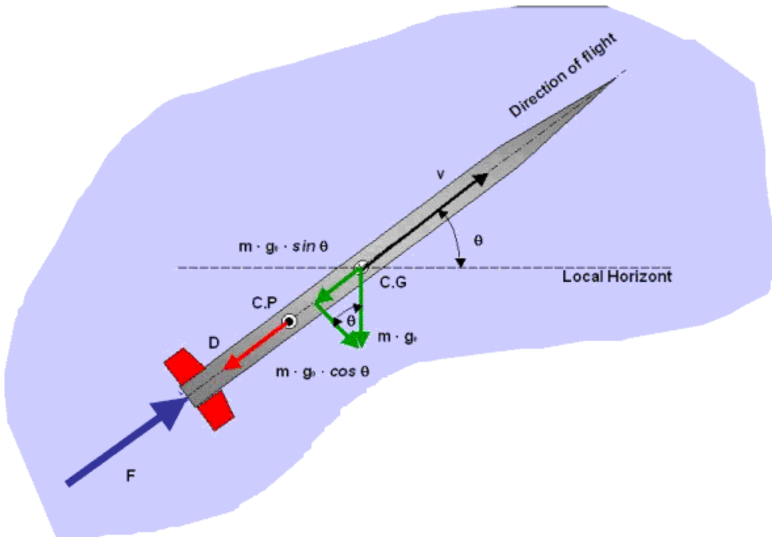
# Rocket Science 103:
# How High will My Rocket Go?

## Newton's Laws as Applied to "Rocket Science"

**... its not just a job ... its an adventure**

# Example Energy Calculation for Suborbital Launch (Ignore Drag)

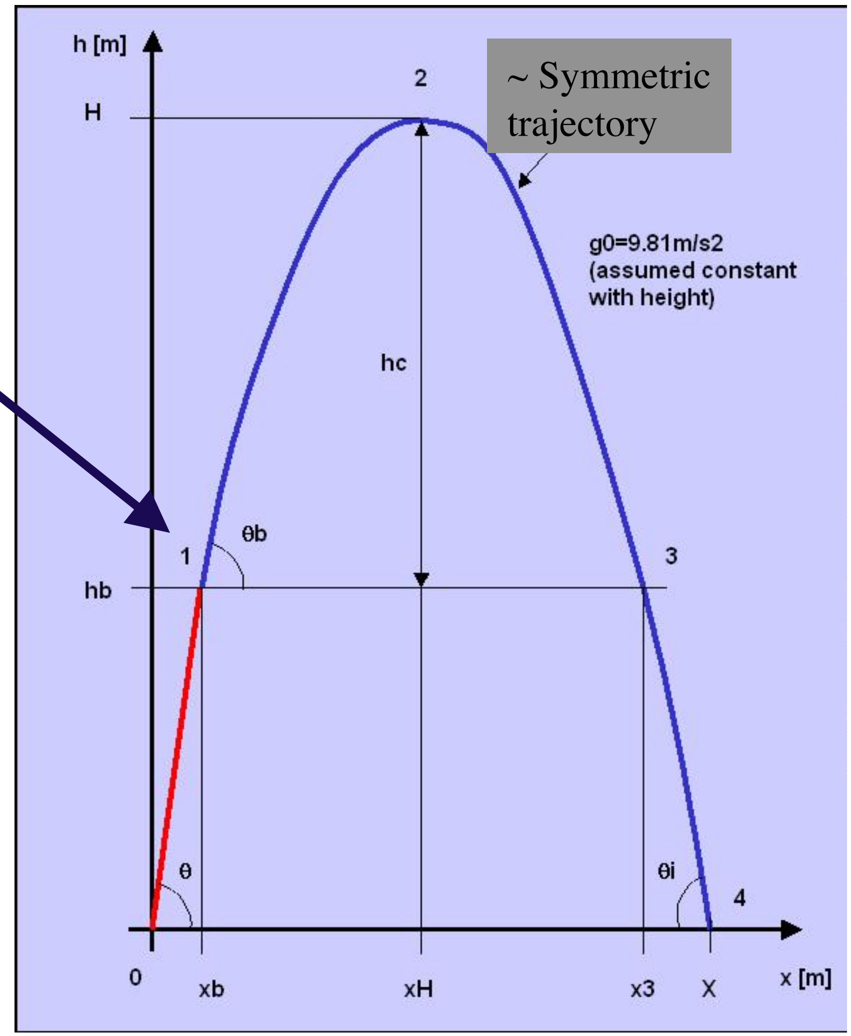



$$\Delta V_{t_{burn}} = g_0 \cdot I_{sp} \left[ \ln\left(1 + P_{mf}\right) \right] - g_0 \cdot \sin\theta_{launch} \cdot t_{burn}$$

$$\rightarrow \quad P_{mf} = \frac{M_{propellant}}{M_{final}}$$

$$t_{burn} = \frac{g_0 \cdot I_{sp} \cdot m_{propellamt}}{F_{thrust}}$$

***How High Will my Rocket Go?***



# Example Energy Calculation for Suborbital Launch (2)

$$Assume\ \theta_{launch} = constant,\ V_0 = 0 \rightarrow at\ time\ t:$$

$$V(t) = \left( g_0 \cdot I_{sp} \ln\left( \frac{M_{initial}}{M_{initial} - \dot{m} \cdot t} \right) - g_0 \cdot \sin\theta_{launch} \cdot t \right)$$

$$\rightarrow \frac{dh}{dt} = V(t) \cdot \sin\theta_{launch} = \left( g_0 \cdot I_{sp} \ln\left( \frac{M_{initial}}{M_{initial} - \dot{m} \cdot t} \right) - g_0 \cdot \sin\theta_{launch} \cdot t \right) \cdot \sin\theta_{launch}$$

- ***Altitude @ Burnout?***

$$\rightarrow h_{burnout} = \left[ \int_0^{t_{burnout}} V(t) \cdot \sin\theta_{launch} \cdot dt \right] =$$

$$\int_0^{t_{burnout}} \left( g_0 \cdot I_{sp} \ln\left( \frac{M_{initial}}{M_{initial} - \dot{m} \cdot t} \right) - g_0 \cdot \sin\theta_{launch} \cdot t \right) \cdot \sin\theta_{launch} \cdot dt$$

# Example Energy Calculation for Suborbital Launch (3)

- ***After a Lot of Arithmetic (see appendix at end of section for derivation)***

$$h_{burnout} = g_0 \cdot \sin\theta_{launch} \cdot t_{\substack{burn\\out}} \left\{ I_{sp} \cdot \left( 1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}} \right) - \frac{1}{2} \sin\theta_{launch} \cdot t_{\substack{burn\\out}} \right\}$$

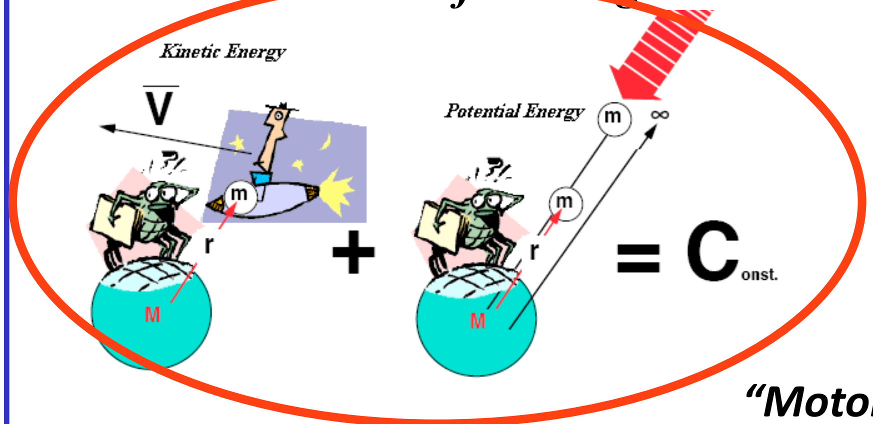
$$V_{burnout} = g_0 \cdot I_{sp} \cdot \ln\left(1 + P_{mf}\right) - g_0 \cdot \sin\theta_{launch} \cdot t_{\substack{burn\\out}}$$

$$\rightarrow \quad P_{mf} = \frac{M_{propellant}}{M_{burnout}}$$

$$t_{burn} = \frac{g_0 \cdot I_{sp} \cdot M_{propellamt}}{F_{thrust}}$$

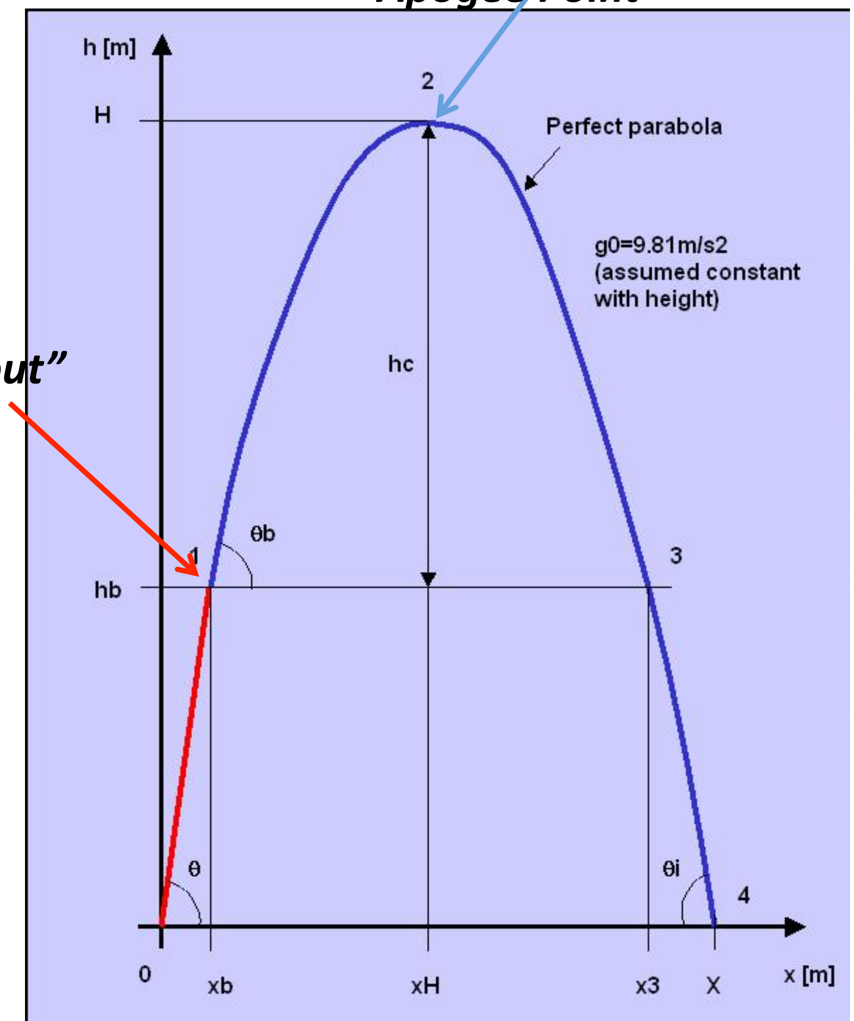
# Example Energy Calculation for Suborbital Launch (4)

***In the absence of dissipative (drag, etc) forces … total mechanical energy of rocket remains constant following motor burnout***




**"Apogee Point"**

**"Motor Burnout"**




***At Motor Burnout …***

$$E_{mech} = M_{final}\left(\frac{(V_{burnout})^2}{2} + g_0 \cdot h_{burnout}\right)$$

***At Apogee Point…***

$$E_{mech} = M_{final}\left(\frac{(V_{apogee})^2}{2} + g_0 \cdot h_{apogee}\right)$$

$$\approx M_{final} \cdot g_0 \cdot h_{apogee}$$

# Example Energy Calculation for Suborbital Launch (5)

## *Equating Total Energy at Burnout and Apogee*

$$\left(\frac{E_{mech}}{M_{final}}\right)_{burnout} = \left(\frac{E_{mech}}{M_{final}}\right)_{apogee} \qquad \frac{\left(V_{burnout}\right)^2}{2\cdot g_0} + h_{burnout} = \left(\frac{\left(V_{apogee}\right)^2}{2\cdot g_0} + h_{apogee}\right) \approx h_{apogee}$$

## *Solving for* h$_{apogee}$

$$h_{apogee} \approx \frac{E_{mech}}{M_{final}\cdot g_0} = \frac{\left(V_{burnout}\right)^2}{2\cdot g_0} + h_{burnout} =$$

$$\frac{\left(g_0\cdot I_{sp}\cdot \ln\left(1+P_{mf}\right) - g_0\cdot \sin\theta_{launch}\cdot t_{\substack{burn\\out}}\right)^2}{2\cdot g_0} + g_0\cdot\sin\theta_{launch}\cdot t_{\substack{burn\\out}}\left\{I_{sp}\cdot\left(1-\frac{\ln\left(1+P_{mf}\right)}{P_{mf}}\right) - \frac{1}{2}\sin\theta_{launch}\cdot t_{\substack{burn\\out}}\right\} =$$

$$\frac{1}{2}g_0\left(I_{sp}\cdot\ln\left(1+P_{mf}\right) - \sin\theta_{launch}\cdot t_{\substack{burn\\out}}\right)^2 + g_0\cdot\sin\theta_{launch}\cdot t_{\substack{burn\\out}}\left\{I_{sp}\cdot\left(1-\frac{\ln\left(1+P_{mf}\right)}{P_{mf}}\right) - \frac{1}{2}\sin\theta_{launch}\cdot t_{\substack{burn\\out}}\right\}$$

# Example Energy Calculation for Suborbital Launch (6)

## *How High Will my Rocket Go?*

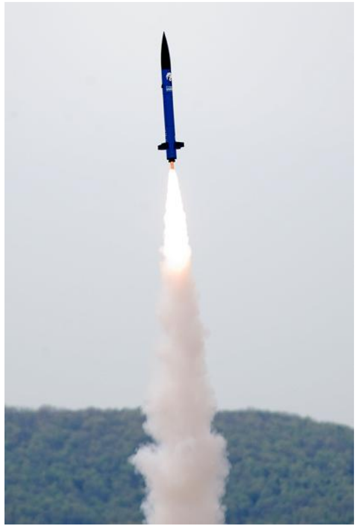
$$h_{burnout} = g_0 \cdot \sin\theta_{launch} \cdot t_{\substack{burn\\out}} \left\{ I_{sp} \cdot \left( 1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}} \right) - \frac{1}{2} \sin\theta_{launch} \cdot t_{\substack{burn\\out}} \right\}$$

$$V_{burnout} = g_0 \cdot I_{sp} \cdot \ln\left(1 + P_{mf}\right) - g_0 \cdot \sin\theta_{launch} \cdot t_{\substack{burn\\out}}$$

$$h_{apogee} \approx \frac{1}{2} g_0 \left( I_{sp} \cdot \ln\left(1 + P_{mf}\right) - \sin\theta_{launch} \cdot t_{\substack{burn\\out}} \right)^2 + g_0 \cdot \sin\theta_{launch} \cdot t_{\substack{burn\\out}} \left\{ I_{sp} \cdot \left( 1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}} \right) - \frac{1}{2} \sin\theta_{launch} \cdot t_{\substack{burn\\out}} \right\}$$

$$V_{apogee} \approx 0$$

# Example Calculation



***2009 USLI Rocket***

***AMW L777 Motor***

**"Dry" vehicle mass : *11.2451 kg*, Propellant mass: 1.7623 kg**

**Propellant $I_{sp}$: 181.49sec, Mean Motor Thrust: 774.475 Newtons**

$$P_{mf} = \frac{m_{propellant}}{M_{final}} = \frac{1.7623}{11.2451} = 0.156717$$

$$t_{burn} = \frac{g_0 \; I_{sp} \; M_{propellamt}}{F_{thrust}} = \frac{9.8067 \cdot 181.49 \cdot 1.7623}{774.475} = 4.04993 \text{ sec}$$

# Example Calculation (2)

$$P_{mf} = \frac{m_{propellant}}{M_{final}} = 0.156717 \qquad t_{burn} = \frac{g_0 \; I_{sp} \; M_{propellamt}}{F_{thrust}} = 4.04993 \text{ sec}$$

$$h_{t_{burn}} = g_0 \cdot t_{burn} \left\{ I_{sp} \cdot \left( 1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}} \right) - \frac{t_{burn}}{2} \right\} =$$

$$9.8067 \cdot 4.04933 \left( 181.49 \left( 1 - \frac{\ln\,(1 + 0.156717)}{0.156717} \right) - \frac{4.04993}{2} \right)$$ **= 431.5 meters**

$$V_{t_{burn}} = g_0 \cdot I_{sp} \left[ \ln\left(1 + P_{mf}\right) \right] - g_0 \cdot t_{burn} =$$

$$9.8067 \cdot 181.49\,(\ln\,(1 + 0.156717)) - 9.8067 \cdot 4.004993$$ **= 219. 5 m/sec**

# Example Calculation (3)

***Calculate Apogee Altitude (above ground level)***

$$h_{apogee} = \frac{E_{mech}}{M_{final} \; g_0} = \frac{\left(V_{burnout}\right)^2}{2 \; g_0} + h_{burnout} =$$

$$\frac{219.5^2}{2 \cdot 9.8067} + 431.5 \quad = 2888 \text{ meters}$$

# Compare to Simulation Results

***We will build this simulation later***

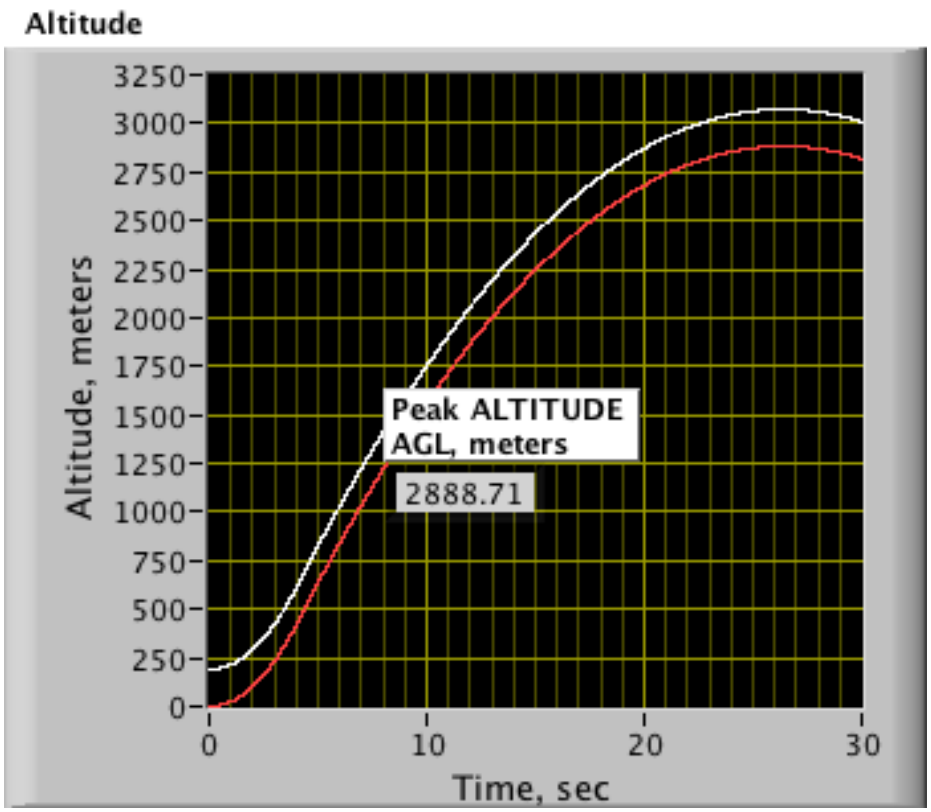
$$h_{apogee} = 2888.71\ m$$
$$V_{burnout} = 219.34\ m/\sec$$

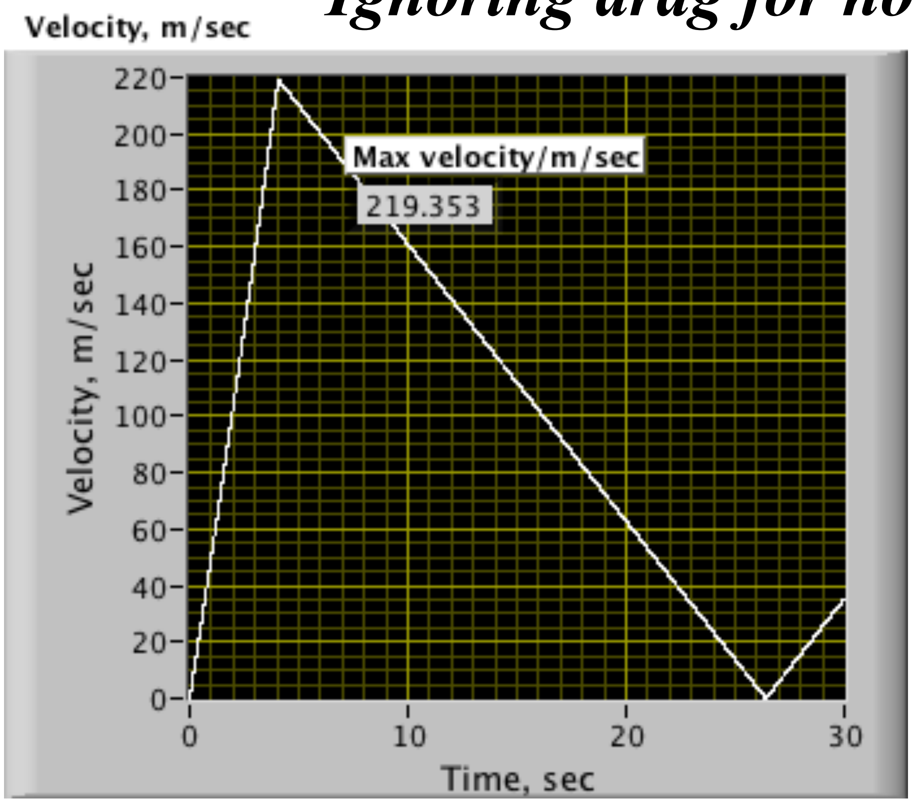
Analytical Solution

***=> 2888 meters***

***= 219. 5 m/sec***

***Ignoring drag for now!***



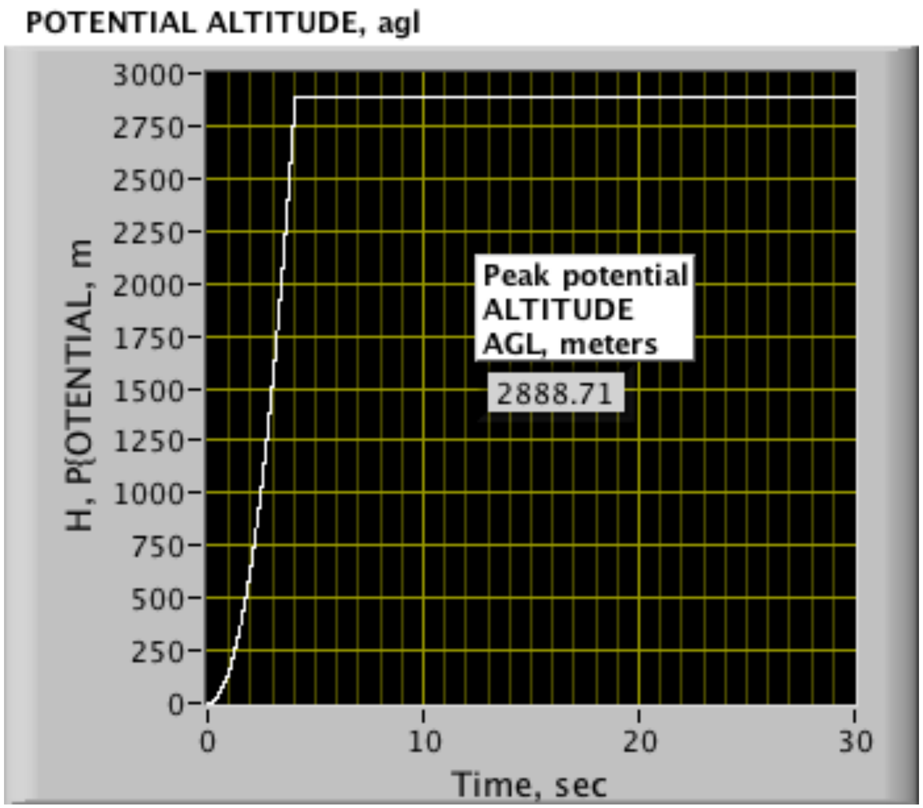




***Better than 0.056%***

# Compare to Simulation Results (2)

$$h_{apogee} = 2888.71\ m$$
$$V_{burnout} = 219.34\ m/\sec$$

***Ignoring drag for now!***




*Ignoring Drag During Burn* @ time $t$:

$$h_{(t)} = g_0 \cdot \sin\theta_{launch} \cdot t \cdot \left\{ I_{sp} \cdot \left[ \left( 1 - \frac{\ln\left(\frac{m_{initial}}{m_{initial} - \dot{m} \cdot t}\right)}{\frac{\dot{m} \cdot t}{m_{initial} - \dot{m} \cdot t}} \right) - \frac{1}{2} \sin\theta_{launch} \cdot t \right] \right\}$$

$$V_{(t)} = g_0 \cdot I_{sp} \cdot \ln\left(\frac{m_{initial}}{m_{initial} - \dot{m} \cdot t}\right) - g_0 \cdot \sin\theta_{launch} \cdot t$$

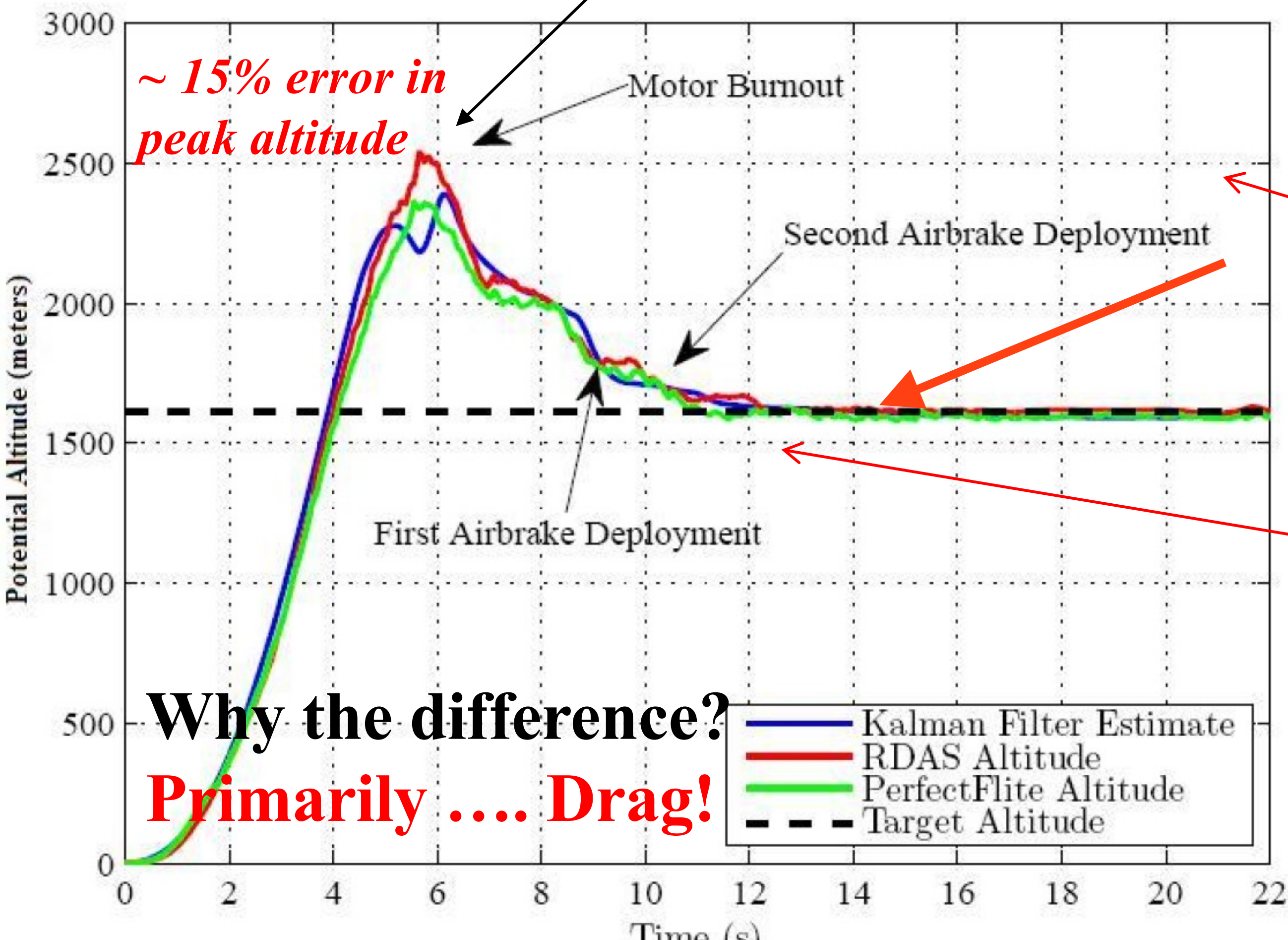
$$h_{potential_{(t)}} \approx \frac{E}{g_0\left(m_{initial} - \dot{m} \cdot t\right)} = \frac{V^2_{(t)}}{2 \cdot g_0} + h_{(t)}$$

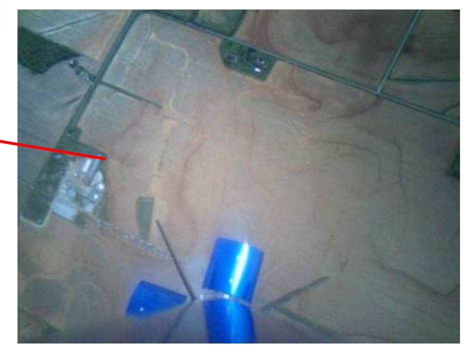
# Compare to Fight Data

***Analytical, Zero Drag***

$$h_{apogee} = 2888.71\ m$$
$$V_{burnout} = 219.34\ m/\sec$$

$$\Delta h_{drag} = \frac{\Delta V_{drag}^{\ 2}}{2\cdot g_0} = \frac{1}{2\cdot g_0\cdot\beta}\left[\int_0^t \rho\cdot V^3\, dt\right]$$



$$h_{potential} = h + \frac{V^2}{2\ g_0}$$

# Compare to Fight Data (2)

**Why the difference? Drag!**

$$\frac{\left(V_{apogee}\right)^2}{2} = \frac{E_{mech}}{M_{final}} \qquad V_{apogee} = \sqrt{\frac{2\ E_{mech}}{M_{final}}} = \sqrt{2\ g_0\ h_{potential}}$$

$$\frac{\left(V_{apogee}\right)_{calc} \quad \left(V_{apogee}\right)_{flight}}{\frac{\left(V_{apogee}\right)_{calc} + \left(V_{apogee}\right)_{flight}}{2}} \cdot 100_{\%} =$$

$$\frac{(2888.71 \cdot 2 \cdot 9.8067)^{0.5} - (2500 \cdot 2 \cdot 9.8067)^{0.5}}{\frac{(2888.71 \cdot 2 \cdot 9.8067)^{0.5} + (2500 \cdot 2 \cdot 9.8067)^{0.5}}{2}} = 7.22\%$$

*~ 7.2 % error in delivered apogee Δ V*

# How Drag Losses Effect Peak Altitude

$$\text{Conservation of Energy:}$$

$$\text{Potential} + \text{Kinetic Energy} = \text{Constant} - \text{Dissipated Energy}$$

$$g \cdot h_{apogee} + \frac{V^2_{apogee}}{2} = h_{burnout} + \frac{V^2_{burnout}}{2} - \int_{t_{burnout}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

$$\rightarrow h_{apogee} = h_{burnout} + \left( \frac{V^2_{burnout}}{2 \cdot g} - \frac{V^2_{apogee}}{2 \cdot g} \right) - \frac{1}{g} \int_{t_{burnout}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

# How Drag Losses Effect Peak Altitude (2)

$$\Delta V_{drag} = \sqrt{2\int_0^t \frac{D_{rag} V}{M} dt} = \sqrt{2\int_0^t \frac{C_D A_{ref} \frac{1}{2}\rho V^2 \cdot V}{M} dt} = \sqrt{\int_0^t \frac{\rho \cdot V^3}{\beta} dt}$$

$$\Delta h_{drag} = \frac{\Delta V_{drag}{}^2}{2 \cdot g_0} = \frac{1}{2 \cdot g_0 \cdot \beta}\left[\int_0^t \rho \cdot V^3 \, dt\right]$$

**Check units!**

$$: \frac{1}{\frac{m}{\sec^2}\,\frac{kg}{m^2}} \frac{kg}{m^3}\left(\frac{m}{\sec}\right)^3 \sec = \frac{\sec^2\ m^5}{m\ \ kg}\,\frac{kg}{m^3}\,\frac{1}{\sec^2} = m$$

***"Rule of thumb" ~ drag loss is about 5-10% of delivered $\Delta V$ from motor***

# Drag Coefficient is Configuration Dependent

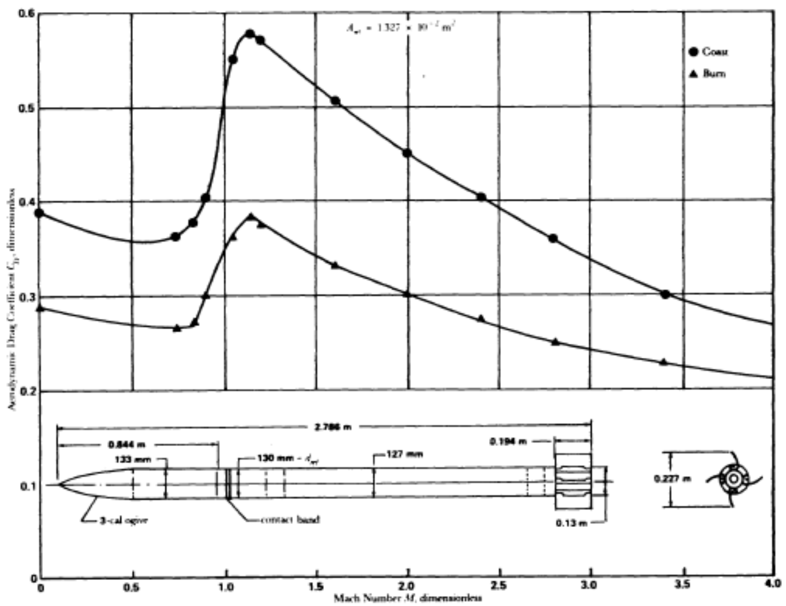



Figure 3-4. Drag Coefficient vs Mach Number-HO-mm Rocket

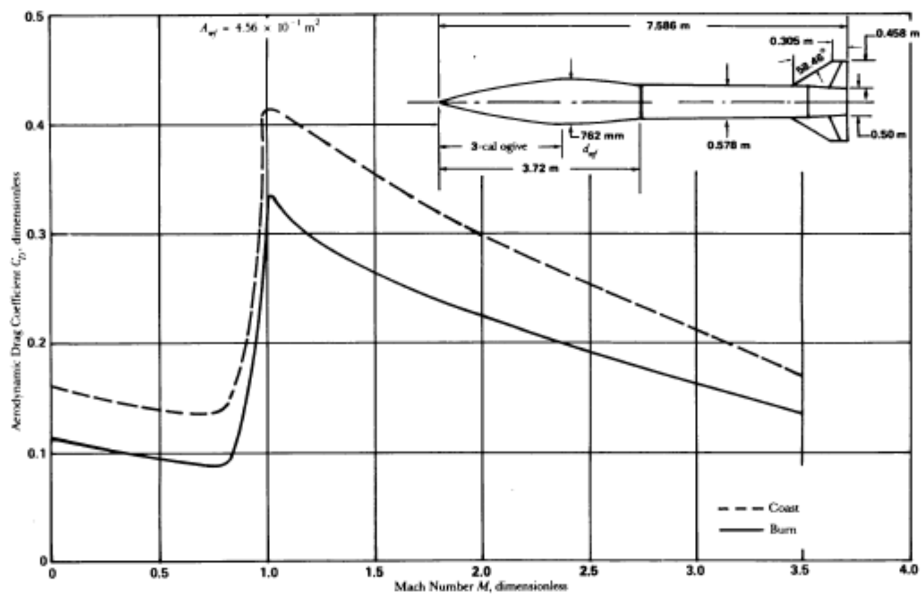



Figure 3-5. Drag Coefficient vs Mach Number—762-mm Rocket

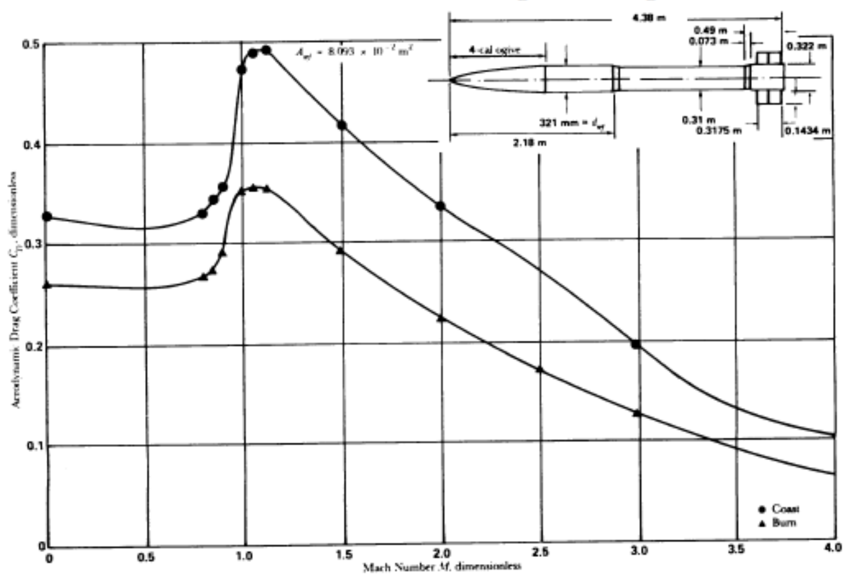
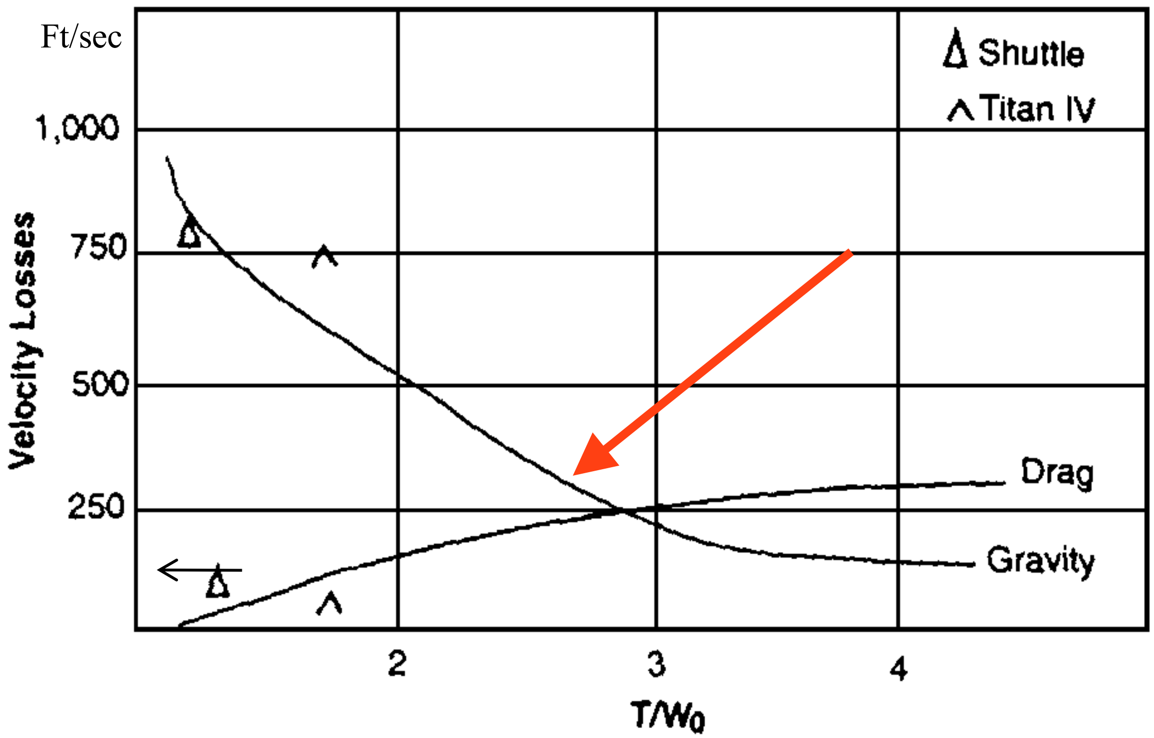



Figure 3-6. Drag Coefficient vs Mach Numbr-321-mm Rocket

But "Rules of Thumb" Apply

# Drag Losses (3)

$$D_{rag} = C_D A_{ref} \frac{1}{2} \rho V^2 \rightarrow \Delta V_{drag} = \sqrt{A_{ref} \int_0^t \frac{C_D \rho V^3}{m} dt} = \sqrt{\int_0^t \frac{\rho V^3}{\beta} dt}$$




Depending
On thrust to-weight
Off of the pad
drag losses
can be significant
During motor burn

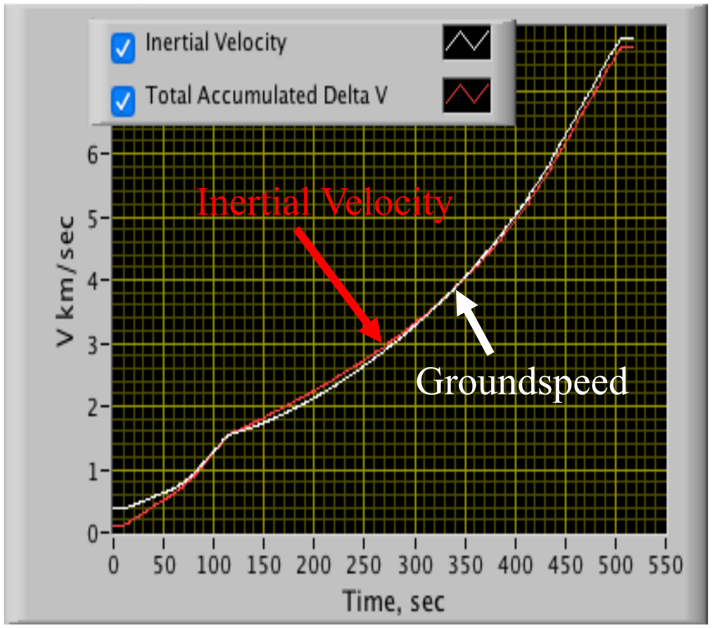
As much as 12-15% of
Potential altitude

***… path dependent!***

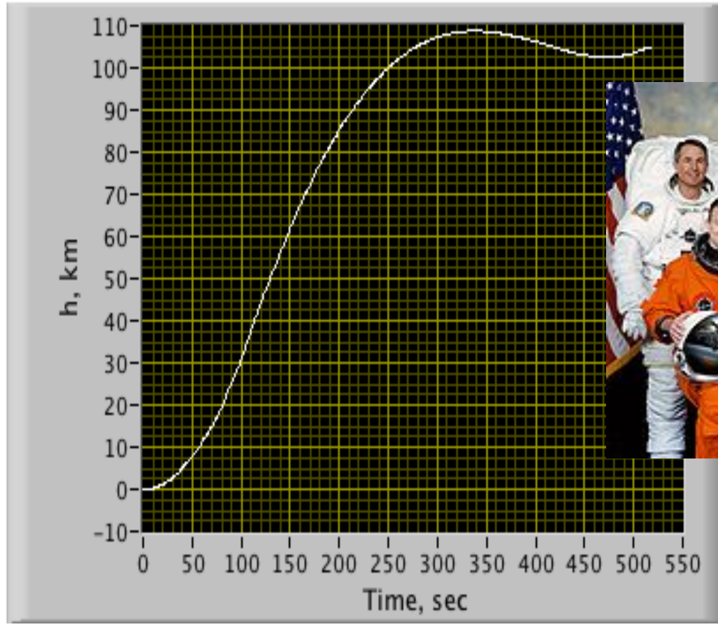
***Must simulate trajectory***

# STS-114 Trajectory Example

Inertial Velocity, Accumulated Kinematic Delta V

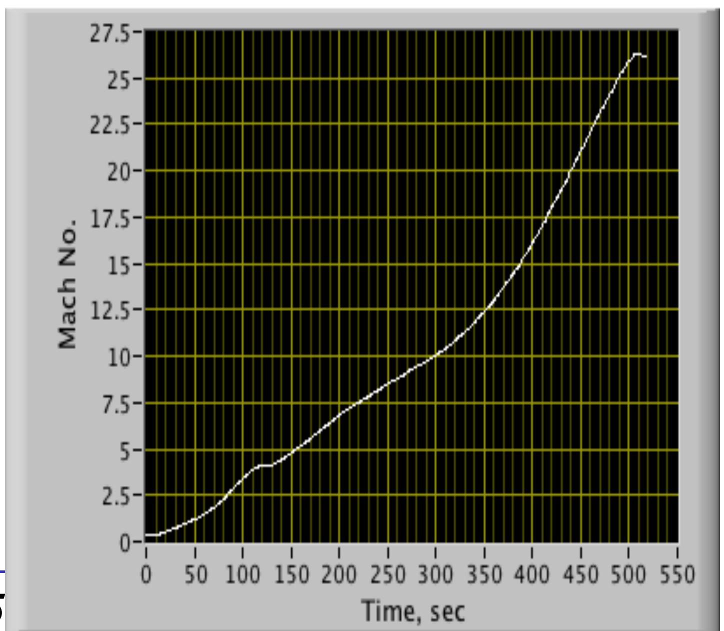



Inertial Altitude from Launch Site

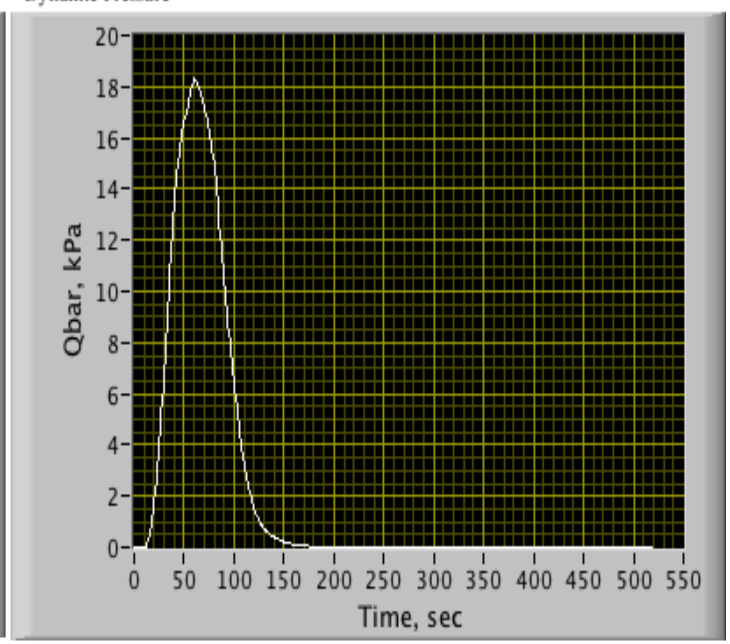



“Return to Flight”

Mach Number




Dynamic Pressure




Launch parameters

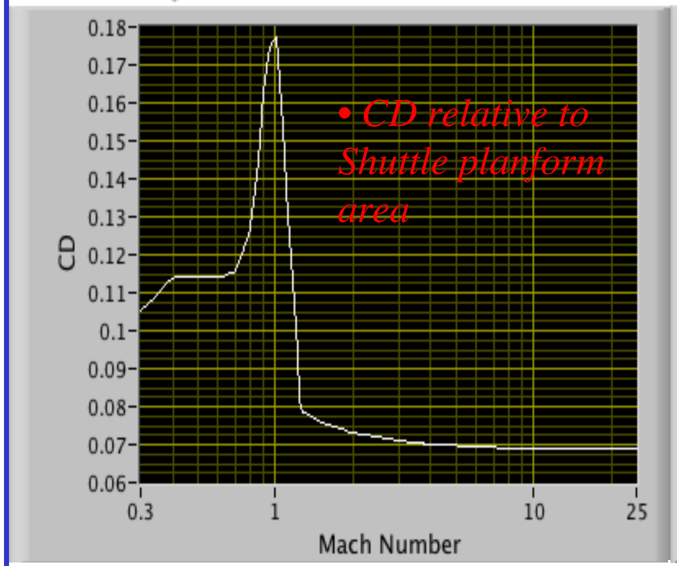
Launch Altitude: 0

Launch Latitude: 28.5

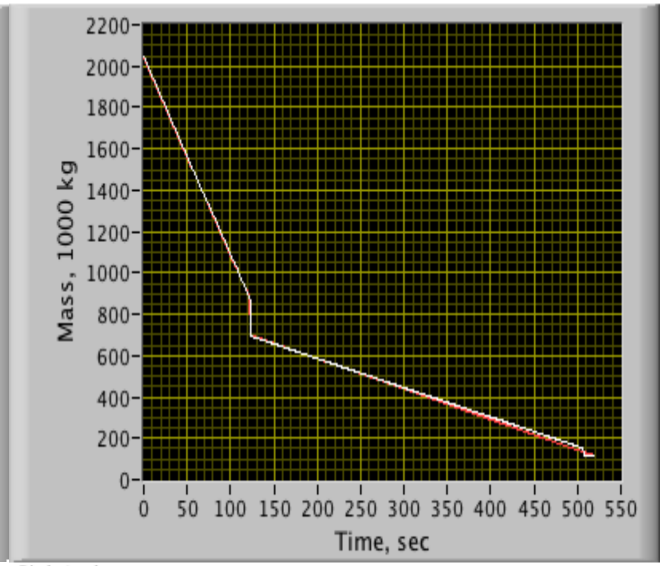
Launch Inclination: 51.6

# STS-114 Trajectory Example (cont'd)

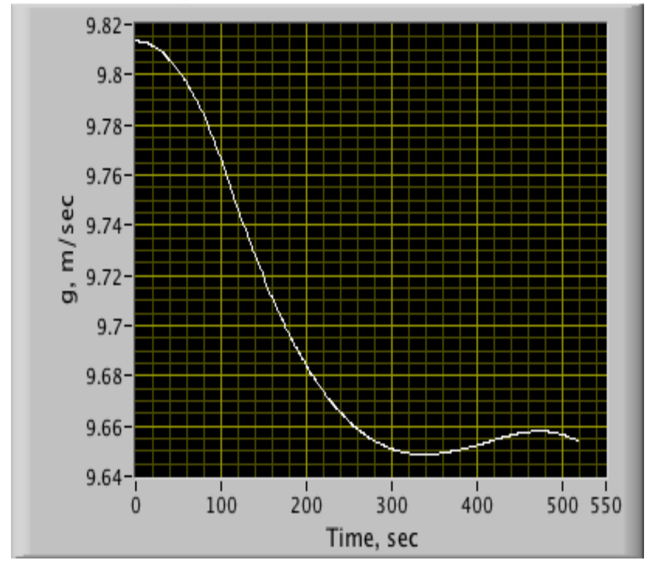
Aero Data Base Drag Coefficient



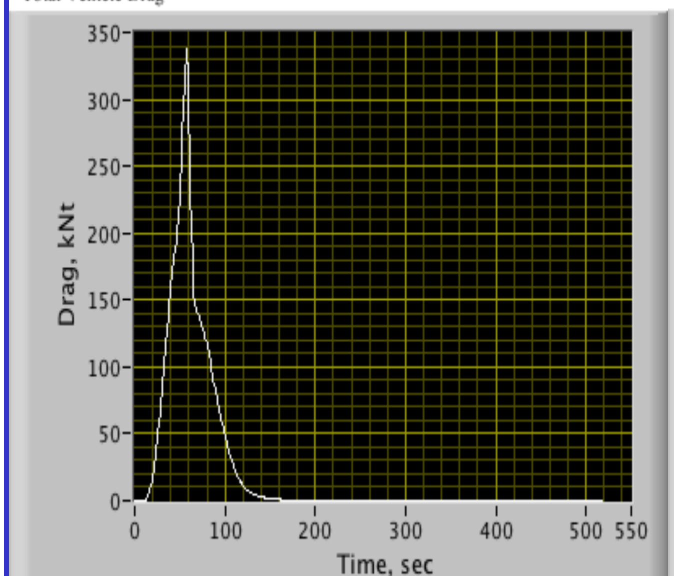

Vehicle Mass



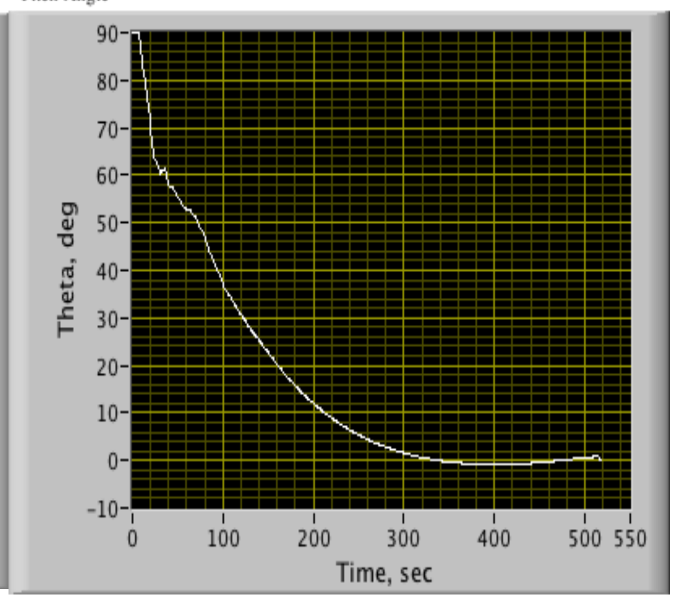

Acceleration of Gravity



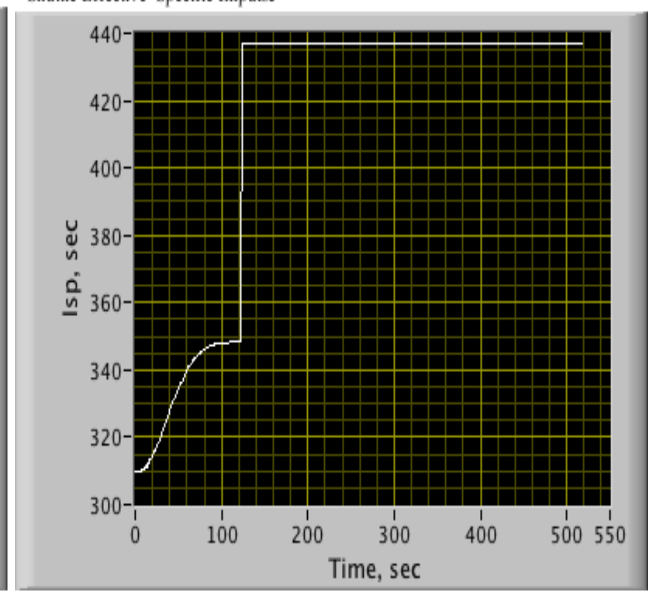

Total Vehicle Drag




Pitch Angle

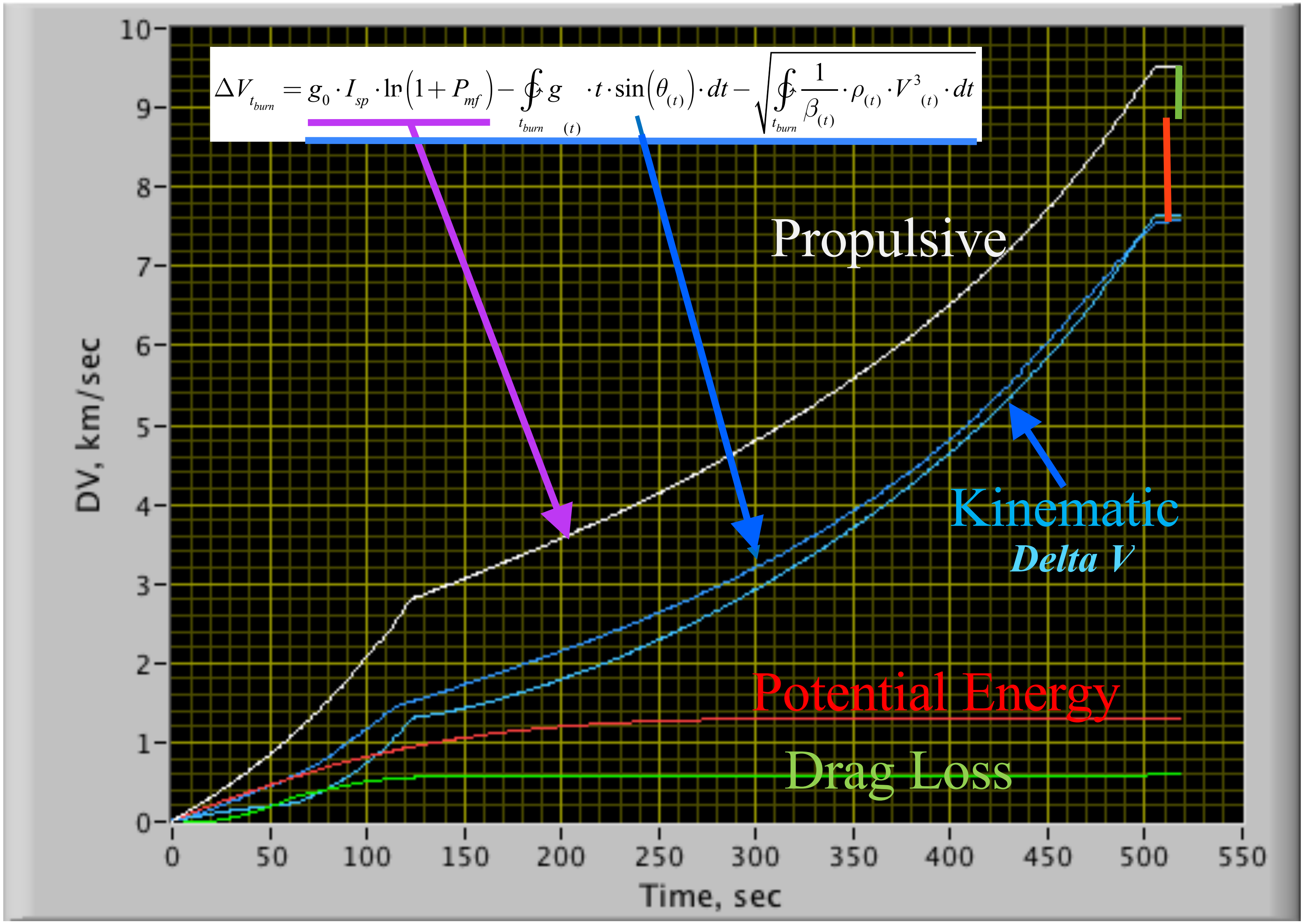



Shuttle Effective Specific Impulse



# STS-114 Trajectory Example (concluded)

Calculated Accumulated Delta V's




Questions??

# Appendix: Integration of the Burnout Out Altitude Equation

$$\rightarrow h_{burnout} = \left[\int_0^{t_{burnout}} V(t)\cdot \sin\theta_{launch}\cdot dt\right]\cdot \sin\theta_{launch}$$

$$\int_0^{t_{burnout}} \left(g_0\cdot I_{sp}\ln\left(\frac{M_{initial}}{M_{initial}-\dot{m}\cdot t}\right) - g_0\cdot\sin\theta_{launch}\cdot t\right)\cdot \sin\theta_{launch}\cdot dt$$

*Simplifying* $\rightarrow$

$$h_{burnout} = \sin\theta_{launch}\cdot \int_0^{t_{burnout}} \left(g_0\cdot I_{sp}\ln\left(\frac{M_{initial}}{M_{initial}-\dot{m}\cdot t}\right) - g_0\cdot\sin\theta_{launch}\cdot t\right)\cdot dt =$$

$$\sin\theta_{launch}\cdot g_0\cdot I_{sp}\cdot \int_0^{t_{burnout}}\left[\ln\left(M_{initial}\right)-\ln\left(M_{initial}-\dot{m}\cdot t\right)\right]\cdot dt - \frac{g_0\cdot\left(\sin\theta_{launch}\cdot t_{\substack{burn\\out}}\right)^2}{2} =$$

$$\sin\theta_{launch}\cdot g_0\cdot I_{sp}\cdot\left\{\ln\left(M_{initial}\right)\cdot t_{\substack{burn\\out}} - \left[\int_0^{t_{burnout}}\left[\ln\left(M_{initial}-\dot{m}\cdot t\right)\right]\cdot dt\right]\right\} - \frac{g_0\cdot\left(\sin\theta_{launch}\cdot t_{\substack{burn\\out}}\right)^2}{2}$$

# Appendix: Integration of the Burnout Out Altitude Equation (2)

*Evaluating the Integral* →

$$h_{burnout} = -\frac{g_0 \cdot \left(\sin\theta_{launch} \cdot t_{\substack{burn\\out}}\right)^2}{2} + \left(\frac{2 \cdot \sin\theta_{launch} \cdot g_0 \cdot I_{sp}}{2 \cdot \dot{m}}\right) \cdot \left(M_{initial} \cdot \ln\left(\frac{M_{initial} - \dot{m} \cdot t_{\substack{burn\\out}}}{M_{initial}}\right) + \dot{m} \cdot t_{\substack{burn\\out}} \cdot \left(1 + \ln\left(\frac{M_{initial}}{M_{initial} - \dot{m} \cdot t_{\substack{burn\\out}}}\right)\right)\right) =$$

$$-\frac{g_0 \cdot \left(\sin\theta_{launch} \cdot t_{\substack{burn\\out}}\right)^2}{2} + \left(\frac{\sin\theta_{launch} \cdot g_0 \cdot I_{sp}}{\dot{m}}\right) \cdot \left(M_{initial} \cdot \ln\left(\frac{M_{initial} - \dot{m} \cdot t_{\substack{burn\\out}}}{M_{initial}}\right) + \dot{m} \cdot t_{\substack{burn\\out}} \cdot \left(1 + \ln\left(\frac{M_{initial}}{M_{initial} - \dot{m} \cdot t_{\substack{burn\\out}}}\right)\right)\right)$$

$$h_{burnout} = -\frac{g_0 \cdot \left(\sin\theta_{launch} \cdot t_{\substack{burn\\out}}\right)^2}{2} + \left(\frac{\sin\theta_{launch} \cdot g_0 \cdot I_{sp}}{\dot{m}}\right) \cdot \left(M_{initial} \cdot \ln\left(\frac{M_{final}}{M_{initial}}\right) + \dot{m} \cdot t_{burn} \cdot \left(1 + \ln\left(\frac{M_{initial}}{M_{final}}\right)\right)\right)$$

# Appendix: Integration of the Burnout Out Altitude Equation (3)

*Rearranging*

$$h_{burnout} = -\frac{g_0 \cdot \left(\sin\theta_{launch} \cdot t_{\substack{burn\\out}}\right)^2}{2} + \left(\sin\theta_{launch} \cdot g_0 \cdot I_{sp}\right) \cdot \left(-\frac{M_{initial}}{\dot{m}} \cdot \ln\left(\frac{M_{initial}}{M_{final}}\right) + t_{\substack{burn\\out}} \cdot \left(1 + \ln\left(\frac{M_{initial}}{M_{final}}\right)\right)\right) =$$

$$-\frac{g_0 \cdot \left(\sin\theta_{launch} \cdot t_{\substack{burn\\out}}\right)^2}{2} + \left(\sin\theta_{launch} \cdot g_0 \cdot I_{sp}\right) \cdot \left(\ln\left(\frac{M_{initial}}{M_{final}}\right)\left(t_{\substack{burn\\out}} - \frac{M_{initial}}{\dot{m}}\right) + t_{\substack{burn\\out}}\right) =$$

$$-\frac{g_0 \cdot \left(\sin\theta_{launch} \cdot t_{\substack{burn\\out}}\right)^2}{2} + \left(\sin\theta_{launch} \cdot g_0 \cdot I_{sp}\right) \cdot t_{\substack{burn\\out}} \left(\ln\left(1 + P_{mf}\right)\left(1 - \frac{M_{initial}}{M_{propellant}}\right) + 1\right)$$

# Appendix: Integration of the Burnout Out Altitude Equation (3)

*Substituting* $\rightarrow \frac{M_{initial}}{M_{final}} = 1 + P_{mf}$ and $1 - \frac{M_{initial}}{\dot{m} \cdot t_{burn\,out}} = 1 - \frac{M_{initial}}{M_{propellant}} = -\frac{1}{P_{mf}}$

$$h_{burnout} = -\frac{g_0 \cdot \left(\sin\theta_{launch} \cdot t_{burn\,out}\right)^2}{2} + \left(\sin\theta_{launch} \cdot g_0 \cdot I_{sp}\right) \cdot t_{burn\,out} \left( \ln\left(1 + P_{mf}\right) \left(1 - \frac{M_{initial}}{M_{propellant}}\right) + 1 \right) =$$

$$-\frac{g_0 \cdot \left(\sin\theta_{launch} \cdot t_{burn\,out}\right)^2}{2} + \left(\sin\theta_{launch} \cdot g_0 \cdot I_{sp}\right) \cdot t_{burn\,out} \left( 1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}} \right)$$

*Rearranging*

$$h_{burnout} = g_0 \cdot \sin\theta_{launch} \cdot t_{burn\,out} \left\{ I_{sp} \cdot \left( 1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}} \right) - \frac{1}{2} \sin\theta_{launch} \cdot t_{burn} \right\}$$