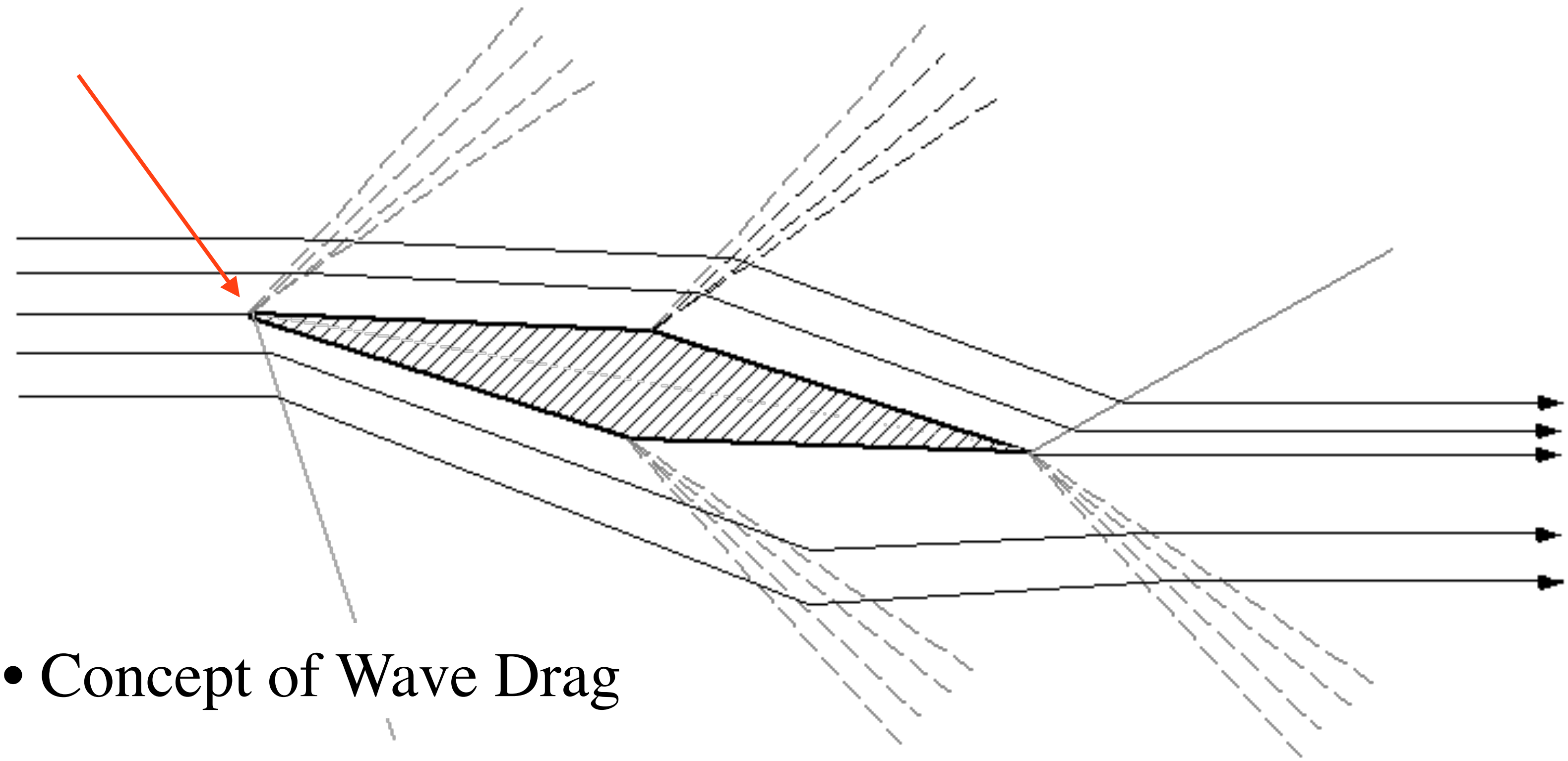
Section 7, Lecture 2:
Supersonic Flow on Finite Thickness Wings

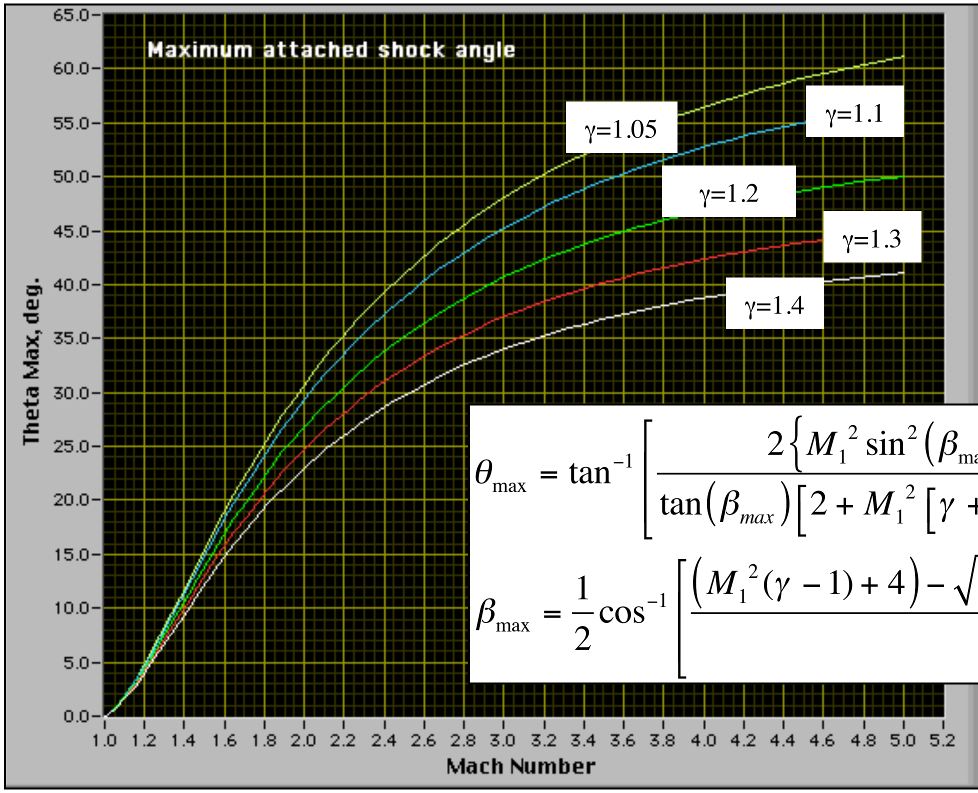


- Concept of Wave Drag

- Anderson, Chapter 4, pp. 174-179

Supersonic Airfoils

- Recall that for a leading edge in Supersonic flow there is a maximum wedge angle at which the oblique shock wave remains attached

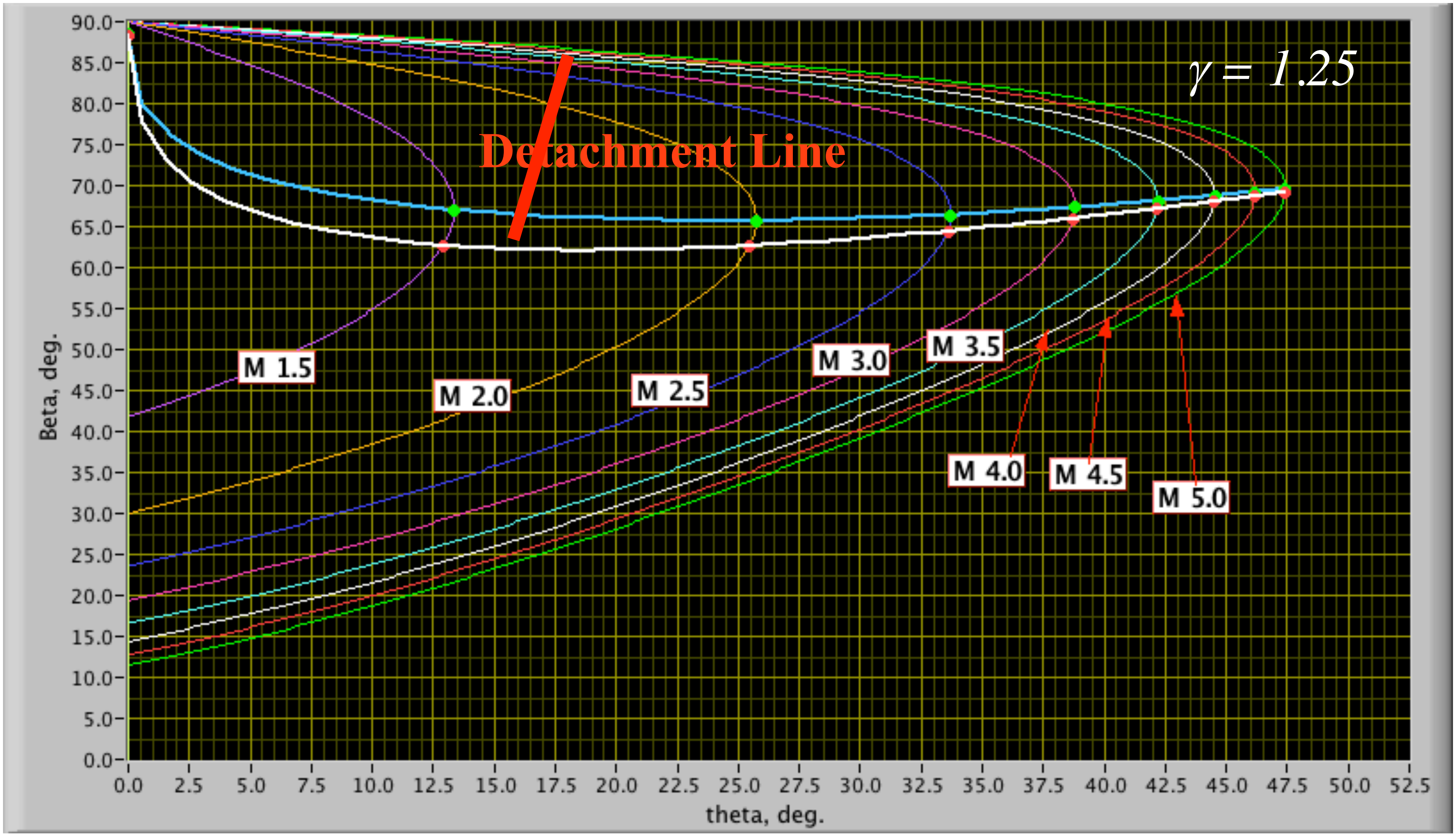


- Beyond that angle shock wave becomes detached from leading edge

$$\theta_{\max} = \tan^{-1}\left[\frac{2\left\{M_1^{\,2}\sin^2\left(\beta_{\max}\right)-1\right\}}{\tan\left(\beta_{max}\right)\left[2+M_1^{\,2}\left[\gamma+\cos\left(2\beta_{\max}\right)\right]\right]}\right]$$

$$\beta_{\max} = \frac{1}{2}\cos^{-1}\left[\frac{\left(M_1^{\,2}(\gamma-1)+4\right)-\sqrt{16(\gamma+1)+8M_1^{\,2}(\gamma^2-1)+M_1^{\,4}(\gamma+1)^2}}{2\gamma M_1^{\,2}}\right]$$

Supersonic Airfoils (continued)



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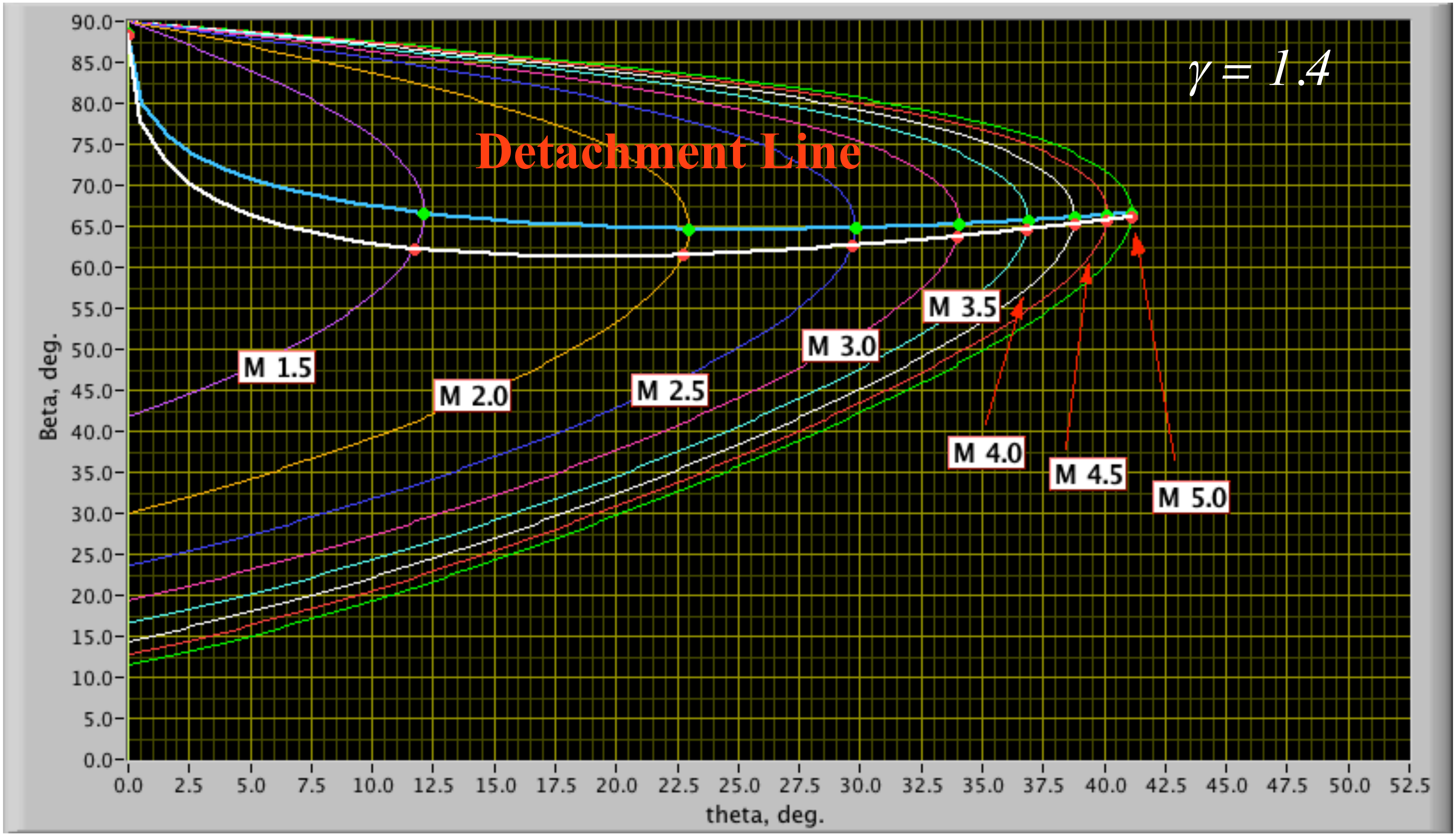
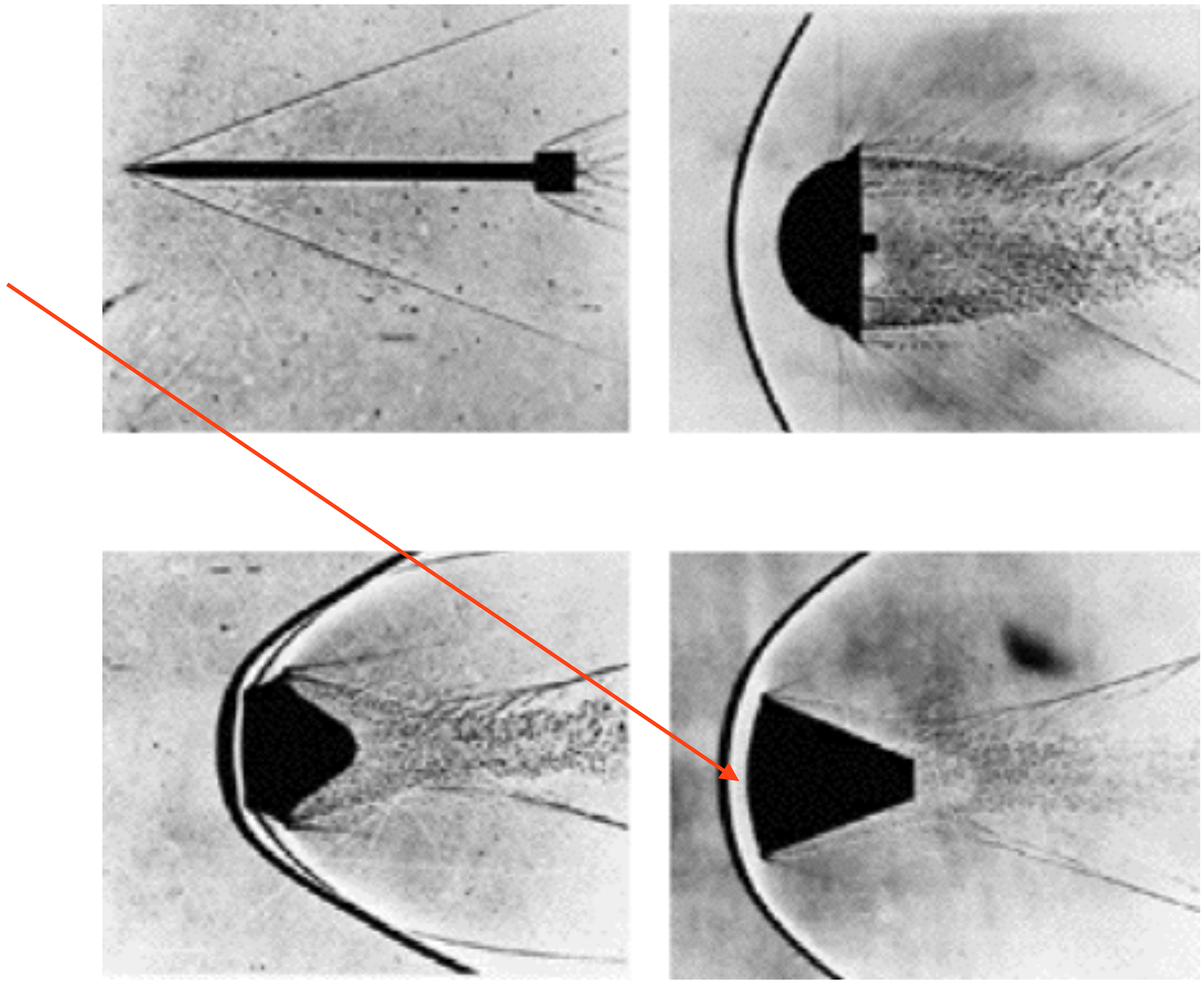
Supersonic Airfoils

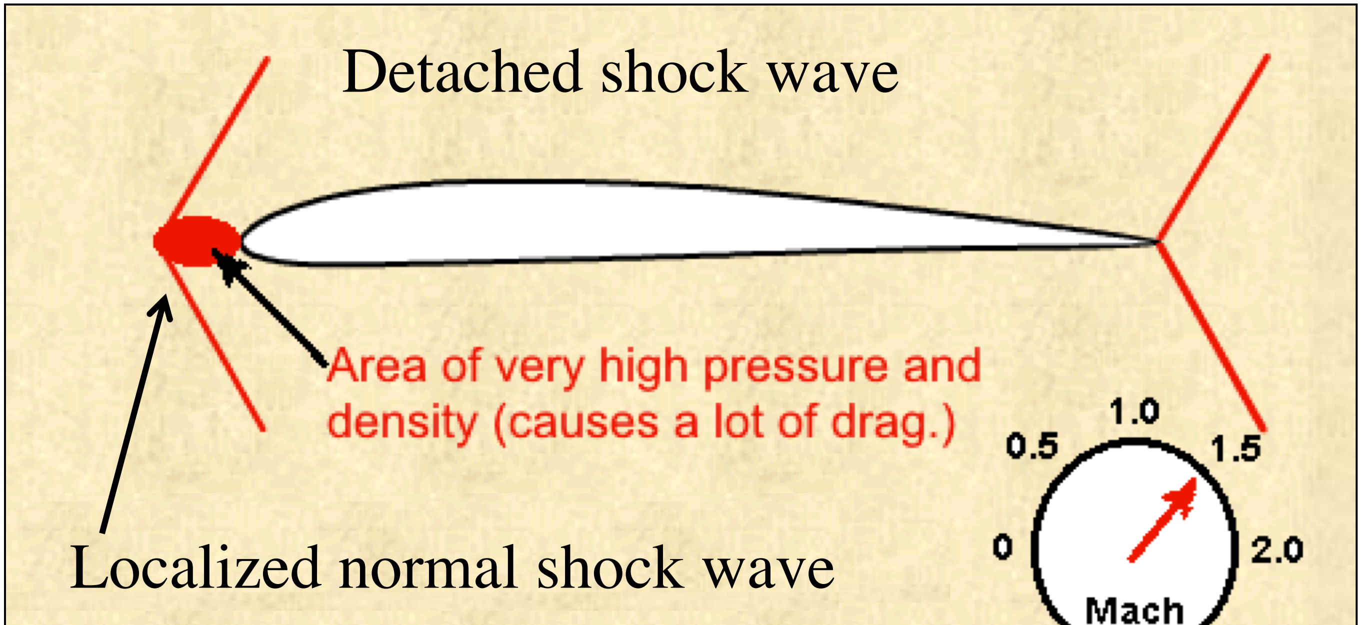
(Part 2 continued)





Supersonic Airfoils (cont'd)

- Normal Shock wave formed off the front of a blunt leading causes significant drag

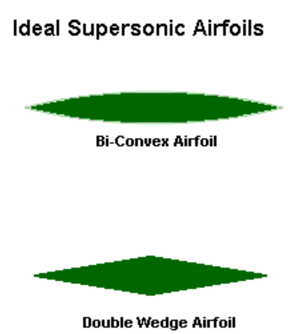


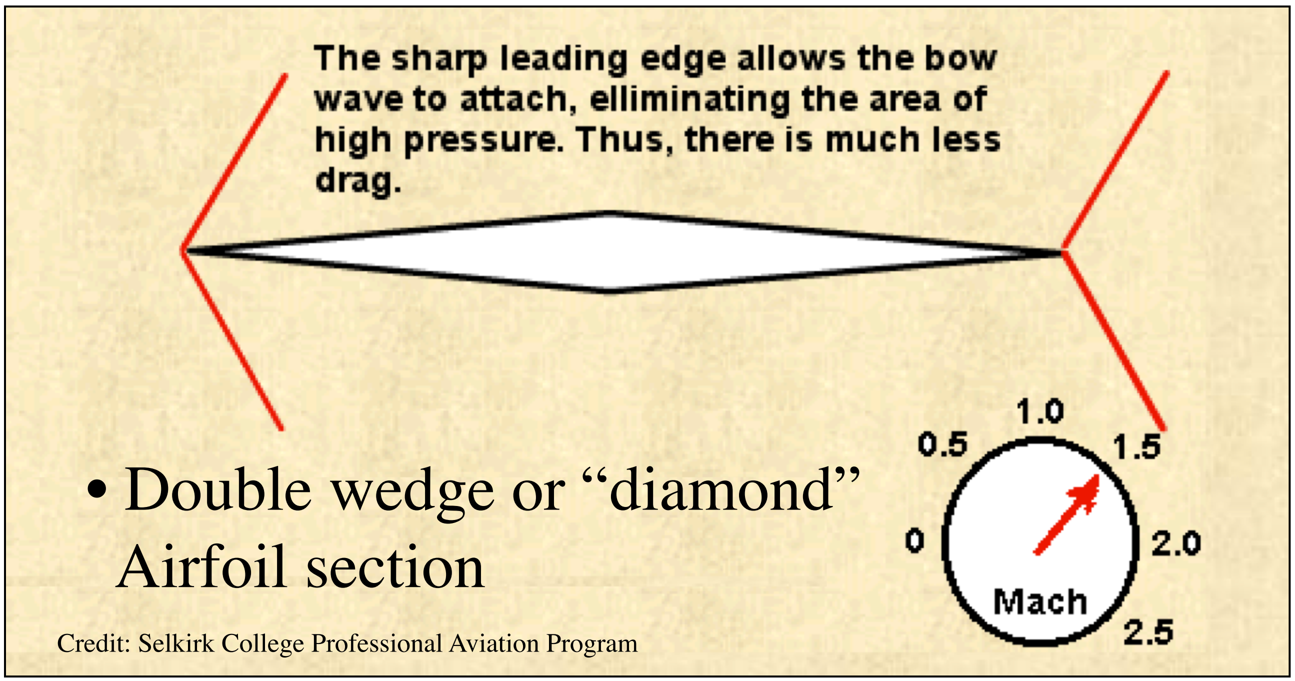
Credit: Selkirk College Professional Aviation Program

Supersonic Airfoils (cont'd)

- **To eliminate this leading edge drag caused by detached bow wave Supersonic wings are typically quite sharp at the leading edge**

- **Design feature allows oblique wave to attach to the leading edge eliminating the area of high pressure ahead of the wing.**

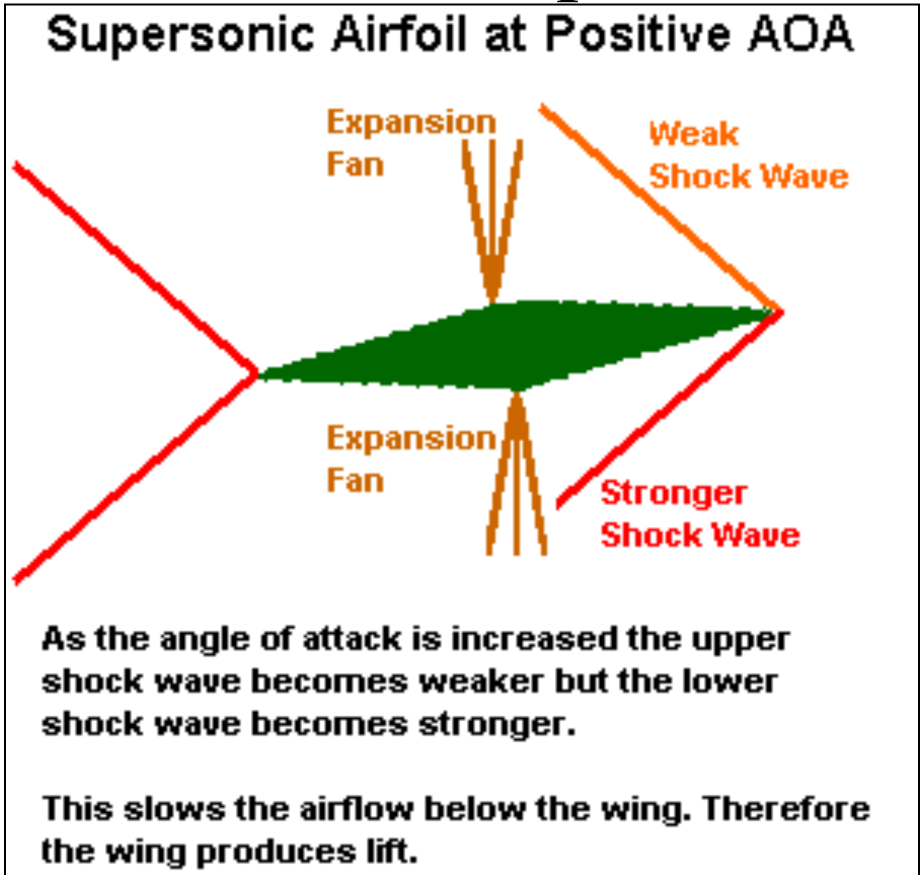




- Double wedge or "diamond" Airfoil section

Credit: Selkirk College Professional Aviation Program

Supersonic Airfoils (cont'd)



Credit: Selkirk College Professional Aviation Program

- **When supersonic airfoil is at positive angle of attack shock wave at the top leading edge is weakened and the one at the bottom is intensified.**
- **Result is the airflow over the top of the wing is now faster.**
- **Airflow will also be accelerated through the expansion fans.**
- **The fan on the top will be stronger than the one on the bottom.**
- **Result is faster flow and therefore lower pressure on the top of the airfoil.**

- We already have all of the tools we need to analyze the flow on this wing

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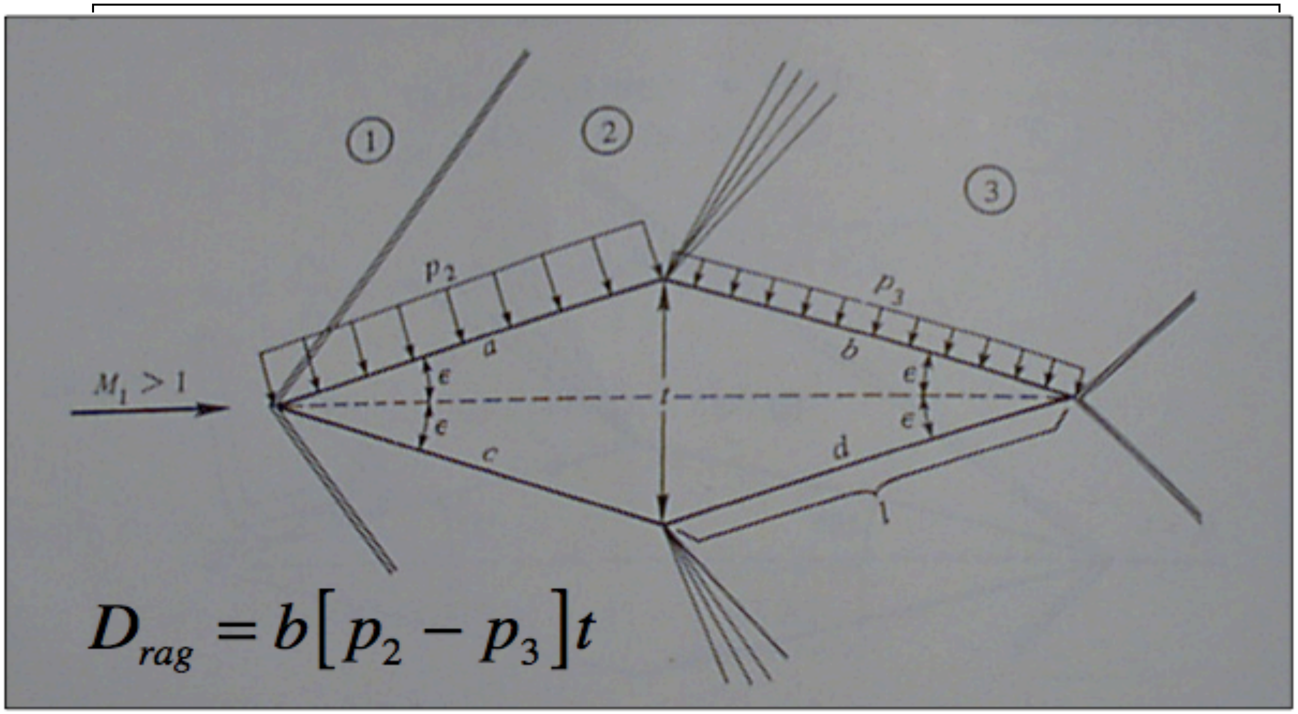
Supersonic Flow on Finite Thickness Wings at zero α

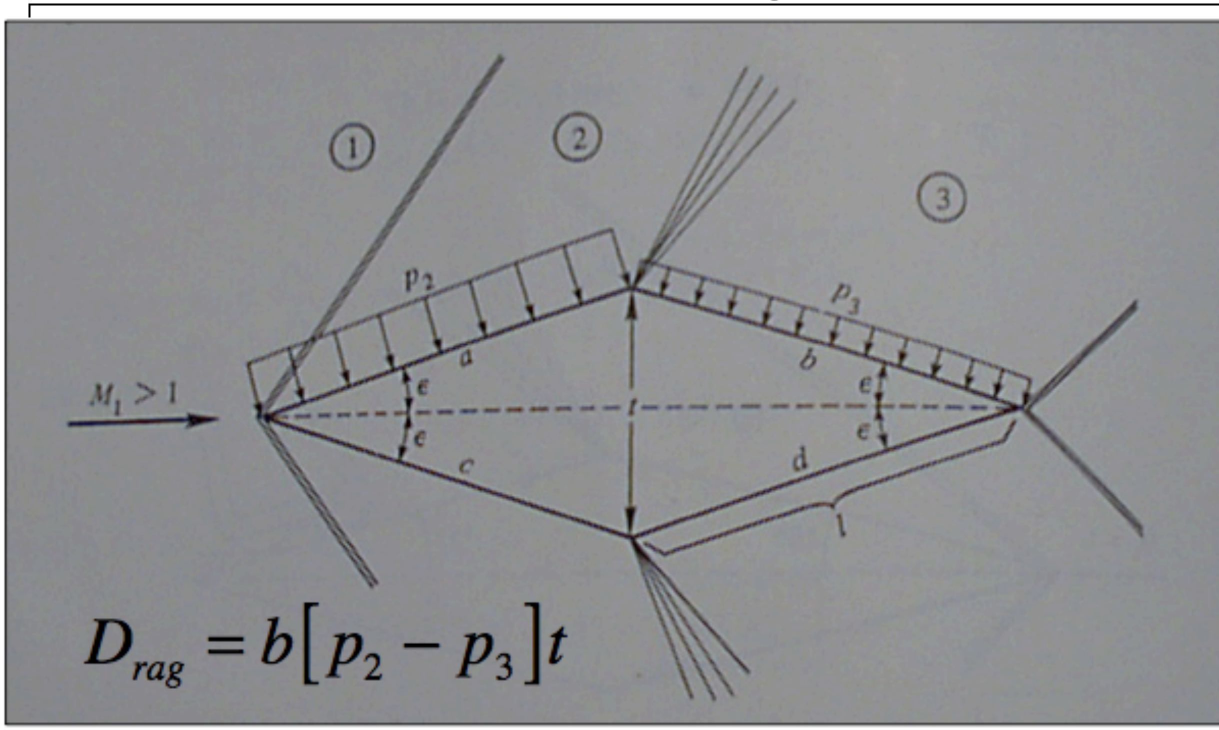


- Symmetrical Diamond-wedge airfoil, zero angle of attack

$$D_{rag} = 2b\left[p_2 l \sin(\varepsilon) - p_3 l \sin(\varepsilon)\right] \Rightarrow \sin(\varepsilon) = \frac{t/2}{l}$$

$$D_{rag} = b\left[p_2 - p_3\right]t$$

Supersonic Flow on Finite Thickness Wings at zero α



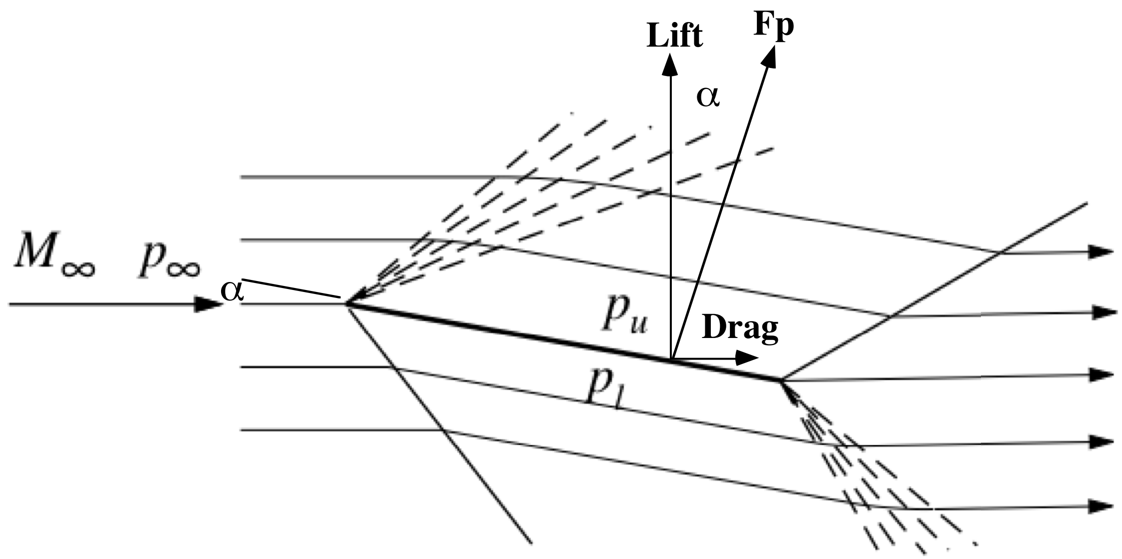
- Look at pressure difference, p_2-p_3

  - *Oblique shock wave from region 1->2 .. $p_2 > p_\infty$*

  ∞ Εξπανσιον φαν φρομ ρεγιον 2---> 3 ... $\pi_3 < \pi_2$

... $p_2 > p_\infty$ --> $\Delta_{\rho\alpha\gamma} > 0$

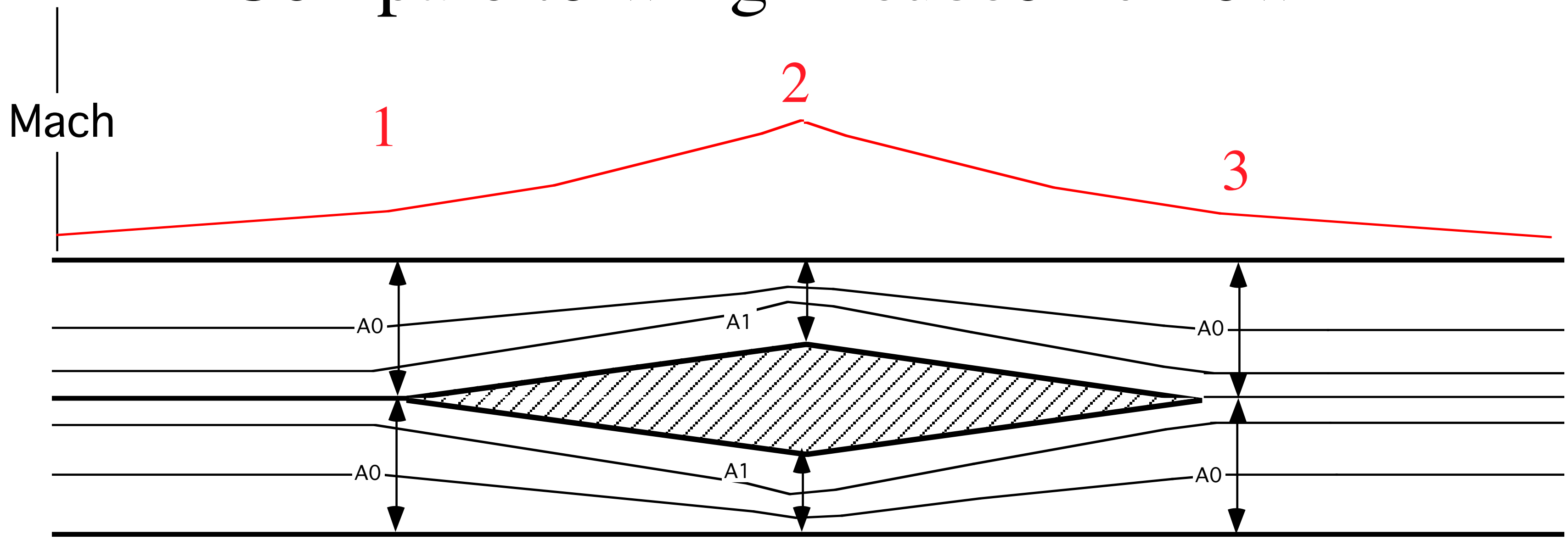
Supersonic Flow on Finite Thickness Wings at zero α



- Compare to Flat Plate

$$C_L = \frac{\left[\frac{p_l}{p_\infty} - \frac{p_u}{p_\infty}\right]}{\left(\frac{\gamma}{2} M_\infty^{\ 2}\right)} \cos\alpha \rightarrow C_D = 0$$

Compare to wing in subsonic flow



- from Bernoulli

$$P_0 = p + Cp_{max}\bar{q} = p\left[1 + Cp_{max}\frac{\gamma}{2}M^2\right] \Rightarrow \int_1^2 p\,dx = P_0\int_1^2 \frac{dx}{f[M]}$$

$$\boxed{p = \frac{P_0}{\left[1 + Cp_{max}\frac{\gamma}{2}M^2\right]} = \frac{P_0}{f[M]}}$$

$$\int_2^3 p\,dx = P_0\int_2^3 \frac{dx}{f[M]} = -P_0\int_1^2 \frac{dx}{f[M]} \Rightarrow \int_1^3 p\,dx = 0$$

Compare to wing in subsonic flow (cont'd)

$$\int_1^2 p\,dx = P_0 \int_1^2 \frac{dx}{f[M]}$$

• at zero lift .. Subsonic wings in inviscid flow have *No-drag!*

$$\int_2^3 p\,dx = P_0 \int_2^3 \frac{dx}{f[M]} = -P_0 \int_1^2 \frac{dx}{f[M]} \Rightarrow \int_1^3 p\,dx = 0$$

• Known as **d'Alembert's Paradox …**

• **Subsonic flow over wings**

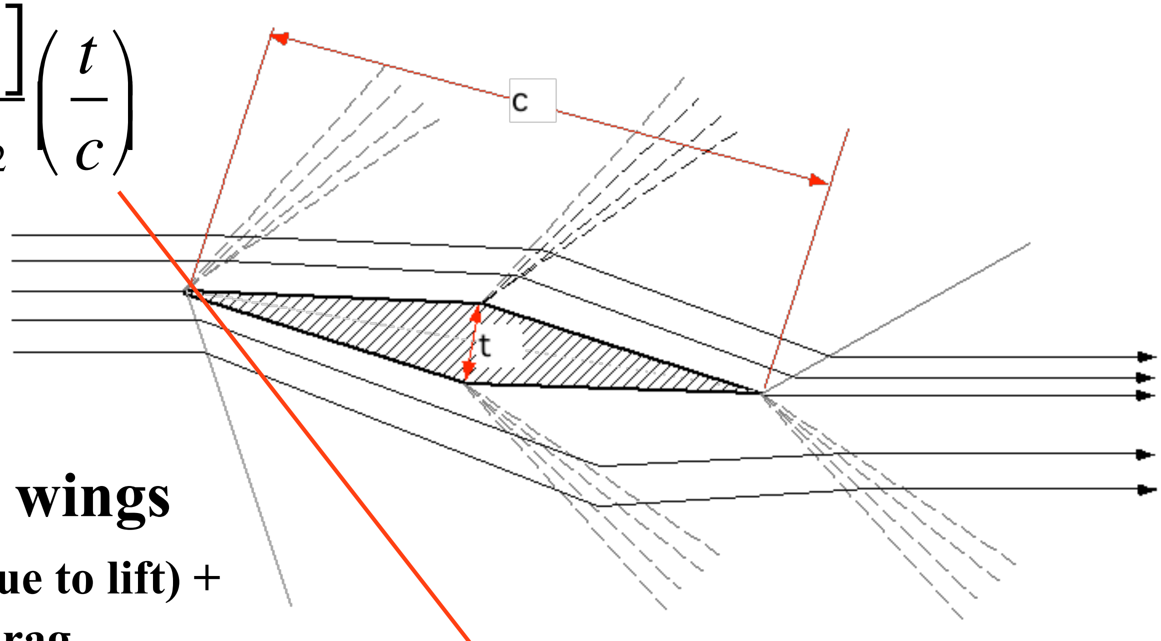
… induced drag (drag due to lift) + viscous drag

Supersonic Wave Drag

- Finite Wings in Supersonic Flow have drag .. Even at zero angle of attack and no lift and no viscosity…. *"wave drag"*

- *Wave* Drag coefficient is proportional to *thickness ratio (t/c)*

$$C_{D_{wave}} \Big|_{alpha = zero} = \frac{D_{rag}}{bc\bar{q}} = \frac{\left[p_2 - p_3\right]}{\frac{\gamma}{2} p_\infty M^2}\left(\frac{t}{c}\right)$$



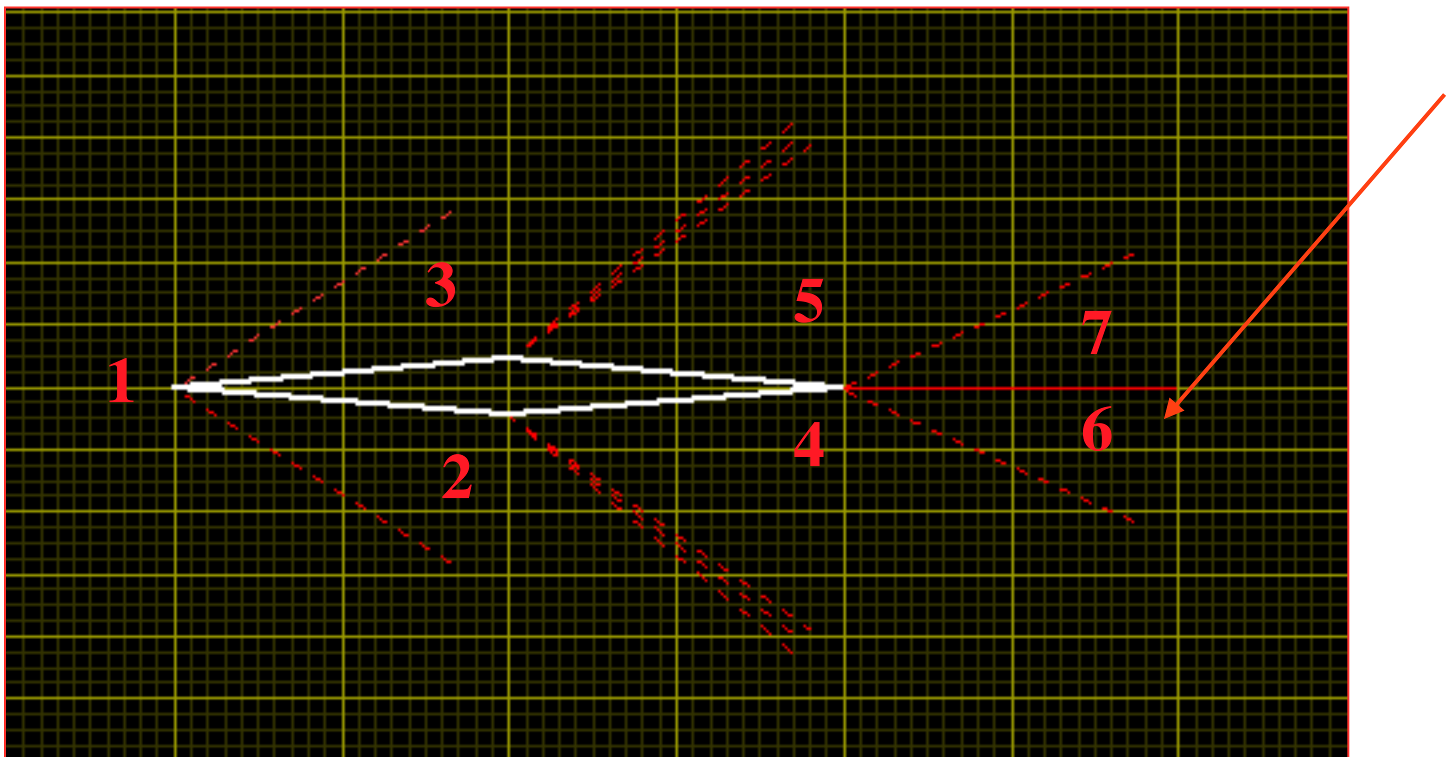
- **Supersonic flow over wings**

  … induced drag (drag due to lift) + viscous drag + wave drag

Example: Symmetric Double-wedge Airfoil

- 5° ramps, $M_\infty = 2.00$
- Total chord: 2 meters
- Fineness ratio (t/c): 0.08749

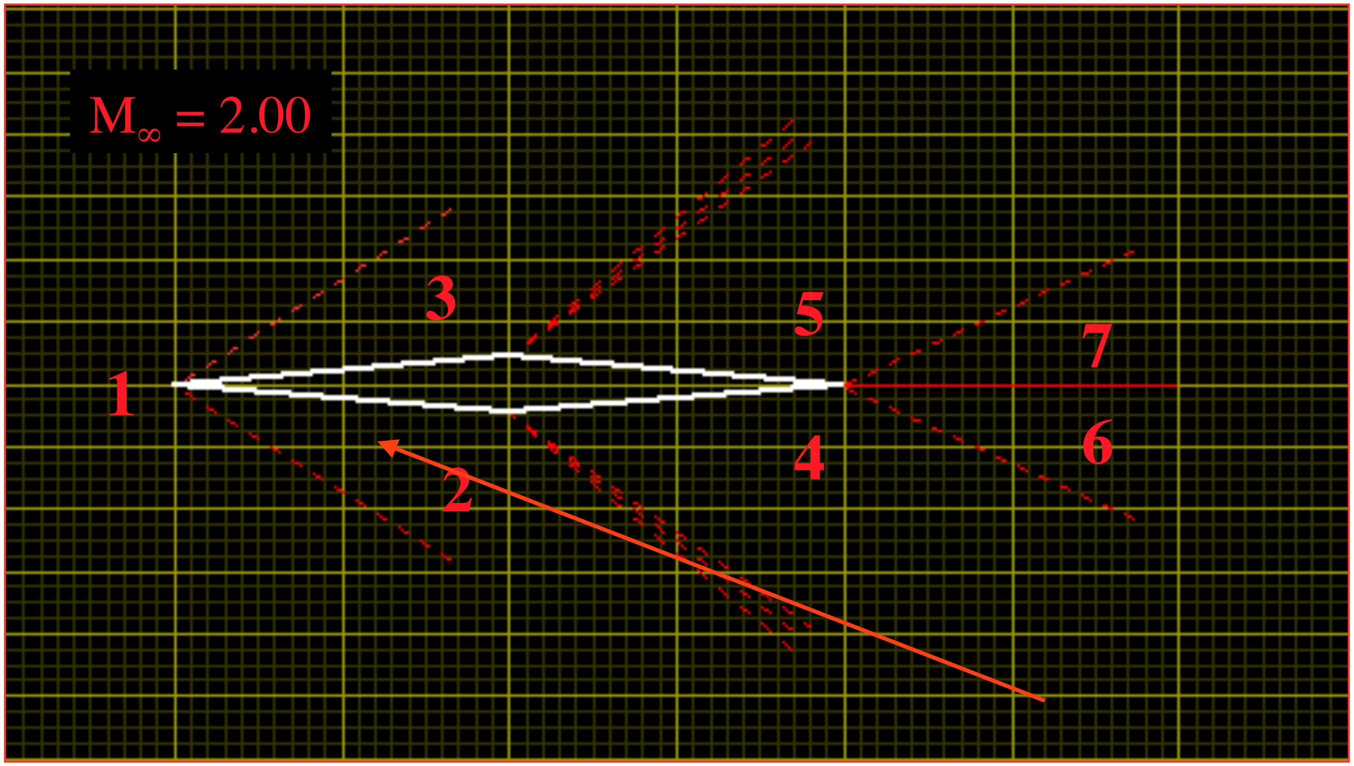
- Look at flow properties across each of the numbered regions



Example: Symmetric Double-wedge Airfoil (cont'd)

- Across region 1-2 … Oblique shock

$\delta_2 = 5° \dashrightarrow \beta_2 = 34.302°$, $p2/p1 = 1.3154$, $M_2 = 1.8213$

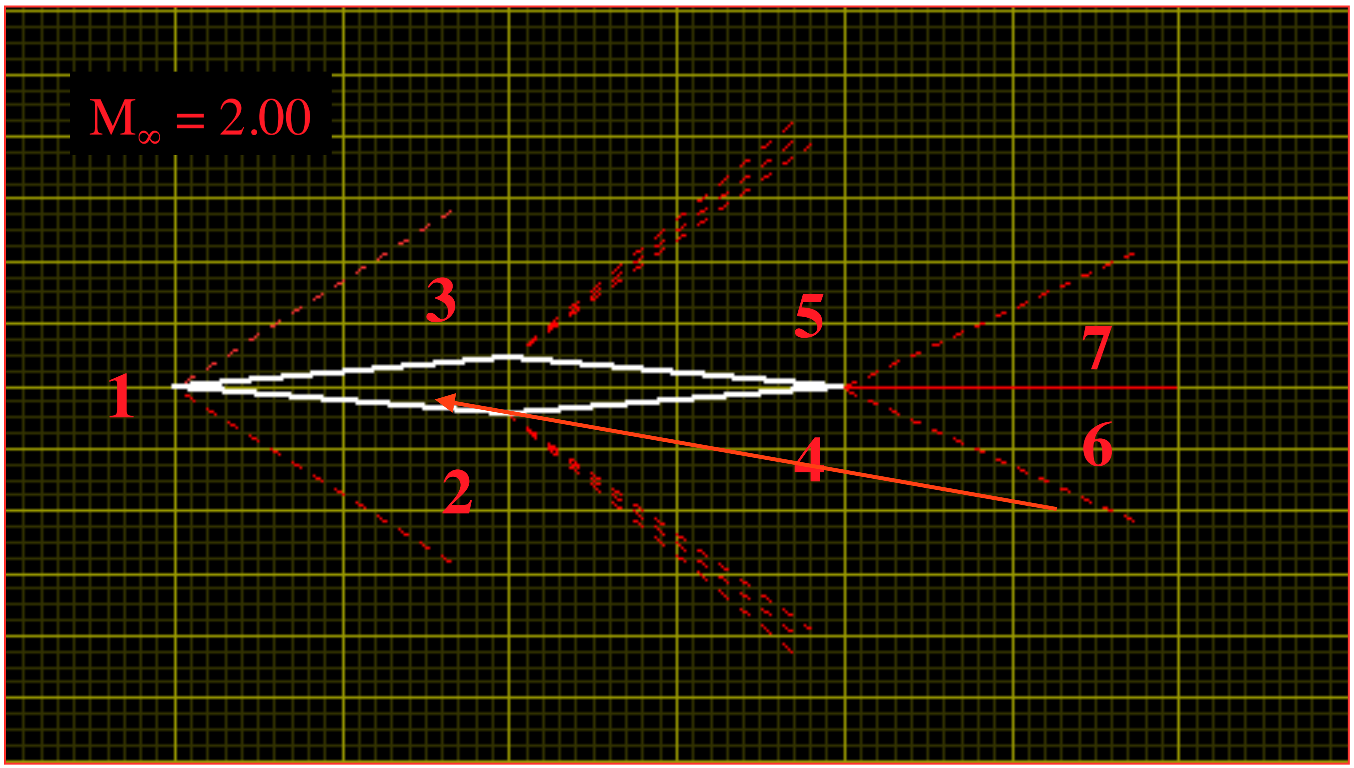


Example: Symmetric Double-wedge Airfoil (cont'd)

• Across region 2-4 … expansion fan

M_2=1.8213

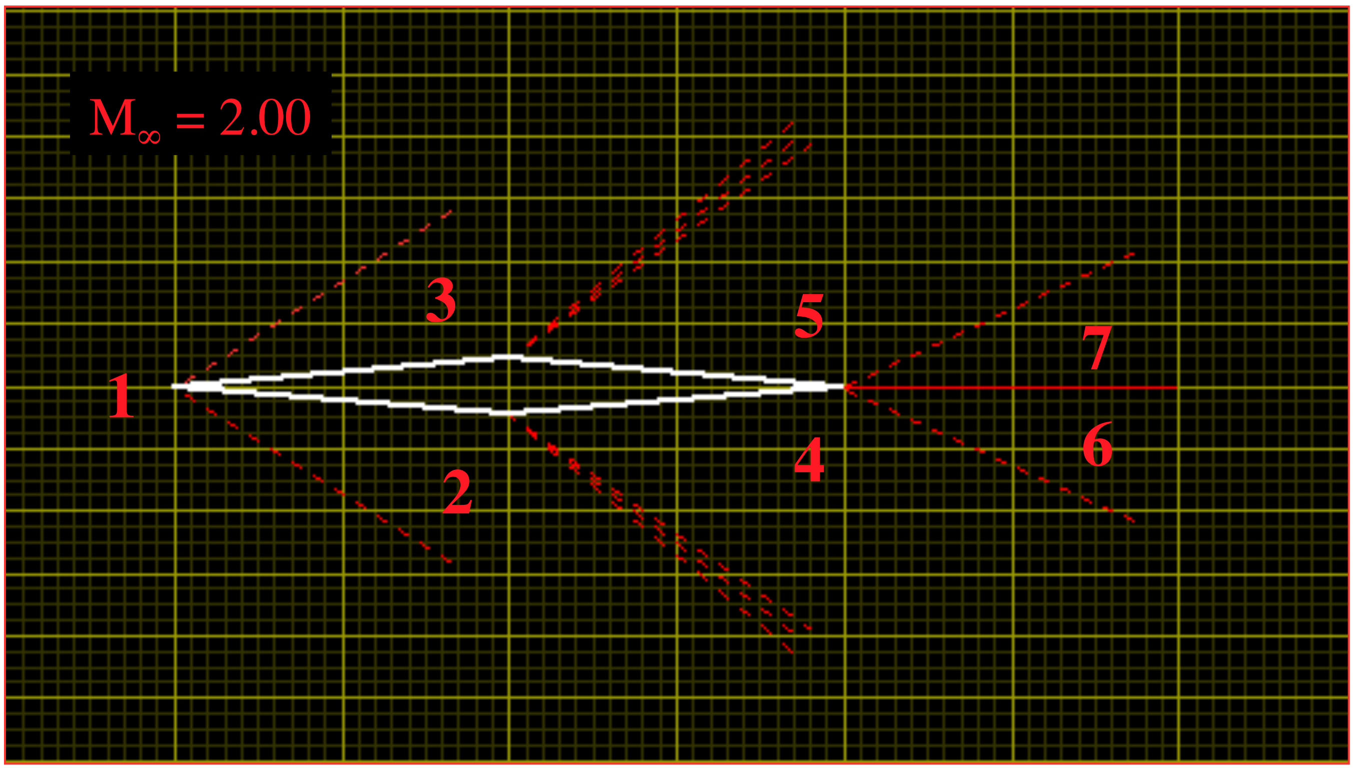
δ_4 = -10°−−> π_4/π_2=0.5685,***p4/p1***=0.7478, M_4=2.1848



Example: Symmetric Double-wedge Airfoil (cont'd)

• Across region 1-3 … Oblique shock

$\delta_3 = 5° \text{-->} \beta_3 = 34.302°$, $\boldsymbol{p3/p1} = 1.3154$, $M_3 = 1.8213$

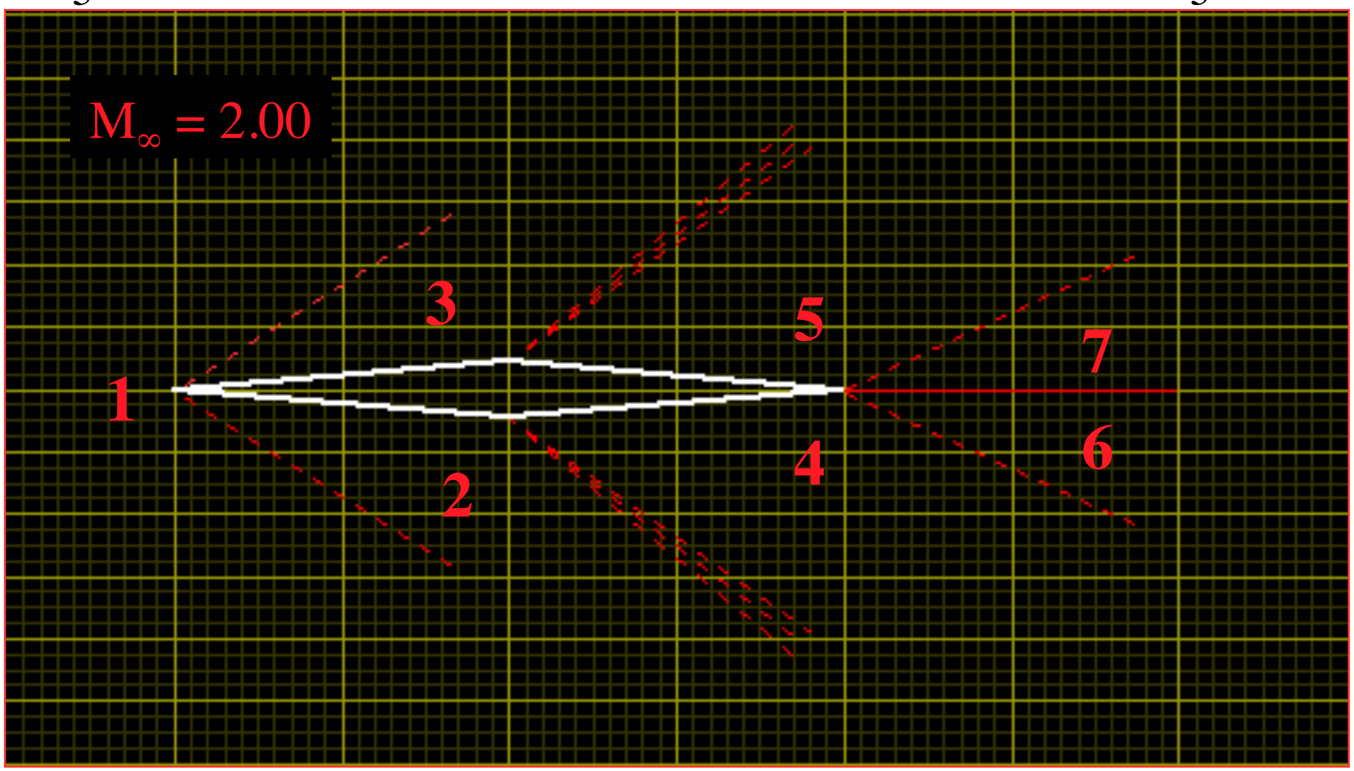


Example: Symmetric Double-wedge Airfoil (cont'd)

• Across region 3-5 … expansion fan

M_3=1.8213

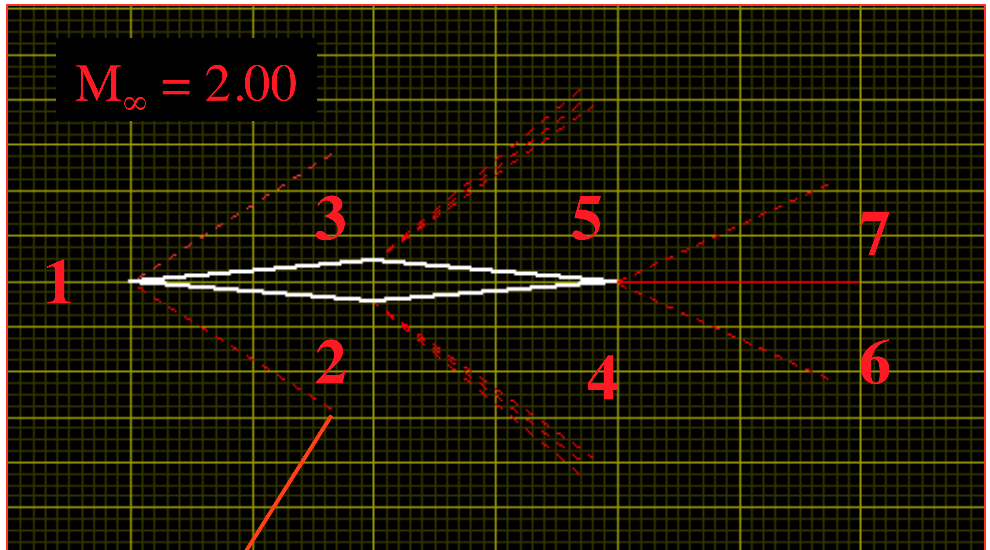
δ_5 = -10°-–> ***p5/p3***=0.5685,***p5/p1***=0.7478, M_5=2.1848



Example: Symmetric Double-wedge Airfoil … Drag

- Calculate Drag Coefficient

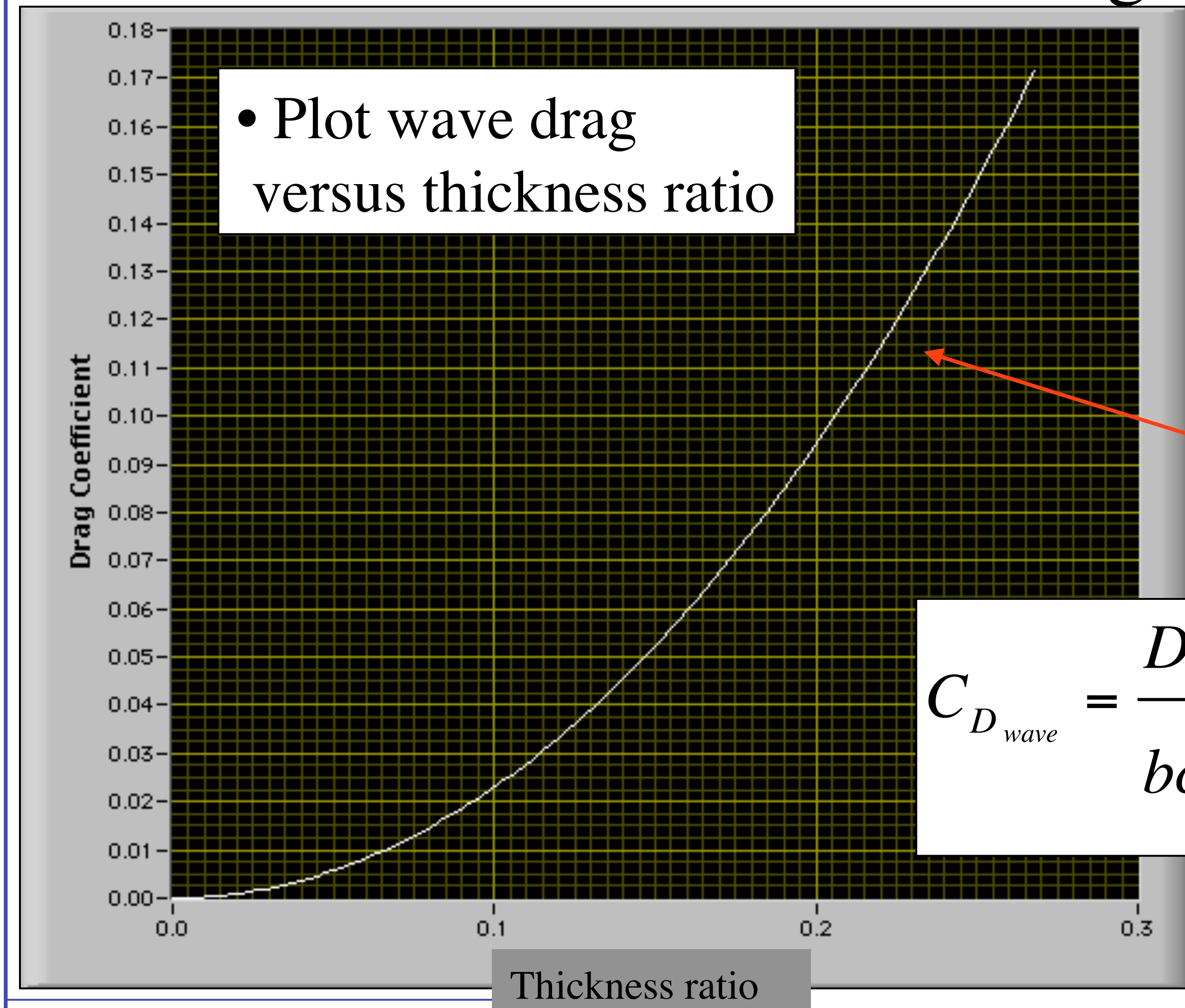
p2/p1=1.3154
p2/p1=1.3154
p4/p1=0.7478
p5/p1=0.7478



$$C_{D_{wave}} = \frac{\left[\frac{1}{2}\left(\frac{p_2}{p_\infty}+\frac{p_3}{p_\infty}\right)-\frac{1}{2}\left(\frac{p_4}{p_\infty}+\frac{p_5}{p_\infty}\right)\right]}{\frac{\gamma}{2}M^2}\left(\frac{t}{c}\right) = \frac{\left(\frac{1.3154+1.3154}{2}-\frac{0.7478+0.7478}{2}\right)0.08749}{\frac{1.4}{2}2^2}$$

=0.01774

Example: Symmetric Double-wedge Airfoil … Drag (cont'd)

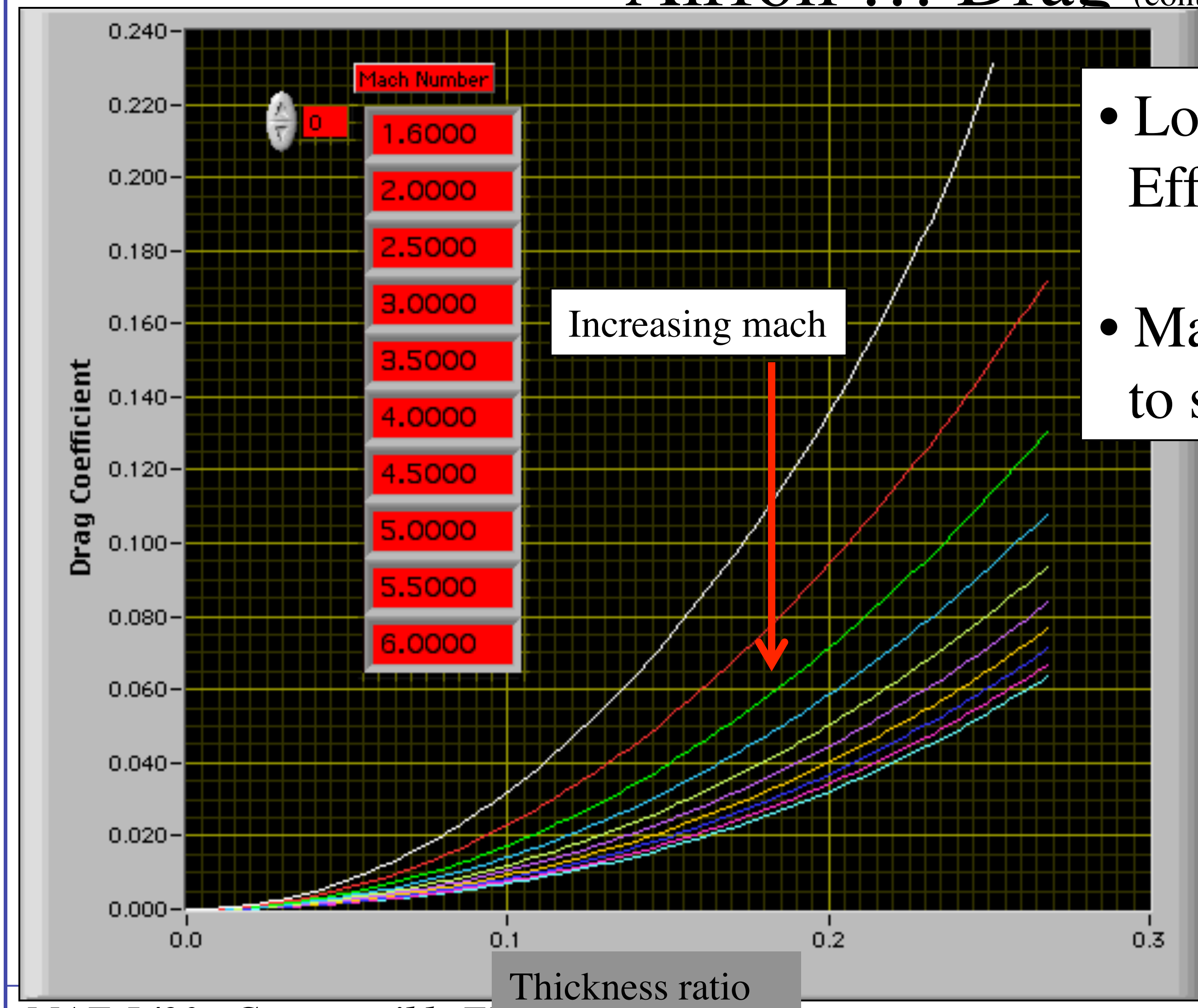


- Plot wave drag versus thickness ratio

Mach 2.0
0° α

$$C_{D_{wave}} = \frac{D_{rag}}{bc\bar{q}} = \frac{\left[p_2 - p_3\right]}{\frac{\gamma}{2} p_\infty M^2}\left(\frac{t}{c}\right)$$

Example: Symmetric Double-wedge Airfoil … Drag (cont'd)

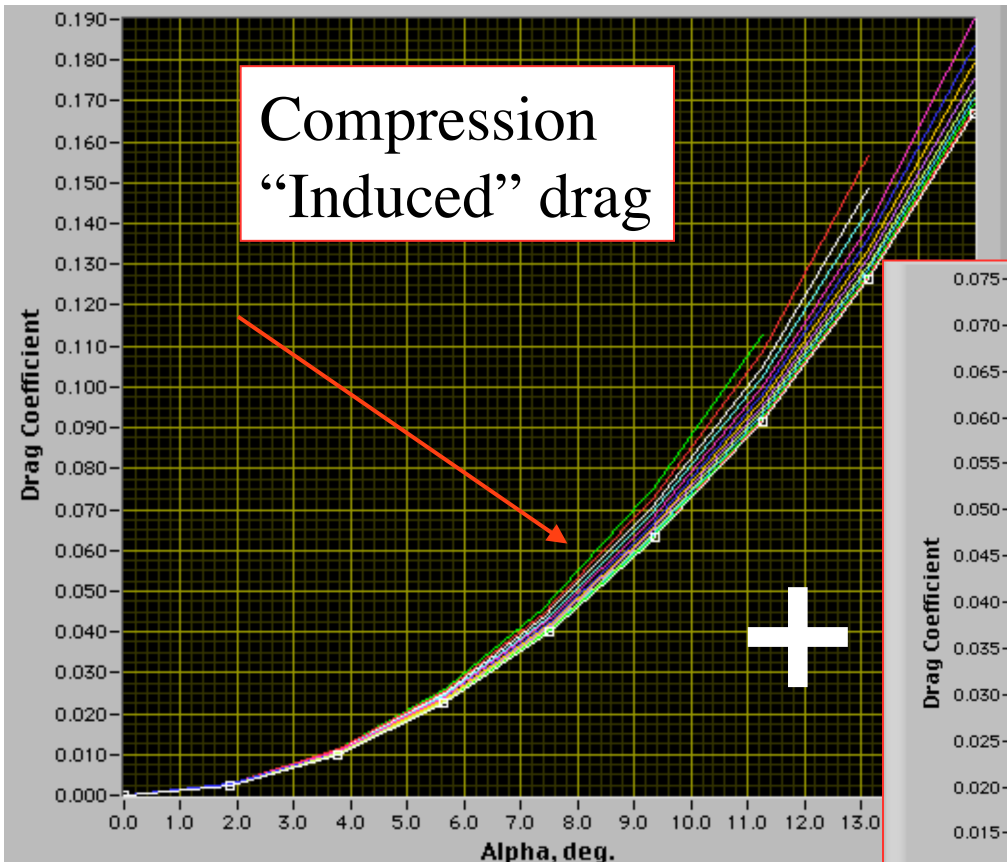


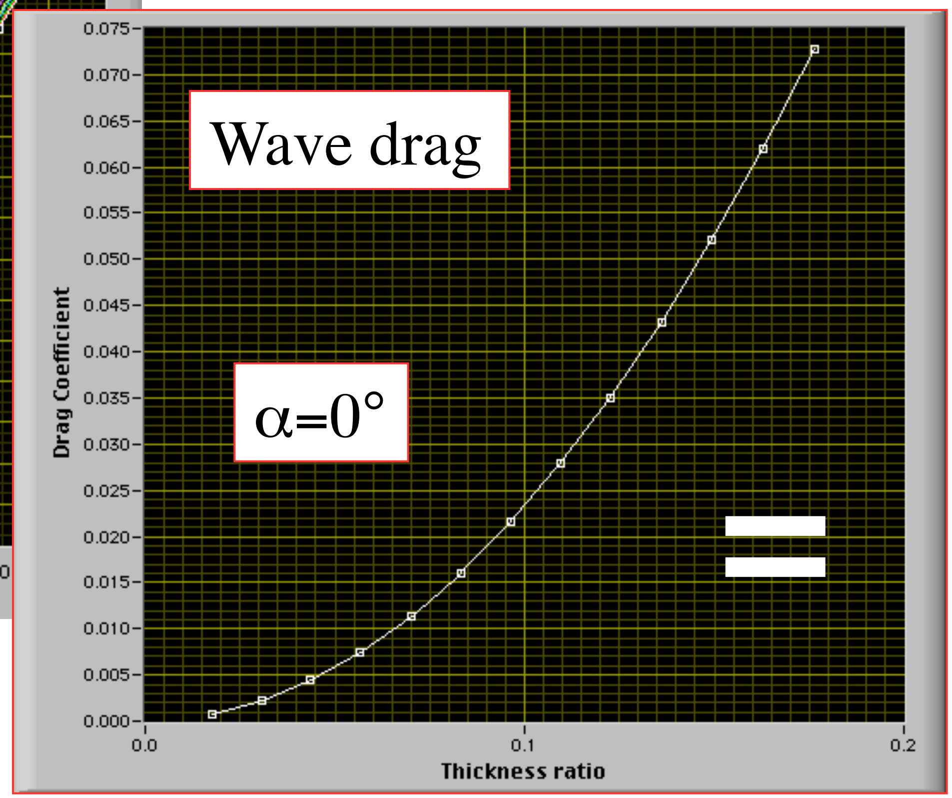
- Look at mach number Effect on wave drag

- Mach Number tends to suppress wave drag

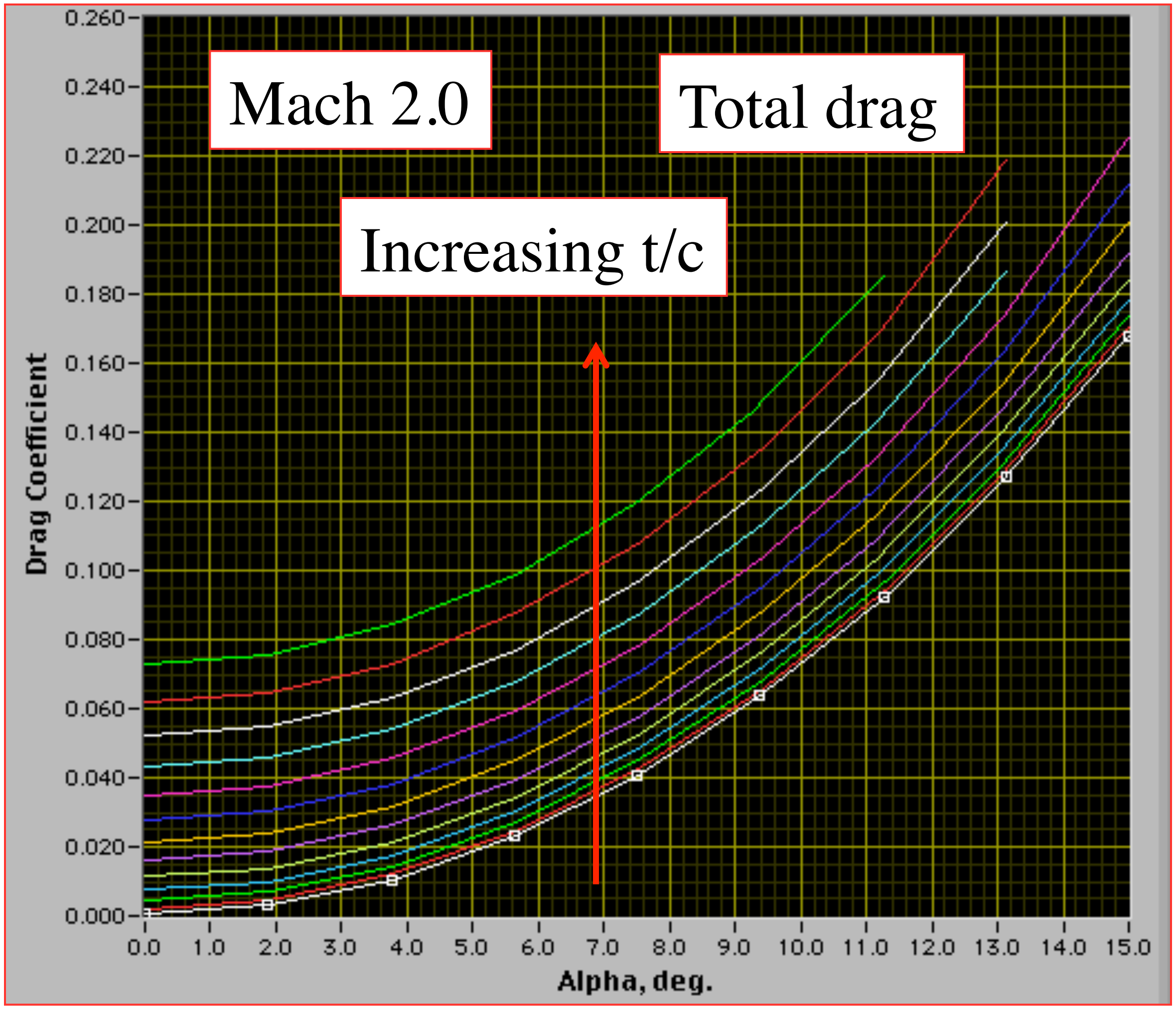
Example: Symmetric Double-wedge Airfoil … Drag (cont'd)

- How About The effect of angle of attack on drag





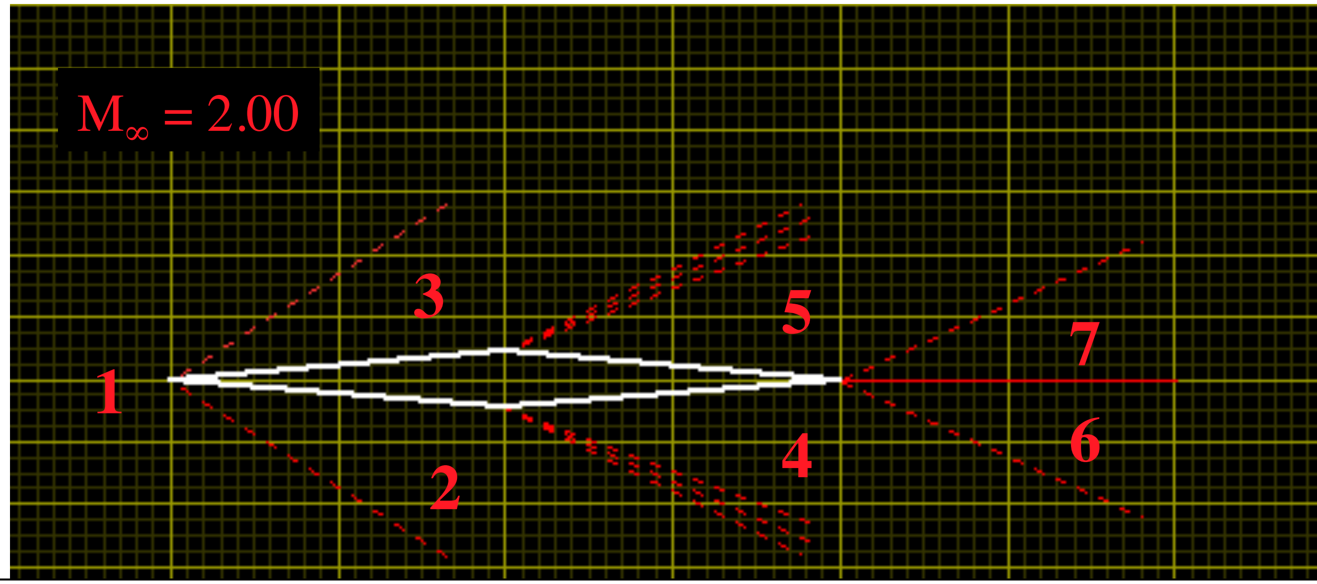
Example: Symmetric Double-wedge Airfoil … Drag (cont'd)



Example: Symmetric Double-wedge Airfoil … Lift

• Calculate Zero α Lift Coefficient

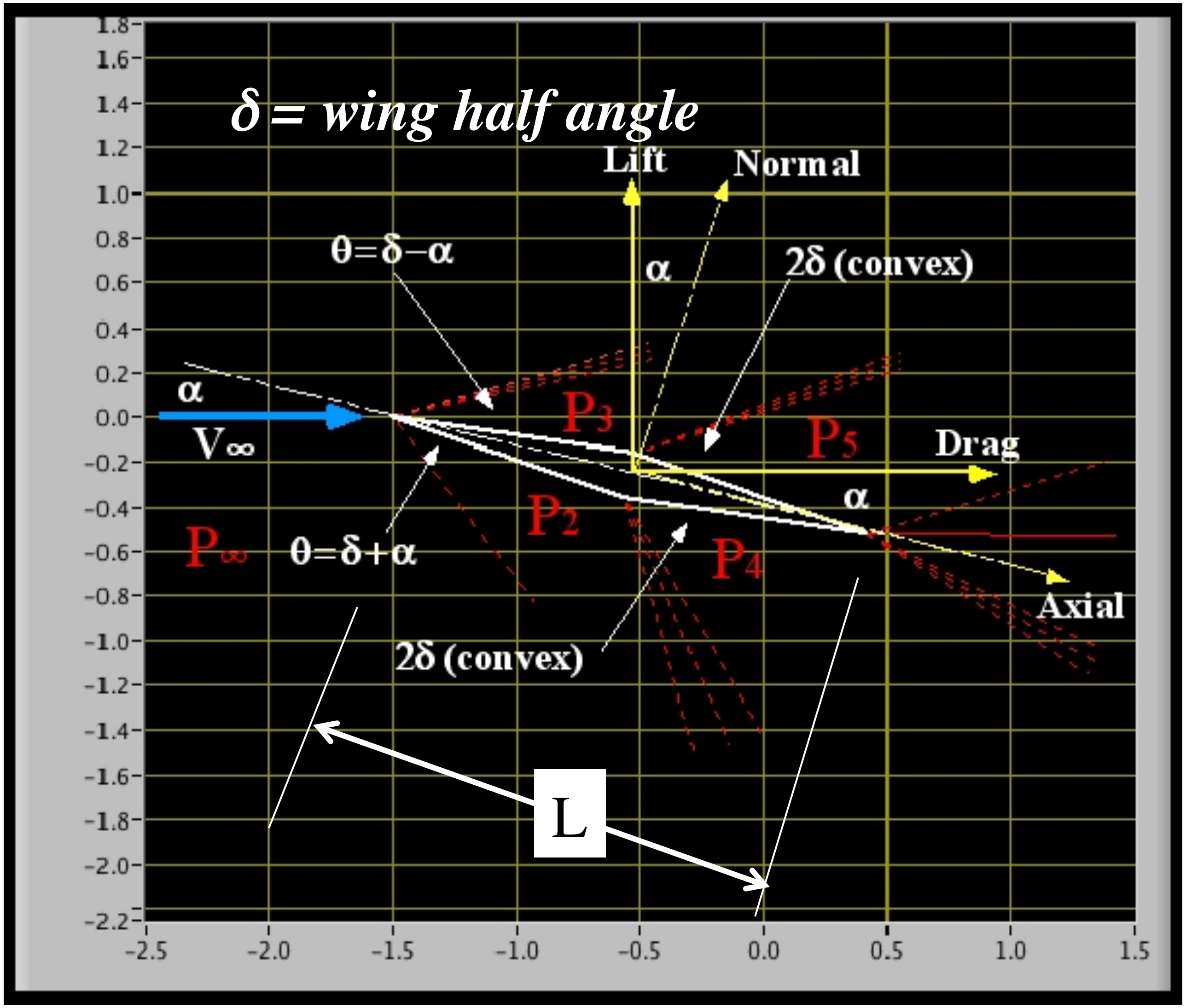
p2/p1=1.3154
p3/p1=1.3154
p4/p1=0.7478
p5/p1=0.7478



$$C_{L_{wave}} = \frac{L_{ift}}{bc\bar{q}} = \frac{\left[\frac{1}{2}\left(\frac{p_2}{p_\infty}+\frac{p_4}{p_\infty}\right)-\frac{1}{2}\left(\frac{p_3}{p_\infty}+\frac{p_5}{p_\infty}\right)\right]}{\frac{\gamma}{2}M^2}\left(\cos[5^o]\right)=0$$

•Makes sense

Pulling it All Together (1)



Consider airfoil @ positive angle of attack, … length … L

- *Lower forward Ramp Angle*

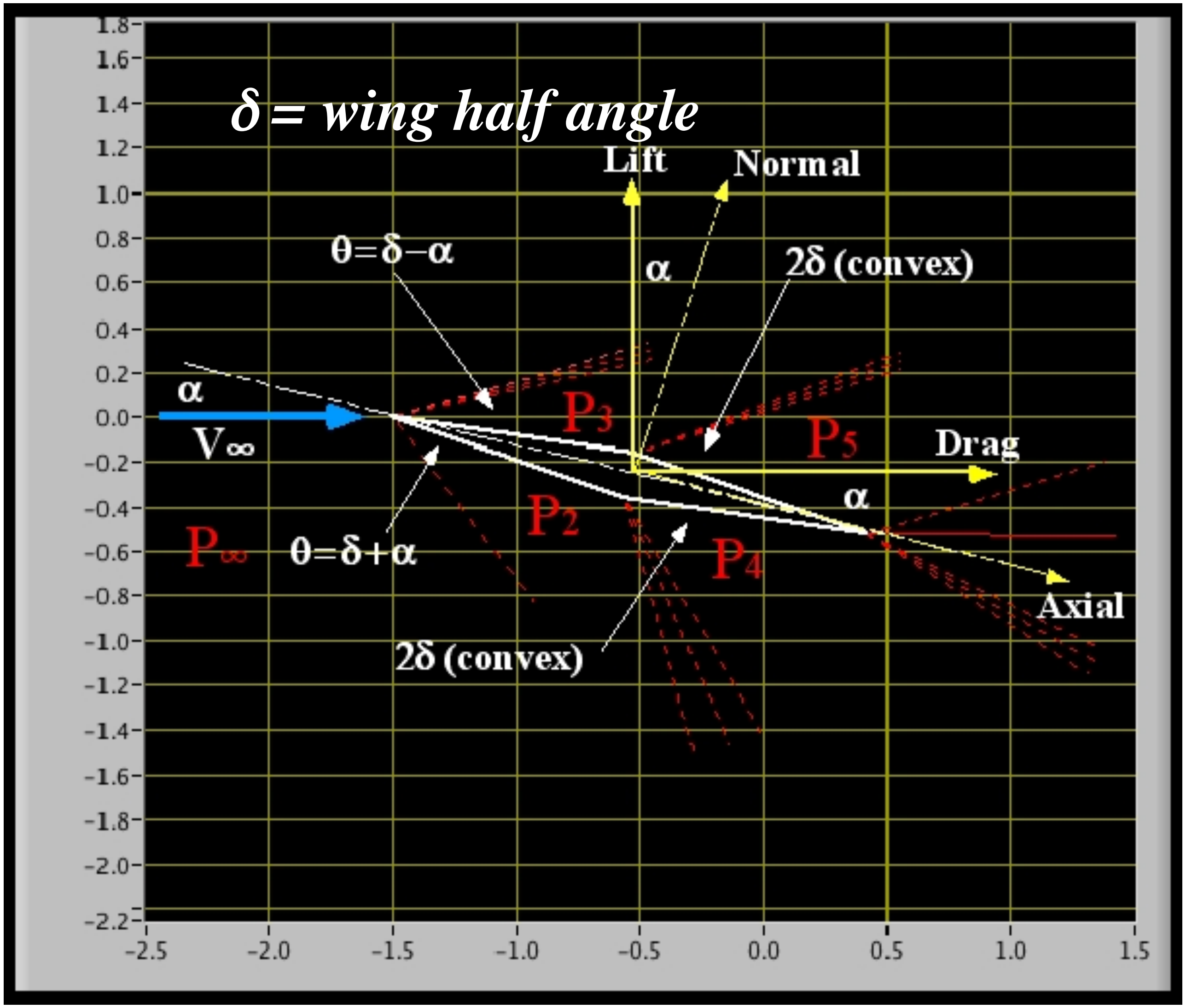
  … $\theta=\delta+\alpha$

- *Upper Forward Ramp Angle*

  … $\theta=\delta-\alpha$

$\theta \geq 0$ "oblique shock"
$\theta < 0$ "expansion fan"

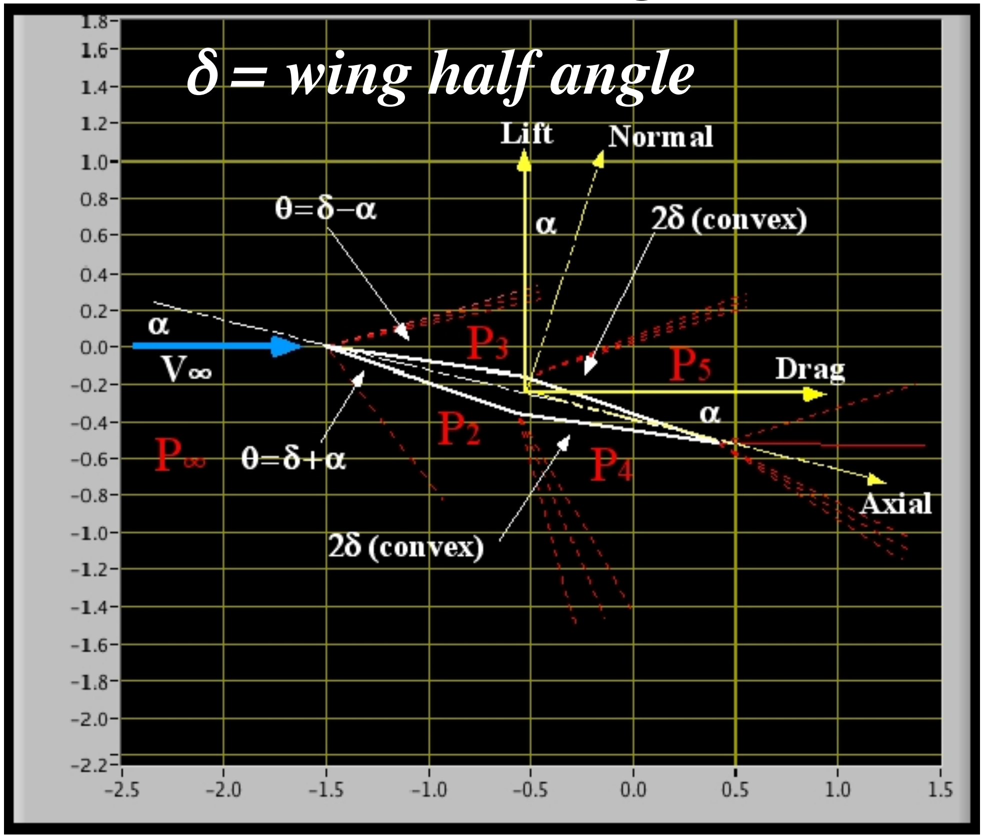
Pulling it All Together (2)



- *Angle from 3-5 and 2-4 Is always convex (expansion), 2δ*
- *Use Shock-Expansion Theory to calculate Ramp pressures*

Upper: { P_3, P_5}
Lower: {P_2, P_4}

Pulling it All Together (3)



- *Upper: { P_3, P_5}*
  Lower: {P_2, P_4}

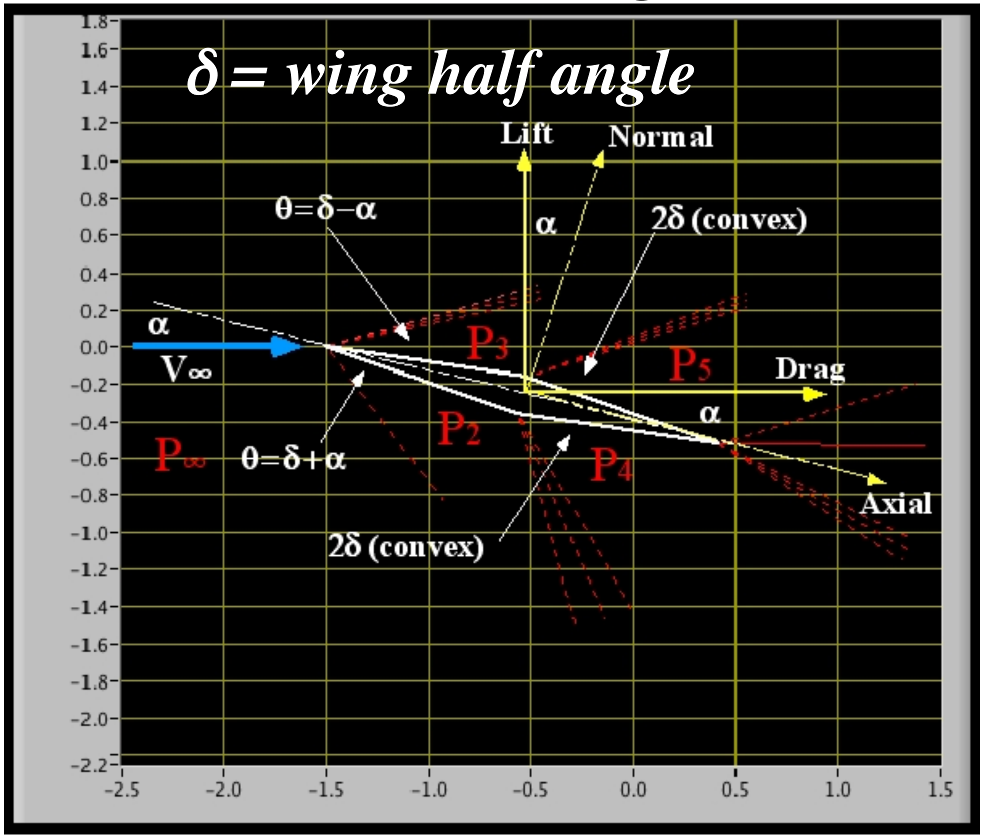
- *Calculate Normal And Axial Pressure Forces on Wing …*

Normal Force

$$F_{normal} = b \cdot \left[\left(P_2 \cdot \frac{L/2}{\cos\delta} - P_3 \cdot \frac{L/2}{\cos\delta} \right) \cos\delta + \left(P_4 \cdot \frac{L/2}{\cos\delta} - P_5 \cdot \frac{L/2}{\cos\delta} \right) \cos\delta \right]$$

$$= b \cdot \left[\left(P_2 - P_3\right)\frac{L}{2} + \left(P_4 - P_5\right)\frac{L}{2} \right] = \left[\left(P_2 + P_4\right) - \left(P_3 + P_5\right) \right] \frac{b \cdot L}{2}$$

Pulling it All Together (4)



- *Upper: { P_3, P_5}*
  Lower: {P_2, P_4}

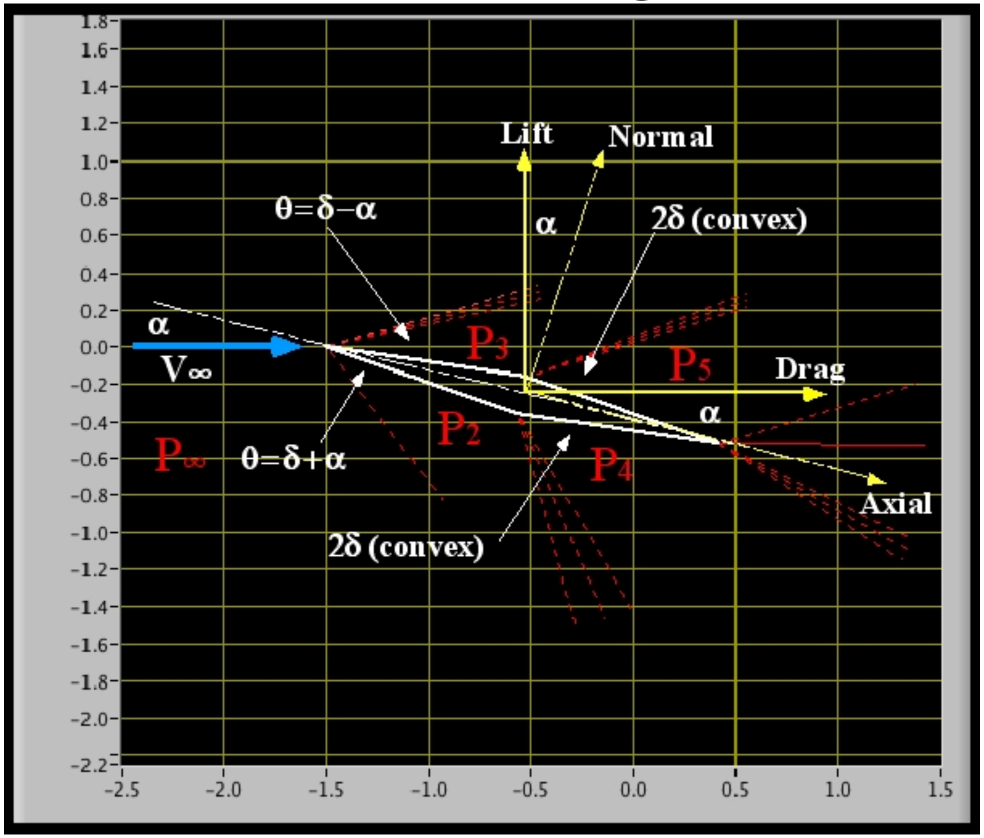
- *Calculate Normal And Axial Pressure Forces on Wing …*

Axial Force

$$F_{axial} = b \cdot \left[\left(P_2 \cdot \frac{L/2}{\cos\delta} - P_4 \cdot \frac{L/2}{\cos\delta} \right) \sin\delta + \left(P_3 \cdot \frac{L/2}{\cos\delta} - P_5 \cdot \frac{L/2}{\cos\delta} \right) \sin\delta \right]$$

$$= b \cdot \left[(P_2 - P_4)\frac{L}{2} + (P_3 - P_5)\frac{L}{2} \right] \tan\delta = \left[(P_2 + P_3) - (P_4 + P_5) \right] \frac{b \cdot L}{2} \tan\delta$$

Pulling it All Together (5)



$$F_{normal} = \left[\left(P_2 + P_4\right) - \left(P_3 + P_5\right)\right]\frac{b \cdot L}{2}$$

$$F_{axial} = \left[\left(P_2 + P_3\right) - \left(P_4 + P_5\right)\right]\frac{b \cdot L}{2}\tan\delta$$

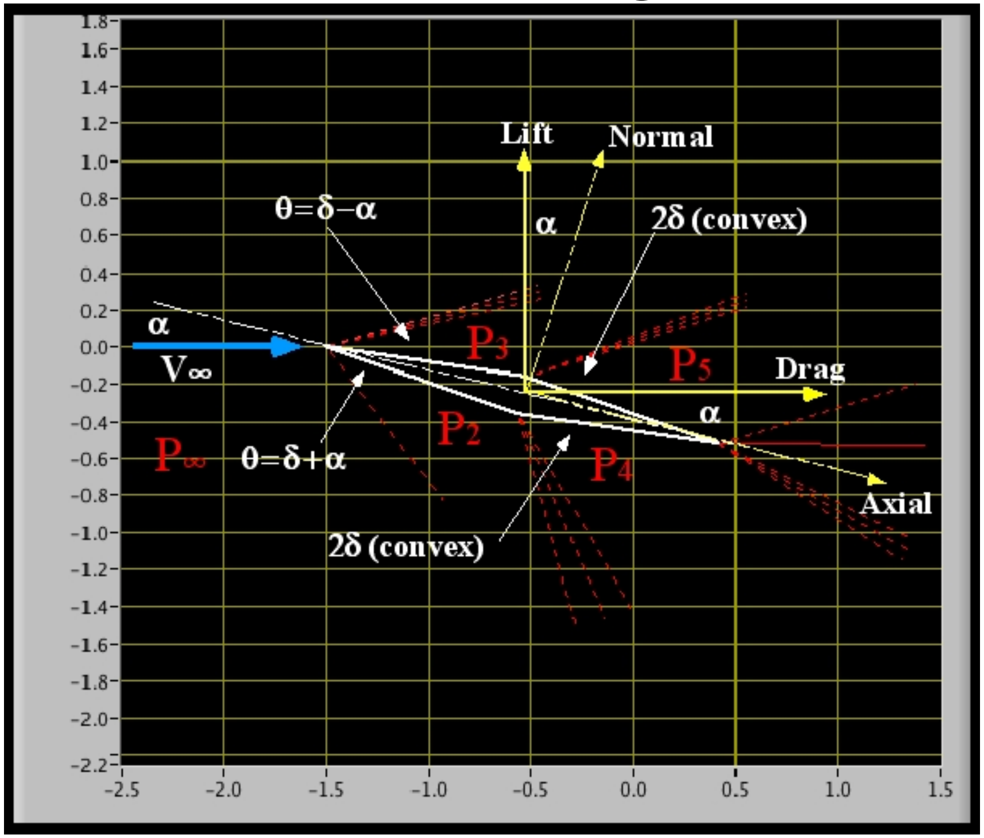
- Transform to wind axis

To calculate *Lift*, *Drag*

$$Drag = F_{axial}\cos(\alpha) + F_{normal}\sin(\alpha)$$

$$Lift = F_{normal}\cos(\alpha) - F_{axial}\sin(\alpha)$$

Pulling it All Together (6)



$$F_{normal} = \left[\left(P_2 + P_4\right) - \left(P_3 + P_5\right)\right]\frac{b \cdot L}{2}$$

$$F_{axial} = \left[\left(P_2 + P_3\right) - \left(P_4 + P_5\right)\right]\frac{b \cdot L}{2}\tan\delta$$

- Transform to wind axis

To calculate *Lift*, *Drag*

$$Drag = F_{axial}\cos(\alpha) + F_{normal}\sin(\alpha)$$

$$Lift = F_{normal}\cos(\alpha) - F_{axial}\sin(\alpha)$$

$$Drag = \left(\left[\left(P_2 + P_3\right) - \left(P_4 + P_5\right)\right]\frac{b \cdot L}{2}\tan\delta\right)\cdot\cos(\alpha) + \left(\left[\left(P_2 + P_4\right) - \left(P_3 + P_5\right)\right]\frac{b \cdot L}{2}\right)\cdot\sin(\alpha)$$

$$Lift = \left(\left[\left(P_2 + P_4\right) - \left(P_3 + P_5\right)\right]\frac{b \cdot L}{2}\right)\cdot\cos(\alpha) - \left(\left[\left(P_2 + P_3\right) - \left(P_4 + P_5\right)\right]\frac{b \cdot L}{2}\tan\delta\right)\cdot\sin(\alpha)$$

Pulling it All Together (7)

- Collect Terms on Drag Equation

$$Drag = \left(\left[\left(P_2 + P_3\right) - \left(P_4 + P_5\right)\right]\frac{b \cdot L}{2}\tan\delta\right)\cdot\cos(\alpha) + \left(\left[\left(P_2 + P_4\right) - \left(P_3 + P_5\right)\right]\frac{b \cdot L}{2}\right)\cdot\sin(\alpha)$$

$$\Downarrow$$

$$Drag = \left(P_2\frac{b \cdot L}{2}\tan\delta\right)\cdot\cos(\alpha) + \left(P_2\frac{b \cdot L}{2}\right)\cdot\sin(\alpha) + \left(P_3\frac{b \cdot L}{2}\tan\delta\right)\cdot\cos(\alpha) - \left(P_3\frac{b \cdot L}{2}\right)\cdot\sin(\alpha)$$

$$+\left(P_4\frac{b \cdot L}{2}\right)\cdot\sin(\alpha) - \left(P_4\frac{b \cdot L}{2}\tan\delta\right)\cdot\cos(\alpha) - \left(P_5\frac{b \cdot L}{2}\tan\delta\right)\cdot\cos(\alpha) - \left(P_5\frac{b \cdot L}{2}\right)\cdot\sin(\alpha) =$$

$$\frac{\left(P_2\frac{b \cdot L}{2}\right)}{\cos\delta}\left[\sin\delta\cdot\cos(\alpha) + \cos\delta\sin(\alpha)\right] + \frac{\left(P_3\frac{b \cdot L}{2}\right)}{\cos\delta}\left[\sin\delta\cdot\cos(\alpha) - \cos\delta\sin(\alpha)\right] +$$

$$\frac{\left(P_4\frac{b \cdot L}{2}\right)}{\cos\delta}\left[\cos\delta\cdot\sin(\alpha) - \sin\delta\cos(\alpha)\right] - \frac{\left(P_5\frac{b \cdot L}{2}\right)}{\cos\delta}\left[\sin(\delta)\cos(\alpha) + \cos(\delta)\cdot\sin(\alpha)\right]$$

Pulling it All Together (8)

- Collect Terms

$$Drag = \frac{\left(P_2 \frac{b \cdot L}{2}\right)}{\cos\delta}\left[\sin\delta \cdot \cos(\alpha) + \cos\delta\sin(\alpha)\right] + \frac{\left(P_3 \frac{b \cdot L}{2}\right)}{\cos\delta}\left[\sin\delta \cdot \cos(\alpha) - \cos\delta\sin(\alpha)\right] +$$

$$\frac{\left(P_4 \frac{b \cdot L}{2}\right)}{\cos\delta}\left[\cos\delta \cdot \sin(\alpha) - \sin\delta\cos(\alpha)\right] - \frac{\left(P_5 \frac{b \cdot L}{2}\right)}{\cos\delta}\left[\sin(\delta)\cos(\alpha) + \cos(\delta) \cdot \sin(\alpha)\right]$$

$$\rightarrow \begin{matrix} \left[\sin\delta \cdot \cos(\alpha) + \cos\delta\sin(\alpha)\right] = \sin\left(\delta + \alpha\right) \\ \left[\cos\delta \cdot \sin(\alpha) - \sin\delta\cos(\alpha)\right] = \sin\left(\delta - \alpha\right) \end{matrix} \rightarrow$$

Trigonometric Identity

$$Drag = \frac{\frac{b \cdot L}{2}}{\cos\delta} P_2 \sin\left(\delta + \alpha\right) + P_3 \sin\left(\delta - \alpha\right) - P_4 \sin(\delta - \alpha) - P_5 \sin\left(\delta + \alpha\right) =$$

$$\boxed{\frac{b \cdot L}{2\cos\delta}\left[\left(P_2 - P_5\right)\sin\left(\delta + \alpha\right) + \left(P_3 - P_4\right)\sin\left(\delta - \alpha\right)\right]}$$

Pulling it All Together (9)

- Similar Reduction for Lift

$$Lift = \left(\left[\left(P_2 + P_4\right) - \left(P_3 + P_5\right)\right]\frac{b \cdot L}{2}\right) \cdot \cos(\alpha) - \left(\left[\left(P_2 + P_3\right) - \left(P_4 + P_5\right)\right]\frac{b \cdot L}{2}\tan\delta\right) \cdot \sin(\alpha)$$

$$\Downarrow$$

$$= \frac{b \cdot L}{2\cos\delta}\begin{bmatrix} P_2 \cdot \left[\cos\delta\cos(\alpha) - \sin\delta \cdot \sin(\alpha)\right] + \\ -P_3\left[\cos\delta\cos(\alpha) + \sin\delta \cdot \sin(\alpha)\right] + \\ P_4\left[\cos\delta\cos(\alpha) + \sin\delta \cdot \sin(\alpha)\right] + \\ -P_5 \cdot \left[\cos\delta\cos(\alpha) - \sin\delta \cdot \sin(\alpha)\right] \end{bmatrix} = \frac{b \cdot L}{2\cos\delta}\begin{bmatrix} P_2 \cdot \cos\left(\delta + \alpha\right) + \\ -P_3\cos\left(\delta - \alpha\right) + \\ P_4\cos\left(\delta - \alpha\right) + \\ -P_5 \cdot \cos\left(\delta + \alpha\right) \end{bmatrix} =$$

$$\frac{b \cdot L}{2\cos\delta}\left[\left(P_2 - P\right)_5 \cdot \cos\left(\delta + \alpha\right) + \left(P_4 - P_3\right)\cos\left(\delta - \alpha\right)\right]$$

$$\rightarrow \begin{matrix} \left[\cos\delta \cdot \cos(\alpha) + \sin\delta\sin(\alpha)\right] = \sin\left(\delta - \alpha\right) \\ \left[\cos\delta \cdot \cos(\alpha) - \sin\delta\sin(\alpha)\right] = \sin\left(\delta + \alpha\right) \end{matrix} \rightarrow$$

Trigonometric Identity

Pulling it All Together (10)

- Lift/Drag Coefficients

$$Drag = \frac{b \cdot L}{2\cos\delta}\left[\left(P_2 - P_5\right)\sin\left(\delta+\alpha\right) + \left(P_3 - P_4\right)\sin\left(\delta-\alpha\right)\right]$$

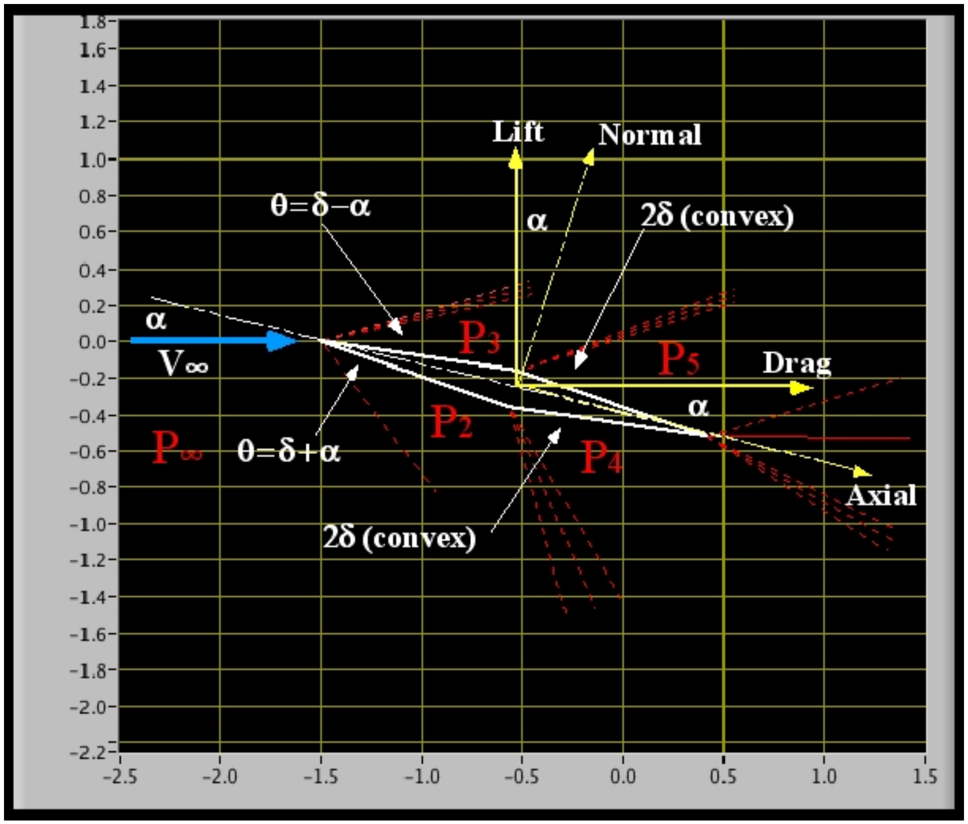
$$Lift = \frac{b \cdot L}{2\cos\delta}\left[\left(P_2 - P_5\right)\cdot\cos\left(\delta+\alpha\right) + \left(P_4 - P_3\right)\cos\left(\delta-\alpha\right)\right]$$

$$\begin{bmatrix} C_D \\ C_L \end{bmatrix} = \frac{1}{\frac{\gamma}{2}P_\infty M_\infty^2 \cdot (b \cdot L)}\begin{bmatrix} Drag \\ Lift \end{bmatrix} = \frac{b \cdot L}{2\cos\delta} \cdot \frac{1}{\frac{\gamma}{2}P_\infty M_\infty^2 \cdot (b \cdot L)}\begin{bmatrix} \left(P_2 - P_5\right)\sin\left(\delta+\alpha\right) + \left(P_3 - P_4\right)\sin\left(\delta-\alpha\right) \\ \left(P_2 - P_5\right)\cdot\cos\left(\delta+\alpha\right) + \left(P_4 - P_3\right)\cos\left(\delta-\alpha\right) \end{bmatrix} =$$

$$\boxed{\begin{bmatrix} C_D \\ C_L \end{bmatrix} = \left(\frac{1}{2\cos\delta} \cdot \frac{1}{\frac{\gamma}{2}M_\infty^2}\right)\begin{bmatrix} \left(\frac{P_2}{P_\infty} - \frac{P_5}{P_\infty}\right)\sin\left(\delta+\alpha\right) + \left(\frac{P_3}{P_\infty} - \frac{P_4}{P_\infty}\right)\sin\left(\delta-\alpha\right) \\ \left(\frac{P_2}{P_\infty} - \frac{P_5}{P_\infty}\right)\cdot\cos\left(\delta+\alpha\right) + \left(\frac{P_4}{P_\infty} - \frac{P_3}{P_\infty}\right)\cos\left(\delta-\alpha\right) \end{bmatrix}}$$

Pulling it All Together (11)

$$\begin{bmatrix} C_D \\ C_L \end{bmatrix} = \left(\frac{1}{2\cos\delta} \cdot \frac{1}{\frac{\gamma}{2} M_\infty^2} \right) \begin{bmatrix} \left(\frac{P_2}{P_\infty} - \frac{P_5}{P_\infty} \right) \sin(\delta+\alpha) + \left(\frac{P_3}{P_\infty} - \frac{P_4}{P_\infty} \right) \sin(\delta-\alpha) \\ \left(\frac{P_2}{P_\infty} - \frac{P_5}{P_\infty} \right) \cdot \cos(\delta+\alpha) + \left(\frac{P_4}{P_\infty} - \frac{P_3}{P_\infty} \right) \cos(\delta-\alpha) \end{bmatrix}$$



Pulling it All Together (12)

$$\begin{bmatrix} C_D \\ C_L \end{bmatrix} = \left(\frac{1}{2\cos\delta} \cdot \frac{1}{\frac{\gamma}{2} M_\infty^2} \right) \begin{bmatrix} \left(\frac{P_2}{P_\infty} - \frac{P_5}{P_\infty} \right) \sin(\delta + \alpha) + \left(\frac{P_3}{P_\infty} - \frac{P_4}{P_\infty} \right) \sin(\delta - \alpha) \\ \left(\frac{P_2}{P_\infty} - \frac{P_5}{P_\infty} \right) \cdot \cos(\delta + \alpha) + \left(\frac{P_4}{P_\infty} - \frac{P_3}{P_\infty} \right) \cos(\delta - \alpha) \end{bmatrix}$$

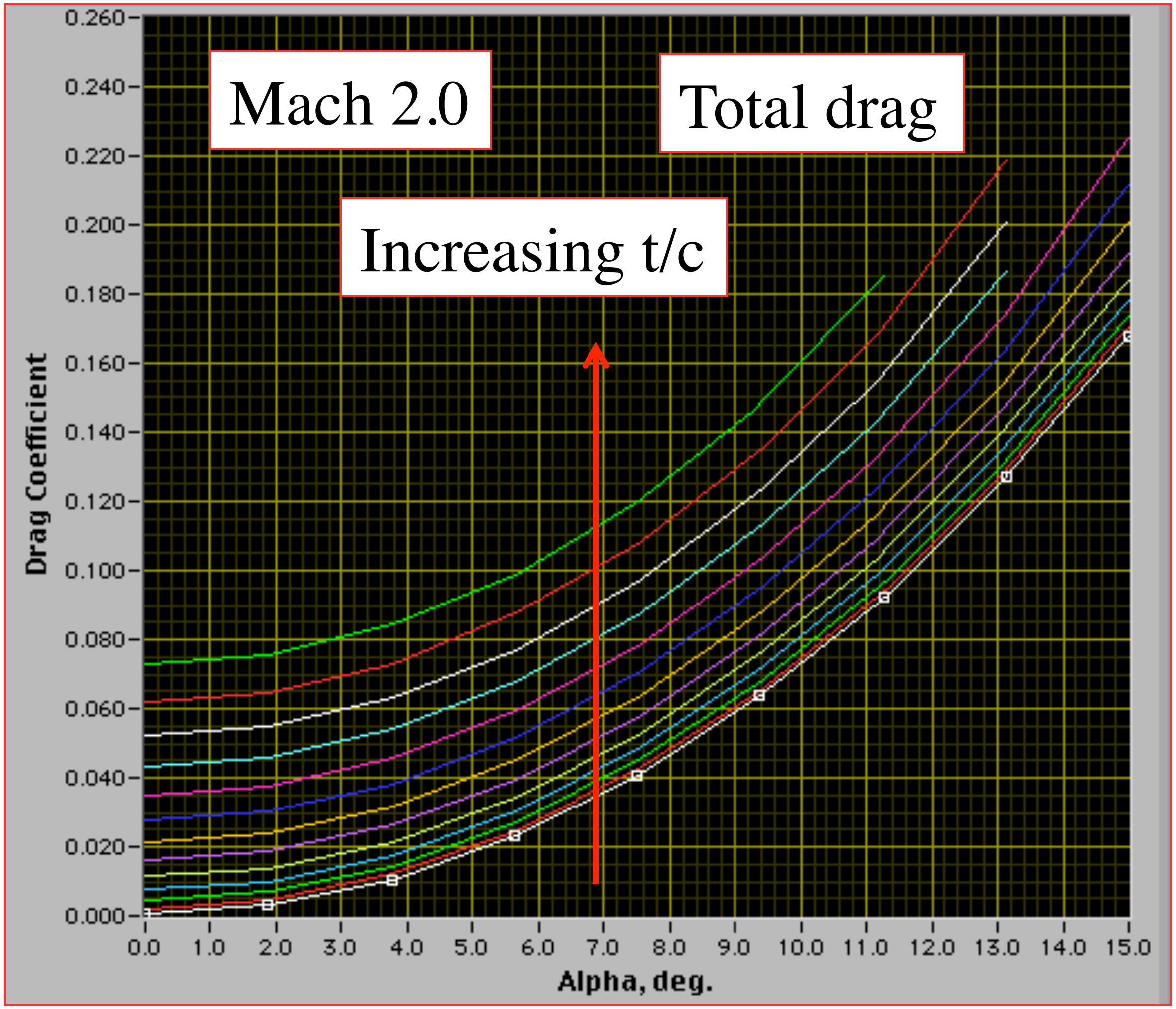
- Check against flat plate

Formulae … *(δ=0)*

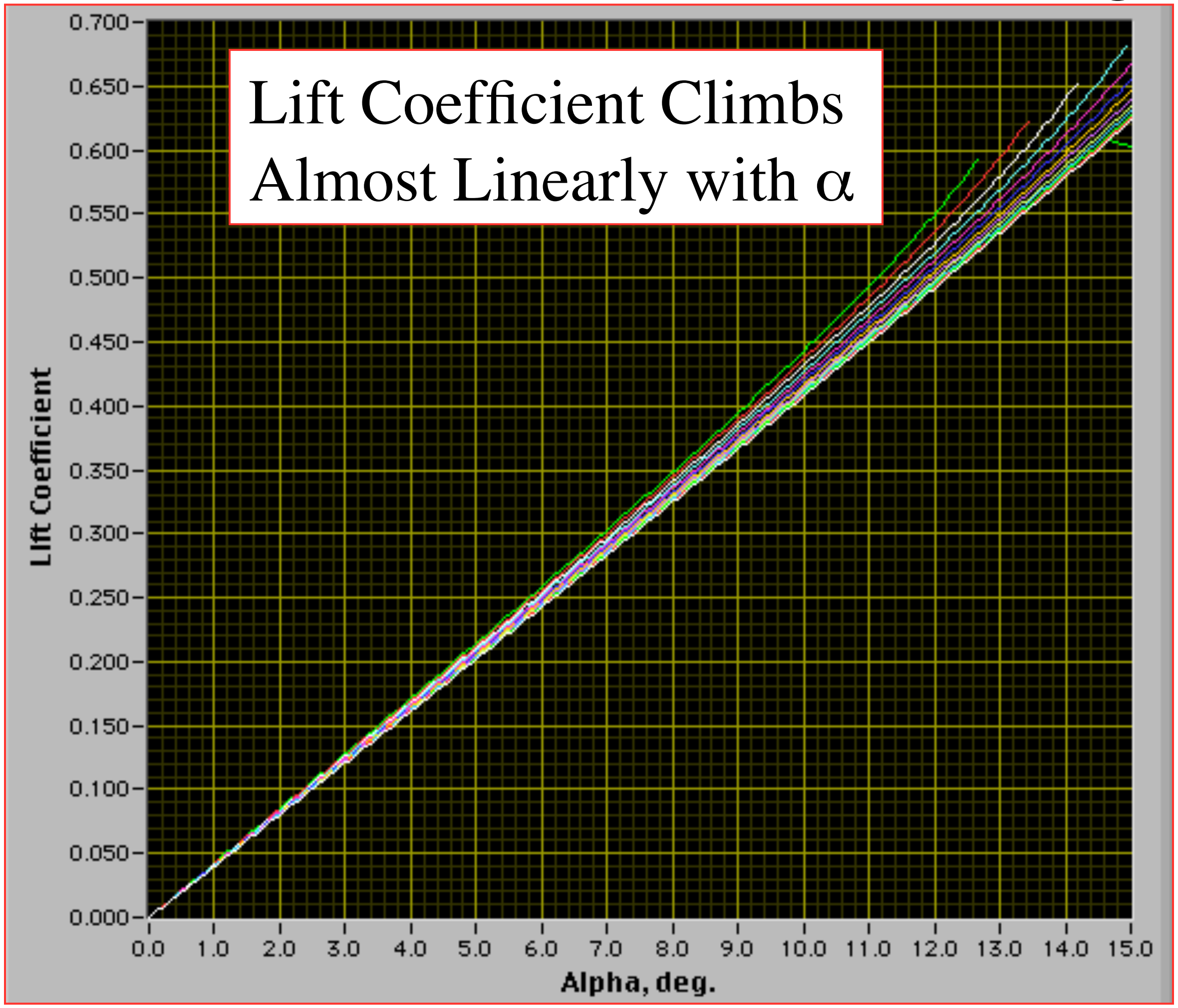
$$\rightarrow \delta = 0 \rightarrow \begin{bmatrix} P_3 = P_5 = P_u \\ P_2 = P_4 = P_l \end{bmatrix} \rightarrow \begin{bmatrix} C_D \\ C_L \end{bmatrix} \rightarrow \frac{1}{2} \cdot \frac{1}{\frac{\gamma}{2} M_\infty^2} \begin{bmatrix} \left(\frac{P_l}{P_\infty} - \frac{P_u}{P_\infty} \right) \sin(\alpha) - \left(\frac{P_u}{P_\infty} - \frac{P_l}{P_\infty} \right) \sin(\alpha) \\ \left(\frac{P_l}{P_\infty} - \frac{P_u}{P_\infty} \right) \cdot \cos(\alpha) + \left(\frac{P_l}{P_\infty} - \frac{P_u}{P_\infty} \right) \cos(\alpha) \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} C_D \\ C_L \end{bmatrix}_{\delta=0} = \frac{1}{\frac{\gamma}{2} M_\infty^2} \begin{bmatrix} \left(\frac{P_l}{P_\infty} - \frac{P_u}{P_\infty} \right) \sin(\alpha) \\ \left(\frac{P_l}{P_\infty} - \frac{P_u}{P_\infty} \right) \cdot \cos(\alpha) \end{bmatrix} \text{.........} Q.E.D$$

Example: Symmetric Double-wedge Airfoil … Drag (cont'd)

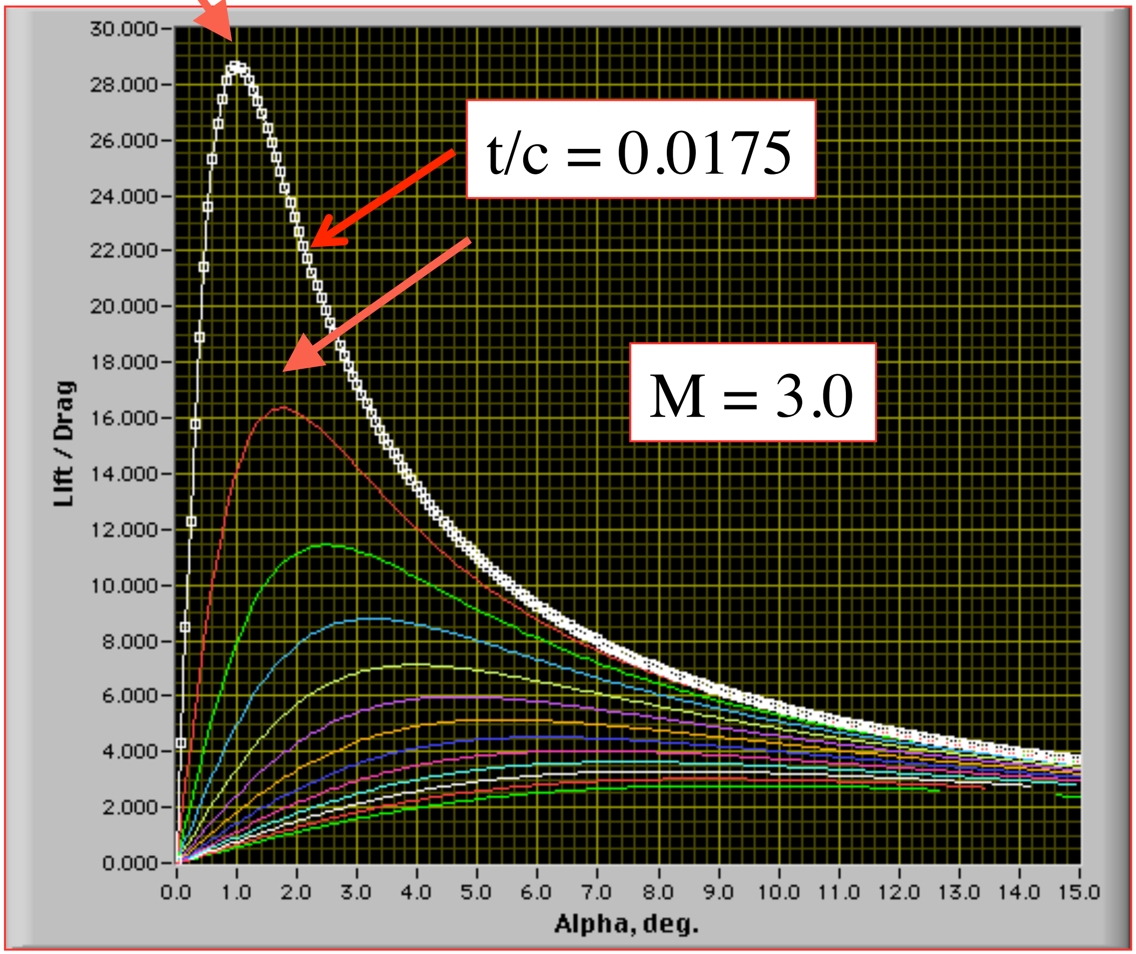


Example: Symmetric Double-wedge Airfoil … Drag (cont'd)



- How About The effect of angle of attack on Lift

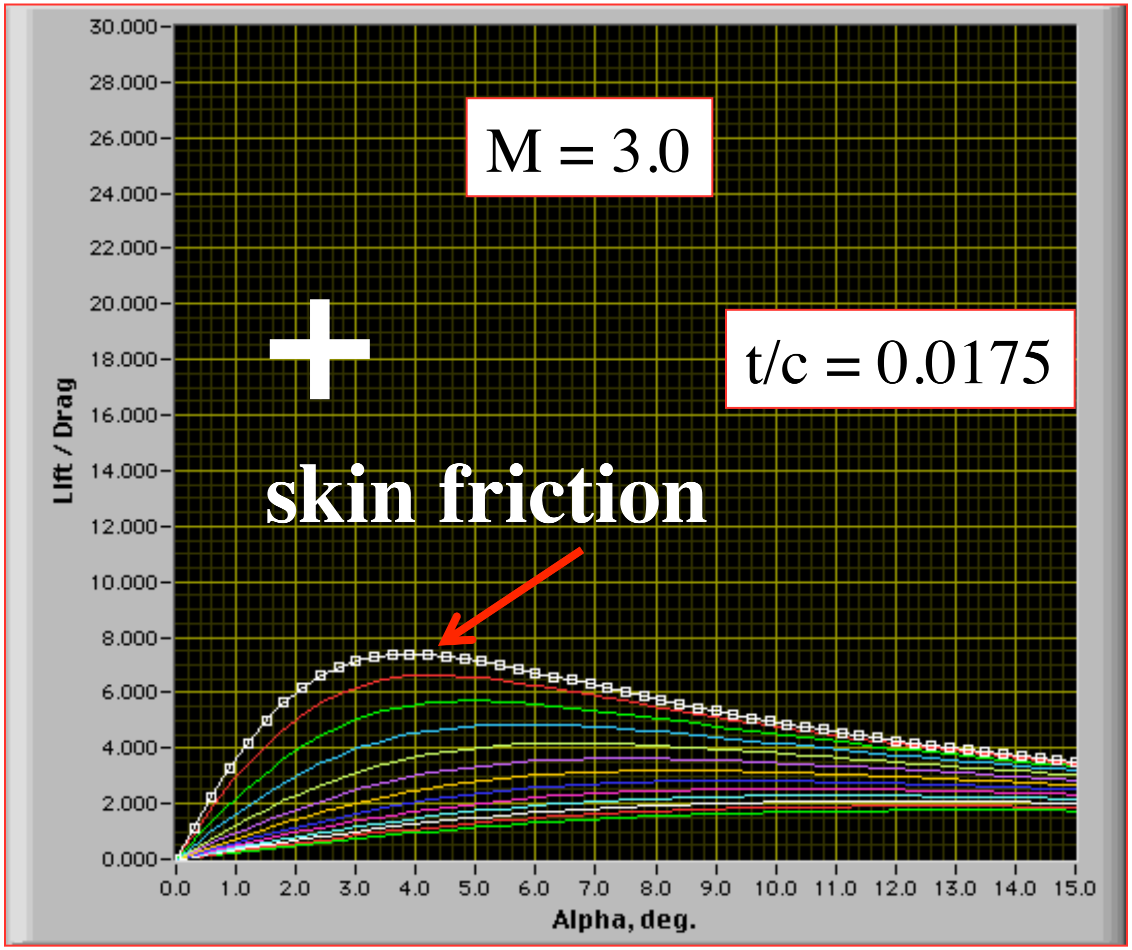
Example: Symmetric Double-wedge Airfoil … L/D



- For Inviscid flow Supersonic Lift to drag ratio almost infinite for very thin airfoil

- But airfoils do not fly in inviscid flows

Example: Symmetric Double-wedge Airfoil … L/D (cont'd)



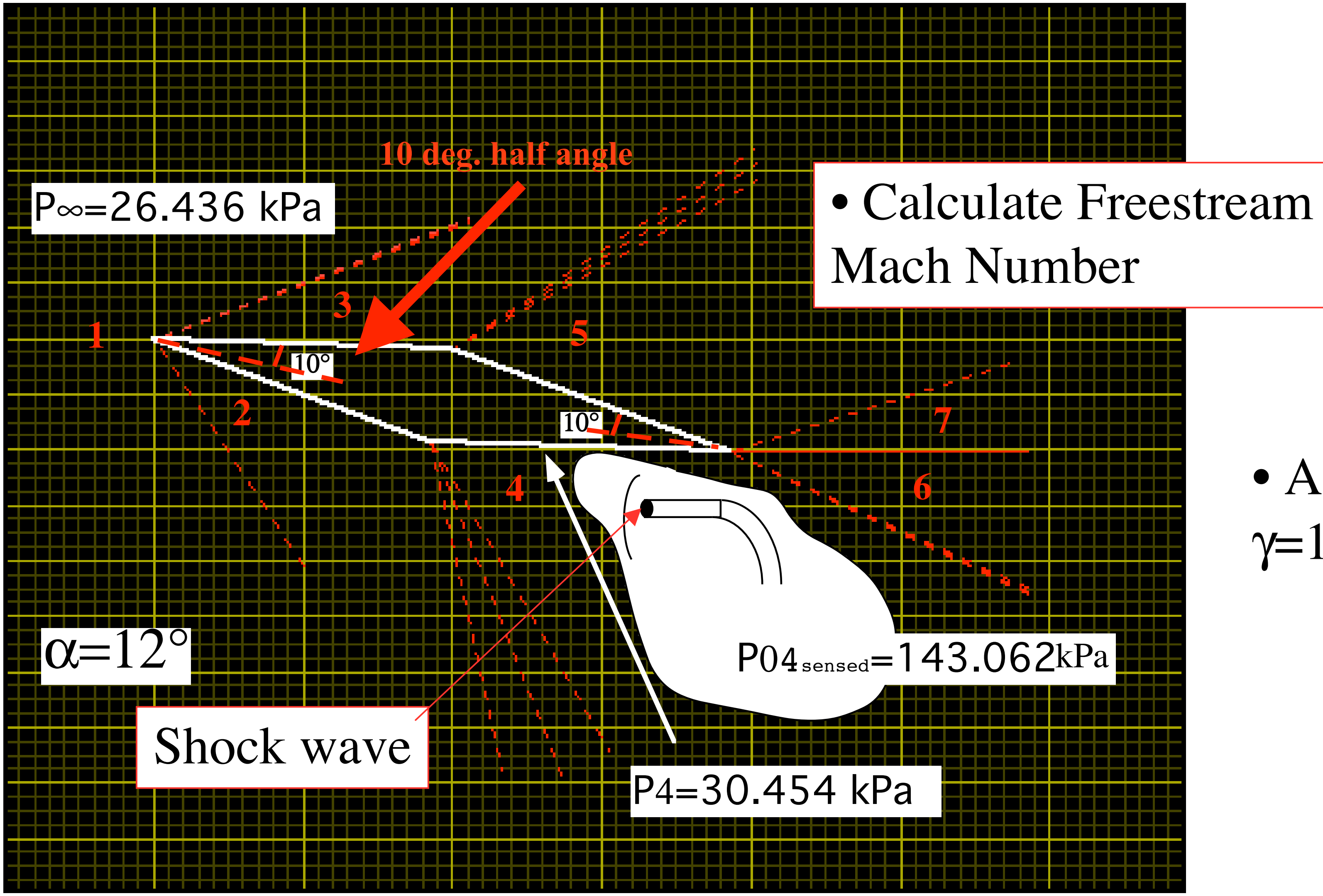
- Friction effects have small effect on Nozzle flow or flow in "large "ducts"

- But contribute significantly to reduce the performance of supersonic wings

Next section: Overview of Turbulent Compressible Boundary Layers and Friction Induced Drag

Due Monday, November 26 2018

Section 7: Home Work



- Calculate Freestream Mach Number

- Assume $\gamma=1.40$