = \frac{1}{M}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Performing some algebraic and trigonometric voodoo

$$\sin(\mu) = \frac{1}{M} \rightarrow \sin^2(\mu) = \frac{1}{M^2} = \frac{\sin^2(\mu)+\cos^2(\mu)}{M^2} \rightarrow$$

$$M^2 = \frac{\sin^2(\mu)+\cos^2(\mu)}{\sin^2(\mu)} = 1 + \frac{1}{\tan^2(\mu)} \rightarrow \frac{1}{\tan^2(\mu)} = M^2 - 1$$

$$\frac{1}{\tan(\mu)} = \sqrt{M^2-1}$$

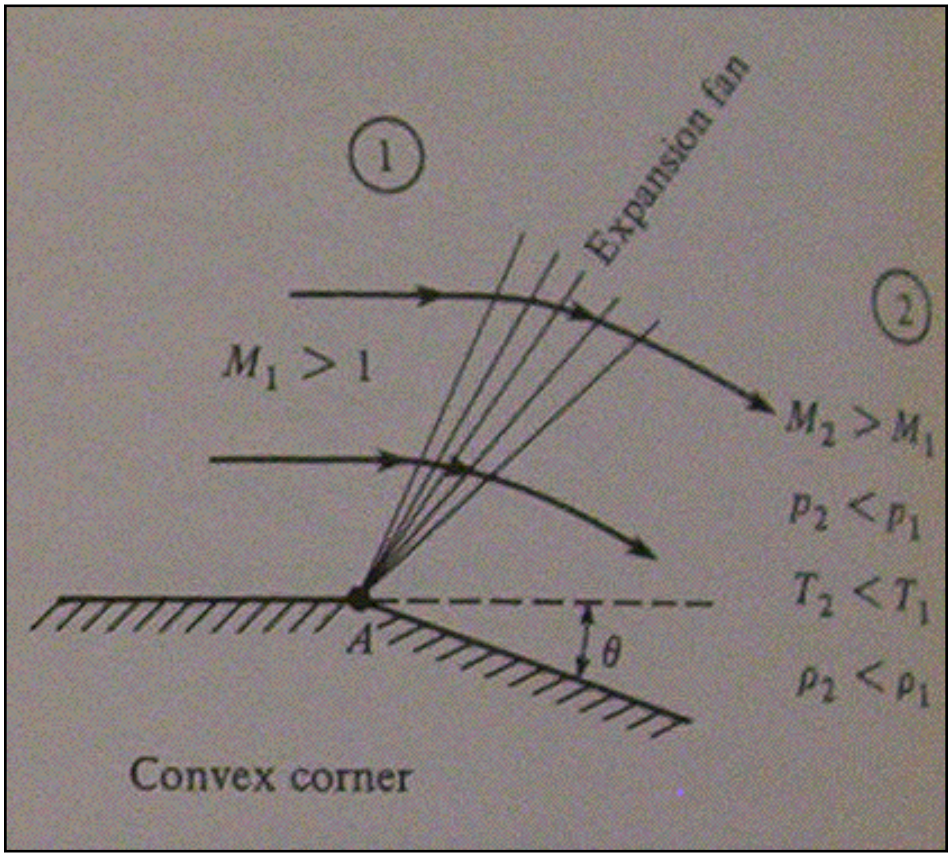
- and ….

$$d\theta = \frac{1}{\tan(\mu)}\frac{dV}{V} = \sqrt{M^2-1}\,\frac{dV}{V}$$

- Valid for Real and ideal gas

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- For a finite deflection the O.D.E is integrated over the complete expansion fan



$$\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\,\frac{dV}{V}$$

- Write in terms of mach by …

$$V = M \times c \rightarrow dV = dM \times c + M \times dc \rightarrow$$

$$\frac{dV}{V} = \frac{dM \times c + M \times dc}{M \times c} = \frac{dM}{M} + \frac{dc}{c}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Substituting in

$$\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\,\frac{dV}{V} = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\left(\frac{dM}{M} + \frac{dc}{c}\right)$$

- For a calorically perfect adiabatic gas flow

$$c_0 = \sqrt{\gamma R_g T_0} \rightarrow \left[\frac{c_0}{c}\right] = \sqrt{\left[\frac{T_0}{T}\right]} = \sqrt{\left(1 + \frac{\gamma - 1}{2} M^2\right)}$$

And T_0 is constant

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Solving for c, differentiating, and normalizing by c

$$\frac{dc}{c} = \frac{\left(-\frac{1}{2}\right)\frac{1}{\sqrt{\left(1+\frac{\gamma-1}{2}M^2\right)}}(\gamma-1)M\left(dM\right)}{\sqrt{\left(1+\frac{\gamma-1}{2}M^2\right)}} = -\frac{\frac{(\gamma-1)}{2}M\left(dM\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Returning to the integral for θ

$$\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\,\frac{dV}{V} = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\left(\frac{dM}{M} + -\frac{\frac{(\gamma - 1)}{2} M\,(dM)}{\left(1 + \frac{\gamma - 1}{2} M^2\right)}\right)$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Simplification gives

$$\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\,\frac{dM}{M}\left(1 + -\frac{\frac{(\gamma-1)}{2}M^2}{\left(1 + \frac{\gamma-1}{2}M^2\right)}\right) =$$

$$\int_{M_1}^{M_2} \sqrt{M^2 - 1}\,\frac{dM}{M}\left(\frac{\left(1 + \frac{\gamma-1}{2}M^2\right) - \frac{(\gamma-1)}{2}M^2}{\left(1 + \frac{\gamma-1}{2}M^2\right)}\right) = \int_{M_1}^{M_2}\left(\frac{\sqrt{M^2 - 1}\,\frac{dM}{M}}{\left(1 + \frac{\gamma-1}{2}M^2\right)}\right)$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Evaluate integral by performing substitution

$$\int \frac{\sqrt{M^2-1}\,\frac{dM}{M}}{\left(1+\frac{\gamma-1}{2}M^2\right)} \rightarrow Ln[M] \equiv u \rightarrow \left\{\frac{dM}{M} = du, M^2 = e^{2u}\right\}$$

$$\int \frac{\sqrt{M^2-1}\,\frac{dM}{M}}{\left(1+\frac{\gamma-1}{2}M^2\right)} = \int \frac{\sqrt{e^{2u}-1}}{\left(1+\frac{\gamma-1}{2}e^{2u}\right)}du$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Standard Integral Table Form

$$\int \frac{\sqrt{e^{2u}-1}}{\left(1+\frac{\gamma-1}{2}e^{2u}\right)}du \rightarrow \int \frac{\sqrt{e^{mx}-1}}{\left(1+be^{mx}\right)}du$$

- From tables (CRC math handbook)

$$\int \frac{\sqrt{e^{mx}-1}}{1+b\cdot e^{mx}}dx = \frac{\frac{2\cdot(b+1)}{m}\tan^{-1}\sqrt{\left(\frac{b}{b+1}\right)\cdot\left(e^{mx}-1\right)}}{\sqrt{b+1}\cdot\sqrt{b}} - \left(\frac{2}{m}\right)\tan^{-1}\sqrt{e^{mx}-1} =$$

$$\frac{2}{m}\left[\left(\sqrt{\frac{b+1}{b}}\right)\tan^{-1}\sqrt{\left(\frac{b}{b+1}\right)\cdot\left(e^{mx}-1\right)} - \tan^{-1}\sqrt{e^{mx}-1}\right]$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Proof by Josh Hodson and Jorge Gonzalez

$$\int \frac{\sqrt{e^{mx}-1}}{1+b\cdot e^{mx}}dx = \int \frac{\left(e^{mx}-1\right)}{\left(1+b\cdot e^{mx}\right)\cdot\sqrt{e^{mx}-1}}dx = \int \frac{\left(e^{mx}+b\cdot e^{mx}\right)-\left(1+b\cdot e^{mx}\right)}{\left(1+b\cdot e^{mx}\right)\cdot\sqrt{e^{mx}-1}}dx =$$

$$\left[\int \frac{\left(e^{mx}+b\cdot e^{mx}\right)}{\left(1+b\cdot e^{mx}\right)\cdot\sqrt{e^{mx}-1}} - \frac{\left(1+b\cdot e^{mx}\right)}{\left(1+b\cdot e^{mx}\right)\cdot\sqrt{e^{mx}-1}}\right]dx = \left[\int \frac{(1+b)\cdot e^{mx}}{\left(1+b\cdot e^{mx}\right)\cdot\sqrt{e^{mx}-1}} - \frac{1}{\sqrt{e^{mx}-1}}\right]dx$$

$\rightarrow Part1:$

$$\int \frac{(1+b)\cdot e^{mx}}{\left(1+b\cdot e^{mx}\right)\cdot\sqrt{e^{mx}-1}}dx \rightarrow$$

$$\vartheta = e^{mx}$$

$$d\vartheta = m\cdot e^{mx}\cdot dx \rightarrow dx = \frac{d\vartheta}{m\cdot e^{mx}} = \frac{1}{m}\cdot\frac{d\vartheta}{\vartheta}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

$$\rightarrow substitute.... \frac{(1+b)\cdot e^{mx}}{(1+b\cdot e^{mx})\cdot\sqrt{e^{mx}-1}}dx = \frac{(1+b)\cdot\vartheta}{(1+b\cdot\vartheta)\cdot\sqrt{\vartheta-1}}\cdot\frac{1}{m}\cdot\frac{d\vartheta}{\vartheta} = \frac{1}{m}\cdot\frac{(1+b)}{(1+b\cdot\vartheta)\cdot\sqrt{\vartheta-1}}\cdot d\vartheta$$

substitute...

$$v = \sqrt{\vartheta-1} \rightarrow \vartheta = v^2+1$$

$$dv = \frac{1}{2}\cdot\frac{d\vartheta}{\sqrt{\vartheta-1}} \rightarrow d\vartheta = 2\cdot v\cdot dv$$

$$\rightarrow \int\frac{(1+b)\cdot e^{mx}}{(1+b\cdot e^{mx})\cdot\sqrt{e^{mx}-1}}dx = \int\frac{1}{m}\cdot\frac{(1+b)}{\left(1+b\cdot(v^2+1)\right)\cdot v}\cdot 2\cdot v\cdot dv = \frac{2}{m}\cdot\int\frac{dv}{\left(1+\left(\frac{b}{1+b}\right)\cdot v^2\right)}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

From previous slide

$$\rightarrow \int \frac{(1+b)\cdot e^{mx}}{(1+b\cdot e^{mx})\cdot\sqrt{e^{mx}-1}}dx = \int \frac{1}{m}\cdot\frac{(1+b)}{(1+b\cdot(v^2+1))\cdot v}\cdot 2\cdot v\cdot dv = \frac{2}{m}\cdot\int\frac{dv}{\left(1+\left(\frac{b}{1+b}\right)\cdot v^2\right)}$$

$$\rightarrow substitute... \left| \begin{array}{l} w = \sqrt{\left(\frac{b}{1+b}\right)\cdot v^2} = \sqrt{\left(\frac{b}{1+b}\right)}\cdot v \\ dw = \sqrt{\left(\frac{b}{1+b}\right)}\cdot dv \rightarrow dw\cdot\sqrt{\frac{b+1}{b}} \end{array} \right.$$

$$\rightarrow \int \frac{(1+b)\cdot e^{mx}}{(1+b\cdot e^{mx})\cdot\sqrt{e^{mx}-1}}dx = \frac{2}{m}\cdot\sqrt{\frac{b+1}{b}}\int\frac{dw}{(1+w^2)}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

$$\rightarrow substitute... \quad w = \sqrt{\left(\frac{b}{1+b}\right)\cdot v^2} = \sqrt{\left(\frac{b}{1+b}\right)}\cdot v$$

$$dw = \sqrt{\left(\frac{b}{1+b}\right)}\cdot dv \rightarrow dw \cdot \sqrt{\frac{b+1}{b}}$$

$$\rightarrow \int \frac{(1+b)\cdot e^{mx}}{(1+b\cdot e^{mx})\cdot\sqrt{e^{mx}-1}}dx = \frac{2}{m}\cdot\sqrt{\frac{b+1}{b}}\int\frac{dw}{(1+w^2)}$$

$$\rightarrow \int \frac{(1+b)\cdot e^{mx}}{(1+b\cdot e^{mx})\cdot\sqrt{e^{mx}-1}}dx = \frac{2}{m}\cdot\sqrt{\frac{b+1}{b}}\tan^{-1}(w) = \frac{2}{m}\cdot\sqrt{\frac{b+1}{b}}\tan^{-1}\left(\sqrt{\left(\frac{b}{1+b}\right)}\cdot v\right) =$$

$$\frac{2}{m}\cdot\sqrt{\frac{b+1}{b}}\tan^{-1}\left(\sqrt{\left(\frac{b}{1+b}\right)}\cdot\sqrt{\vartheta-1}\right) = \boxed{\frac{2}{m}\cdot\sqrt{\frac{b+1}{b}}\tan^{-1}\left(\sqrt{\left(\frac{b}{1+b}\right)}\cdot\sqrt{e^{mx}-1}\right)}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

$\rightarrow Part2:$

$$\int \frac{1}{\sqrt{e^{mx}-1}}dx \rightarrow \left| \begin{array}{l} \vartheta = e^{mx} \\ d\vartheta = m \cdot e^{mx} \cdot dx \rightarrow dx = \frac{d\vartheta}{m \cdot e^{mx}} = \frac{1}{m} \cdot \frac{d\vartheta}{\vartheta} \end{array} \right.$$

$$\int \frac{1}{\sqrt{e^{mx}-1}}dx = \frac{1}{m}\int \frac{1}{\vartheta \cdot \sqrt{\vartheta - 1}}d\vartheta \rightarrow \left| \begin{array}{l} \nu = \sqrt{\vartheta - 1} \rightarrow \vartheta = \nu^2 + 1 \\ d\nu = \frac{1}{2} \cdot \frac{d\vartheta}{\sqrt{\vartheta - 1}} \rightarrow d\vartheta = 2 \cdot \nu \cdot d\nu \end{array} \right.$$

$$\int \frac{1}{\sqrt{e^{mx}-1}}dx = \frac{1}{m}\int \frac{2 \cdot \nu}{\left(\nu^2+1\right) \cdot \nu}d\nu = \frac{2}{m}\int \frac{1}{\left(\nu^2+1\right)}d\nu = \frac{2}{m}\tan^{-1}\nu$$

$$= \frac{2}{m}\tan^{-1}\sqrt{\vartheta - 1} = \frac{2}{m}\tan^{-1}\sqrt{e^{mx}-1}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

$$\rightarrow \textit{collect parts } 1 \textit{ and } 2$$

$$\int \frac{\sqrt{e^{mx}-1}}{1+b\cdot e^{mx}}dx = \frac{2}{m}\cdot\left[\sqrt{\frac{b+1}{b}}\tan^{-1}\left(\sqrt{\left(\frac{b}{1+b}\right)}\cdot\sqrt{e^{mx}-1}\right)-\tan^{-1}\sqrt{e^{mx}-1}\right] \ldots Q.E.D!$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Substituting $\left\{ m = 2, b = \frac{\gamma - 1}{2}, e^{mx} = M^2 \right\}$

$$\int \frac{\sqrt{e^{mx} - 1}}{1 + b \cdot e^{mx}} dx = \int \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma - 1}{2} \cdot M^2} \frac{dM}{M} =$$

$$\frac{2}{2}\left[\left(\sqrt{\frac{\frac{\gamma-1}{2}+1}{\frac{\gamma-1}{2}}}\right)\tan^{-1}\sqrt{\left(\frac{\frac{\gamma-1}{2}}{\frac{\gamma-1}{2}+1}\right)\cdot\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1}\right] =$$

$$\boxed{\left[\left(\sqrt{\frac{\gamma+1}{\gamma-1}}\right)\tan^{-1}\sqrt{\frac{\gamma+1}{\gamma-1}\cdot\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1}\right]}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Collected Equations

$$\theta = \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_2^{\,2}-1\right)}\right\}-\tan^{-1}\sqrt{M_2^{\,2}-1}\right]-\left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_1^{\,2}-1\right)}\right\}-\tan^{-1}\sqrt{M_1^{\,2}-1}\right]$$

- Or more simply

$$\theta = \nu(M_2)-\nu(M_1) \rightarrow \nu(M)=\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right\}-\tan^{-1}\sqrt{M^2-1}$$

"Prandtl-Meyer Function"

Implicit function … more Newton!

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- And we already know the derivative

$$\frac{d}{dM}\left[\nu(M)\right] = \frac{1}{M}\frac{\sqrt{M^2-1}}{\left(1+\frac{\gamma-1}{2}M^2\right)}$$

Newton Solver Algorithm

$$\theta = \nu(M_2) - \nu(M_1) \rightarrow \nu(M_2) = \theta + \nu(M_1)$$

- Expand in Taylor's series

$$\theta + \nu(M_1) = \nu(M_2) = \nu(M_{2(j)}) + \left(\frac{\partial \nu}{\partial M}\right)_{(j)} \left(M_2 - M_{2(j)}\right) + O(M_2) + \ldots.$$

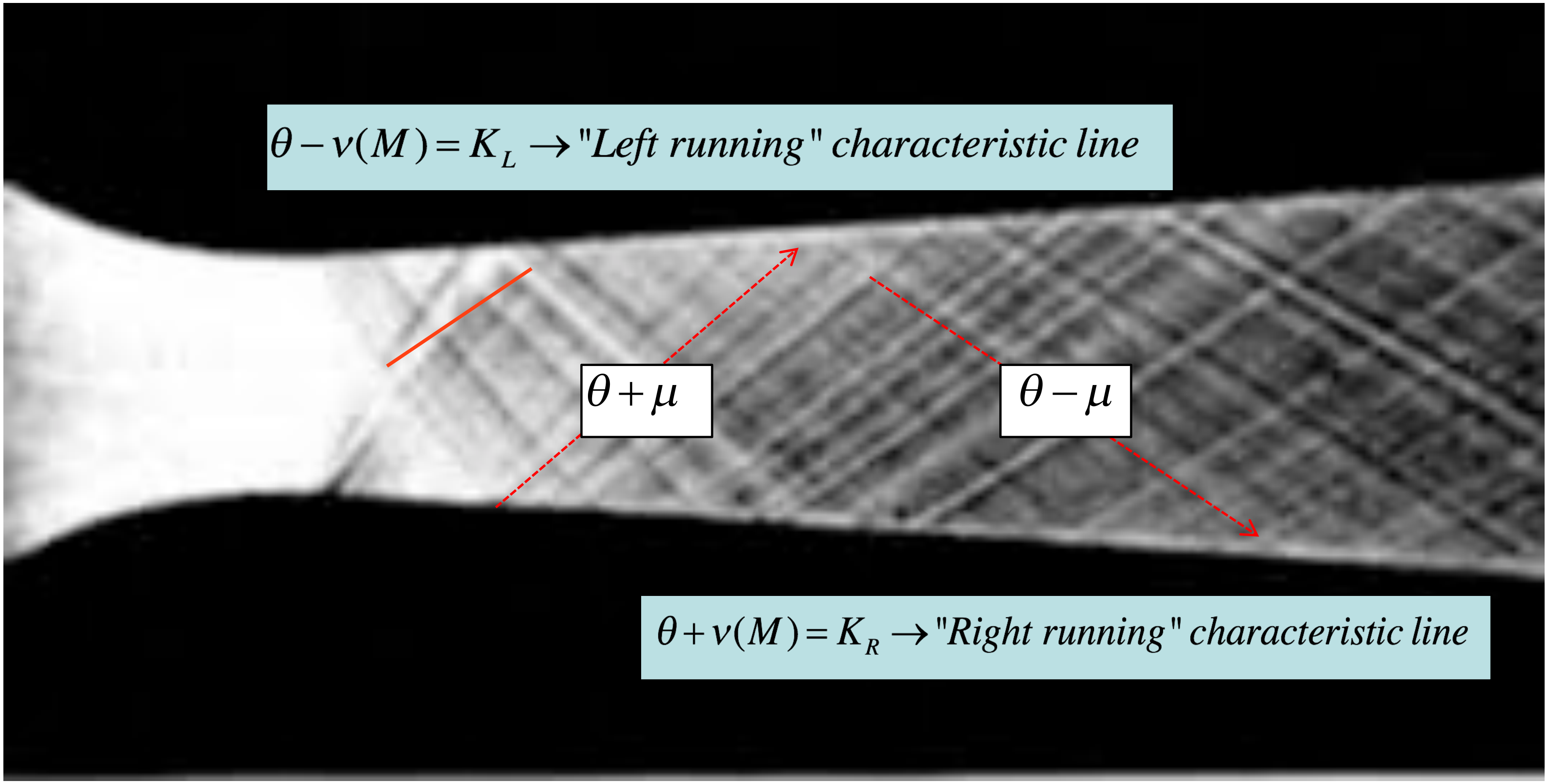
- Truncate after first order terms and solve for $M_{2(j)}$

$$M_{2(j+1)} = \frac{\left[\theta + \nu(M_1) - \nu(M_{2(j)})\right]}{\left(\frac{\partial \nu}{\partial M}\right)_{(j)}} + M_{2(j)}$$

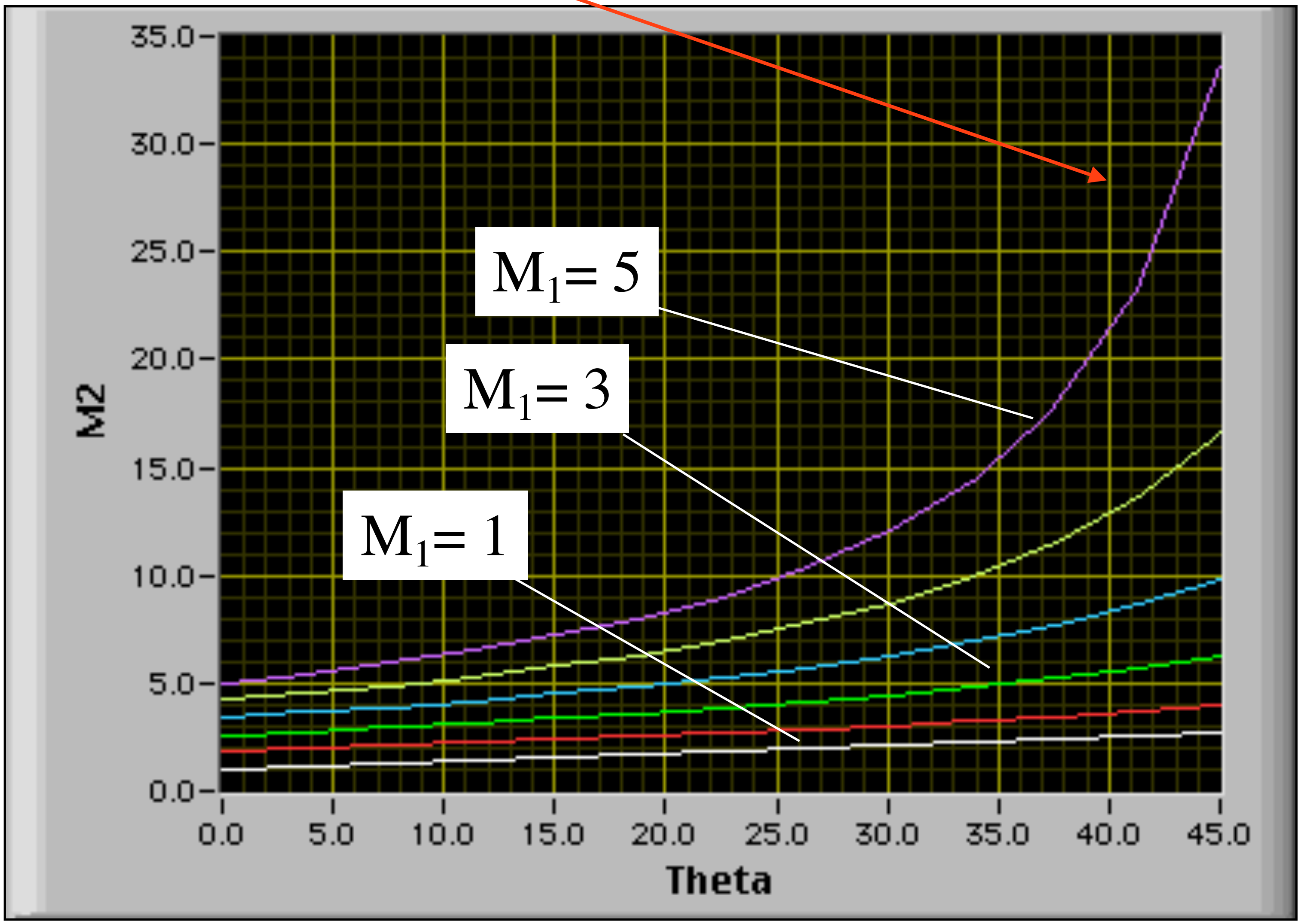
- Note: use radians!

$\theta + \nu(M) \rightarrow$ "Right running" characteristic line

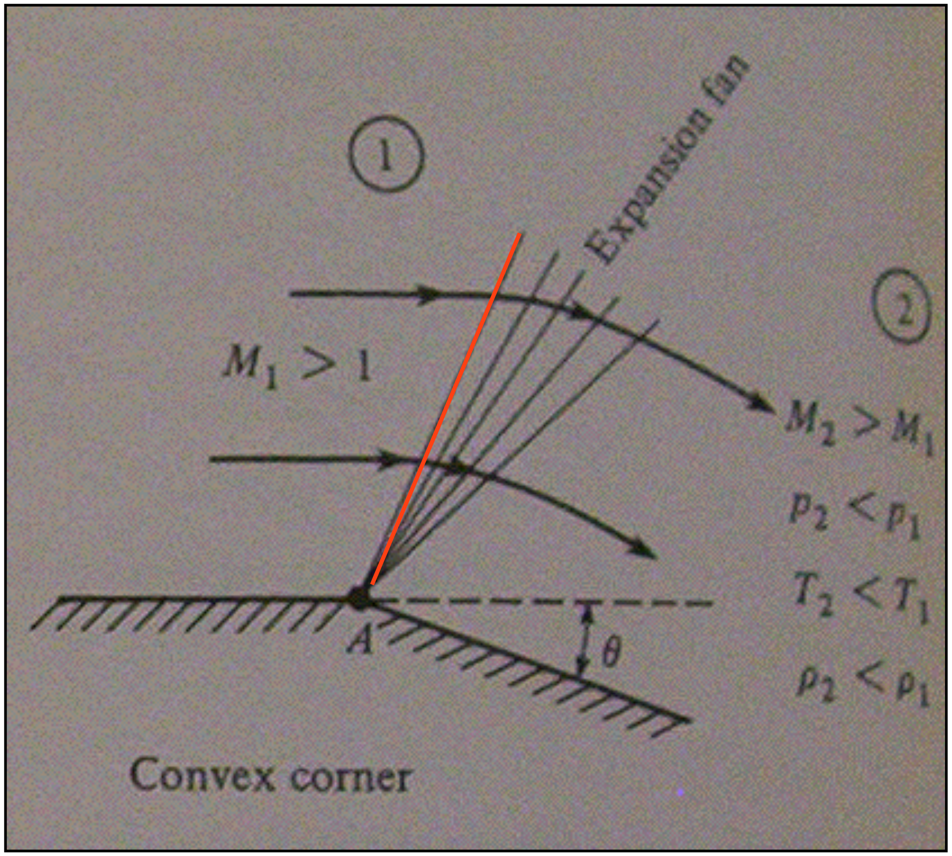
Shadow Graph Showing Supersonic Characteristic Lines (Prantl-Meyer Lines)



M_2 versus M_1, θ



Pressure and Temperature Change Across Expansion Fan



- Because each mach wave is infinitesimal, expansion is isentropic

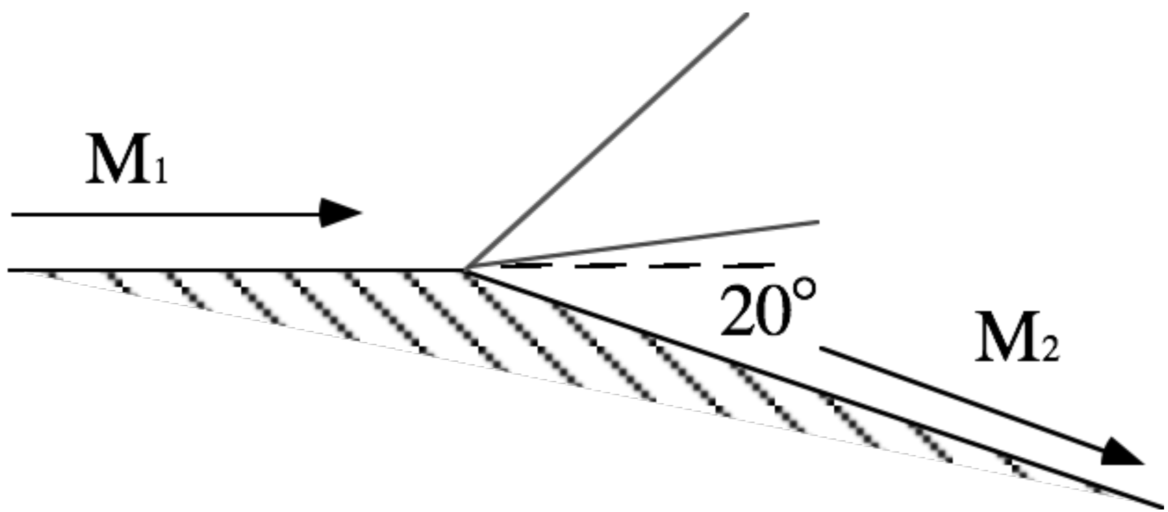
- $P_{0_2} = P_{0_1}$
- $T_{0_2} = T_{0_1}$

$$\frac{p_2}{p_1} = \frac{P_{0_1}}{p_1} \times \frac{p_2}{P_{0_2}} = \left[\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_2}{T_1} = \frac{T_{0_1}}{T_1} \times \frac{T_2}{T_{0_2}} = \left[\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right]$$

Numerical Example

- M_1=1.5, p_1=81.4 kPa, T1=255.6°K, γ =1.4, θ =20°



- [illegible]

Numerical Example (cont'd)

- $M_1=1.5$ ⟶

Corner entrance mach angle

$$\mu_1 = \sin^{-1}\left[\frac{1}{M_1}\right] = \text{asin}\left(\frac{1}{1.5}\right)\cdot\frac{180}{\pi} = 41.81°$$

$$\nu(M_1) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_1^{\,2}-1\right)}\right\} - \tan^{-1}\sqrt{M_1^{\,2}-1}$$

$$\left(\frac{1.4+1}{1.4-1}\right)^{0.5}\text{atan}\left(\left(\frac{1.4-1}{1.4+1}\,(1.5^2-1)\right)^{0.5}\right) - \text{atan}\left(\,\left(1.5^2-1\right)^{0.5}\right)$$

=0.207785 radians

Prandtl-Meyer Function

Numerical Example (cont'd)

- Compute ν (M_2)

$$\nu(M_2) = \theta + \nu(M_1) = 20\frac{\pi}{180} + 0.207785 = 0.5569$$

- Use Iterative Solver to Compute M_2

$$M_{2(j+1)} = \frac{\left[\theta + \nu(M_1) - \nu(M_{2(j)})\right]}{\left(\frac{\partial \nu}{\partial M}\right)_{(j)}} + M_{2(j)} \rightarrow M_2 = 2.2067$$

Numerical Example (cont'd)

- Pressure
  - Because each mach wave is infinitesimal, expansion is isentropic

    - $P_{0_2} = P_{0_1}$
    - $T_{0_2} = T_{0_1}$

$$p_2 = p_1 \left[\frac{1 + \frac{\gamma - 1}{2} M_1^{\ 2}}{1 + \frac{\gamma - 1}{2} M_2^{\ 2}} \right]^{\frac{\gamma}{\gamma - 1}} =$$

$$\left(\frac{1 + \frac{1.4 - 1}{2} 1.5^2}{1 + \frac{1.4 - 1}{2} 2.2067^2} \right)^{\left(\frac{1.4}{1.4 - 1}\right)} 81.4 = 27.655 \text{ kPa}$$

Numerical Example (cont'd)

- Temperature
  - Because each mach wave is infinitesimal, expansion is isentropic

    - $P_{0_2} = P_{0_1}$
    - $T_{0_2} = T_{0_1}$

$$T_2 = T_1 \left[\frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_2^2} \right] =$$

$$\left(\frac{1 + \frac{1.4 - 1}{2} 1.5^2}{1 + \frac{1.4 - 1}{2} 2.2067^2} \right) 255.6 \quad = 187.76°K$$

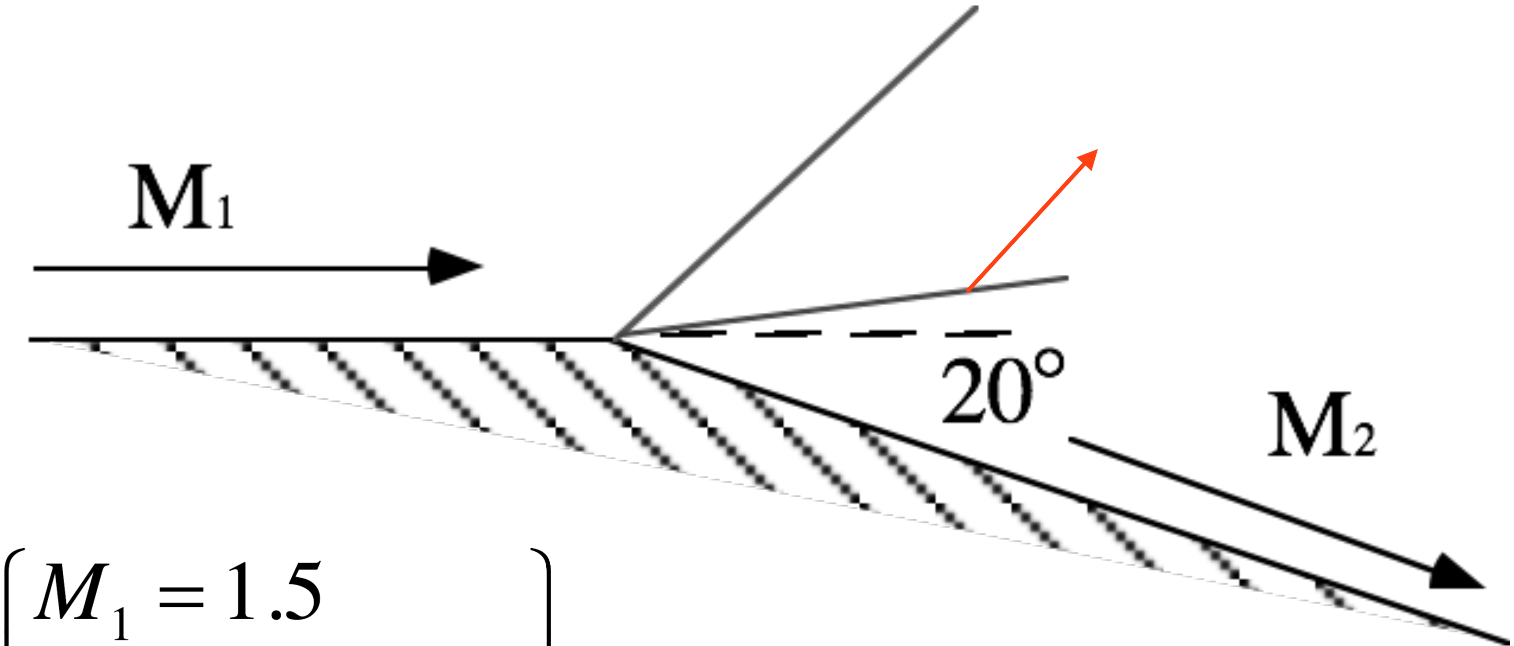
Numerical Example (cont'd)

- Exit Mach Angle

$$\mu_2 = \sin^{-1}\left[\frac{1}{M_2}\right] - \theta =$$

$$\operatorname{asin}\left(\frac{1}{2.2067}\right)\cdot\frac{180}{\pi} - 20 \quad = 6.947°$$

Numerical Example (concluded)



$$\left\{\begin{array}{l} M_1 = 1.5 \\ p_1 = 81.4\,kPa \\ T_1 = 255.56^o K \\ \mu_1 = 41.81^o \end{array}\right\}$$

$$\left\{\begin{array}{l} M_2 = 2.2067 \\ p_2 = 27.655\,kPa \\ T_2 = 187.76^o K \\ \mu_2 = 6.947^o \end{array}\right\}$$

Maximum Turning Angle

$$\frac{p_2}{p_1} = \frac{P0_1}{p_1} \times \frac{p_2}{P0_2} = \left[\frac{1 + \frac{\gamma - 1}{2} M_1^{\ 2}}{1 + \frac{\gamma - 1}{2} M_2^{\ 2}}\right]^{\frac{\gamma}{\gamma - 1}}$$

$$p_2 \Rightarrow 0 \rightarrow M_2 \Rightarrow \infty$$

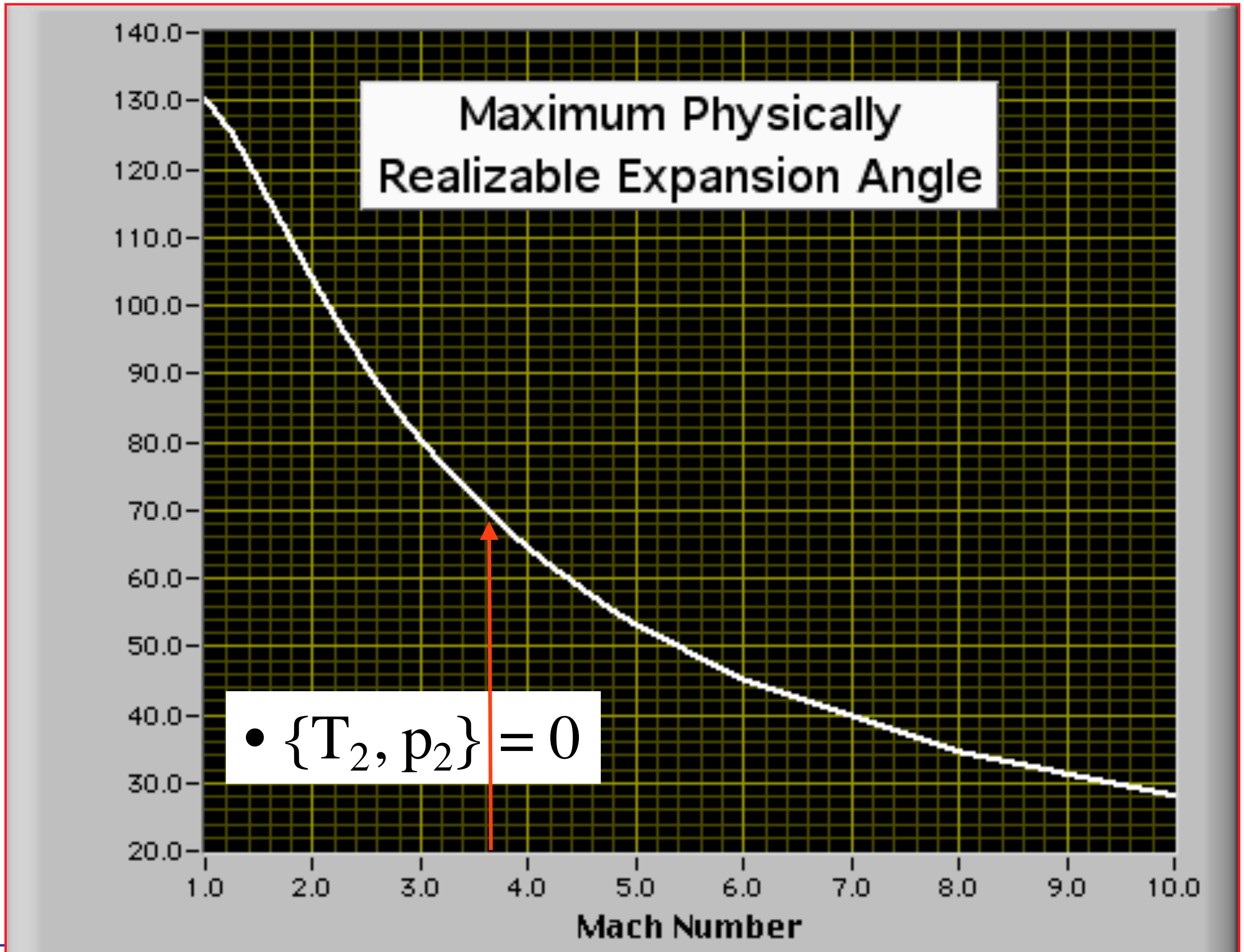
$$\nu(\infty) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}}\left(\infty^2 - 1\right)\right\} - \tan^{-1}\sqrt{\infty^2 - 1} = \left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right)\frac{\pi}{2}$$

$$\nu(\infty) = \theta_{max} + \nu(M_1) \longrightarrow \theta_{max} = \nu(\infty) - \nu(M_1)$$

$$\theta_{max} = \left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right)\frac{\pi}{2} - \left[\sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}}\left(M_1^{\ 2} - 1\right)\right\} - \tan^{-1}\sqrt{M_1^{\ 2} - 1}\right]$$

Maximum Turning Angle (cont'd)

- Plotting as a θ_{max} function of Mach number



- $\{T_2, p_2\} = 0$

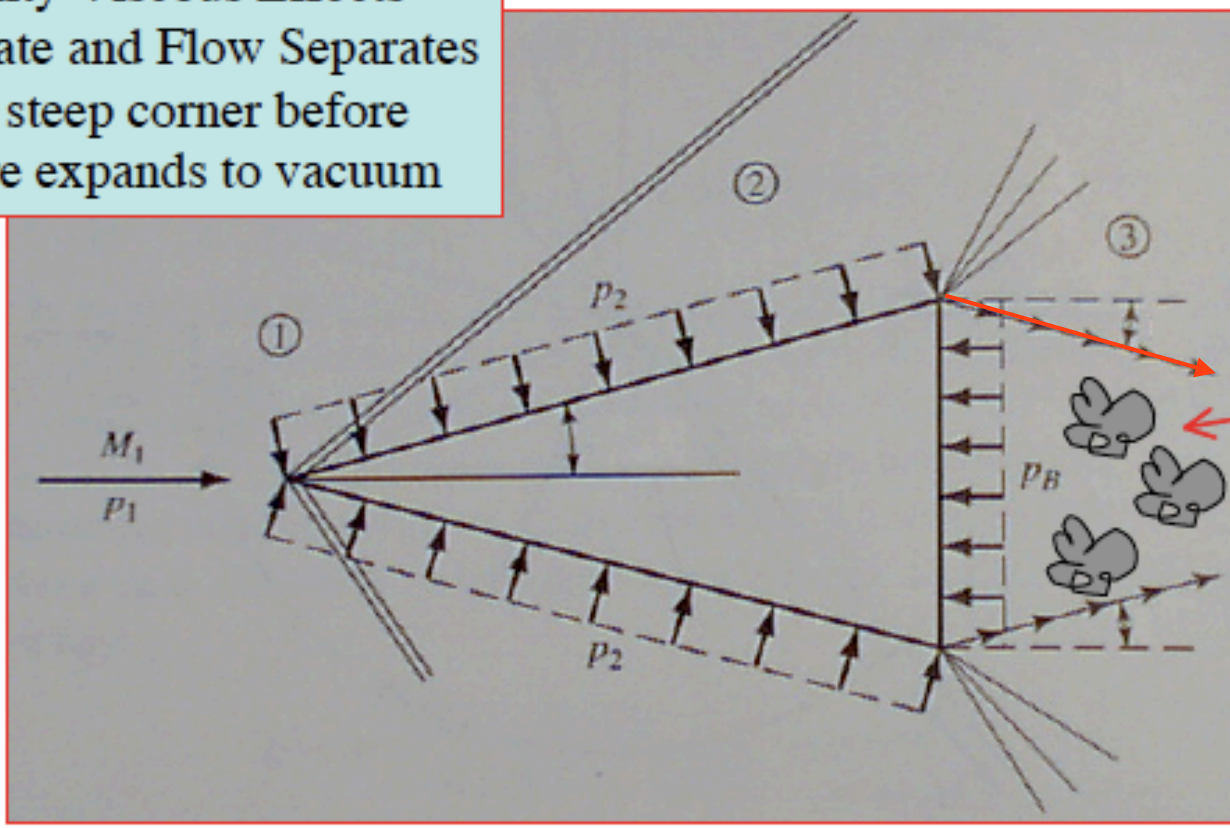
Maximum Turning Angle

F-117 Stealth Fighter

Subsonic Vehicle with Supersonic Features

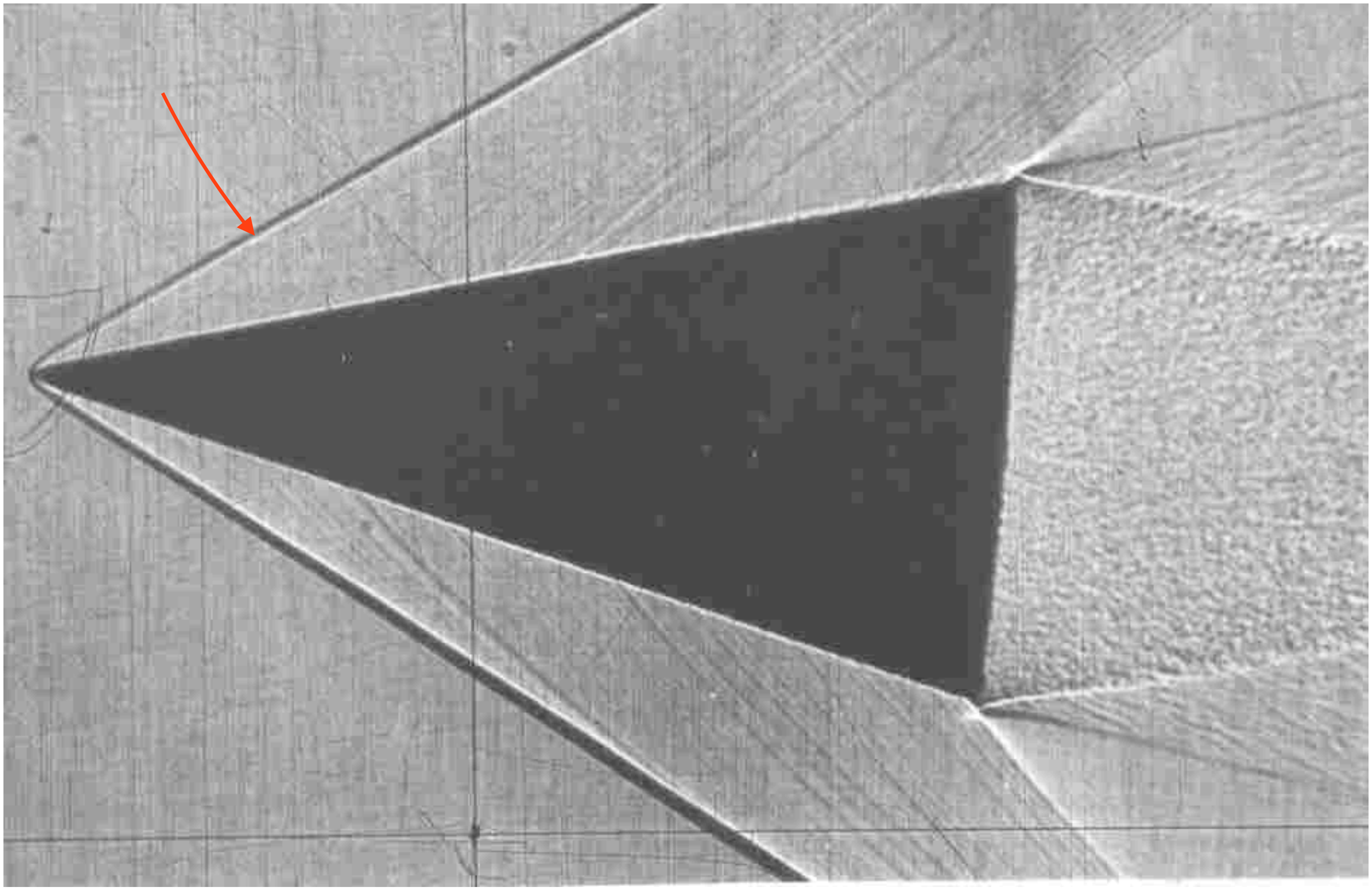
Maximum Turning Angle (continued)

- In Reality Viscous Effects Dominate and Flow Separates around steep corner before pressure expands to vacuum

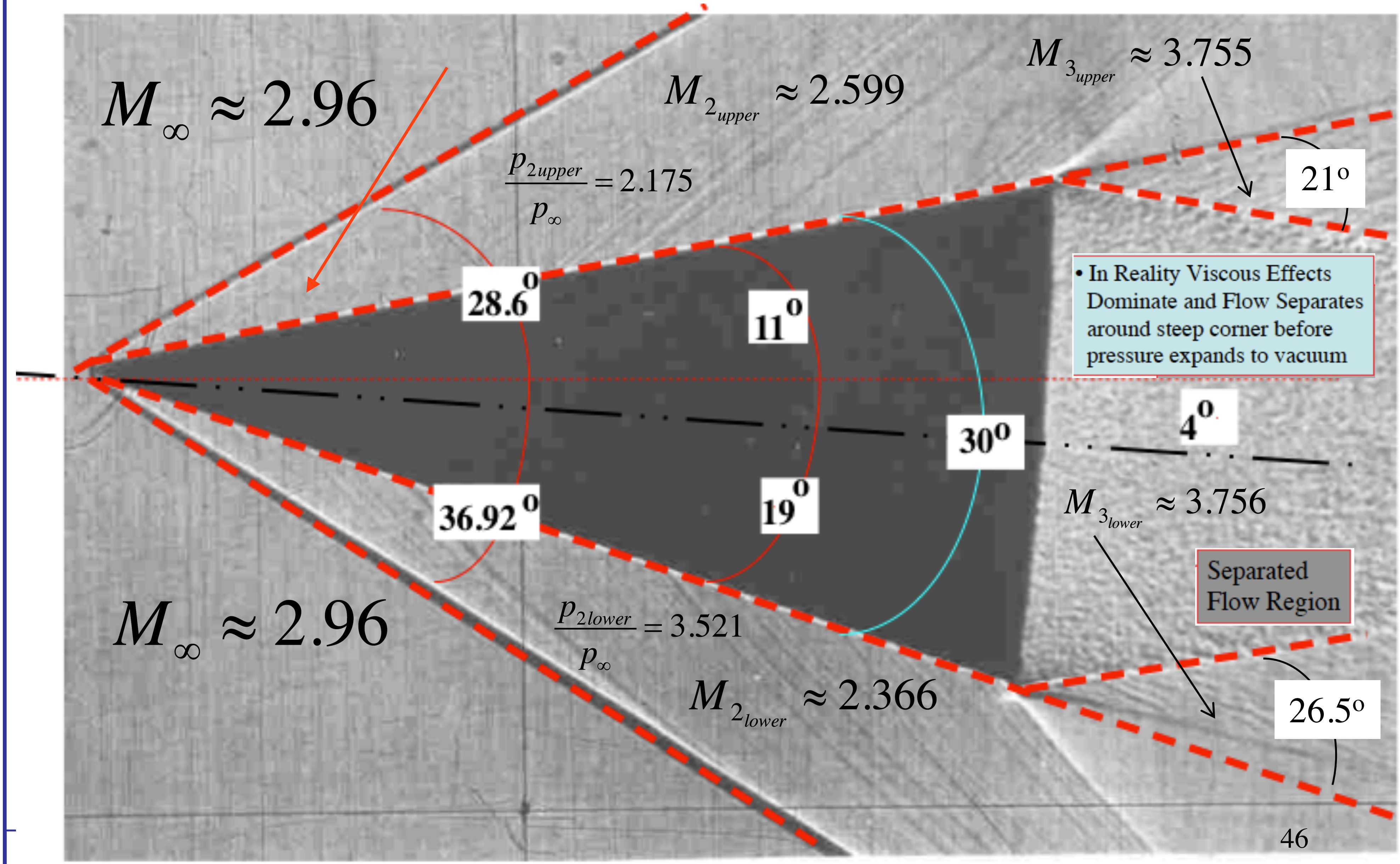


Separated Flow Region

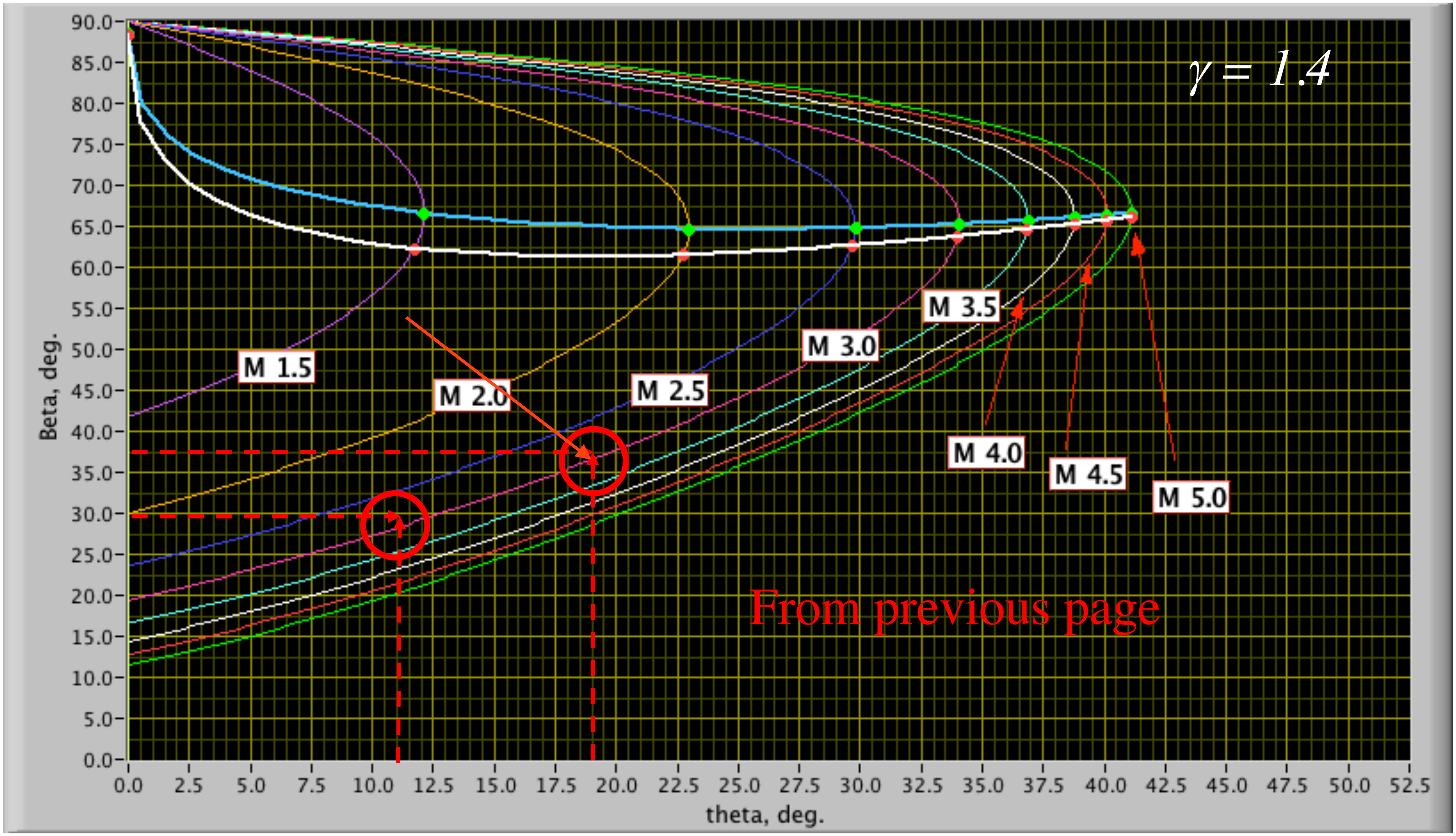
Maximum Turning Angle (concluded)



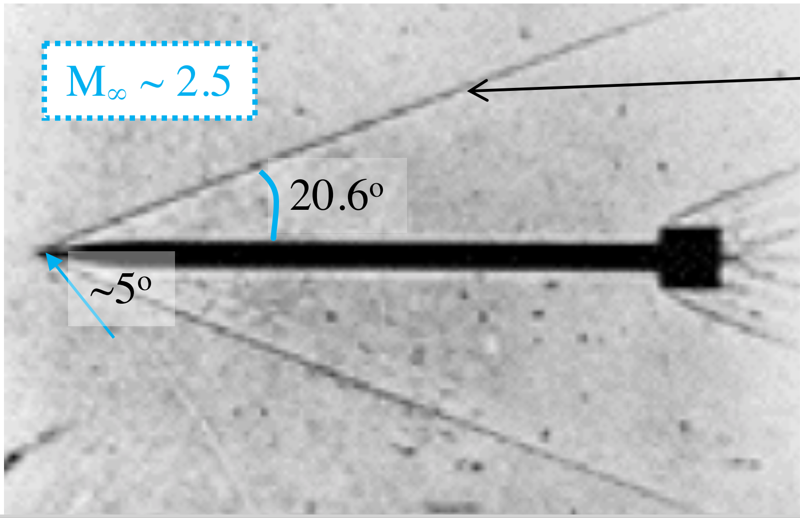
Blunt Wedge Example (continued)



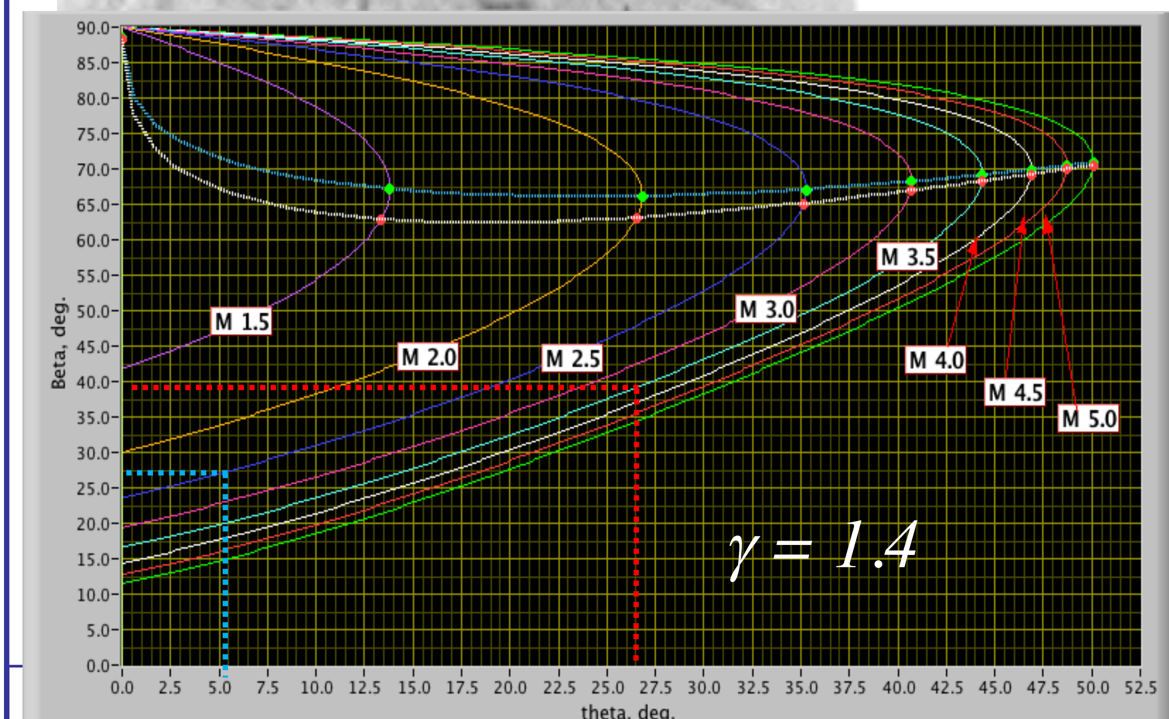
Blunt Wedge Example *(continued)*

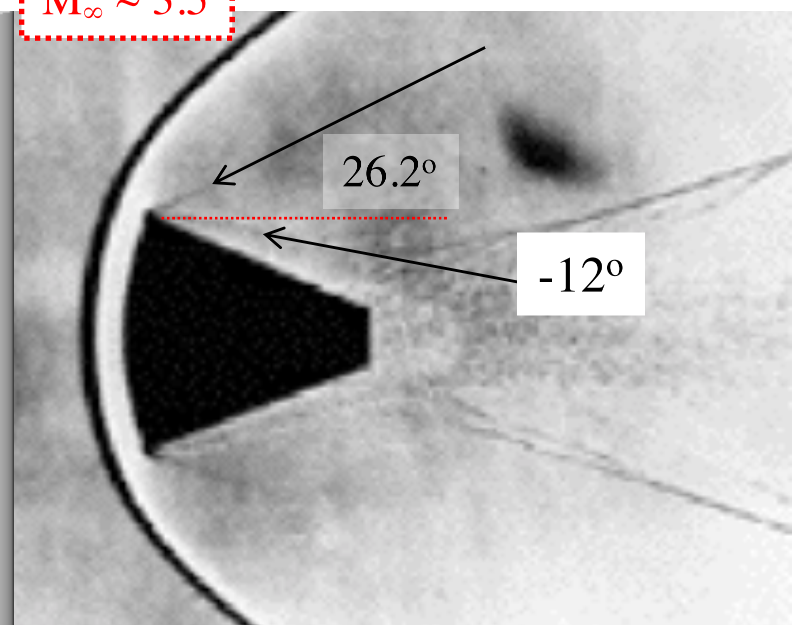


Concluding Example Weak, Strong, and Detached Shockwaves

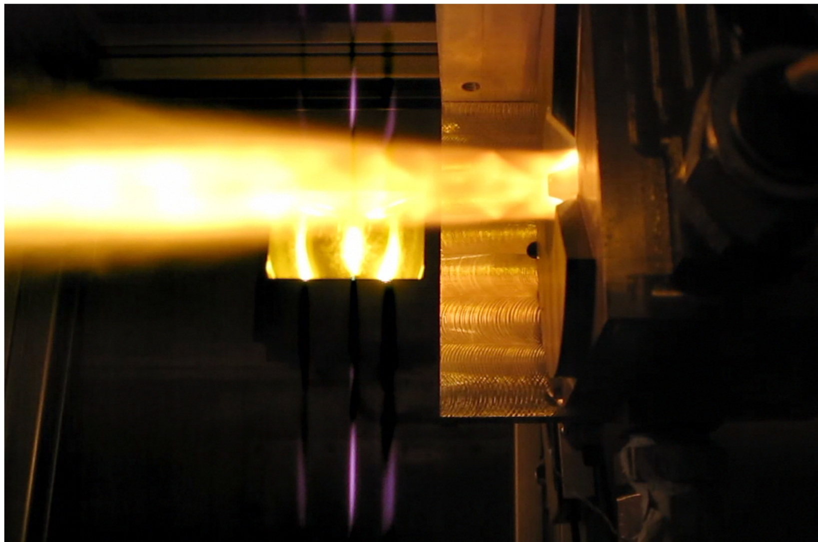


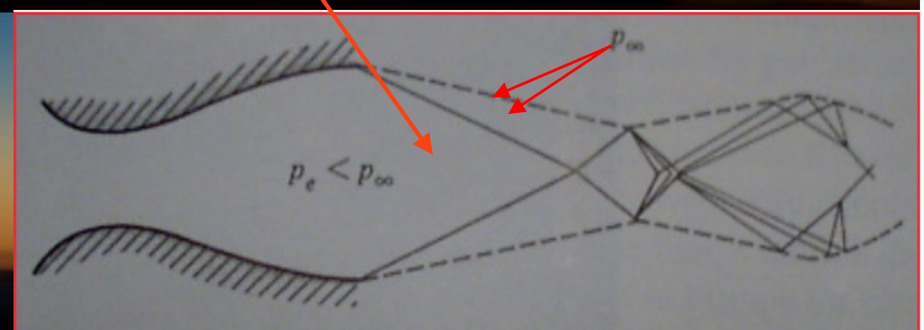
What is the approximate Free stream Mach Number?





Concluding Example, Shock Diamonds and Shock-Expansion Wave Theory





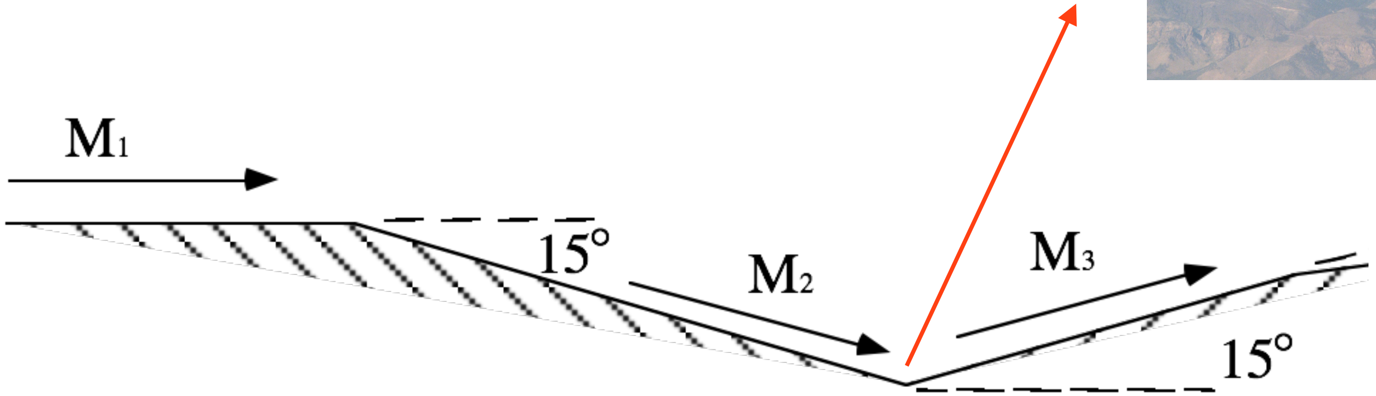
Schematic of the diamond wave pattern in the exhaust from a supersonic nozzle.

Section 6: Home Work #7

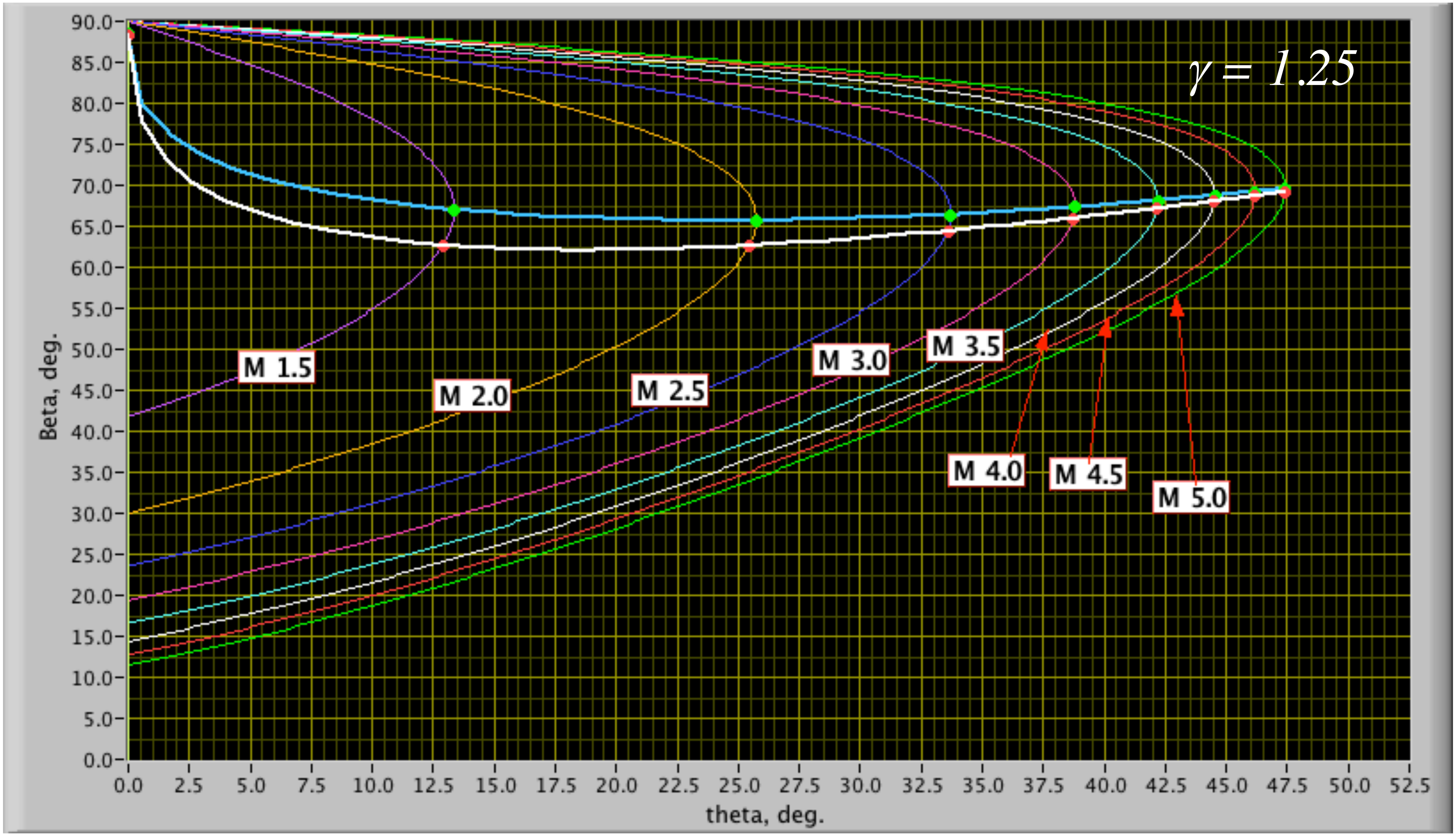
Assigned Wednesday November 3, Due Friday November 12

- M_1=4, p_1=0.01 atm, T_1=217°K, γ =1.25, θ_1=15°, θ_2=15°

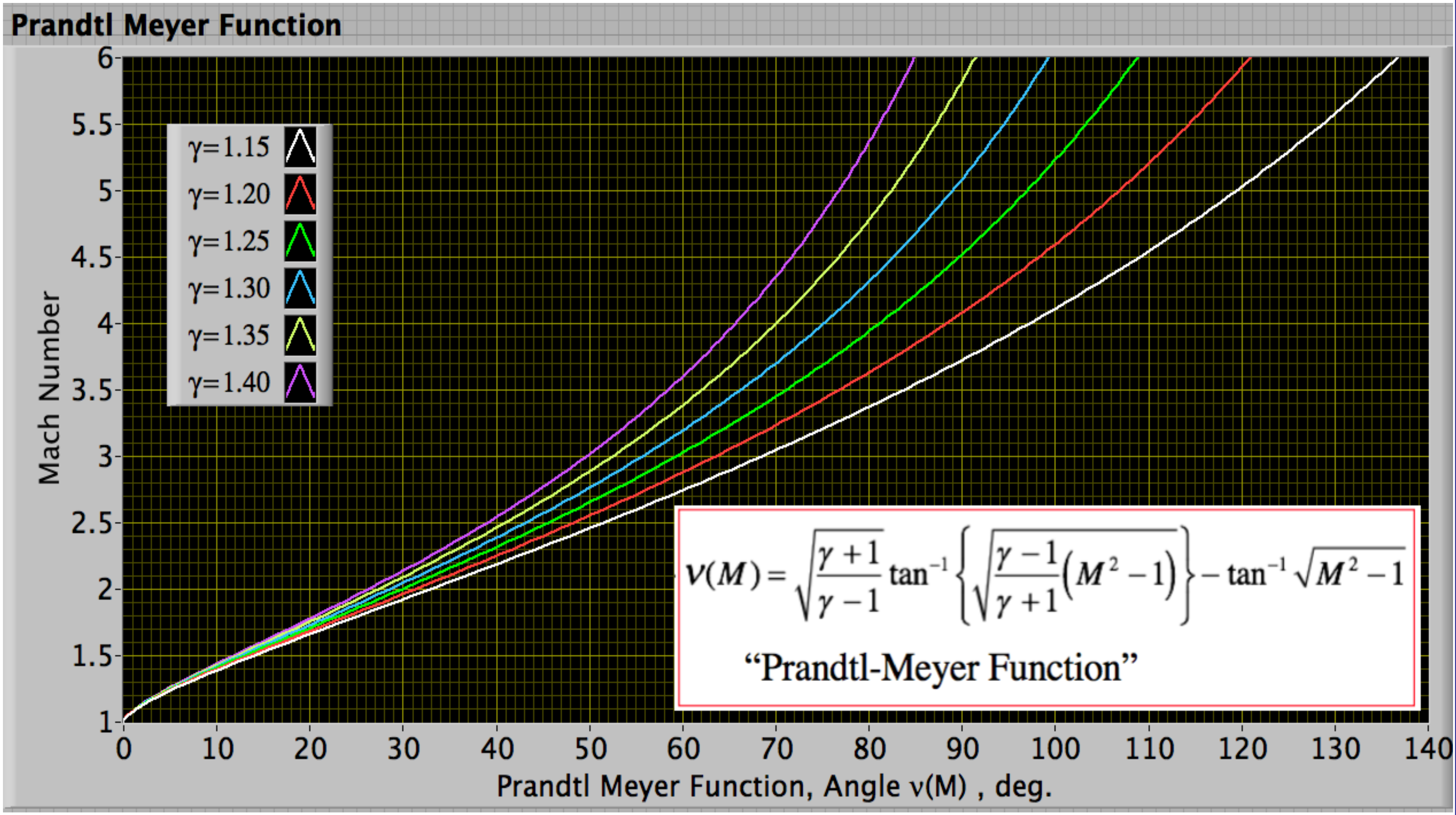
- Compute conditions after each corner
  - Entry and exit Mach wave angles or shock angles
  - Mach number
  - static & total pressure
  - temperature



Section 6: Home Work #7 (continued)

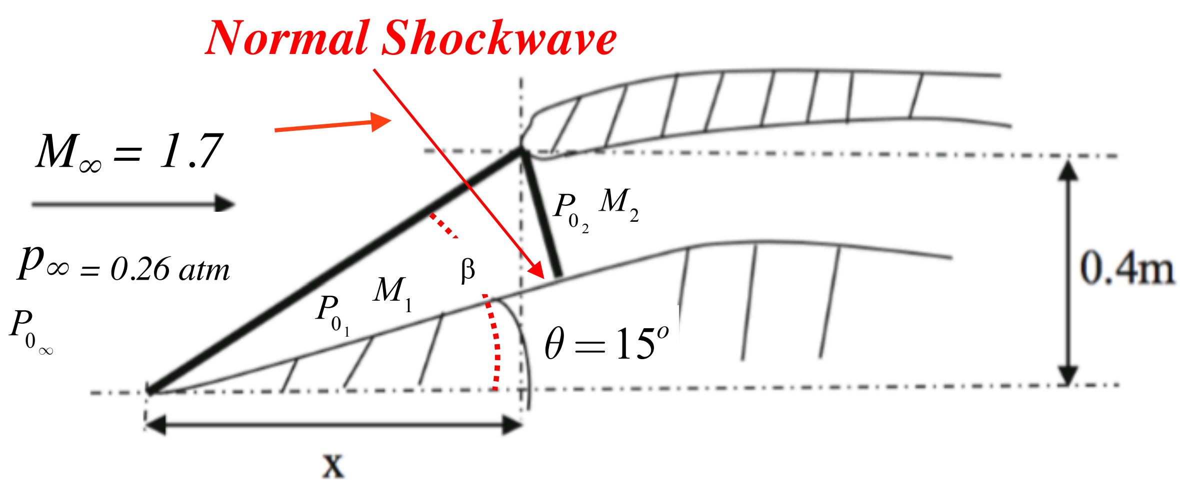


Section 6: Home Work #7 *(continued)*



Section 6: Home Work #7 Part2

A ramjet operates at an altitude of 10,000 m ($T_a = 223\ K$, $p_\infty = 0.26\ atm$, $\gamma = 1.4$) at a Mach number of 1.7. The external diffusion is based on an oblique shock and on a normal shock, as described in the shown figure.



Hint →

$$\beta \rightarrow f(M_\infty, \theta)$$

$$\left\{\frac{P_{0_1}}{P_{0_\infty}}, M_1\right\} \rightarrow f(M_\infty, \beta)$$

$$\frac{P_{0_2}}{P_{0_1}} \rightarrow f(M_1)$$

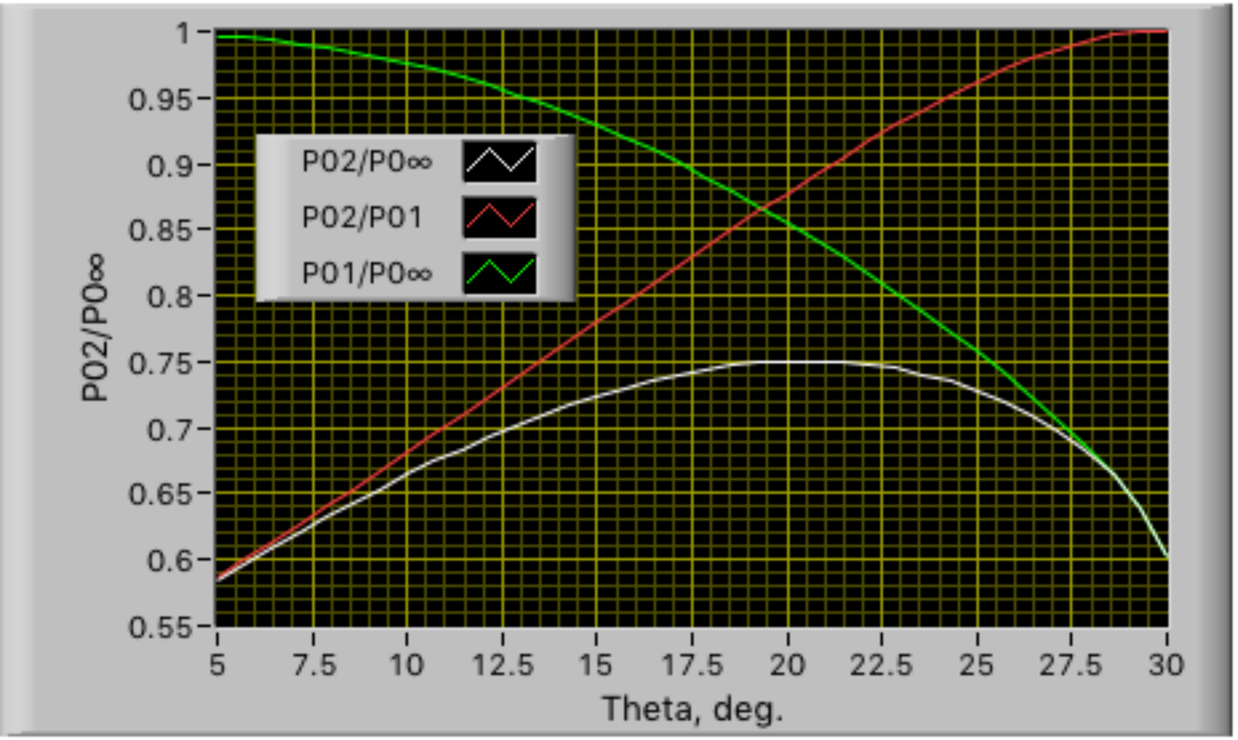
$$\frac{P_{0_2}}{P_{0_\infty}} = \frac{P_{0_2}}{P_{0_1}} \times \frac{P_{0_1}}{P_{0_\infty}}$$

...Plot $\frac{P_{0_2}}{P_{0_\infty}}$ vs θ

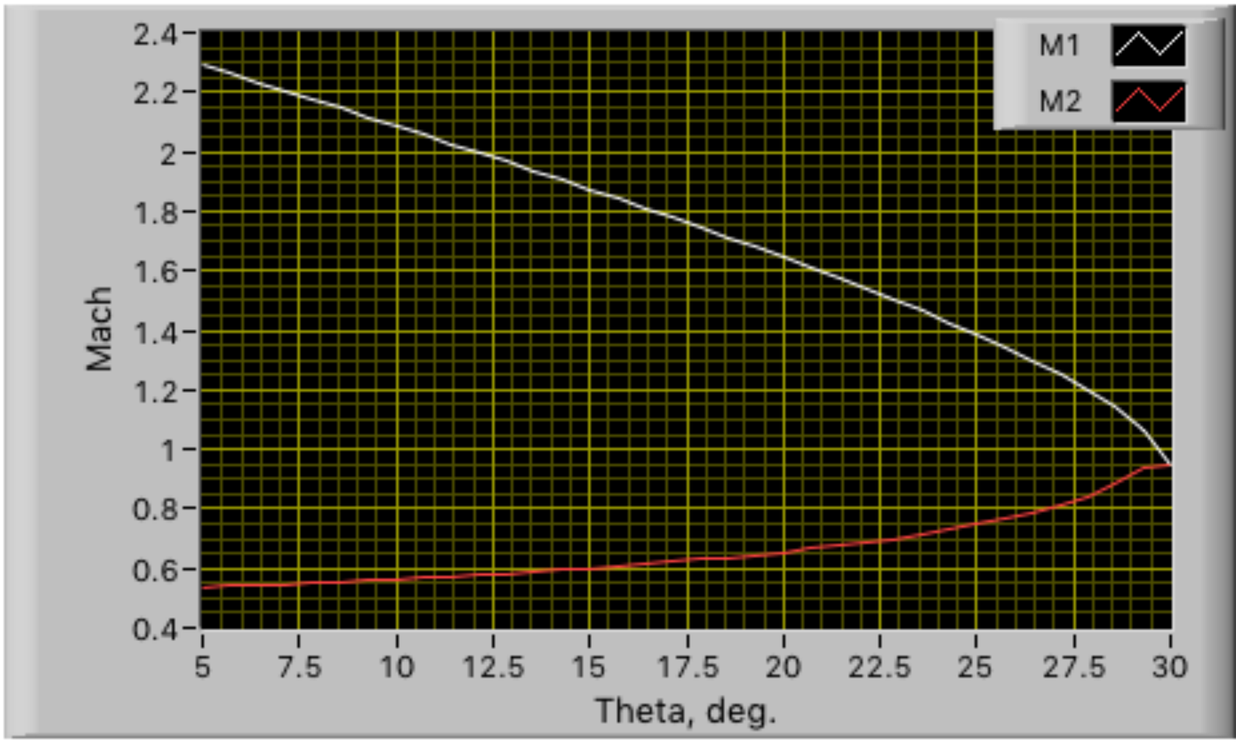
Calculate

- Stagnation pressure recovery, $\frac{P_{0_2}}{P_{0_\infty}}$?
- At what Mach number does the oblique shock become detached?
- What is the distance x, from the cone tip to the outer inlet lip, for the condition described in the figure? $\{M_\infty = 1.7, \theta = 15^o\}$
- What is the best turning angle θ in terms of highest pressure ratio, $\frac{P_{0_2}}{P_{0_\infty}}$?

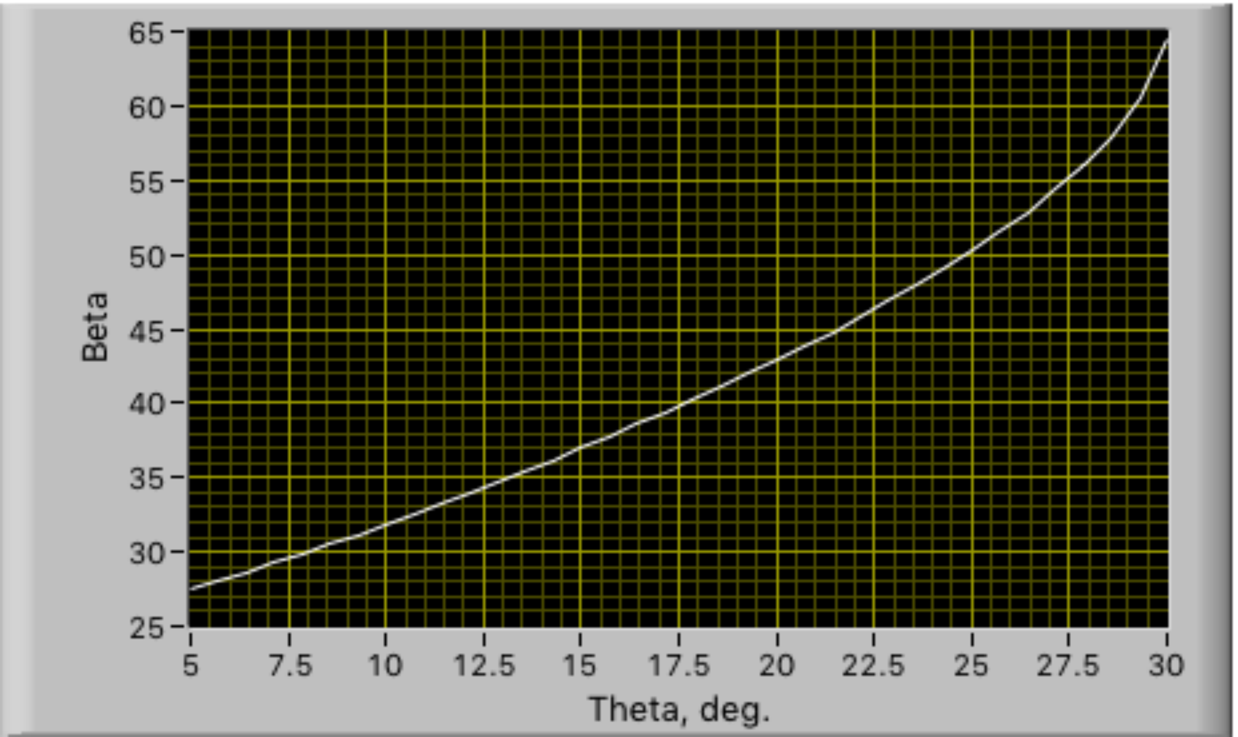
Stagnation Pressure Ratio



Mach Number



Obliaque Shock Angle



Input data

theta max: 30.00000

theta min: 5.000000

# of points: 35

gamma: 1.40

Mach Nu mber: 2.5000

EXAMPLE OPTIMIZATION

Section 6: Home Work #7 (Part 2 continued)