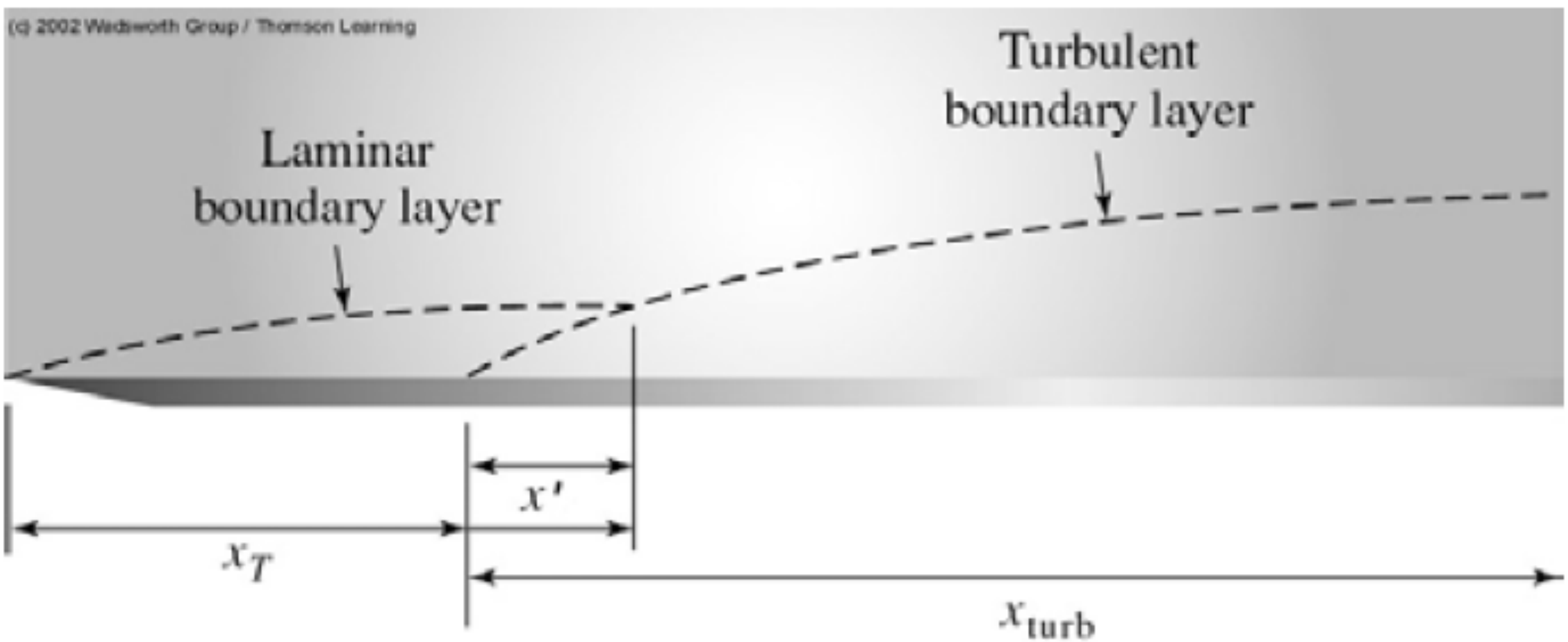
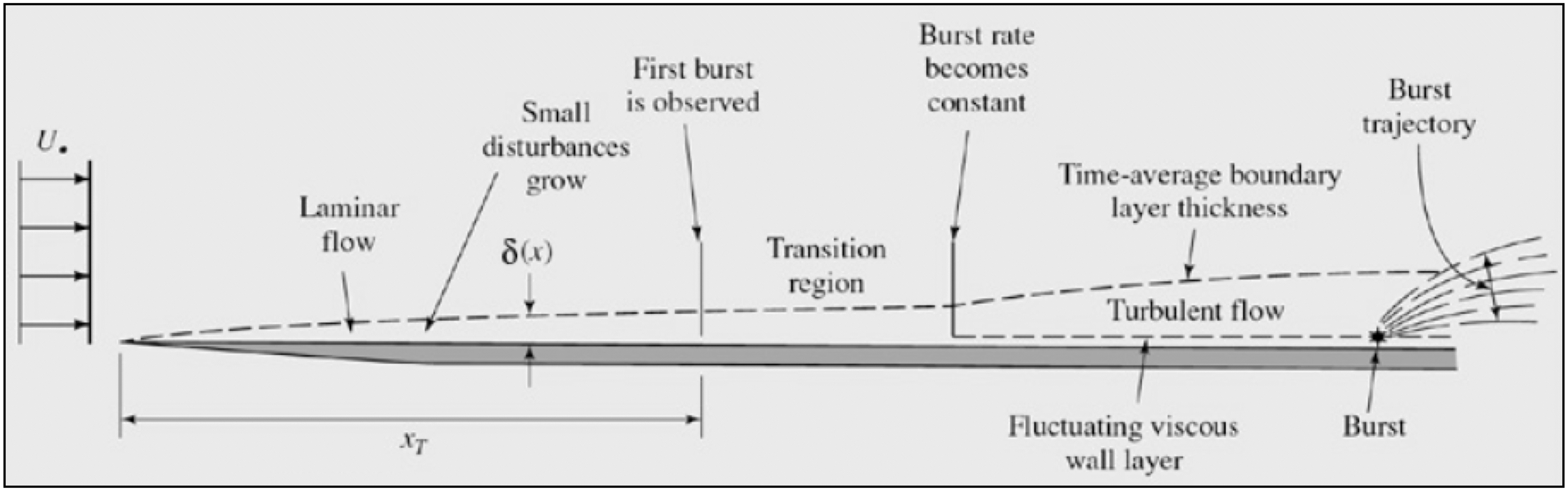
Section 8, Lecture 1, Supplemental

Effect of Pressure Gradients on Boundary layer



- Not in Anderson

Displacement and Momentum Thickness

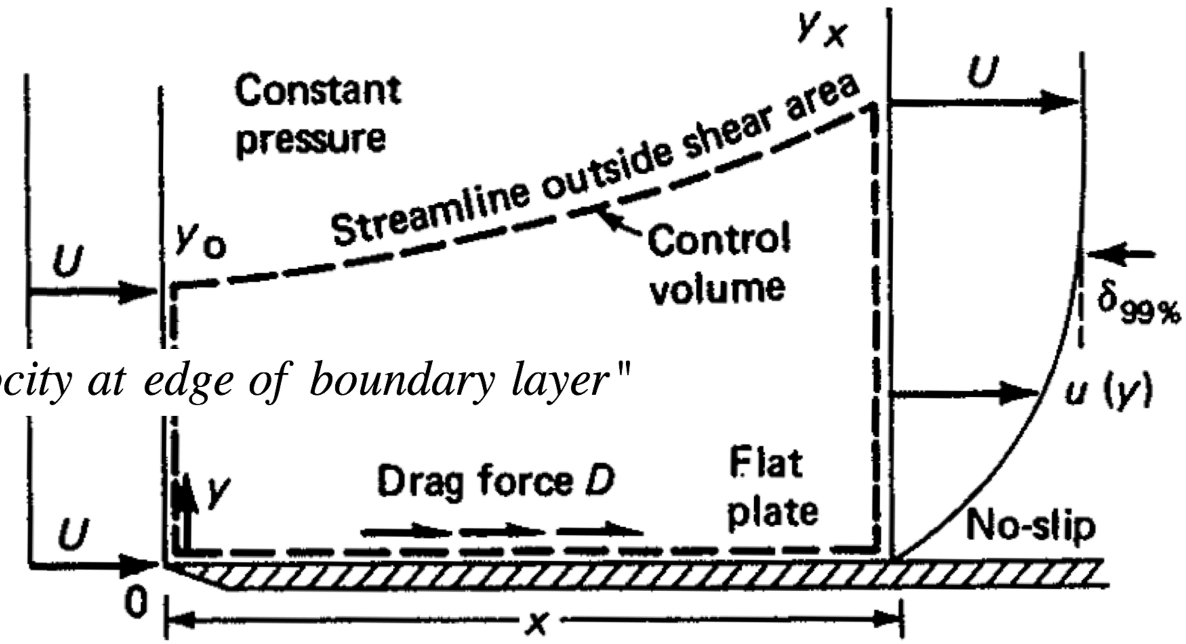


- **Fluid sticks to wall .. Creating boundary layer**
- **Local thickness is a function of downstream distance**
- **Local effects *(at each x)***

-- External streamlines are displaced *(displacement thickness, δ^*)*
-- Momentum is list to friction *(momentum thickness, Θ)*

Displacement Thickness

- How are external streamlines displaced by boundary layer?



$U = V_e \rightarrow$ *"velocity at edge of boundary layer"*

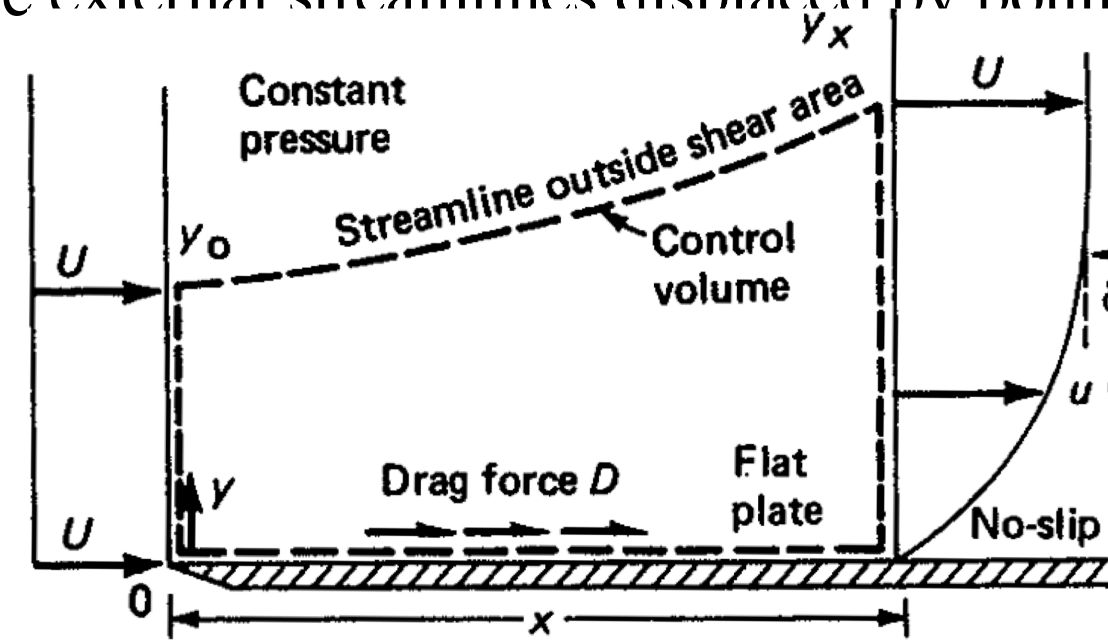
- Steady Mass Flow Across Control Volume

“0”

$$-\iint_{C.S.}\left(\rho \vec{V} \bullet \vec{ds}\right) = \frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho dv\right)$$

Displacement Thickness (2)

- How are external streamlines displaced by boundary layer?

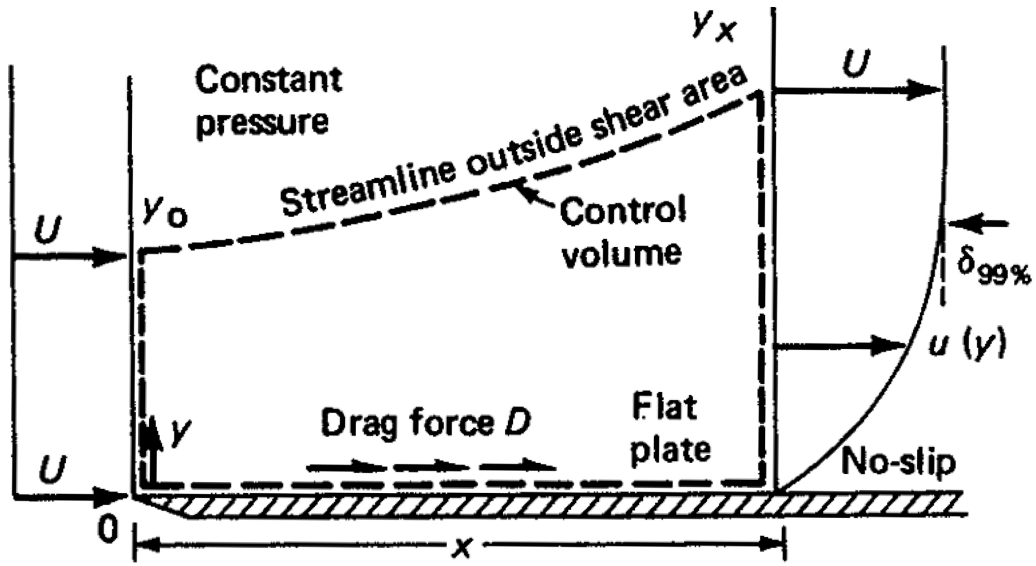


$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) = 0 = b \cdot \left[\int_0^{y_x} \rho u(y) dy - \int_0^{y_0} \rho u(y) dy \right] \rightarrow$$

$$for\ incompressible\ \rho = const \rightarrow \int_0^{y_x} u(y) dy = \int_0^{y_0} u(y) dy$$

Displacement Thickness (3)

- How are external streamlines displaced by boundary layer?



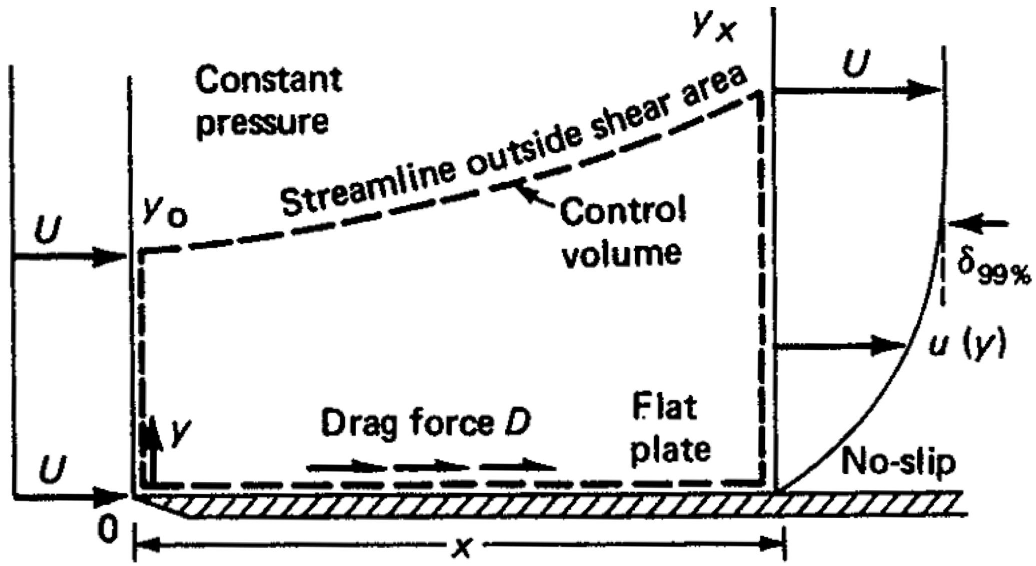
- At leading edge of plate *u(y) = U* --->

$$\int_0^{y_0} u(y)dy = U \cdot y_0 \rightarrow$$

$$U \cdot y_0 = \int_0^{y_x} u(y)dy = \int_0^{y_x} \left[U + u(y) - U\right] dy = U \cdot y_x + \int_0^{y_x} \left[u(y) - U\right] dy$$

Displacement Thickness (4)

- How are external streamlines displaced by boundary layer?

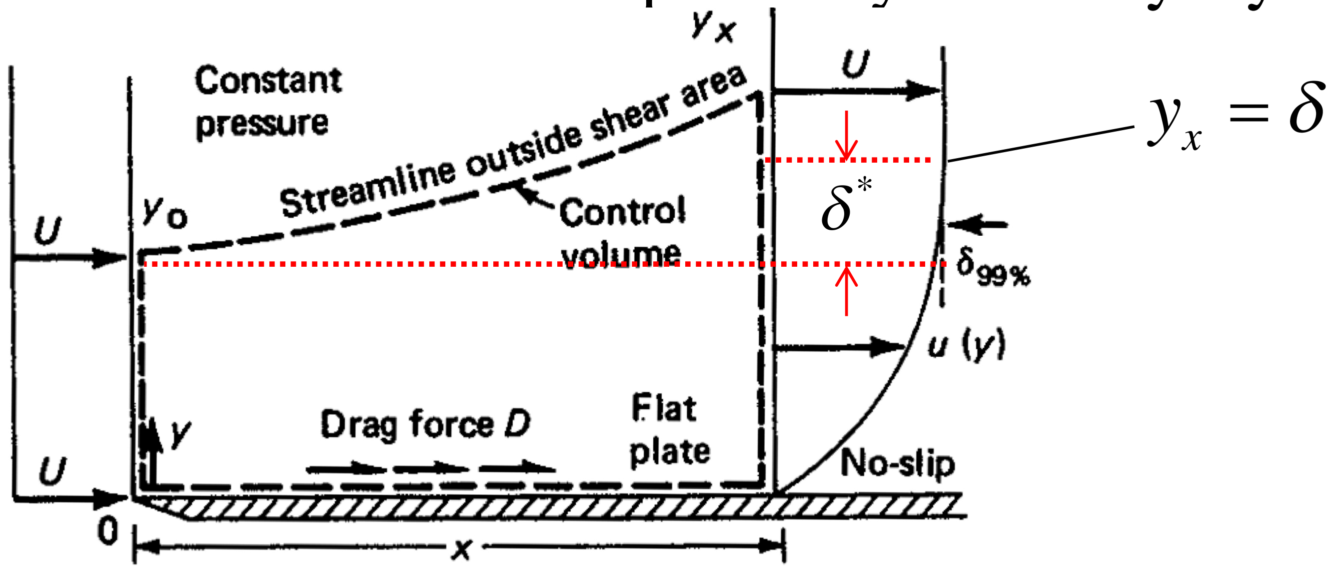


- Simplifying …

$$\left(y_x - y_0\right) = \int_0^{y_x} \left[u(y) - U\right] dy \rightarrow \boxed{y_x - y_0 = \int_0^{y_x} \left[1 - \frac{u(y)}{U}\right] dy}$$

Displacement Thickness (5)

- How are external streamlines displaced by boundary layer?



- Defining … $\delta^* = y_x - y_0$ @ $y_x = \delta$ … *edge of the boundary layer*

$$\cdot\left(y_x - y_0\right) = \int_0^{y_x} \left[u(y) - U\right] dy \rightarrow \boxed{\delta^* = \int_0^{\delta} \left[1 - \frac{u(y)}{U}\right] dy}$$

"Displacement Thickness"

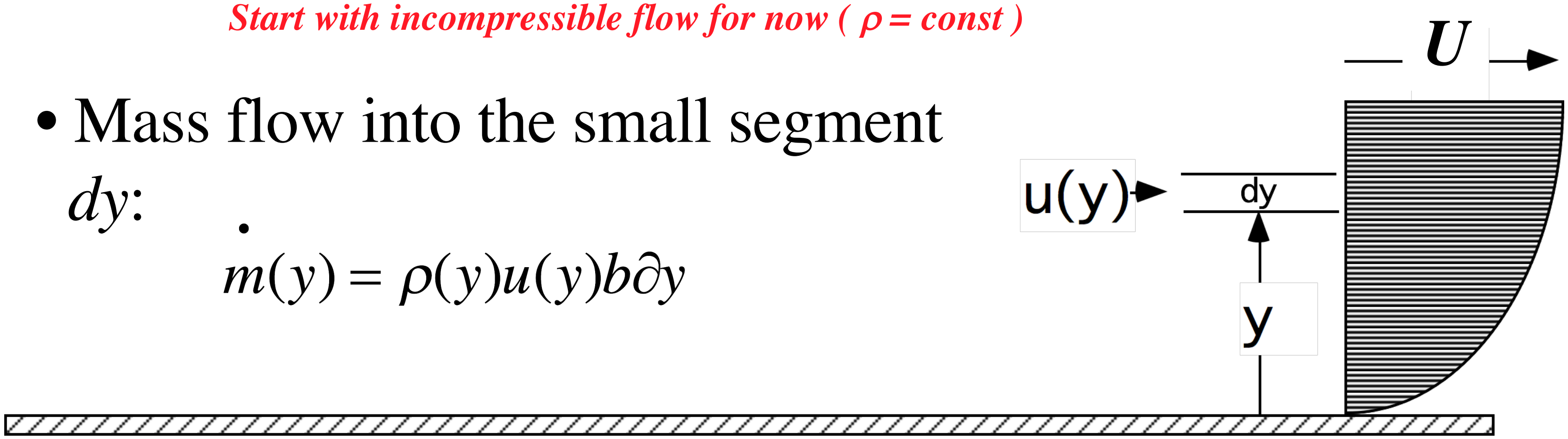
Momentum Thickness

- Look at the flow near the aft end of a flat plate with span, *b*

Start with incompressible flow for now ($\rho = const$)

- Mass flow into the small segment *dy*:

$$\dot{m}(y) = \rho(y) u(y) b \partial y$$



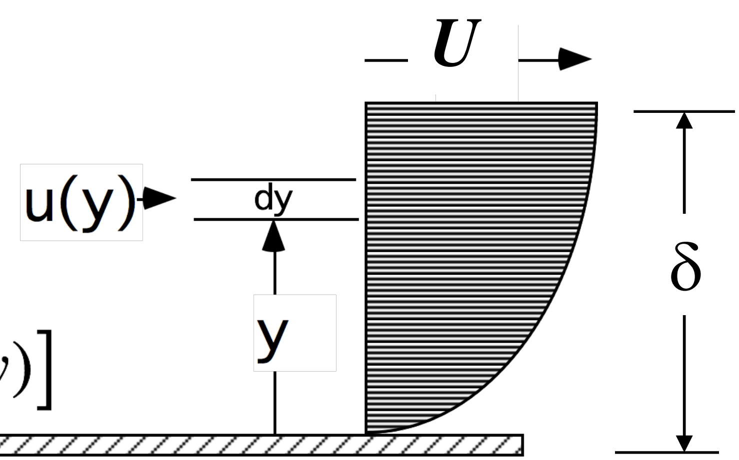
- Momentum defect from freestream across segment *dy:*

$$\partial M_{omentum} = \dot{m}(y)\left[U - u(y)\right]$$

Momentum Thickness (2)

- Assuming no pressure gradient along plate

Drag = loss in momentum $F = \frac{\partial}{\partial t}\left[M_{omentum}\right]$



$$\partial D = \dot{m}\left[U - u(y)\right] = \rho\left[b\partial y\right]u(y)\left[U - u(y)\right]$$

- Integrating across the depth of the boundary layer

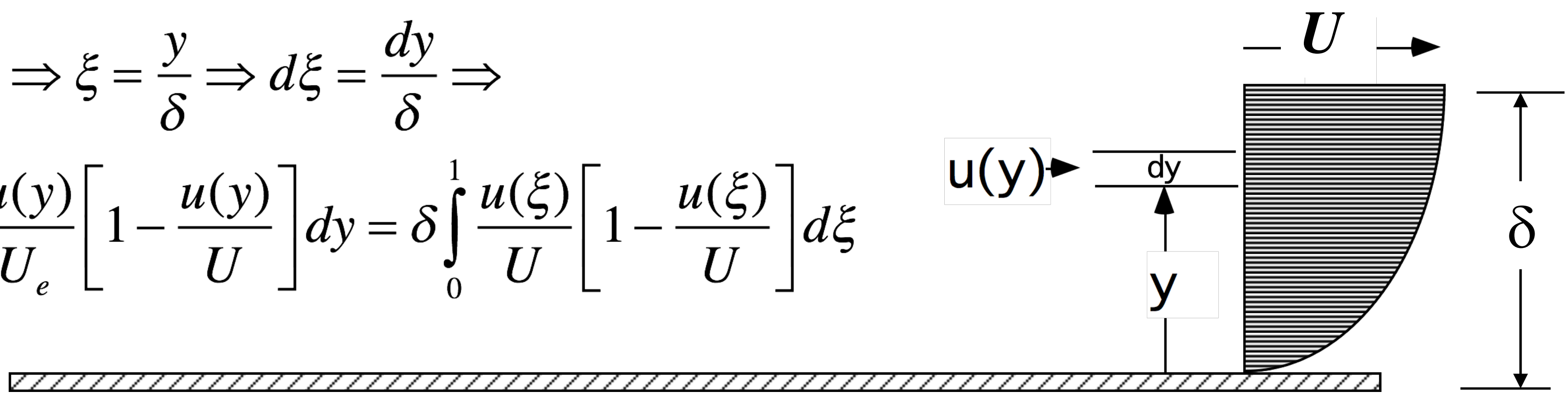
Momentum Thickness (3)

- Integrating across the depth of the boundary layer

$$D = \rho U^2 b \int_0^{\delta} \frac{u(y)}{U}\left[1 - \frac{u(y)}{U}\right] dy \Rightarrow$$

$$let \Rightarrow \xi = \frac{y}{\delta} \Rightarrow d\xi = \frac{dy}{\delta} \Rightarrow$$

$$\int_0^{\delta} \frac{u(y)}{U_e}\left[1 - \frac{u(y)}{U}\right] dy = \delta \int_0^{1} \frac{u(\xi)}{U}\left[1 - \frac{u(\xi)}{U}\right] d\xi$$



$$D_{fric} = \frac{1}{2}\rho U^2 b \cdot (2 \cdot \Theta) \rightarrow \boxed{\Theta \equiv \delta \int_0^{1} \frac{u(\xi)}{U}\left[1 - \frac{u(\xi)}{U}\right] d\xi}$$

"momentum thickness"

Displacement and Momentum Thickness (2)

- **Displacement Thickness …**

  -- How far external streamlines are displaced *by local boundary layer* (δ^*)

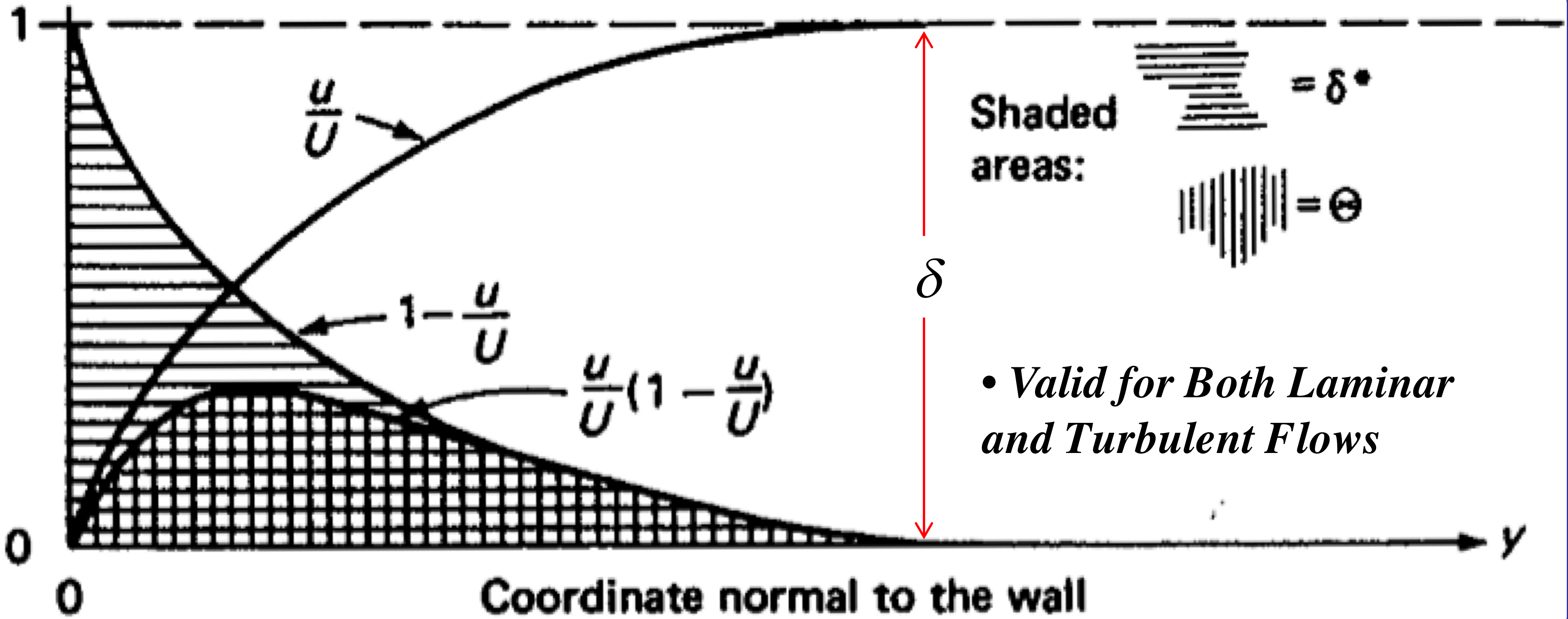
$$\delta^* = \delta \int_0^1 \left[1 - \frac{u(\xi)}{U} \right] d\xi \rightarrow \left| \xi = \frac{y}{\delta} \right.$$

- **Momentum Thickness …**

  -- Momentum Loss in Boundary Layer from front of plate to local station, x (Θ)

$$\Theta \equiv \delta \int_0^1 \frac{u(\xi)}{U} \left[1 - \frac{u(\xi)}{U} \right] d\xi$$

Displacement and Momentum Thickness (3)

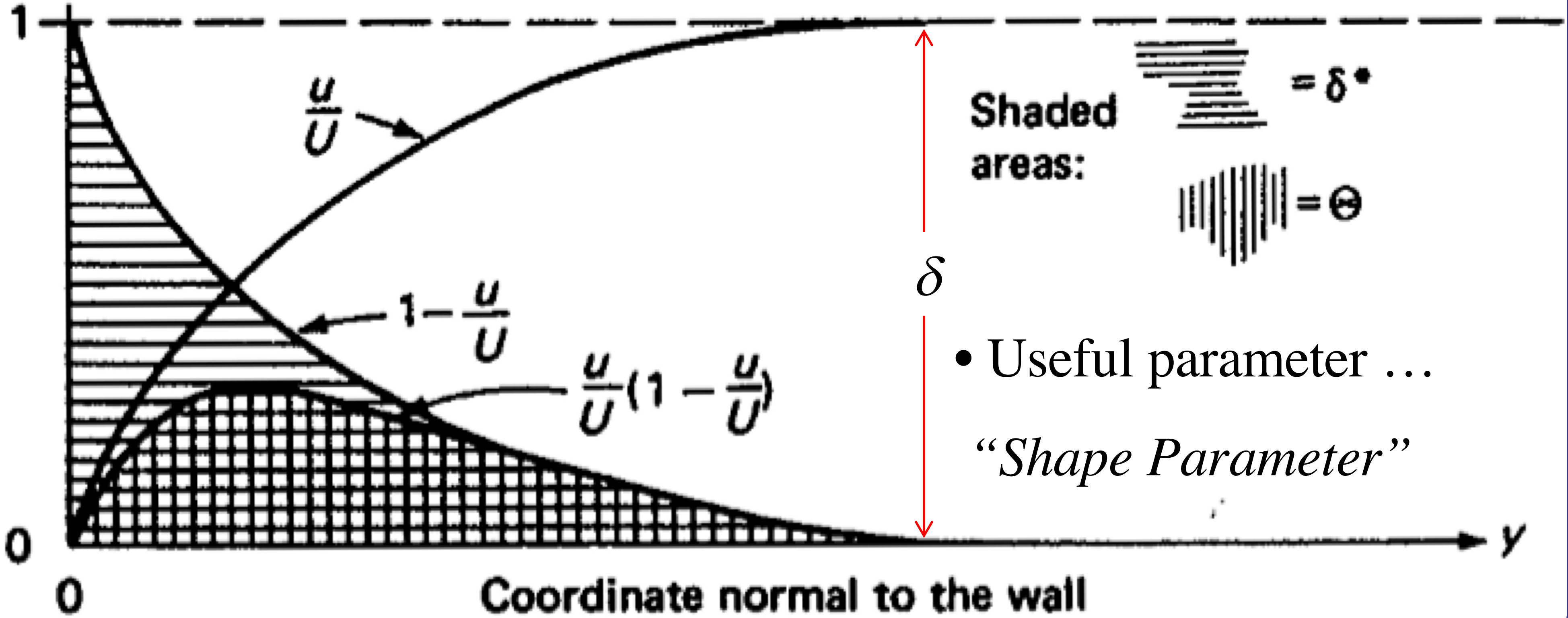


• *Valid for Both Laminar and Turbulent Flows*

$$\delta^* = \delta \int_0^1 \left[1 - \frac{u(\xi)}{U}\right] d\xi \rightarrow \xi = \frac{y}{\delta}$$

$$\Theta \equiv \delta \int_0^1 \frac{u(\xi)}{U}\left[1 - \frac{u(\xi)}{U}\right] d\xi$$

Displacement and Momentum Thickness (3)



- Useful parameter …

"Shape Parameter"

- ***More on This later***

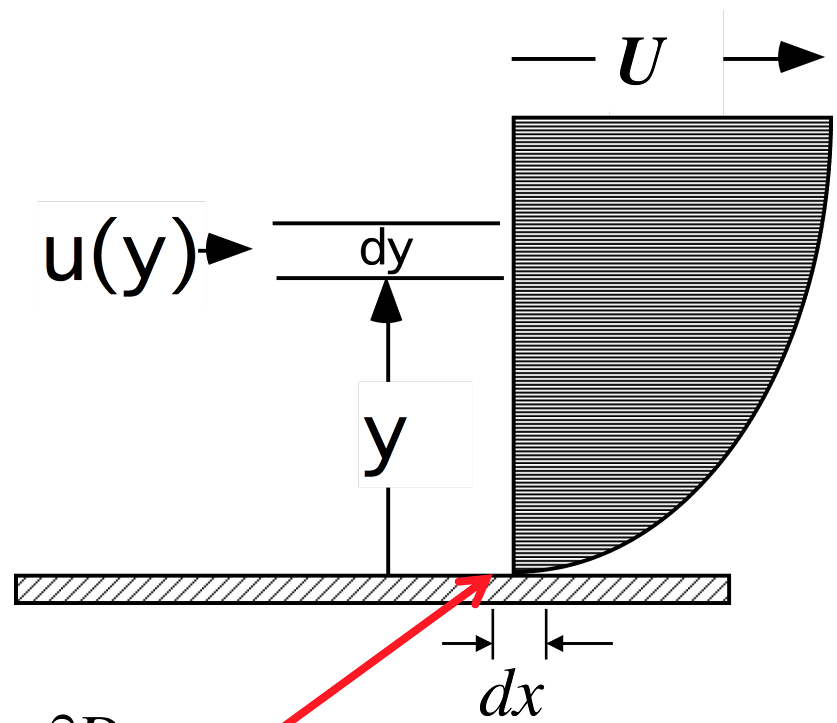
$$H = \frac{\delta^*}{\Theta} = \frac{\int_0^1 \left[1 - \frac{u(\xi)}{U}\right] d\xi}{\int_0^1 \frac{u(\xi)}{U}\left[1 - \frac{u(\xi)}{U}\right] d\xi} \rightarrow H > 1 \text{ "}always\text{"}$$

Local Skin Friction Coefficient and Total Skin-Drag Coefficient

- Per Earlier Discussion
  - *Start with Wall Shear Stress*

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0}$$

"dynamic viscosity"



$$\tau_w = \frac{\partial D_{fric}}{b dx}$$

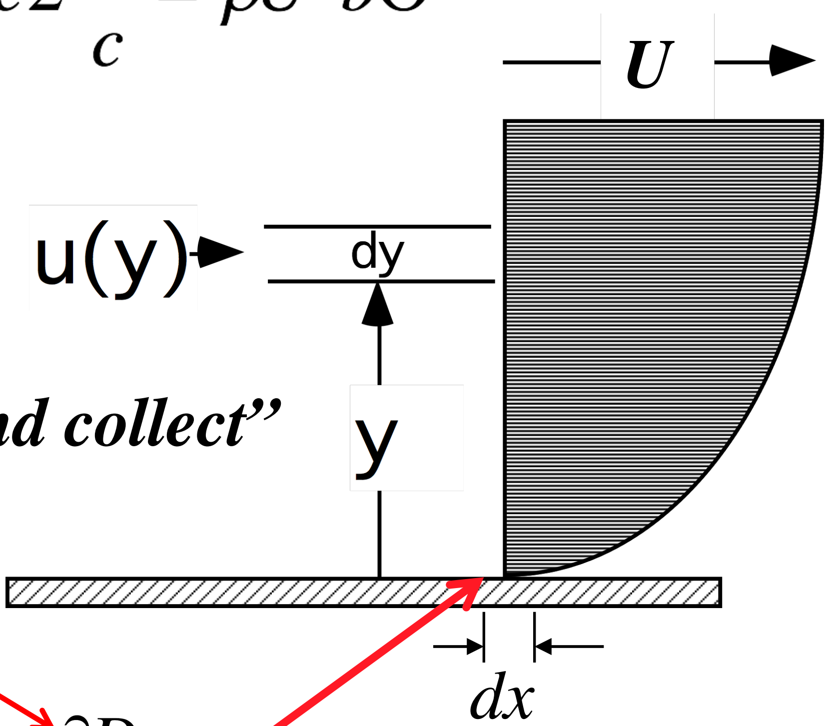
"incremental drag /unit surface area"

Local Skin Friction Coefficient and Total Skin-Drag Coefficient (2)

- From Earlier derivation of momentum thickness

$$D_{fric} = \frac{1}{2}\rho U^2 bc C_{D_{fric}} = \frac{1}{2}\rho U^2 bc 2\frac{\Theta}{c} = \rho U^2 b\Theta$$

$$\frac{\partial D_{fric}}{\partial x} = \rho U^2 b \frac{\partial \Theta}{\partial x}$$



"equate and collect"

$$\tau_w = \rho U^2 \frac{\partial \Theta}{\partial x}$$

$$\tau_w = \frac{\partial D_{fric}}{b \partial x}$$

Local Skin Friction Coefficient and Total Skin-Drag Coefficient (3)

- ***Von Karman Momentum law for boundary layer***

$$b \cdot \frac{\tau_w}{\frac{1}{2}\rho U^2} = b \cdot 2 \frac{\partial \Theta}{\partial x} \rightarrow \boxed{c_{f_x} \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2} = 2 \frac{\partial \Theta}{\partial x}}$$

- ***Valid for flat Plate with no pressure gradient***

$c_{f_x} \rightarrow$ "*local skin friction coefficient*"

- Contrast with

"Total Plate Skin Drag Coefficient"

$$\boxed{C_{D\,fric} = \frac{D_{fric}}{\frac{1}{2}\rho U^2 \cdot c \cdot b} = 2 \cdot \frac{\Theta(c)}{c}}$$

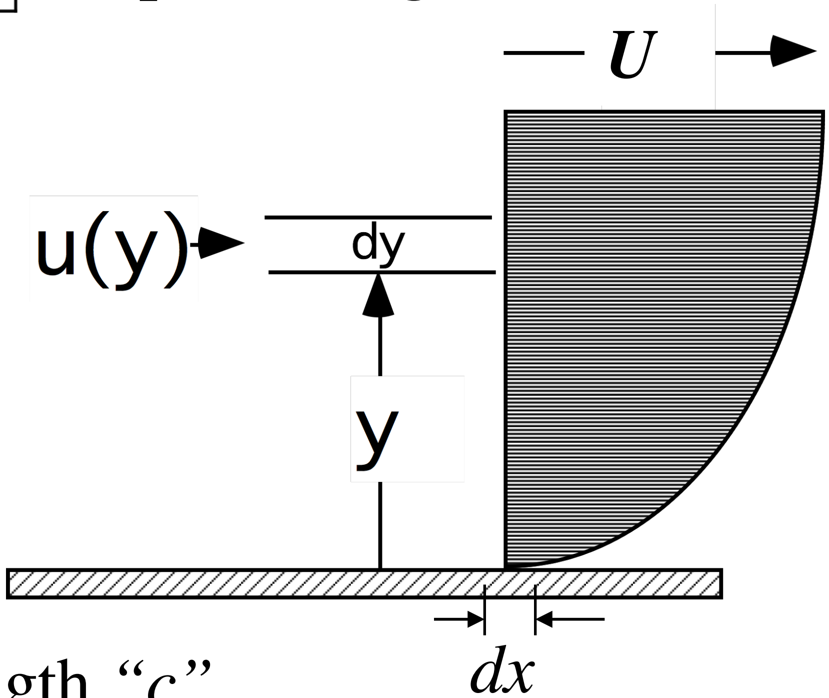


Plate length *"c"*

Local Skin Friction Coefficient and Total Skin-Drag Coefficient (4)

$$\boxed{c_{f_x} \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2} = 2\frac{\partial \Theta}{\partial x}} \ldots\ldots c_{f_x} \rightarrow \text{"local skin friction coefficient"}$$

- ***Valid for flat Plate with no pressure gradient***

"Total Plate Skin Drag Coefficient"

Plate length "c

$$\boxed{C_{D\,fric} = \frac{D_{fric}}{\frac{1}{2}\rho U^2 \cdot c \cdot b} = 2 \cdot \frac{\Theta(c)}{c}} \ldots C_{D\,fric} \rightarrow \text{"total skin drag coefficient"}$$

"... logically ...

$$C_{D\,fric} = \frac{1}{c}\int_0^c c_{f_x}\,dx = \frac{2}{c}\int_0^c \frac{\partial \Theta}{\partial x}\,dx$$

Local Skin Friction Coefficient and Total Skin-Drag Coefficient (5)

$$C_{D\,fric} = \frac{2}{c}\left(\int_0^1 \frac{u(\xi)}{U}\left[1-\frac{u(\xi)}{U}\right]d\xi\right)\int_0^c \frac{\partial\delta}{\partial x}dx = \frac{2}{c}\left(\int_0^1 \frac{u(\xi)}{U}\left[1-\frac{u(\xi)}{U}\right]d\xi\right)\cdot\delta(c)$$

Laminar

$$\frac{u_{(y)}}{V_e} = \left[\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right]$$

$$\delta(x) = \frac{4.98\cdot x}{\left[\mathrm{R}_{e_x}\right]^{\frac{1}{2}}}$$

$$\frac{2}{c}\int_0^1 \frac{u(\xi)}{U}\left[1-\frac{u(\xi)}{U}\right]d\xi = \frac{2}{c}\int_0^1\left[2\xi-\xi^2\right]\left[1-\left[2\xi-\xi^2\right]\right]d\xi = \frac{4}{15c}$$

↓

$$C_{D\,fric} = \frac{4}{15c}\frac{4.98\cdot c}{\left[\mathrm{R}_{e_c}\right]^{\frac{1}{2}}} = \frac{1.328}{\left[\mathrm{R}_{e_c}\right]^{\frac{1}{2}}}$$

Local Skin Friction Coefficient and Total Skin-Drag Coefficient (6)

$$c_{f_x} = 2\frac{\partial \Theta}{\partial x} = 2\left(\int_0^1 \frac{u(\xi)}{U}\left[1-\frac{u(\xi)}{U}\right]d\xi\right)\frac{\partial \delta}{\partial x} = \frac{4}{15}\frac{\partial \delta}{\partial x}$$

↓

Laminar Flow

$$\frac{u_{(y)}}{V_e} = \left[\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right]$$

$$\frac{\partial \delta(x)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{4.98\cdot x}{\left[\mathrm{R}_{e_x}\right]^{\frac{1}{2}}}\right) = \frac{\partial}{\partial x}\left(\frac{4.98}{\left[\frac{\rho U}{\mu}\right]^{\frac{1}{2}}}\left[x\right]^{\frac{1}{2}}\right) = \frac{1}{2}\cdot\frac{4.98}{\left[\frac{\rho U}{\mu}\right]^{\frac{1}{2}}}\frac{1}{\left[x\right]^{\frac{1}{2}}} = \frac{2.49}{\left[\mathrm{R}_{e_x}\right]^{\frac{1}{2}}}$$

$$\delta(x) = \frac{4.98\cdot x}{\left[\mathrm{R}_{e_x}\right]^{\frac{1}{2}}}$$

↓

$$c_{f_x} = \frac{4}{15}\frac{\partial \delta}{\partial x} = \frac{4}{15}\frac{2.49}{\left[\mathrm{R}_{e_x}\right]^{\frac{1}{2}}} = \frac{0.664}{\left[\mathrm{R}_{e_x}\right]^{\frac{1}{2}}}$$

- Similar process for turbulent flow

Incompressible Flat Plate Summary

Laminar		*Turbulent*
$\frac{u_{(y)}}{U} = \left[\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right]$	***Velocity profile***	$\frac{u_{(y)}}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{n}}$
$C_{D\,fric} = \frac{1.328}{\left[R_{e_c}\right]^{\frac{1}{2}}}$	***Total Skin Drag Coefficient***	$C_{D_{fric}} = \frac{\left[\frac{0.32n}{(n+1)(n+2)}\right]}{\left[R_{e_c}\right]^{\frac{1}{n}}}$
$c_{f_x} = \frac{0.664}{\left[R_{e_x}\right]^{\frac{1}{2}}}$	***Local Skin Friction Coefficient***	$c_{f_x} = \left(\frac{n-1}{n}\right)\frac{\left[\frac{0.32n}{(n+1)(n+2)}\right]}{\left[R_{e_x}\right]^{\frac{1}{n}}}$

For most applications ***n = 7***

Incompressible Flat Plate Summary (2)

Laminar		*Turbulent*
$\delta(x) = \frac{4.98 \cdot x}{\left[R_{e_x}\right]^{\frac{1}{2}}}$	***Boundary Layer Thickness***	$\delta(x) = 0.16 \frac{x}{\left[R_{e_x}\right]^{\frac{1}{n}}}$
$\delta^*(x) = 1.721 \frac{x}{\left[R_{e_x}\right]^{\frac{1}{2}}}$	***Displacement Thickness***	$\delta^* = \frac{0.16}{[n+1]} \frac{x}{\left[R_{e_x}\right]^{\frac{1}{n}}}$
$\Theta(x) = 0.664 \frac{x}{\left[R_{e_x}\right]^{\frac{1}{2}}}$	***Momentum Thickness***	$\Theta(x) = \left[\frac{0.16n}{(n+1)(n+2)}\right] \frac{x}{\left[R_{e_x}\right]^{\frac{1}{n}}}$
$H = 2.592$	***Shape Parameter***	$H = \frac{n+2}{n}$

What Happens when pressure (or velocity) along plate is not constant?

- Modified Von-Karman Momentum Equation

-- Flat Plate (no gradient)

$$\frac{\partial \Theta}{\partial x} = \frac{c_{f_x}}{2}$$

-- with pressure (velocity) gradient

$$\frac{\partial \Theta}{\partial x} + (2 + H)\frac{\Theta}{U}\frac{\partial U}{\partial x} = \frac{c_{f_x}}{2}$$

What Happens when pressure (or velocity) along plate is not constant? (2)

- Write in terms of pressure gradient

…. Bernoulli … $$p + \frac{1}{2}\rho U^2 = const \rightarrow \frac{\partial p}{\partial x} = -\rho U \frac{\partial U}{\partial x}$$

$$\frac{\partial U}{\partial x} = -\left(\frac{1}{\rho U}\right)\left(\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial \Theta}{\partial x} - \left(1 + \frac{H}{2}\right)\frac{\Theta}{\frac{1}{2}\rho U^2}\left(\frac{\partial p}{\partial x}\right) = \frac{c_{f_x}}{2}$$

What Happens when pressure (or velocity) along plate is not constant? (3)

- Express in terms of local skin friction coefficient

-->Let $$\beta = \frac{\delta^*}{\frac{1}{2}\rho U^2 \cdot c_{f_x}} \frac{\partial p}{\partial x}$$

$$\frac{\Theta}{\frac{1}{2}\rho U^2}\left(\frac{\partial p}{\partial x}\right) = \frac{\delta^*}{H \cdot \frac{1}{2}\rho U^2}\left(\frac{\partial p}{\partial x}\right) = \frac{\beta}{H} c_{f_x}$$

$$\frac{\partial \Theta}{\partial x} - \left(H + \frac{1}{2}\right)\beta c_{f_x} = \frac{c_{f_x}}{2}$$

What Happens when pressure (or velocity) along plate is not constant? (4)

- Collect Terms, solve for c_{fx}

$$\frac{\partial\Theta}{\partial x} = c_{f_x}\left[\frac{1}{2} + \left(H + \frac{1}{2}\right)\beta\right] = \frac{c_{f_x}}{2}\left[1 + \left(2H + 1\right)\beta\right]$$

With Pressure gradient

$$c_{f_x} = \frac{2\dfrac{\partial\Theta}{\partial x}}{\left[1 + \left(2H + 1\right)\beta\right]}$$

Flat Plate (no pressure gradient)

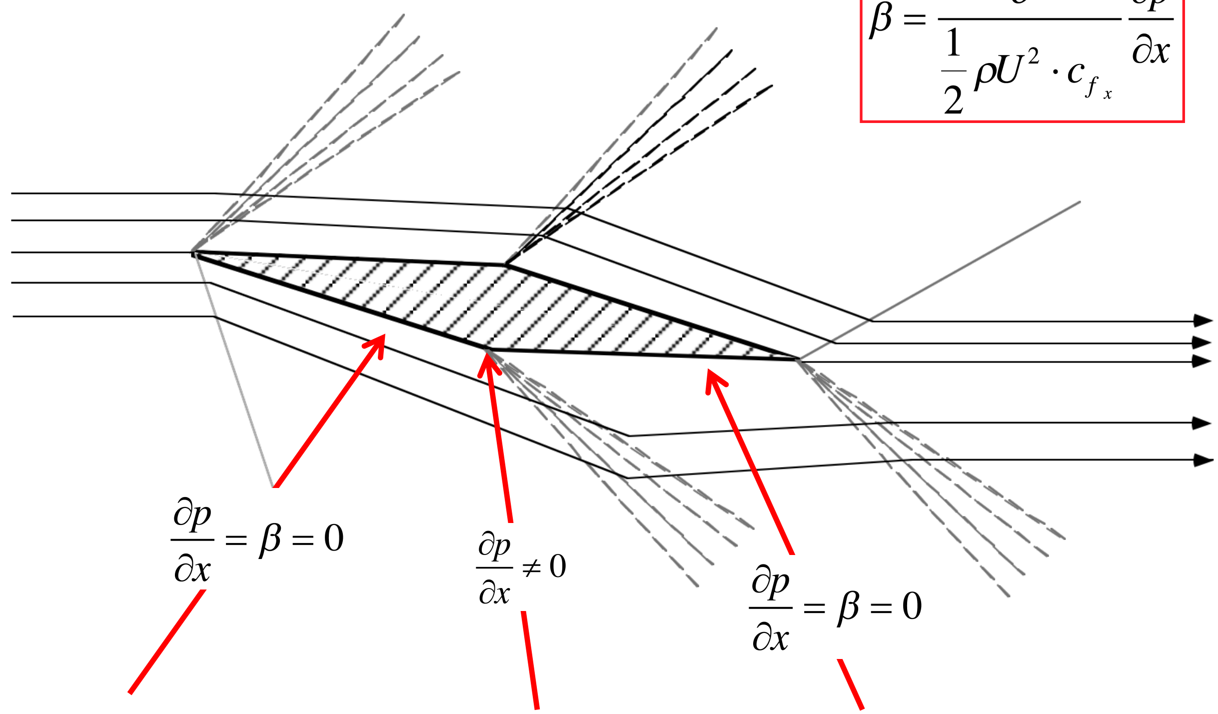
$$c_{f_x} = 2\frac{\partial\Theta}{\partial x}$$

What Happens when pressure (or velocity) along plate is not constant? (5)

- Apply to Diamond -wedge airfoil

$$\beta = \frac{\delta^*}{\frac{1}{2}\rho U^2 \cdot c_{f_x}} \frac{\partial p}{\partial x}$$

$$c_{f_x} = \frac{2\frac{\partial \Theta}{\partial x}}{\left[1+(2H+1)\beta\right]}$$



Along lower surface

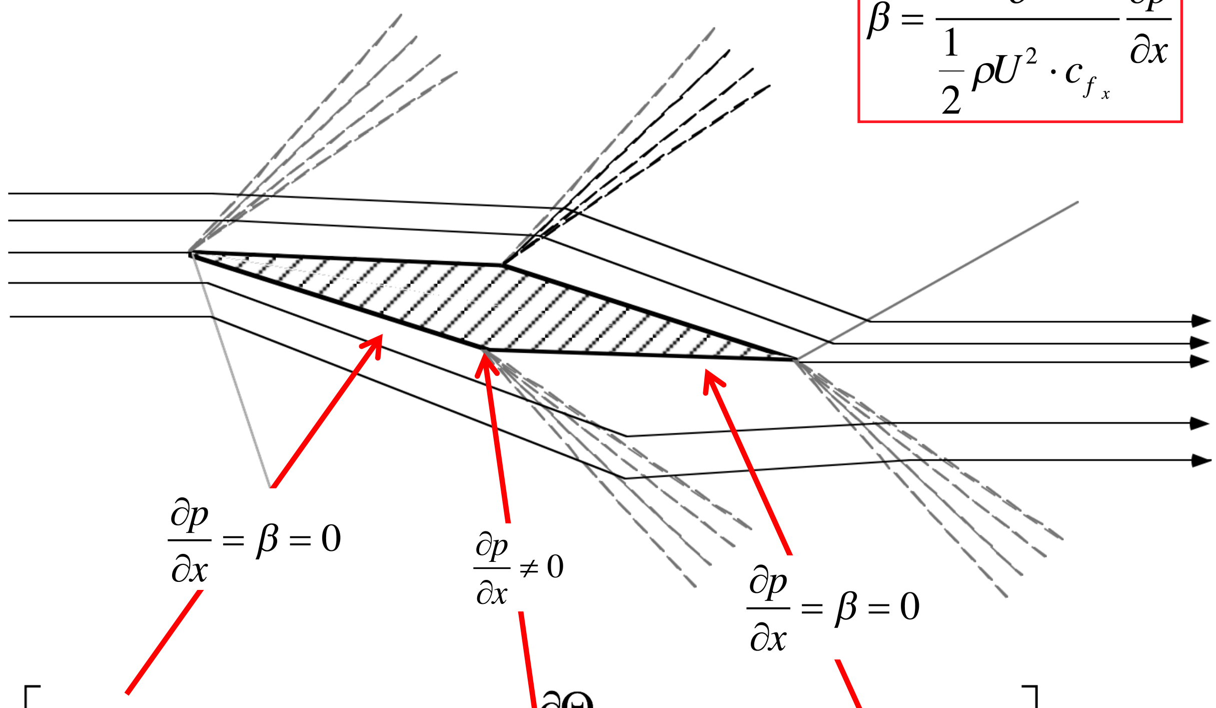
$$C_{D_{fric}} = \frac{1}{c}\int_0^c c_{f_x} dx = \frac{1}{c}\int_0^{c/2(-)} c_{f_x} dx + \int_{c/2(-)}^{c/2(+)} c_{f_x} dx + \int_{c/2(+)}^{c} c_{f_x} dx$$

Apply to Diamond Airfoil

- Along lower surface

$$c_{f_x} = \frac{2\frac{\partial \Theta}{\partial x}}{\left[1+\left(2H+1\right)\beta\right]}$$

$$\beta = \frac{\delta^*}{\frac{1}{2}\rho U^2 \cdot c_{f_x}}\frac{\partial p}{\partial x}$$



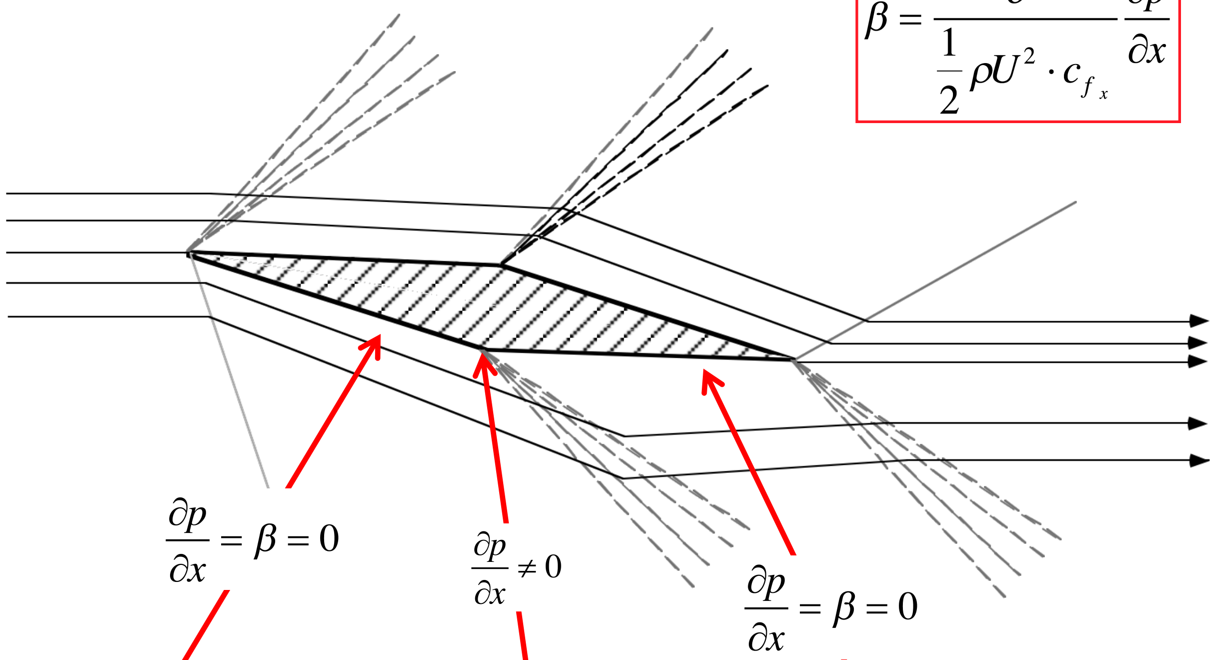
$$C_{D\,fric} = \frac{1}{c}\left[\int_0^{c/2(-)} 2\frac{\partial \Theta}{\partial x}dx + \int_{c/2(-)}^{c/2(+)} \frac{2\frac{\partial \Theta}{\partial x}}{\left[1+\left(2H+1\right)\beta\right]}dx + \int_{c/2(+)}^{c} 2\frac{\partial \Theta}{\partial x}dx\right]$$

Apply to Diamond Airfoil (2)

- Evaluate Integrals

$$\beta = \frac{\delta^*}{\frac{1}{2}\rho U^2 \cdot c_{f_x}} \frac{\partial p}{\partial x}$$

$$c_{f_x} = \frac{2\dfrac{\partial \Theta}{\partial x}}{\left[1+\left(2H+1\right)\beta\right]}$$



$$C_{D_{fric}} = \frac{1}{c}\left[2\Theta_{c/2(-)} + \int_{c/2(-)}^{c/2(+)} \frac{2\dfrac{\partial \Theta}{\partial x}}{\left[1+\left(2H+1\right)\beta\right]} dx + \left(2\Theta_c - 2\Theta_{c/2(+)}\right)\right]$$

Apply to Diamond Airfoil (3)

- Distance from *c/2(-)* to *c/2(+)* .. Almost *infinitesimal*

$$C_{D\,fric} \Rightarrow \lim_{c/2(-)\to c/2(+)} C_{D\,fric} \to \frac{1}{c}\left[2\Theta_{c/2} + \int_{c/2}^{c/2} \frac{2\dfrac{\partial \Theta}{\partial x}}{\left[1+\left(2H+1\right)\beta\right]}dx + \left(2\Theta_c - 2\Theta_{c/2}\right)\right] = \frac{2}{c}\Theta_c$$

- Which is exactly the flat plate formula … !

- So we see that the pressure gradient due to
The expansion wave has negligible effect on the skin drag coefficient for this airfoil!

Apply to Diamond Airfoil (4)

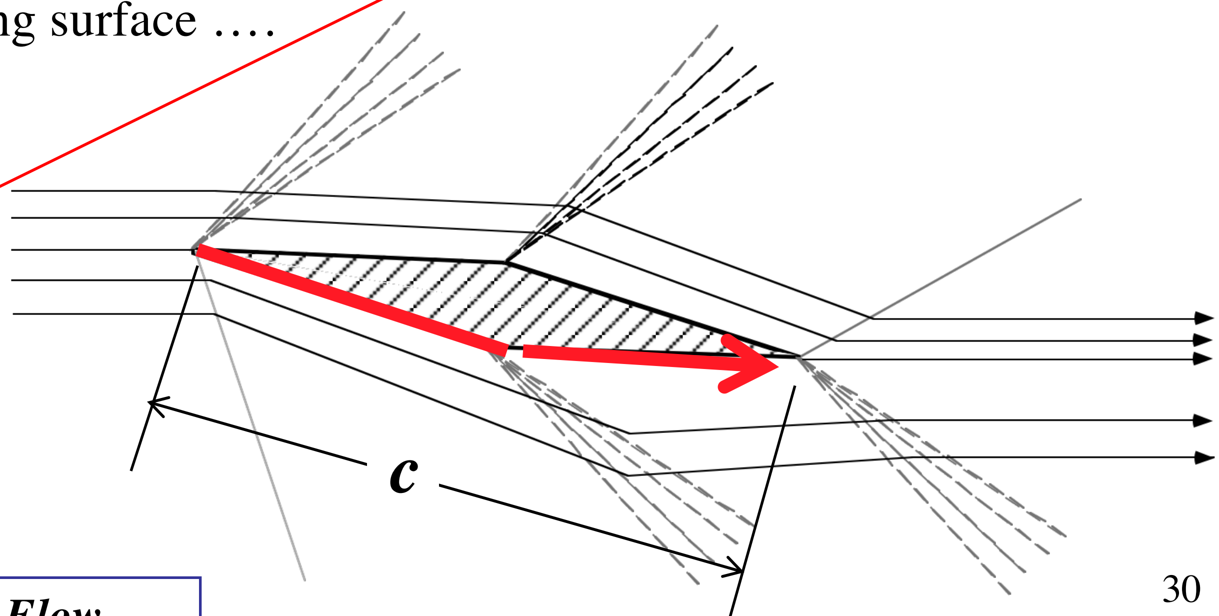
- Calculate Θ_c for lower surface, Turbulent flow

$$C_{D\,fric} \approx \frac{2}{c_{"effective"}}\Theta_c \qquad \Theta(x) = \left[\frac{0.16n}{(n+1)(n+2)}\right]\frac{x}{\left[\mathrm{Re}_x\right]^{\frac{1}{n}}}$$

- To "map" to flat plate we base calculation on "run length" along surface ….

$x_{run} = c/cos(\delta_{wedge})$

$c_{"effective"} = c/cos(\delta_{wedge})$



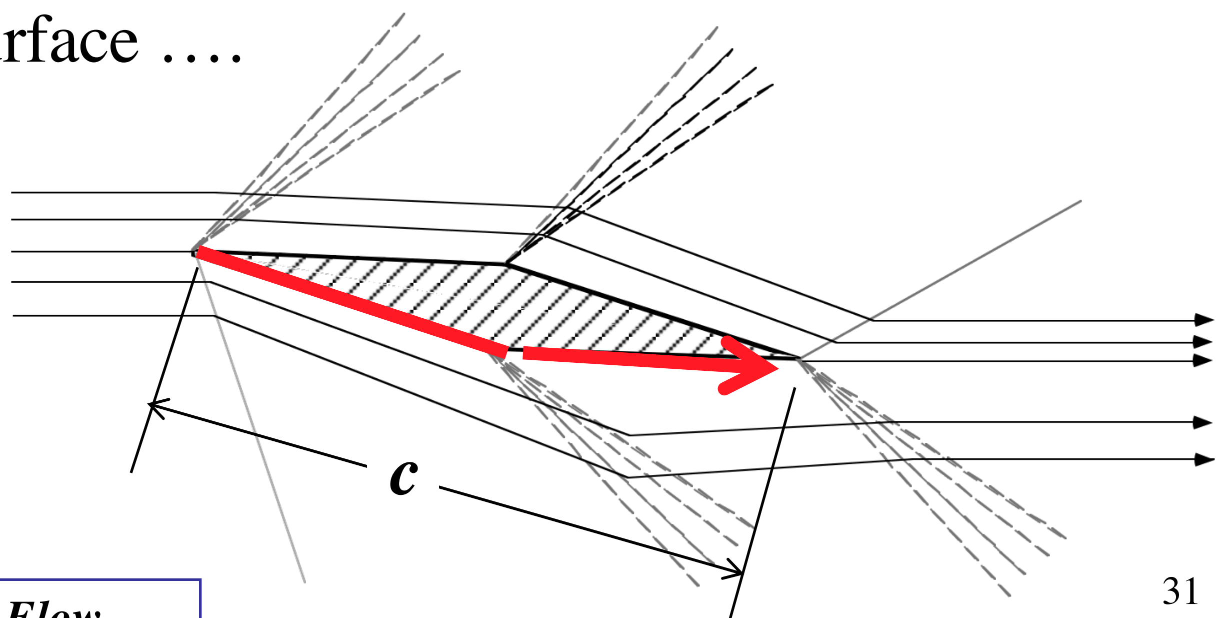
Apply to Diamond Airfoil (5)

- To calculate drag we apply to BOTH upper and lower surfaces

$$C_{D\,fric} \approx \frac{2}{c_{"effective"}} \Theta_c \qquad \Theta(x) = \left[\frac{0.16n}{(n+1)(n+2)}\right] \frac{x}{\left[\mathrm{Re}_x\right]^{\frac{1}{n}}}$$

- To "map" to flat plate we base calculation on "run length" along surface ....

$$x_{run} = c/\cos(\delta_{wedge})$$



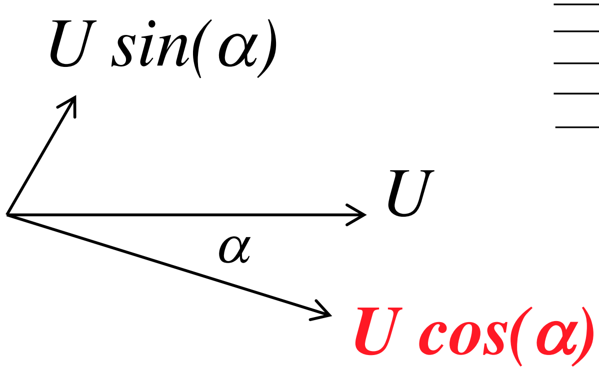
Apply to Diamond Airfoil (6)

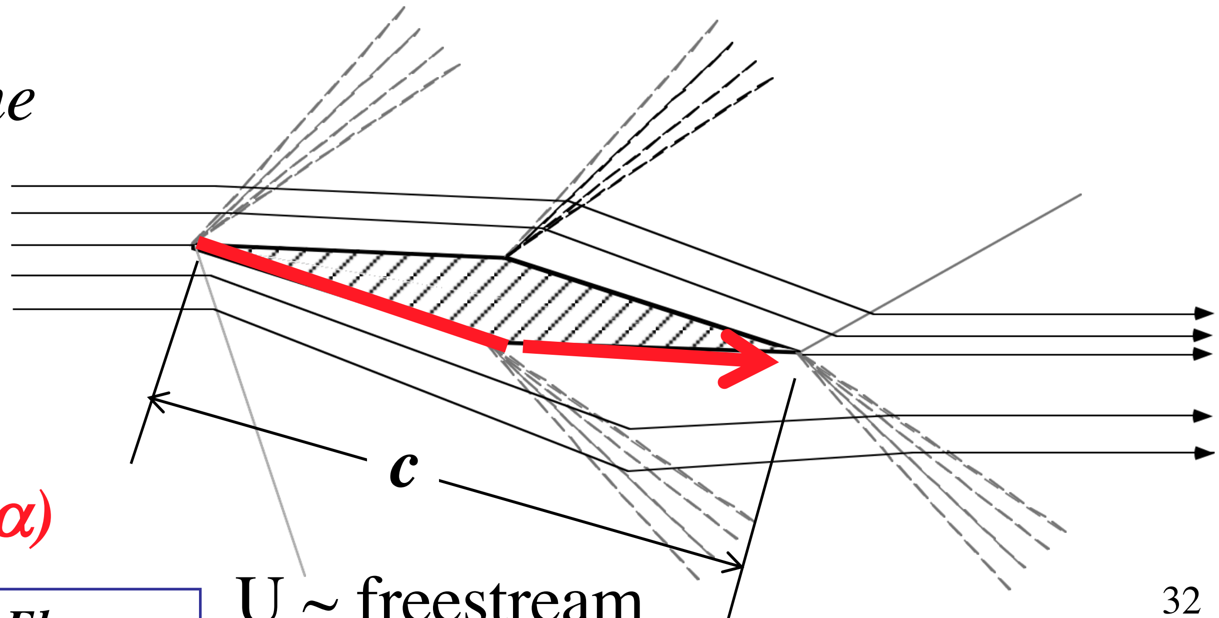
- Calculate Θ_c for lower surface, Turbulent flow

$$C_{D\,fric} \approx \frac{2}{c_{"effective"}} \Theta_c \qquad \Theta(x) = \left[\frac{0.16n}{(n+1)(n+2)}\right] \frac{x}{\left[\mathrm{Re}_x\right]^{\frac{1}{n}}}$$

- *To account for angle Of attack we use only Velocity component Along wing centerline*

$$x_{run} = c/cos(\delta_{wedge})$$





Apply to Diamond Airfoil (7)

- We have to account for BOTH sides of the wing

$$x_{run} = c/cos(\delta_{wedge}) \qquad R_{e_c} = \frac{\rho \cdot (U\cos\alpha) \cdot \left(c/\cos(\delta_{wedge})\right)}{\mu}$$

$$C_{D\,fric} \approx \frac{2}{c/\cos(\delta_{wedge})}\Theta_c =$$

$$\frac{2}{c/\cos(\delta_{wedge})}\left[\frac{0.16n}{(n+1)(n+2)}\right]\frac{c/\cos(\delta_{wedge})}{\left[\mathrm{R}_{e_c}\right]^{\frac{1}{n}}} = 2\cdot\left[\frac{0.16n}{(n+1)(n+2)}\right]\frac{1}{\left[\mathrm{R}_{ec}\right]^{\frac{1}{n}}}$$

$$D_{fric} = C_{D\,fric}\cdot\bar{q}\cdot A_{wet} = C_{D\,fric}\cdot\bar{q}\cdot\left[\left(b\cdot\frac{c}{\cos(\delta_{wedge})}\right)_{upper} + \left(b\cdot\frac{c}{\cos(\delta_{wedge})}\right)_{lower}\right]$$

Apply to Diamond Airfoil (8)

- Normalize by planform area *(b x c)*

$$\left(C_{D\,fric}\right)_{A_{plan}} = \frac{D_{fric}}{\bar{q}\cdot A_{plan}} = C_{D\,fric}\cdot\frac{A_{wet}}{A_{plan}} = \frac{C_{D\,fric}\cdot\bar{q}\cdot\left[\left(b\cdot\frac{c}{\cos(\delta_{wedge})}\right)_{upper} + \left(b\cdot\frac{c}{\cos(\delta_{wedge})}\right)_{lower}\right]}{\bar{q}\cdot b\cdot c} =$$

$$\rightarrow\left(C_{D\,fric}\right)_{A_{plan}} = \frac{2\cdot C_{D\,fric}}{\cos(\delta_{wedge})} = \left(\frac{4}{\cos(\delta_{wedge})}\right)\left[\frac{0.16n}{(n+1)(n+2)}\right]\frac{1}{\left[\mathrm{R}_{ec}\right]^{\frac{1}{n}}}$$

- Apply compressibility correction

Summary for Turbulent Flow, *n=7*

$$\left(C_{D\,fric}\right) = \frac{7}{225 \cdot \left[\mathrm{R}_{ec}\right]^{\frac{1}{7}}}$$

$$\left(C_{D\,fric}\right)_{A_{plan}} = \frac{2 \cdot C_{D\,fric}}{\cos(\delta_{wedge})} = \frac{14}{225 \cdot \cos(\delta_{wedge}) \cdot \left[\mathrm{R}_{ec}\right]^{\frac{1}{7}}}$$

Referenced To planform

$$\left[C_{D_{fric}}\right]_{compressible} = \frac{\left[C_{D_{fric}}\right]_{incompressible}}{\left[\left(\frac{T_\infty}{T_{avg}}\right)^{5/2}\left(\frac{T_{avg}+\mathrm{C}_s}{T_\infty+\mathrm{C}_s}\right)\right]^{\frac{1}{7}}} \rightarrow C_s = 120^0 K \; for\; air$$

$$T_{avg} \approx T_\infty\left[1+\frac{2}{9}\left(\frac{\gamma-1}{2}M_\infty^{\;2}\right)\right]$$