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Relationship of Eccentric Anomaly to the Elliptic Area Integral

- Eccentric anomaly, E, is related to area swept by geometry
- Assuming a starting point is at perigee, the sweep area at some true anomaly, ν is given by the integral

$$A_{[\nu-0]} = \int_0^{\nu} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^{\nu} \left[a \frac{1-e^2}{[1+e\cos(\theta)]} \right]^2 d\theta =$$

$$\frac{1}{2} \frac{a^2\sqrt{1-e^2}}{1+e\cos(\nu)} \left\{ -2\tanh^{-1}\left[\frac{-(1-e)\tan\left(\frac{\nu}{2}\right)}{j\sqrt{1-e^2}} \right] [1+e\cos(\nu)] + e\, j\sqrt{1-e^2}\sin(\nu) \right\}$$

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Relationship of Eccentric Anomaly to the Elliptic Area Integral

(cont'd)

- **The integral can be simplified considerably by subsituting in the expression for *Eccentric anomaly***

$$\tan\left[\frac{\nu}{2}\right] = \sqrt{\frac{1+e}{1-e}}\ \tan\left[\frac{E}{2}\right]$$

- **When this substitution is performed**

$$\frac{-(1-e)\tan\left(\frac{\nu}{2}\right)}{j\sqrt{1-e^2}} = \frac{-(1-e)\sqrt{\frac{1+e}{1-e}}\ \tan\left[\frac{E}{2}\right]}{j\sqrt{1-e^2}} = \frac{j\sqrt{1-e}\sqrt{1+e}\ \tan\left[\frac{E}{2}\right]}{\sqrt{1-e^2}} = j\tan\left[\frac{E}{2}\right]$$

Relationship of Eccentric Anomaly to the Elliptic Area Integral
(cont'd)

- When this result is substituted into the area integral the area integral reduces to

$$A_{[\nu\text{-}0]} =$$

$$\frac{1}{2}\frac{a^2 j\sqrt{1-e^2}}{1+e\cos(\nu)}\left\{-2\tanh^{-1}\left[j\tan\left[\frac{E}{2}\right]\right][1+e\cos(\nu)]+e j\sqrt{1-e^2}\sin(\nu)\right\}$$

- Buuuut...

$$\frac{a^2 j\sqrt{1-e^2}}{1+e\cos(\nu)} = j\frac{a}{\sqrt{1-e^2}}\frac{a\left[1-e^2\right]}{1+e\cos(\nu)} = j\frac{a}{\sqrt{1-e^2}}\, r$$

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Relationship of Eccentric Anomaly to the Elliptic Area Integral

(cont'd)

- When this equation is substituted result is

$$A_{[v-0]} =$$

$$-\frac{1}{2} a \left\{ \frac{2j}{\sqrt{1-e^2}}\, r\, \tanh^{-1}\left[j \tan\left[\frac{E}{2}\right] \right] [1 + e \cos(v)] + r\, e \sin(v) \right\} =$$

$$-\frac{1}{2} a \left\{ \frac{2j}{\sqrt{1-e^2}}\, a\left(1 - e^2\right) \tanh^{-1}\left[j \tan\left[\frac{E}{2}\right] \right] + r\, e \sin(v) \right\} =$$

$$-\frac{1}{2} a \left\{ 2j\, a \sqrt{1-e^2}\, \tanh^{-1}\left[j \tan\left[\frac{E}{2}\right] \right] + r\, e \sin(v) \right\}$$

Relationship of Eccentric Anomaly to the Elliptic Area Integral

(cont'd)

$$A_{[v-0]} = -\frac{1}{2} a \left\{ 2j\, a \sqrt{1-e^2}\, \tanh^{-1}\left[j \tan\left[\frac{E}{2}\right] \right] + r\, e \sin(v) \right\}$$

- But from Trigonometric Identity

$$j \tan\left[\frac{E}{2}\right] = j \frac{\sin\left[\frac{E}{2}\right]}{\cos\left[\frac{E}{2}\right]} = j \frac{\dfrac{e^{j\left[\frac{E}{2}\right]} - e^{-j\left[\frac{E}{2}\right]}}{2j}}{\dfrac{e^{j\left[\frac{E}{2}\right]} + e^{-j\left[\frac{E}{2}\right]}}{2}} = \frac{\dfrac{e^{j\left[\frac{E}{2}\right]} - e^{-j\left[\frac{E}{2}\right]}}{2}}{\dfrac{e^{j\left[\frac{E}{2}\right]} + e^{-j\left[\frac{E}{2}\right]}}{2}} = \frac{\sinh\left[j \frac{E}{2}\right]}{\cosh\left[j \frac{E}{2}\right]} = \tanh\left[j \frac{E}{2}\right]$$

$$\boxed{\tanh^{-1}\left[j \tan\left[\frac{E}{2}\right] \right] = j \left[\frac{E}{2}\right]}$$

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Relationship of Eccentric Anomaly to the Elliptic Area Integral (cont'd)

$$\tanh^{-1}\left[j \tan\left[\frac{E}{2}\right] \right] = j\left[\frac{E}{2}\right]$$

$$A_{[v-0]} = -\frac{1}{2} a \left\{ 2j\ a\sqrt{1-e^2}\ \tanh^{-1}\left[j \tan\left[\frac{E}{2}\right] \right] + r\, e \sin(v) \right\}$$

- And the Area Integral reduces to

$$A_{[v-0]} = -\frac{1}{2} a \left\{ 2j\ a\sqrt{1-e^2}\ j\left[\frac{E}{2}\right] + r\, e \sin(v) \right\} =$$

$$\frac{1}{2} a \left\{ a\sqrt{1-e^2}\ E - r\, e \sin(v) \right\}$$

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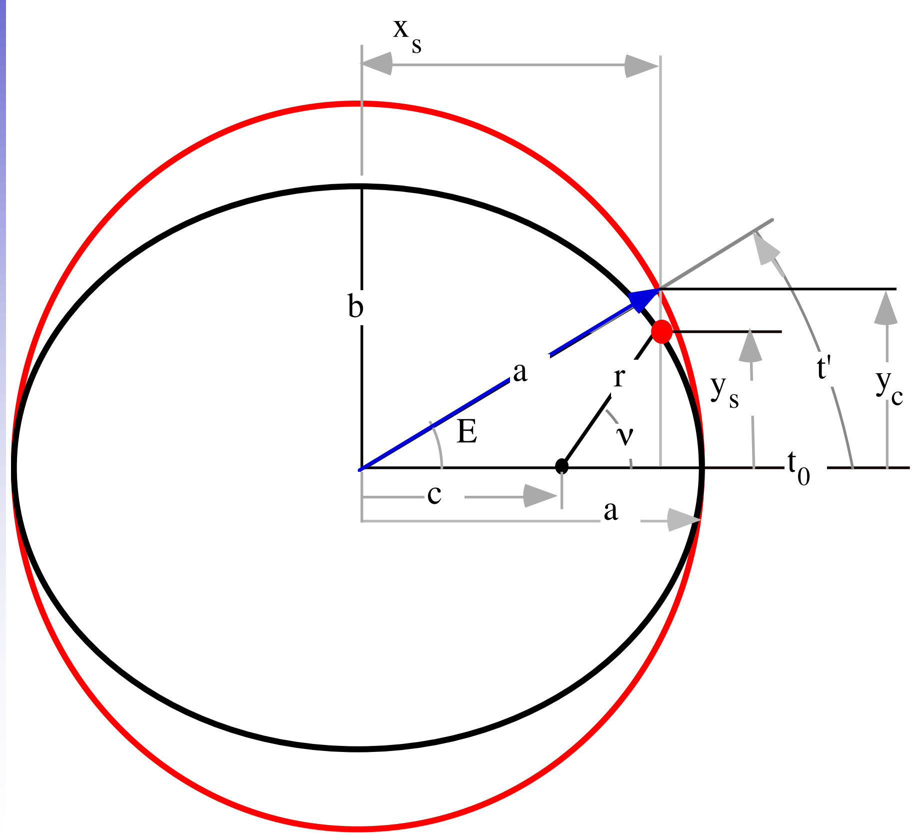
Relationship of Eccentric Anomaly to the Elliptic Area Integral
(cont'd)



Anomaly: True and Eccentric

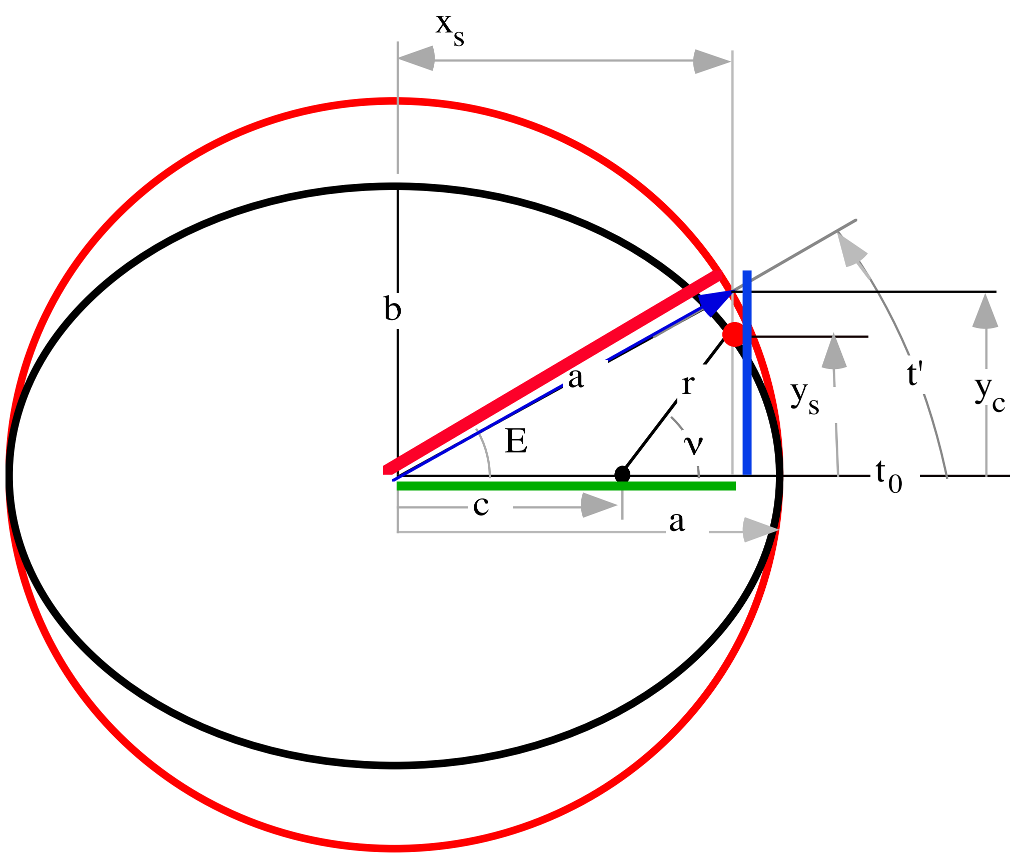
- **From Cartesian forms for Circle and Ellipse**

$$\text{Ellipse:}\left[\frac{x_s^2}{a^2}+\frac{y_s^2}{b^2}=1\right]\Rightarrow$$

$$\left[\frac{x_s^2}{a^2}+\frac{y_c^2}{a^2}=1\right]\text{: Circle}$$

$$\frac{y_s^2}{b^2}=\frac{y_c^2}{a^2}\Rightarrow\boxed{\frac{y_s}{y_c}=\frac{b}{a}}$$

Relationship of Eccentric Anomaly to Elliptic Area Integral
(cont'd)



Anomaly: True and Eccentric

- Evaluating the geometric projections

$$\frac{y_s}{y_c} = \frac{b}{a} = \frac{r\sin(\nu)}{a\sin(E)}$$

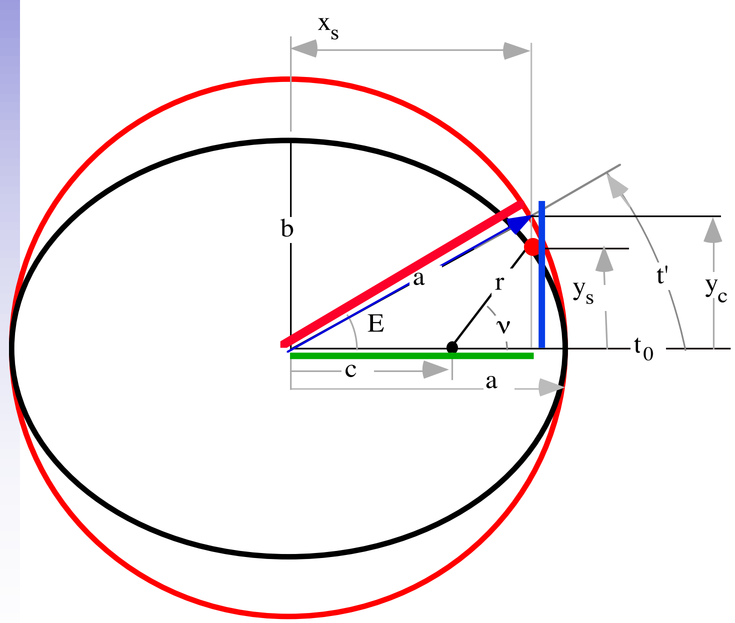
- Rearranging gives

$$\boxed{r\sin(\nu) = b\sin(E)}$$

Relationship of Eccentric Anomaly to Elliptic Area Integral
(cont'd)

- **Substituting into the area integral**

$$\boxed{r \sin(\nu) = b \sin(E)}$$

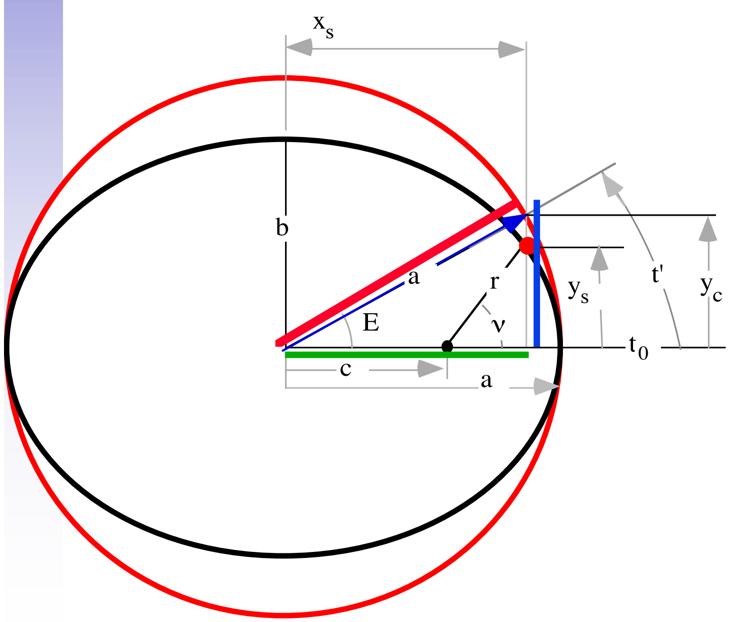


Anomaly: True and Eccentric

$$A_{[\nu - 0]} = \frac{1}{2} a \left\{ a \sqrt{1-e^2}\, E - r\, e \sin(\nu) \right\} =$$

$$\frac{1}{2} a \left\{ a \sqrt{1-e^2}\, E - e\, b \sin(E) \right\}$$

Relationship of Eccentric Anomaly to Elliptic Area Integral
(cont'd)



Anomaly: True and Eccentric

- Simplifying the expression gives

$$A_{[\nu-0]} = \frac{1}{2} a \left\{ a\sqrt{1-e^2}\, E - e\, b \sin(E) \right\} =$$

$$\frac{1}{2} a^2 \left\{ \sqrt{1-e^2}\, E - e\frac{b}{a}\sin(E) \right\} = \frac{1}{2} a^2 \sqrt{1-e^2} \left\{ E - e\sin(E) \right\}$$

$$\frac{b}{a} = \sqrt{1-e^2}$$

Relationship of Eccentric Anomaly to Elliptic Area Integral
(cont'd)

•But from kepler's second law ... total area of Ellipse is:

$$A_{\substack{ellipse \\ total}} = \int_0^{2\pi} \left[\frac{1}{2} r(\nu)^2 \, d\nu \right] = \boxed{a^2 \pi \sqrt{1 - e^2}}$$

$$A_{[\nu - 0]} = \frac{1}{2} a^2 \sqrt{1-e^2} \left(E - e \sin(E) \right) =$$

$$\frac{1}{2\pi} a^2 \pi \sqrt{1-e^2} \left(E - e \sin(E) \right) = \frac{A_{\substack{ellipse \\ total}}}{2\pi} \left\{ E - e \sin(E) \right\}$$

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Relationship of Eccentric Anomaly to Elliptic Area Integral

(concluded)

- Solving for the Area ratio gives:

$$\frac{A_{[\nu - 0]}}{A_{\substack{ellipse \\ total}}} = \frac{1}{2\pi}\left(E - e \sin(E) \right)$$

- Applying Kepler's second law

$$\frac{A_{\nu - 0}}{A_{\substack{ellipse \\ total}}} = \frac{t - t_0}{T}$$

- Defining the *Mean Anomaly* as

$$M_{t-0} = 2\pi \left[\frac{t - t_0}{T}\right]$$

$$2\pi \frac{A_{[\nu - 0]}}{A_{\substack{ellipse \\ total}}} = 2\pi \left[\frac{t - t_0}{T}\right] = M_{t-0} = \left\{ E - e \sin(E) \right\}$$

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FINALLY ... (WHEW!) KEPLER'S EQUATION

$$2\pi\left[\frac{t - t_0}{T}\right] = M_{t-0} = \{ E - e \sin(E) \}$$

- Where ..... t_0 is the time of perapsis passage and

$$\tan\left[\frac{\nu}{2}\right] = \sqrt{\frac{1+e}{1-e}} \tan\left[\frac{E}{2}\right]$$

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