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Appendix II to Lecture 2.2

Solving and Applying Kepler's Equation

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Numerical Solution of Kepler's Equation

- OK, So How we Extract Numerical Solutions of

Kepler's Revenge!

Kepler

$$M_{t-0} = \left\{ E_t - e \sin\left(E_t\right) \right\}$$

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Anatomy of the Solver Algorithm

$$E^{(j+1)} = E^{(j)} + \frac{M - \left[E^{(j)} - e \sin\left(E^{(j)}\right)\right]}{1 - e \cos\left(E^{(j)}\right)}$$

- *True Mean Anomaly*: M
- *Iteration index*: $(j+1)$, (j)
- *Current estimate of mean anomaly*: $E^{(j)} - e \sin\left(E^{(j)}\right)$
- *Refined estimate*: $E^{(j+1)}$
- *Current estimate*: $E^{(j)}$
- *Steepest descent*: $1 - e \cos\left(E^{(j)}\right)$

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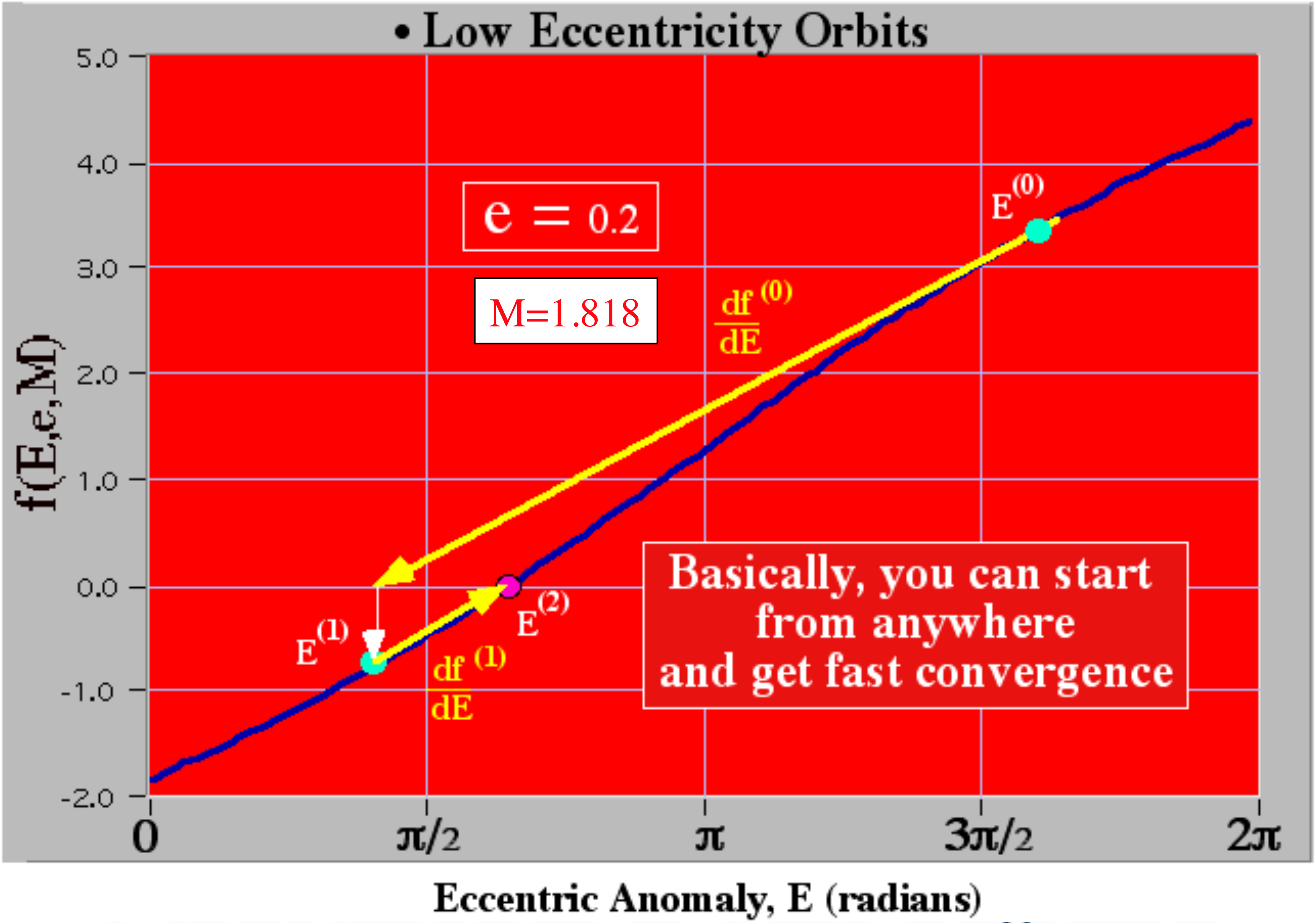
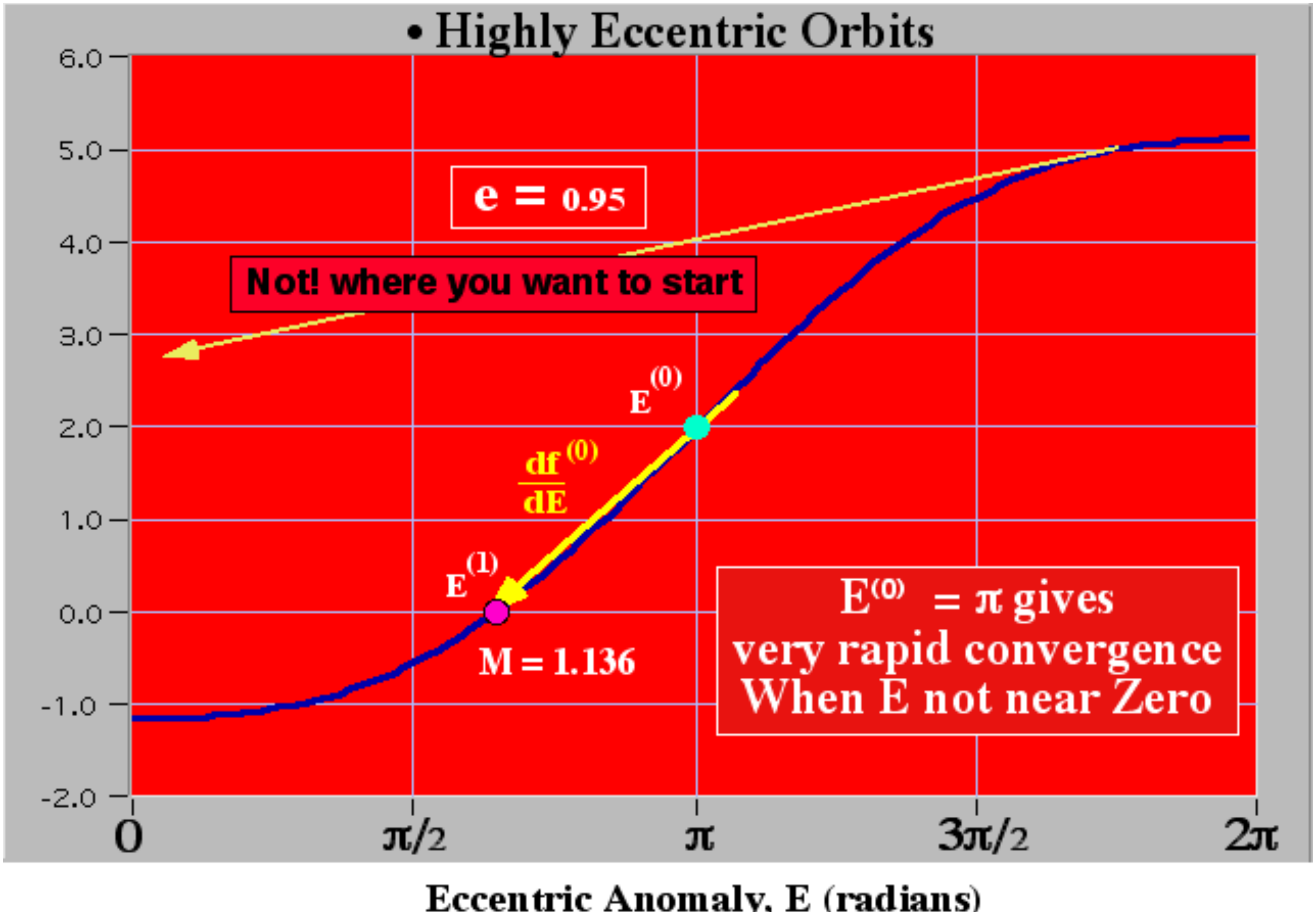
Starting Value

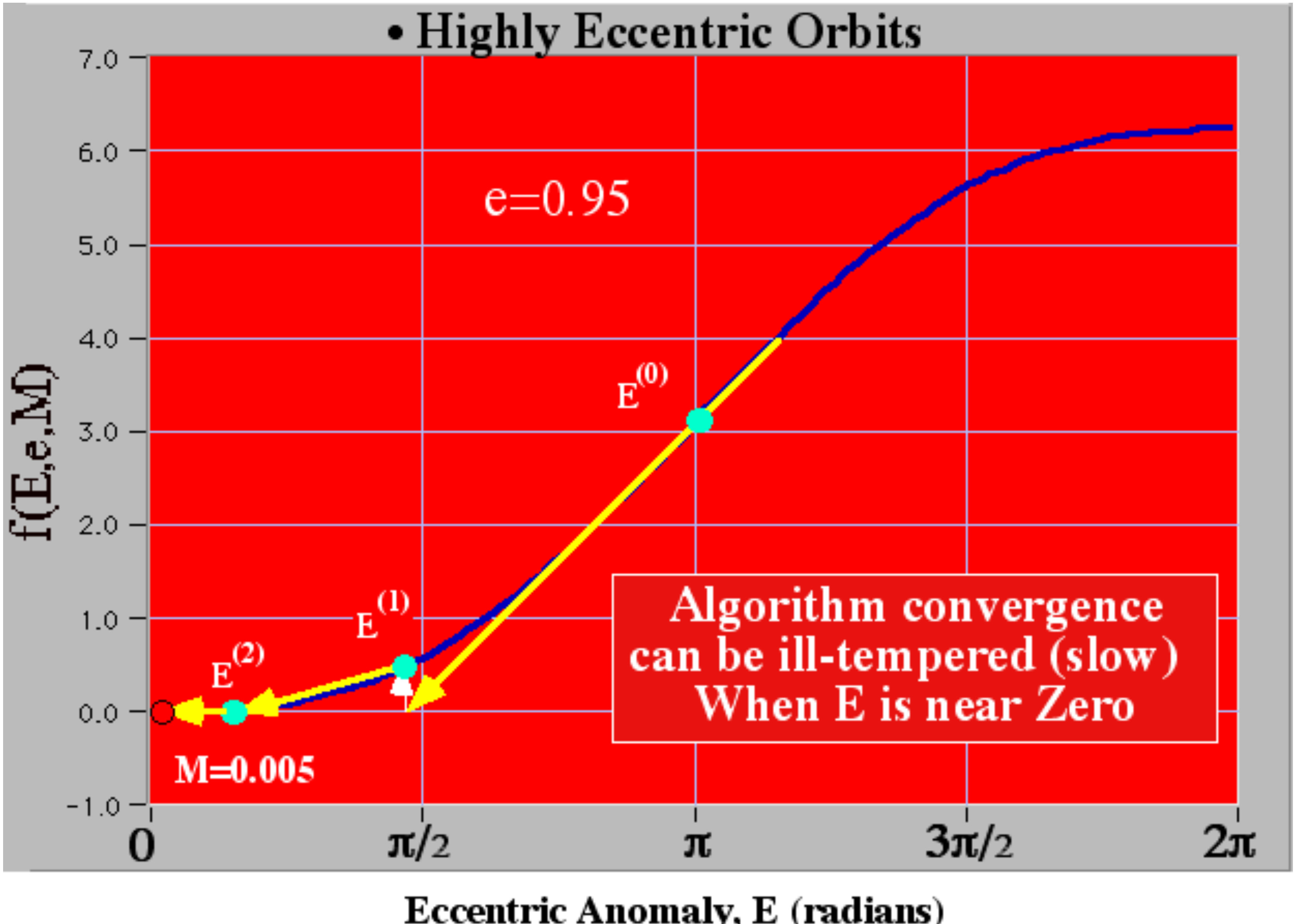


Starting Value (cont'd)



Starting Value (cont'd)

- Highly Eccentric Orbits



Starting Value (cont'd)

• Highly Eccentric Orbits

- Why is convergence "ill-tempered" (slow) near perigee?

$$E^{(j+1)} = E^{(j)} + \frac{M - \left[E^{(j)} - e \sin\left(E^{(j)}\right)\right]}{1 - e \cos\left(E^{(j)}\right)}$$

$$E^{(j+1)} = .11 + \frac{0.0051583 - \left[.11 - 0.95 \sin\left(.11\right)\right]}{1 - 0.95 \cos\left(.11\right)} =$$

$$0.11 + \frac{-.0005523}{.05574} = .10009154$$

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Starting Value (cont'd)

- Highly Eccentric Orbits

Why is convergence "ill-tempered" (slow) near perigee?

- Normalized Change convergence criterion

$$\left| \frac{E^{(j+1)} - E^{(j)}}{E^{(j+1)} + E^{(j)}} \right| = 2 \left| \frac{.10009154 - 0.11}{.10009154 + 0.11} \right| = .0943$$

only a 9.4% change

... even though we are only 0.00009154 radians (.0052°) off in our estimate

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Starting Value (concluded)

• Since we are solving Kepler's Equation for each and every Time point in our propagation of the orbit … convergence speed becomes critically important

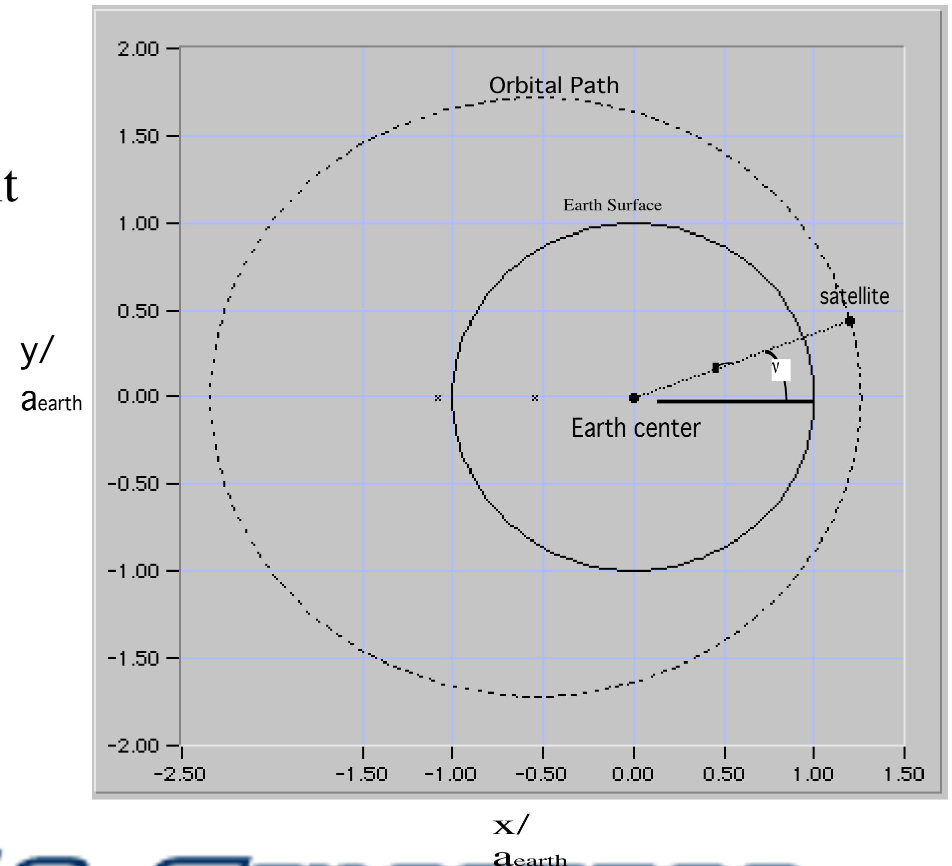
• Clearly, since convergence near the Perigee of an eccentric orbit can be a bit Of a problem …

Its clear that we need ….

• a better way to start each iteration

• Next time …"Startup Algorithms"

Figure 5: Sample Orbit Calculation using Kepler's Equation



OK … so we need a better startup algorithm

• So where do we start?… with a really Simple ad-hoc solution …..

• When M ~ 0, we simply "kick it off zero" by adding or subtracting **… e** **(Vallado Algorithm)**

$$M = E - e \sin(E)$$

$$\Downarrow$$

$$\text{Startup Method 1}: \begin{Bmatrix} 0 \le M \le \pi: \; E_0 = M + e \\ \pi < M < 2\pi: \; E_0 = M - e \end{Bmatrix}$$

Derivation of Vallado Startup Algorithm*

- Regroup Kepler's Equation and Expand in a Taylor's series

$$M = E - e \sin[E] \Rightarrow E(M) = M + e \sin[E]$$

$$E(M) = E(0) + M \left[\frac{\partial [M + e \sin[E]]}{\partial M} \right]_{|0} + O(M^2)$$

- But

$$\text{when } M = 0 \Rightarrow E(0) = e \sin[E(0)] \Rightarrow E(0) = 0$$

* Proof is Courtesy of Brian M. Moore
CPT, US Army, Naval Postgraduate School
Space Systems Engineering, bmmoore@nps.navy.mil

Derivation of Vallado Startup Algorithm*(cont'd)

- and the Taylor's series reduces to

$$E(M) = M\left[1 + e\cos[E]\frac{\partial E}{\partial M}\right]_{|0} + O(M^2)$$

- But

$$\frac{\partial\left[M + e\sin[E]\right]}{\partial M} = \frac{\partial E}{\partial M} \Rightarrow \frac{\partial E}{\partial M} = 1 + e\cos[E]\frac{\partial E}{\partial M}$$

$$\frac{\partial E}{\partial M} = \frac{1}{1 - e\cos[E]} \Rightarrow \boxed{1 + e\cos[E]\frac{\partial E}{\partial M} = 1 + \frac{e\cos[E]}{[1 - e\cos[E]]}}$$

* Proof is Courtesy of Brian M. Moore
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Derivation of Vallado Startup Algorithm*(cont'd)

- Solving for the derivative

$$\frac{\partial E}{\partial M} = \frac{1}{1 - e\cos[E]} \Rightarrow \boxed{1 + e\cos[E]\,\frac{\partial E}{\partial M} = 1 + \frac{e\cos[E]}{[1 - e\cos[E]]}}$$

- and the Taylor's series further reduces to

$$E = \quad M\left[1 + \frac{e\cos[E]}{[1 - e\cos[E]]}\right]_{|0} + O(M^2)$$

$$E = \quad M\left[1 + \frac{e}{[1 - e]}\right] + O(M^2)$$

* Proof is Courtesy of Brian M. Moore
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Space Systems Engineering, bmmoore@nps.navy.mil

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Derivation of Vallado Startup Algorithm*(cont'd)

- Truncating the series after first order

$$\hat{E} \approx M + e\left[\frac{M}{[1-e]}\right] \Rightarrow \begin{bmatrix} 0 \le M < \pi & \Rightarrow \frac{M}{[1-e]} > 0 \\ -\pi \le M < 0^{(-)} & \Rightarrow \frac{M}{[1-e]} < 0 \end{bmatrix}$$

- and now the "*Vallado approximation*"

$$\boxed{\left|\frac{M}{[1-e]}\right| \approx 1} \Rightarrow \begin{bmatrix} 0 \le M < \pi & \Rightarrow \hat{E} \approx M + e \\ -\pi \le M < 0^{(-)} & \Rightarrow \hat{E} \approx M - e \end{bmatrix}$$

* Proof is Courtesy of Brian M. Moore
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Space Systems Engineering, bmmoore@nps.navy.mil

Derivation of Vallado Startup Algorithm*(cont'd)

• "*Vallado approximation*"

$$\left|\frac{M}{[1-e]}\right| \approx 1$$

• Startup is most accurate for M ~ 1-e -> for highly eccentric Orbits … this value for M also happens to be near Perigee

• For low eccentricity orbits … we've already seen that It really doesn't matter where you start

• But how about intermediate eccentricity orbits, I.e. e=0.5
M -> 1-e = 0.5 radians ~28.65° ... ισ τηισ γοοδ ενουγη?

* Proof is Courtesy of Brian M. Moore
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Space Systems Engineering, bmmoore@nps.navy.mil

Linear Interpolation as a Startup Algorithm

$$M_{t-0} = \{ E_t - e \sin(E_t) \}$$

• Exploit the fact that Kepler's equation is *explicit* in the "forward direction"

* Proof is Courtesy of Brian M. Moore
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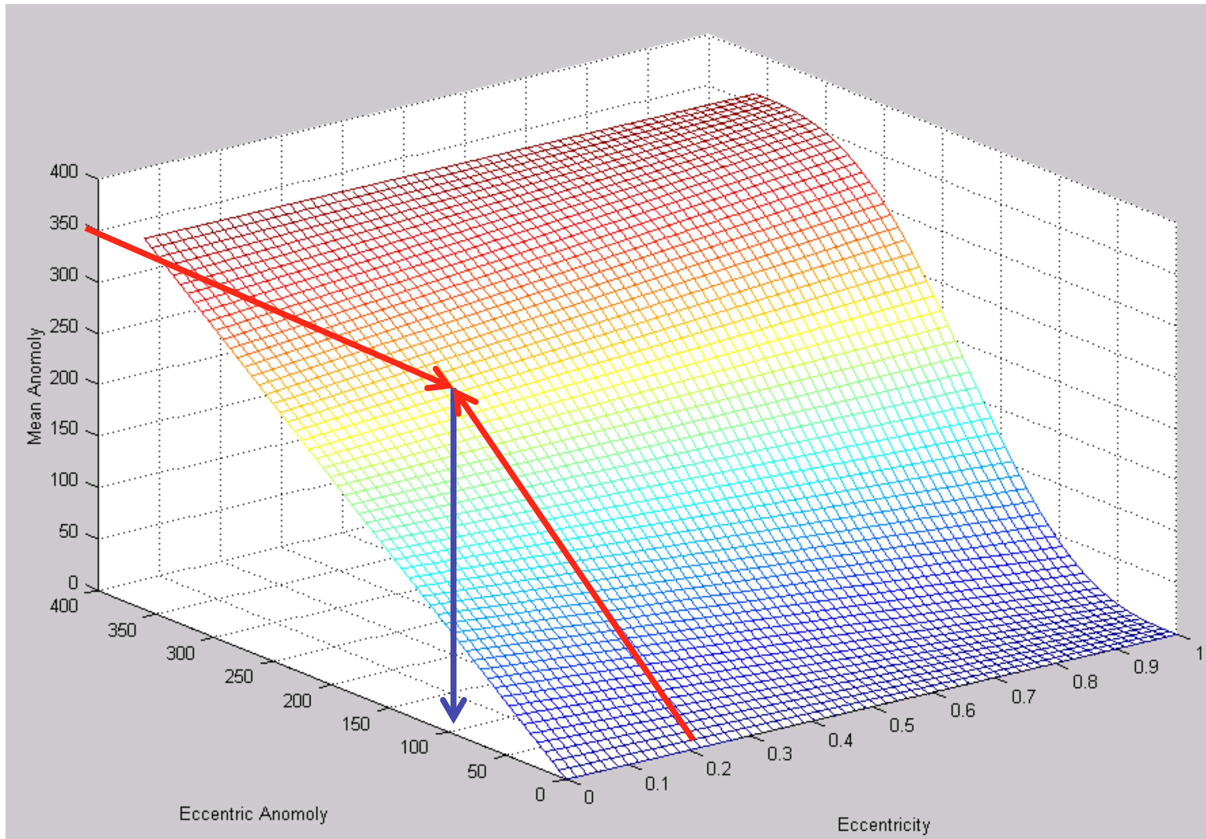
Linear Interpolation as a Startup Algorithm (cont'd)

- Generate 3-D Space of {M, e, E} that Satisfy Kepler's Equation

- Use 2-D Interpolation Of {M, e, E} To Generate Starting Guess



* Proof is Courtesy of Brian M. Moore
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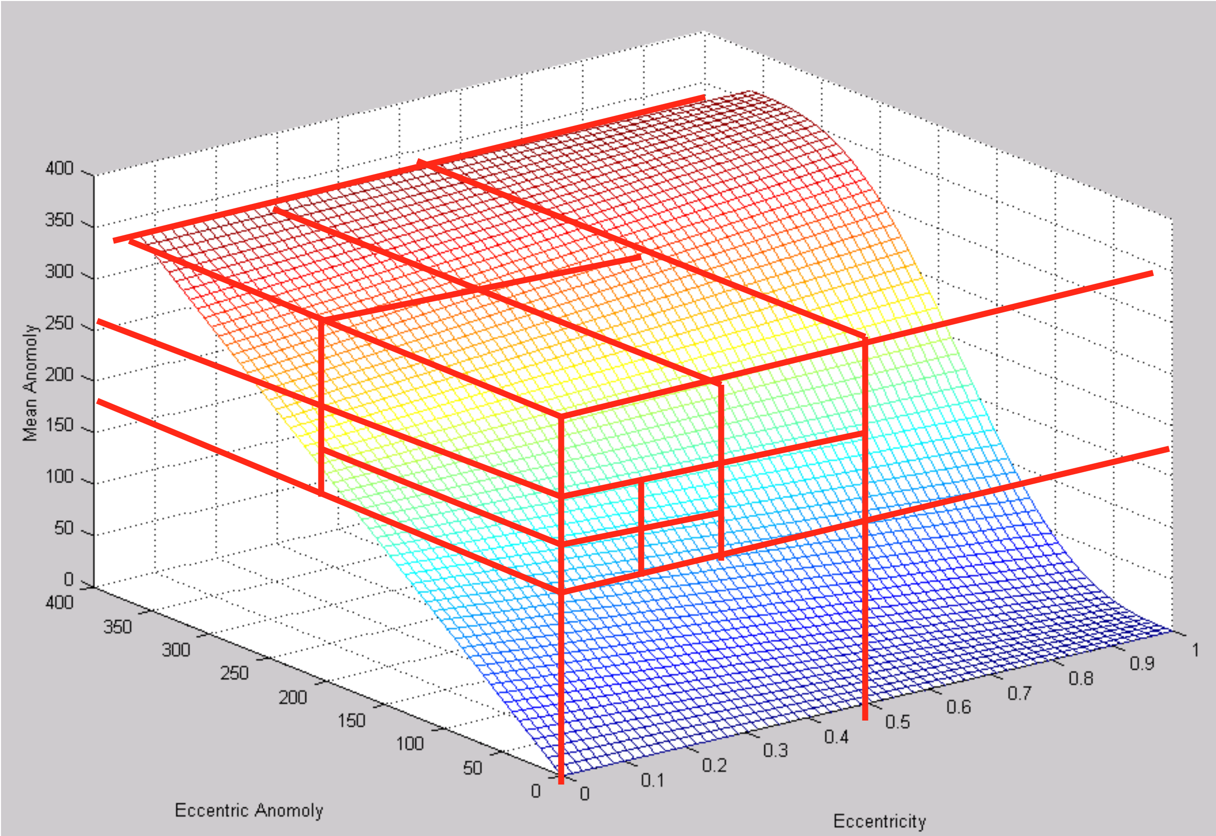
Linear Interpolation as a Startup Algorithm (cont'd)

- Using a "quaternary" Search pattern Interpolation is fast

- But stored Data base is Rather large

- Mesh Mesh Density fixed By maximum Eccentricity of Orbits to be analyzed



* Proof is Courtesy of Brian M. Moore
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OK … can we develop another startup algorithm?

- **So where do we start?… with the** *Fourier series solution* **of Kepler's Equation …..***

*Dörrie, H. ``The Kepler Equation.'' 100 Great Problems of Elementary Mathematics: Their History and Solutions. New York: Dover, pp. 330-334, 1965.

Fourier Series Solution

$$M = E - e\ \sin(E)$$

$$\Downarrow$$

$$E = M + \sum_{n=1}^{\infty}\left[\frac{2}{n} J_n(n\,e)\,\sin(n\,M)\right]$$

$$J_n(n\,e) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(n+j)!}\left(\frac{n\,e}{2}\right)^{n+2j} \Rightarrow \left\{\begin{matrix}\text{Bessel Function}\\ \text{of the First Kind}\end{matrix}\right\}$$

- **Very inefficient way to solve Kepler's equation**

Fourier Series Solution (cont'd)

- Buuuuttt … if we expand out the terms in the series

$$E \approx M + \left[2\, J_1(e) \sin(M) \right] + \left[J_2(2e) \sin(2\,M) \right] + \left[\frac{2}{3} J_3(3e) \sin(3\,M) \right] + \ldots$$

$$J_1(e) \approx \frac{e}{2} - \frac{e^3}{16} + \frac{e^5}{384} + \ldots$$

$$J_2(e) \approx \frac{e^2}{2} - \frac{e^4}{6} + \frac{e^6}{48} + \ldots$$

$$J_3(e) \approx \frac{9\,e^3}{16} - \frac{81 e^5}{256} + \frac{729\,e^7}{10240} + \ldots$$

Fourier Series Solution (cont'd)

- Substituting in for the J_i's and collecting terms:

$$E \approx M + \left(e - \frac{e^3}{8} + \frac{e^5}{192}\right)\sin(M) +$$

$$\left(\frac{e^2}{2} - \frac{e^4}{6} + \ldots\right)\sin(2M) +$$

$$\left(\frac{3e^3}{8} - \frac{27e^5}{128} + \ldots\right)\sin(3M) + \ldots$$

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Near the Troublesome point at perigee

- at perigee … sin(μ M) ~ μ M, { μ = 0, 1, 2, ...}

$$E_0 \approx M + \left(e - \frac{e^3}{8} + \dots\right) M +$$

$$\left(\frac{e^2}{2} - \frac{e^4}{6} + \dots\right) 2\, M +$$

$$\left(\frac{3\, e^3}{8} - \frac{27 e^5}{128} + \dots\right) 3\, M + \dots \approx$$

$$M\left[1 + e + e^2 + e^3 + \dots\right]$$

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Fourier Series Method
Near Perigee

Startup

Method 2

$$E_0 \approx M\left[1 + e + e^2 + e^3\right]$$

• Higher Order Method … but still simple to implement

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Taylor Series Method

- **Kepler's Equation**

$$E - e \sin(E) - M = 0$$

- **Expand "Sine" term in Taylor's series**

$$\sin(E) = E - \frac{E^3}{3!} + \frac{E^5}{5!} \ldots$$

- **Truncate : At '5th" Order term**

Fourth-order accurate

Taylor Series Method (cont'd)

- Substitute Truncated Taylor's Series into Kepler's Equation

$$\hat{E}^{(0)} - e\left[\hat{E}^{(0)} - \frac{\left(\hat{E}^{(0)}\right)^3}{3!}\right] - M = 0$$

- Solve for E^3 term ... to give startup algorithm

$$\left(\hat{E}^{0}\right)^3 + \frac{6\,(1-e)}{e}\,\hat{E}^{0} - \frac{6\,M}{e} = 0$$

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Taylor Series Method (cont'd)

$$\left(\hat{E}^0\right)^3 + \frac{6\,(1-e)}{e}\,\hat{E}^0 - \frac{6\,M}{e} = 0$$

- "Resolvent Cubic" Equation has Closed-form Solution

  .... two complex roots (complex root 1)

$$-\frac{3 * 2^{1/3}\left(+\cdot j\sqrt{3}\right)(-1+e)}{\left(162\,e^2 M + \sqrt{-23328\,(-1+e)^3 e^3 + 26244\,e^4 M^2}\right)^{1/3}} - \frac{\left(1 + j\sqrt{3}\right)\left(162\,e^2 M + \sqrt{-23328\,(-1+e)^3 e^3 + 26244\,e^4 M^2}\right)^{1/3}}{6 * 2^{1/3}\,e}$$

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Taylor Series Method (cont'd)

$$\left(\hat{E}^0\right)^3 + \frac{6\,(1-e)}{e}\,\hat{E}^0 - \frac{6\,M}{e} = 0$$

- "Resolvent Cubic" Equation has Closed-form Solution

  .... two complex roots (complex root 2)

$$-\frac{3 * 2^{1/3}\left(1 - j\sqrt{3}\right)(-1+e)}{\left(162\,e^2 M + \sqrt{-23328\,(-1+e)^3 e^3 + 26244\,e^4 M^2}\right)^{1/3}} - \frac{\left(1 + j\sqrt{3}\right)\left(162\,e^2 M + \sqrt{-23328\,(-1+e)^3 e^3 + 26244\,e^4 M^2}\right)^{1/3}}{6 * 2^{1/3}\,e}$$

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Taylor Series Method (cont'd)

$$\left(\hat{E}^0\right)^3 + \frac{6(1-e)}{e}\hat{E}^0 - \frac{6M}{e} = 0$$

- "Resolvent Cubic" Equation has Closed-form Solution

....and one REAL root ... which is the one we want ...

Startup Method 3:

$$\hat{E}^{(0)} = \frac{-2e + 2e^2 + \left(3e^2 M + \sqrt{e^3\left(-8(-1+e)^3 + 9eM^2\right)}\right)^{2/3}}{e\left(3e^2 M + \sqrt{e^3\left(-8(-1+e)^3 + 9eM^2\right)}\right)^{1/3}}$$

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OK .... How Good are these Startup Values

- From Vallado (Ad-Hoc Solution) ... the Traditional Startup Assumption is

$$\hat{E}^{(0)} = \begin{matrix} M + e & (0 \le M \le \pi) \\ M - e & (\pi \le M \le 2\pi) \end{matrix}$$

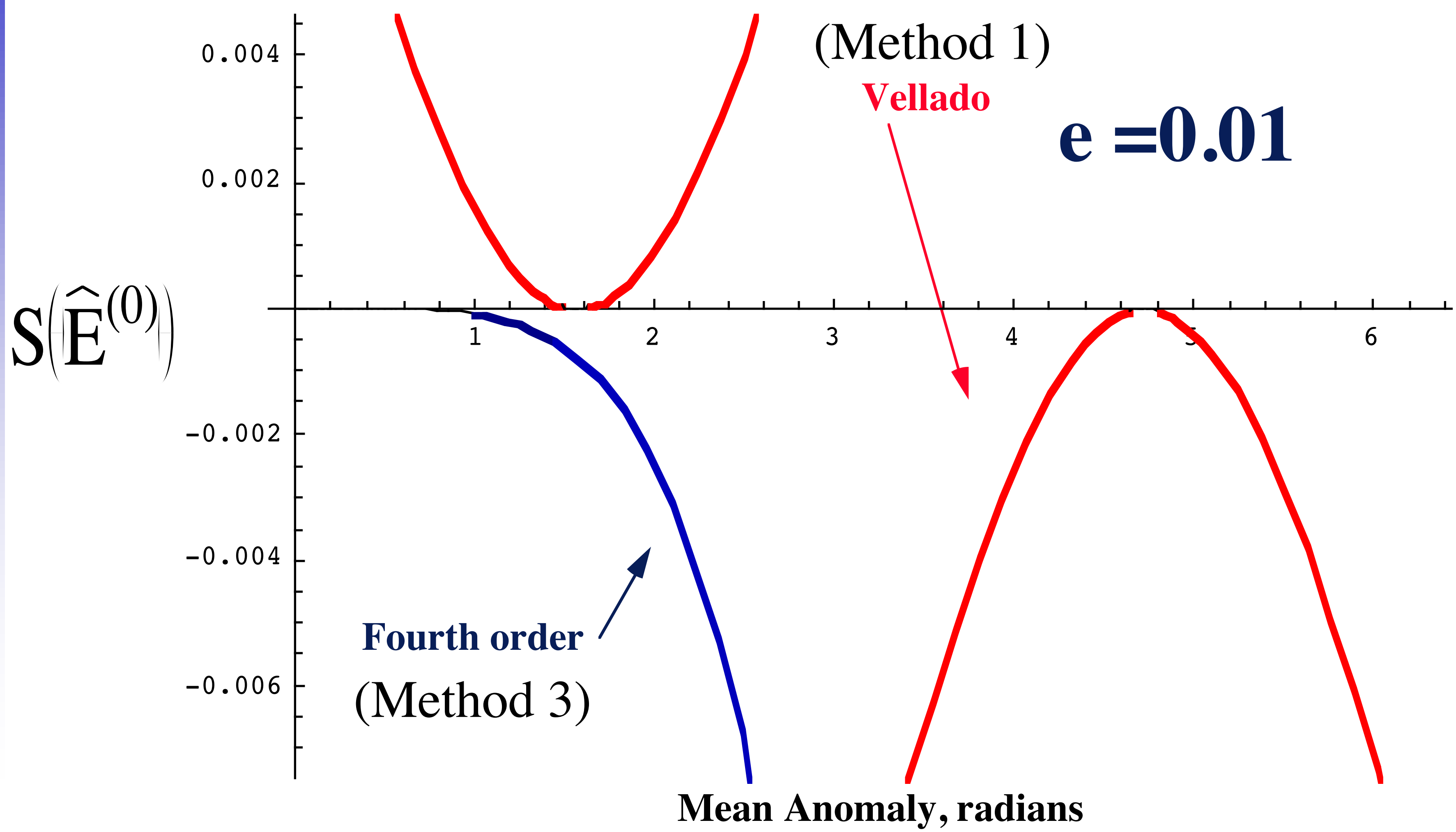
Startup Accuracy Metric

- The Closer $S\left(\hat{E}^{(0)}\right)$ is to "zero", then the more accurate the startup value

$$S\left(\hat{E}^{(0)}\right) \equiv \hat{E}^{(0)} - e \sin\left(\hat{E}^{(0)}\right) - M$$

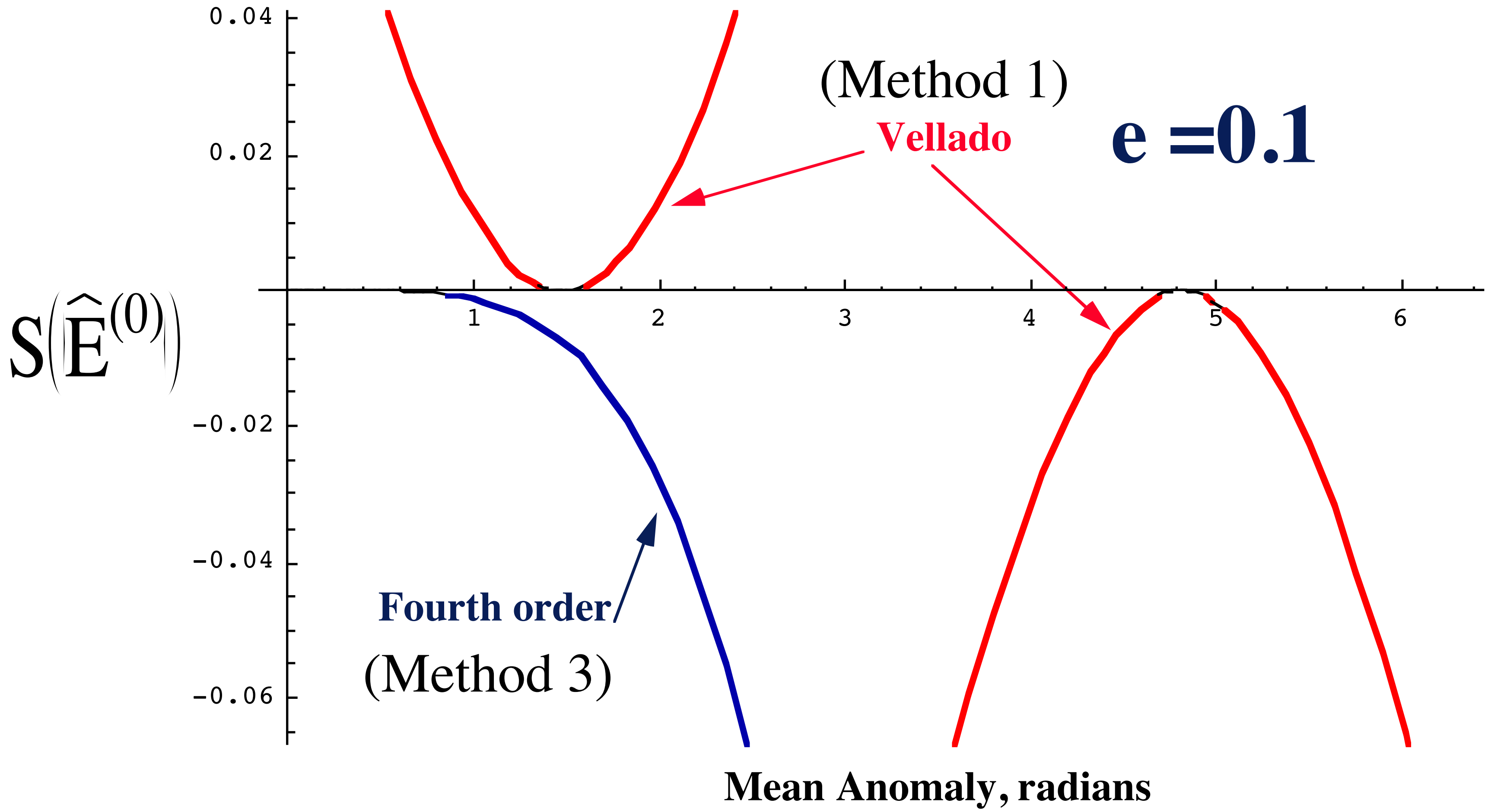
$$\boxed{\text{when } \hat{E}^{(0)} = E \Rightarrow S\left(\hat{E}^{(0)}\right) = 0}$$

Comparison of Start-up Values

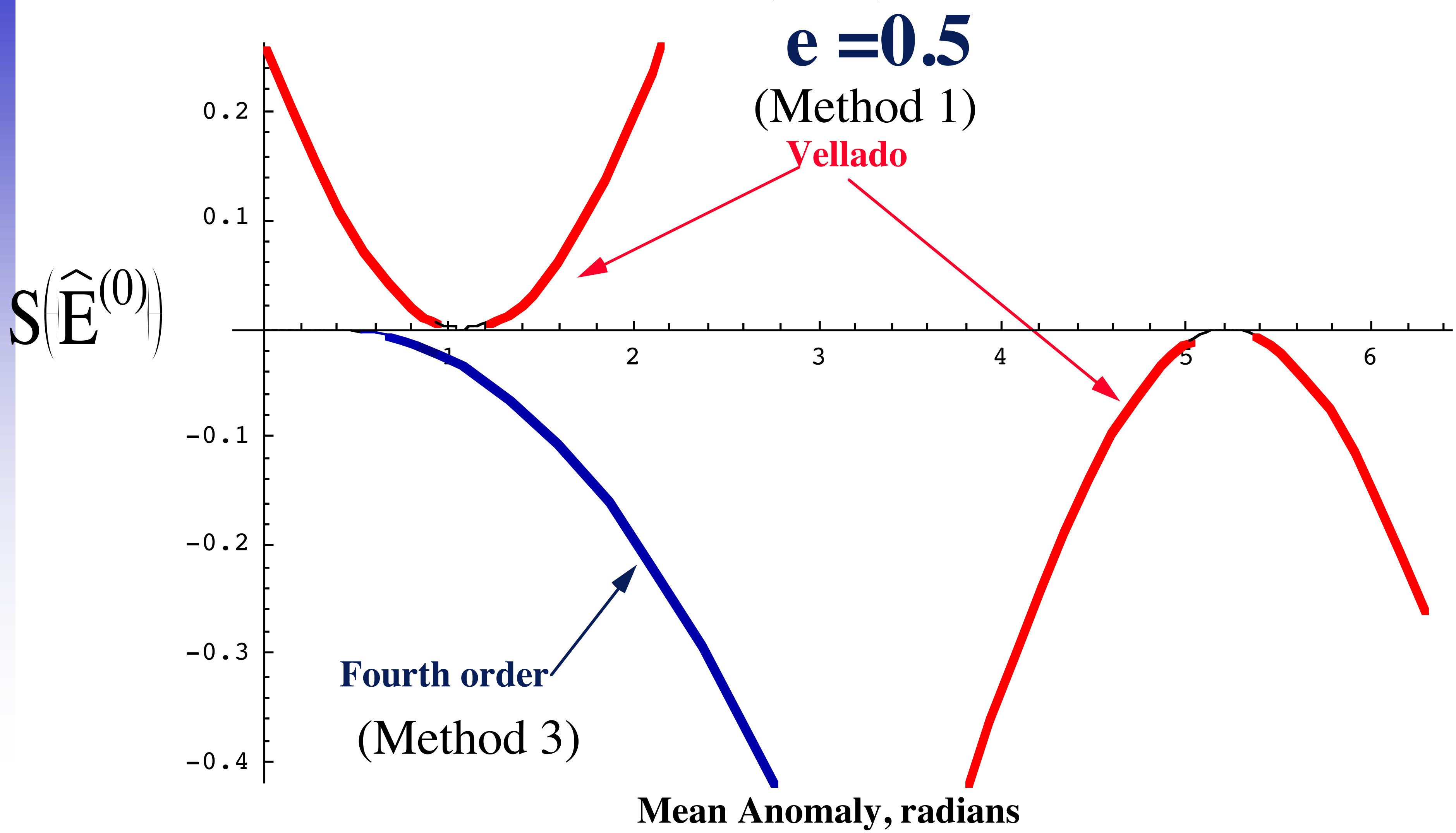


Comparison of Start-up Values

(cont'd)

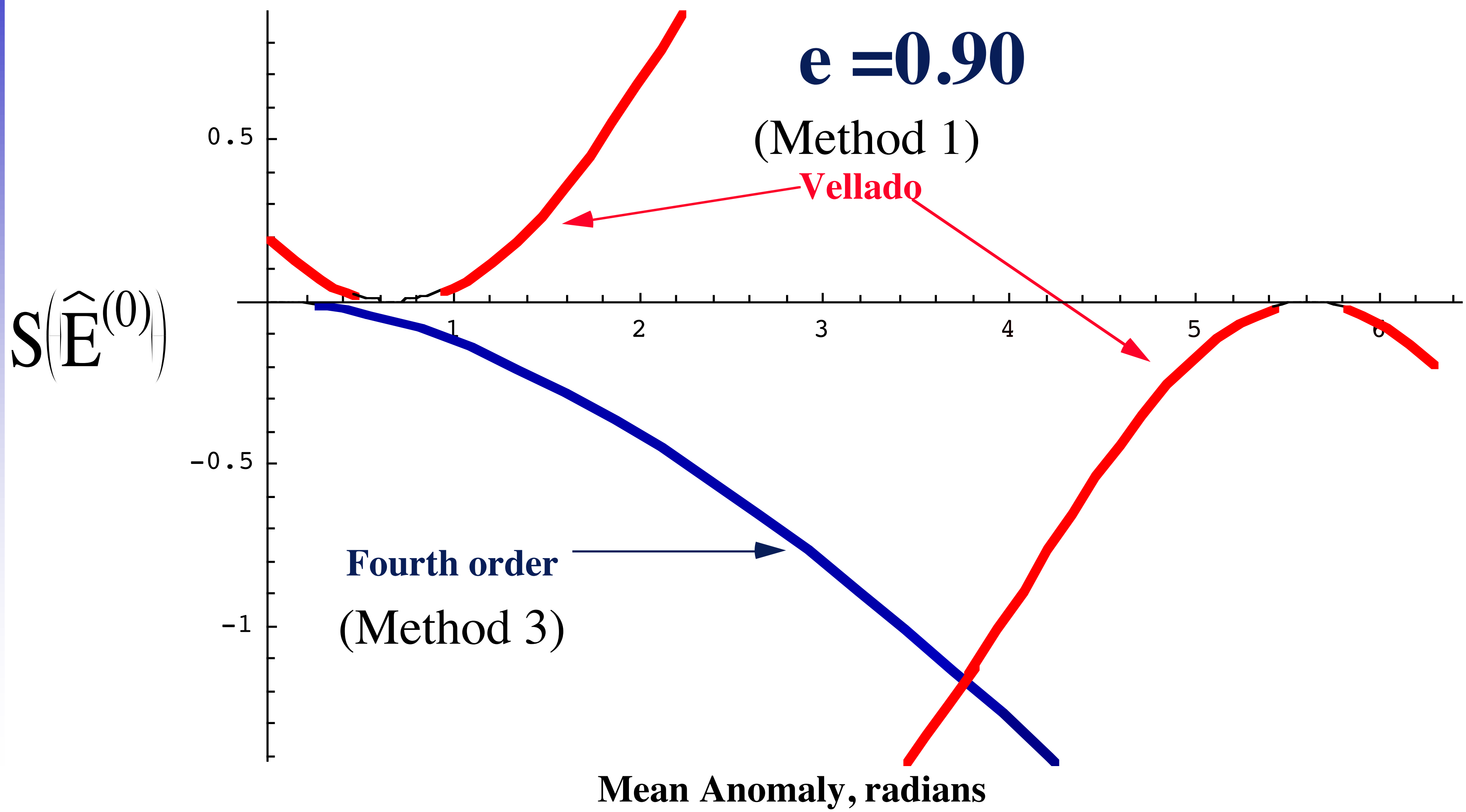


Comparison of Start-up Values (cont'd)

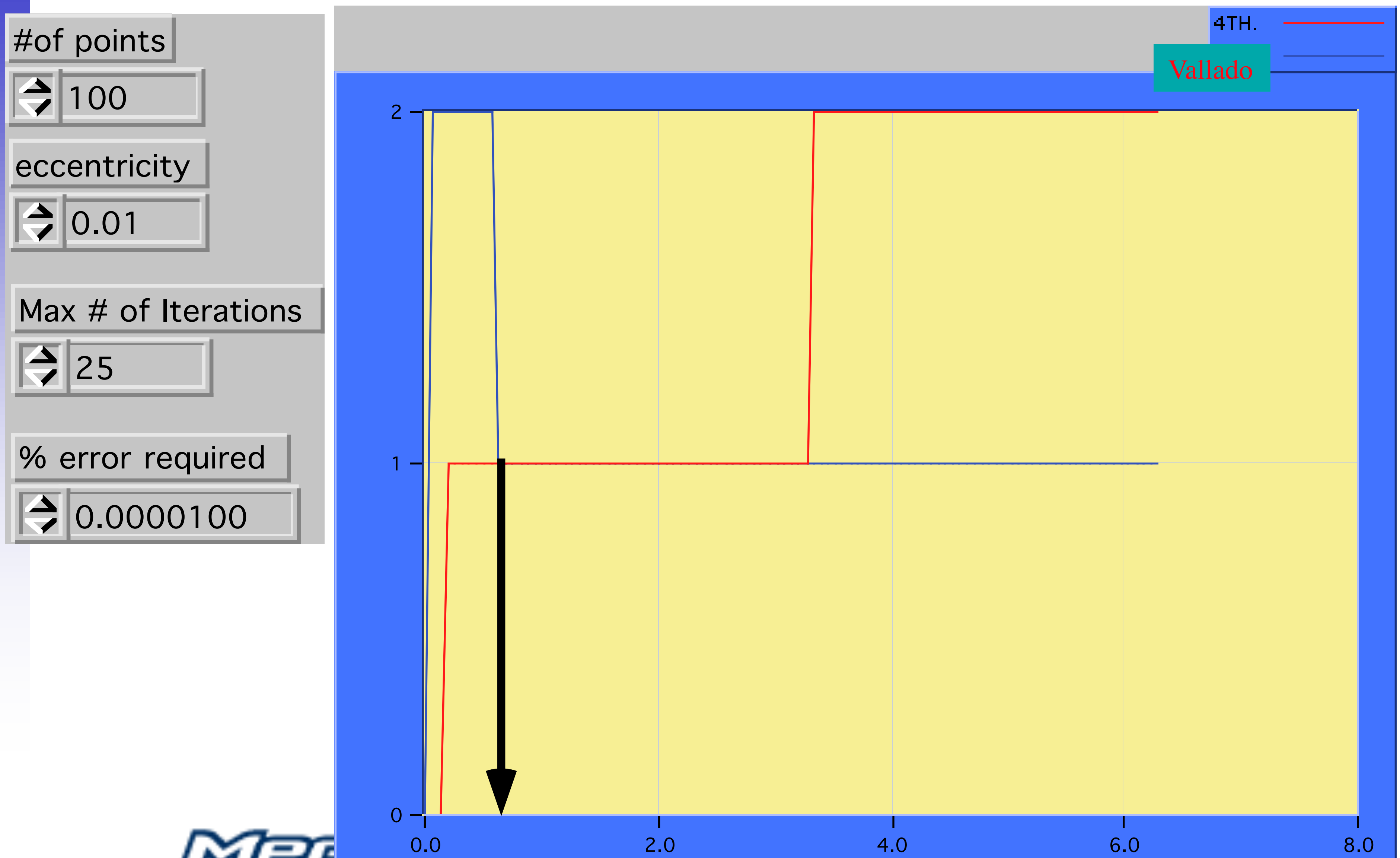


Comparison of Start-up Values

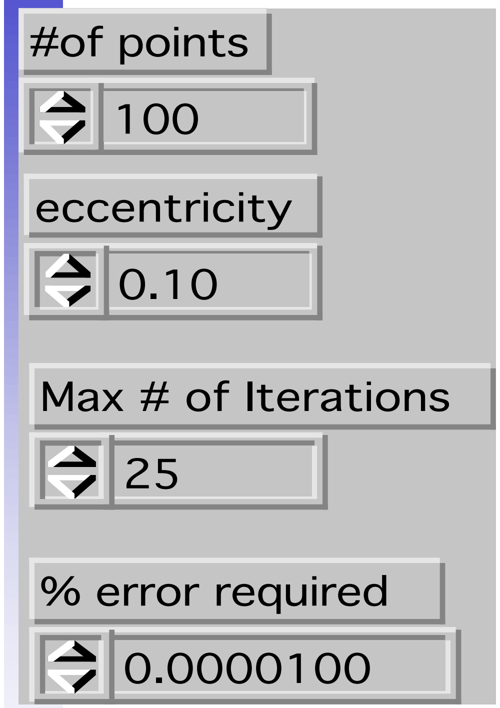
(cont'd)

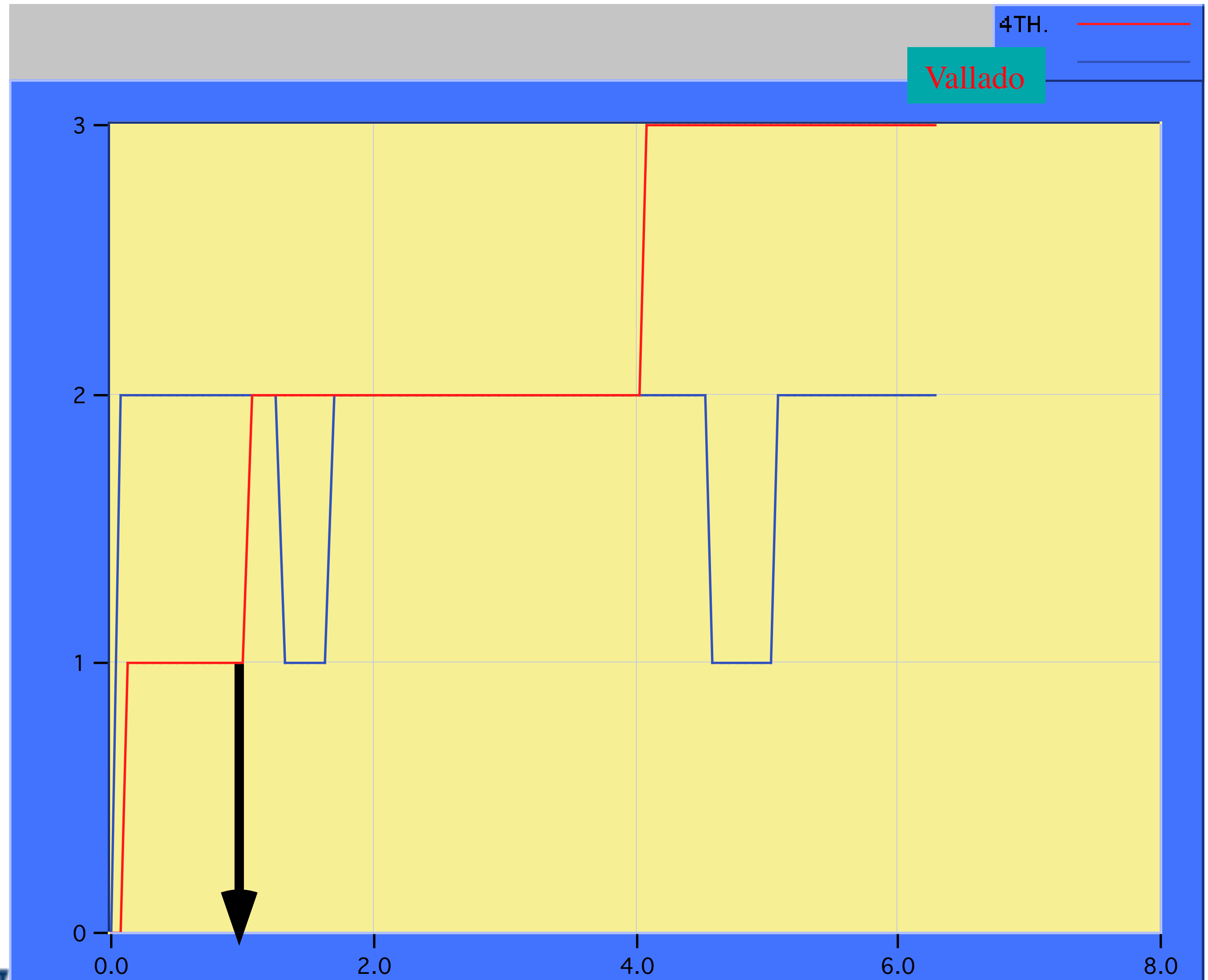


Iteration Plots (e=0.01)

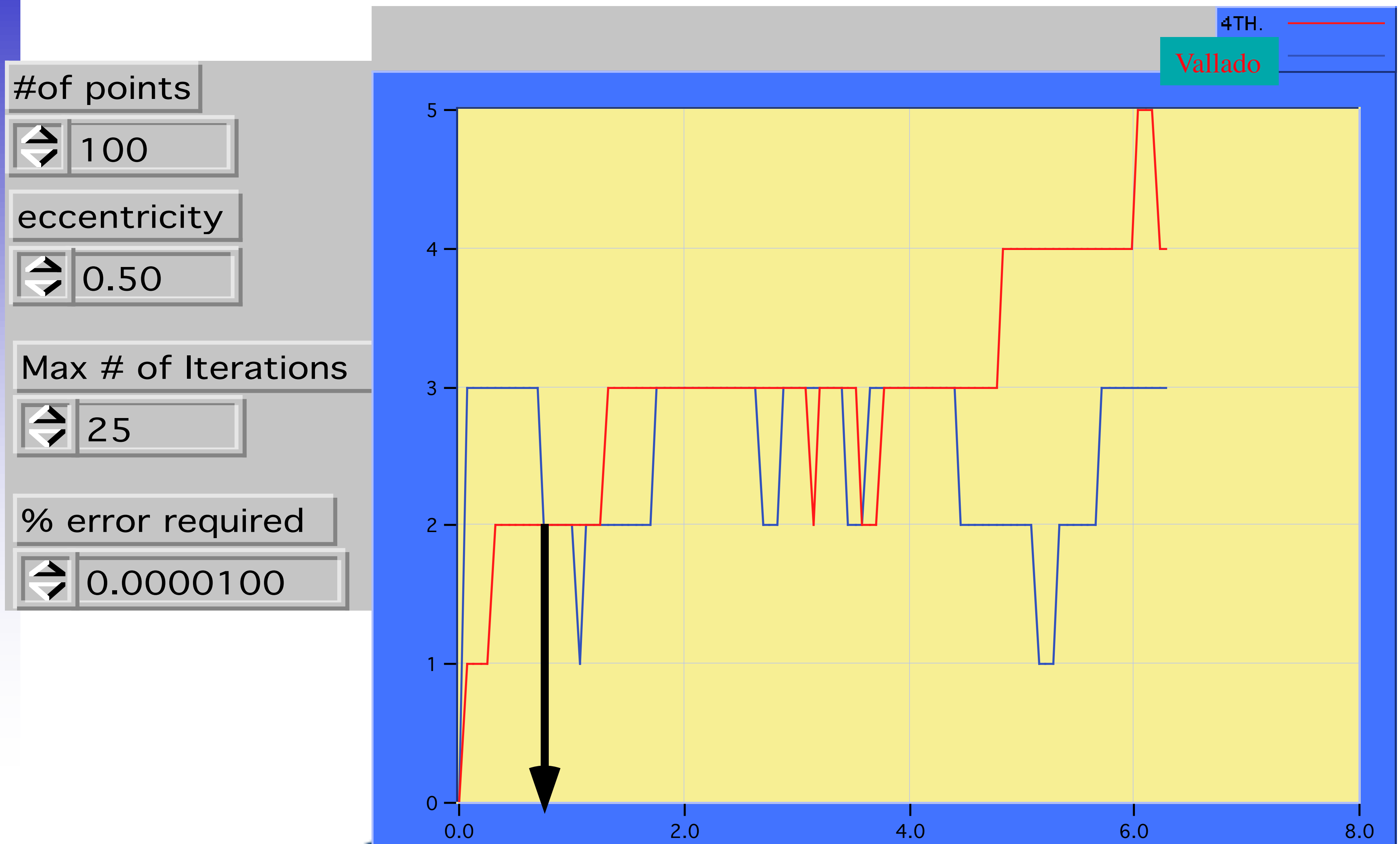


Iteration Plots (e=0.1)





Iteration Plots (e=0.5)



Iteration Plots (e=0.9)

#of points

100

eccentricity

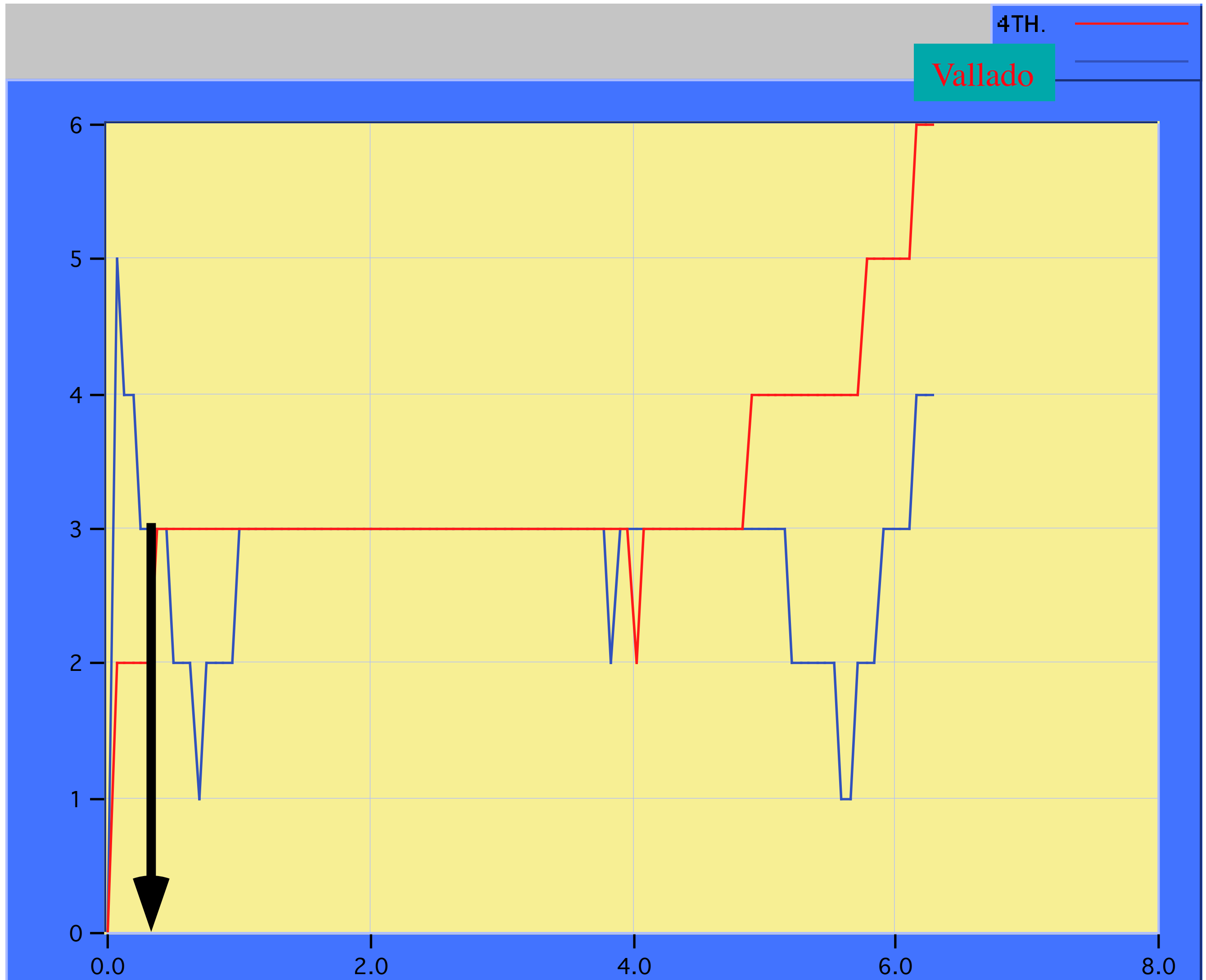
0.90

Max # of Iterations

25

% error required

0.0000100



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Conclusions?

- **For Most Conditions the Simple "Vallado" Startup Method gives Superior connvergence**
  ... simplicity of the algorithm clearly justified its use

- **Near the perigee, the 4th order (Taylor's series) startup gives better convergence**

- **Where's the "push" point ... it appears that for all eccentricities the Taylor's serie startup method offers convergence advantages for**

$$\nu < 0.25 radians\ (\sim 15^\circ)$$

- ***Gives a convergence aid for solutions near perigee***

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Soooo... its looks like

M > 0.25 radians ... use

$$\hat{E}^{(0)} = \begin{cases} M + e & (0 \le M \le \pi) \\ M - e & (\pi \le M \le 2\pi) \end{cases}$$

Otherwise .... use

$$\hat{E}^{(0)} = \frac{-2e + 2e^2 + \left(3e^2 M + \sqrt{e^3\left(-8(-1+e)^3 + 9eM^2\right)}\right)^{2/3}}{e\left(3e^2 M + \sqrt{e^3\left(-8(-1+e)^3 + 9eM^2\right)}\right)^{1/3}}$$

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