Addendum to Sections 2.1-2.3
…. The Open Conic Sections

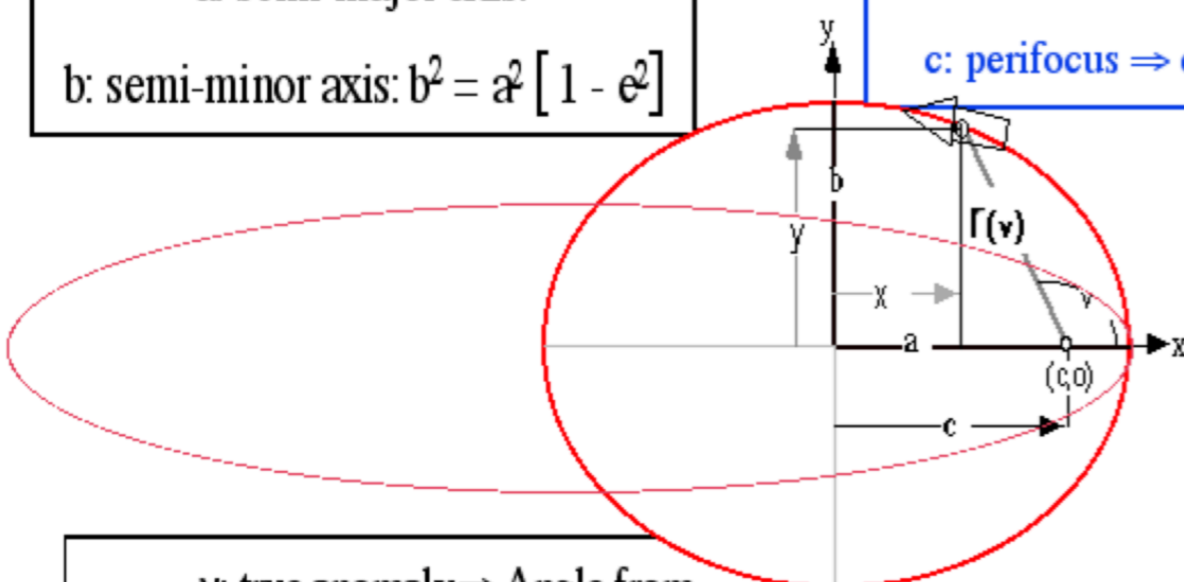
What happens when a->∞ in an elliptical orbit?

a: semi-major axis:

b: semi-minor axis: $b^2 = a^2 [1 - e^2]$

e: orbital eccentricity ⇒ $e = \sqrt{1 - \left[\frac{b}{a}\right]^2}$

c: perifocus ⇒ $c = a\sqrt{1 - \left[\frac{b}{a}\right]^2} = a\,e$



$a \rightarrow \infty$

$e \rightarrow 1$

$$r(\nu) = \frac{a[1 - e^2]}{[1 + e\cos(\nu)]}\ ?$$

ν: true anomaly ⇒ Angle from perapsis to satellite

(ν): orbital radius ⇒ $r(\nu) = \frac{a[1 - e^2]}{[1 + e\cos(\nu)]}$

Addendum to Sections 2.1-2.3 …. (cont'd) The Open Conic Sections

What happens when a->∞ in an elliptical orbit?

$$R_{a \to \infty} = \left[\lim_{\substack{a \to \infty \\ e \to 1}} \frac{a\left[1 - e^2\right]}{1 + e \cos\left[\nu_i\right]} \right] \text{"indeterminant"}$$

- **But** $a\left[1 - e^2\right] = \underline{a\left[1 - e\right]}\left[1 + e_T\right] = \underline{R_{perigee}}\left[1 + e_T\right]$

$$R_{a \to \infty} = \left[\lim_{\substack{a \to \infty \\ e \to 1}} \frac{a\left[1 - e^2\right]}{1 + e \cos\left[\nu_i\right]} \right] = \frac{2\, R_{perigee}}{1 + \cos\left[\nu_i\right]}$$

Addendum to Sections 2.1-2.3 …. The Open Conic Sections

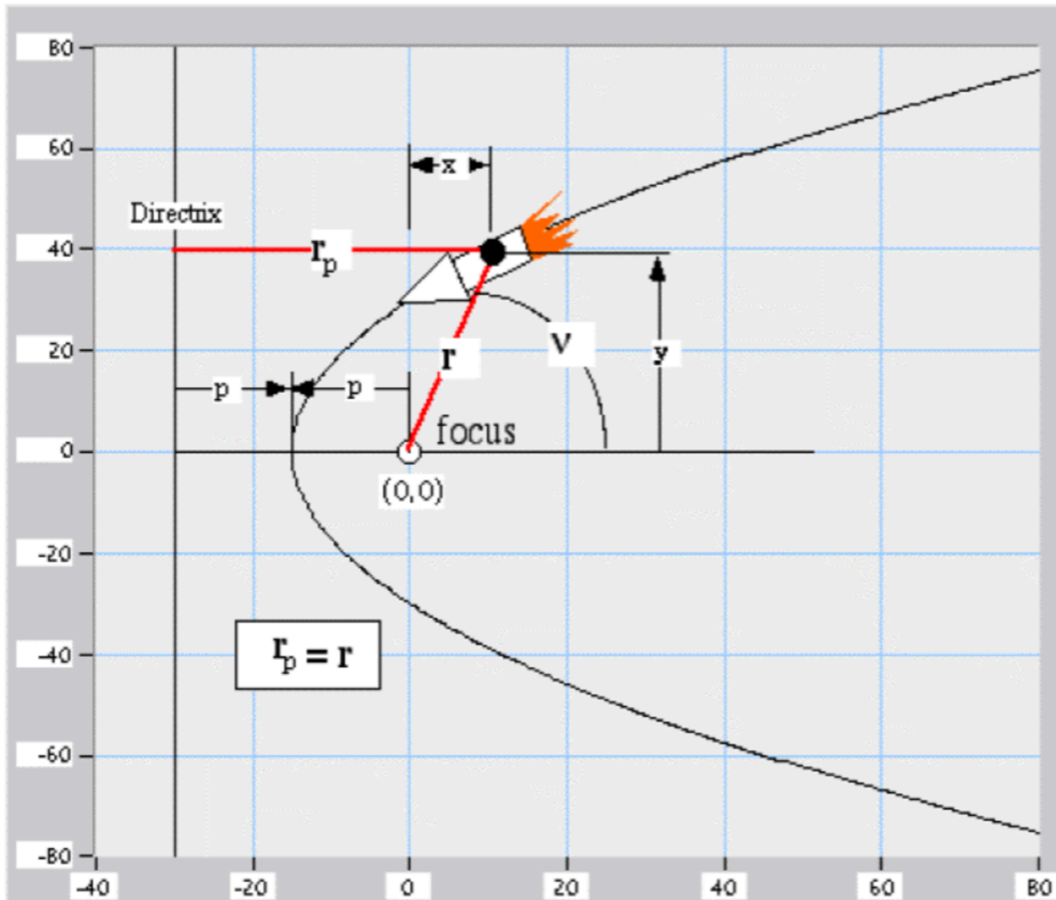
- a->∞ implies an "open" parabolic trajectory

$$R_{a\to\infty} = \frac{2\,R_{perigee}}{1 - \cos\left[\nu_i\right]}$$

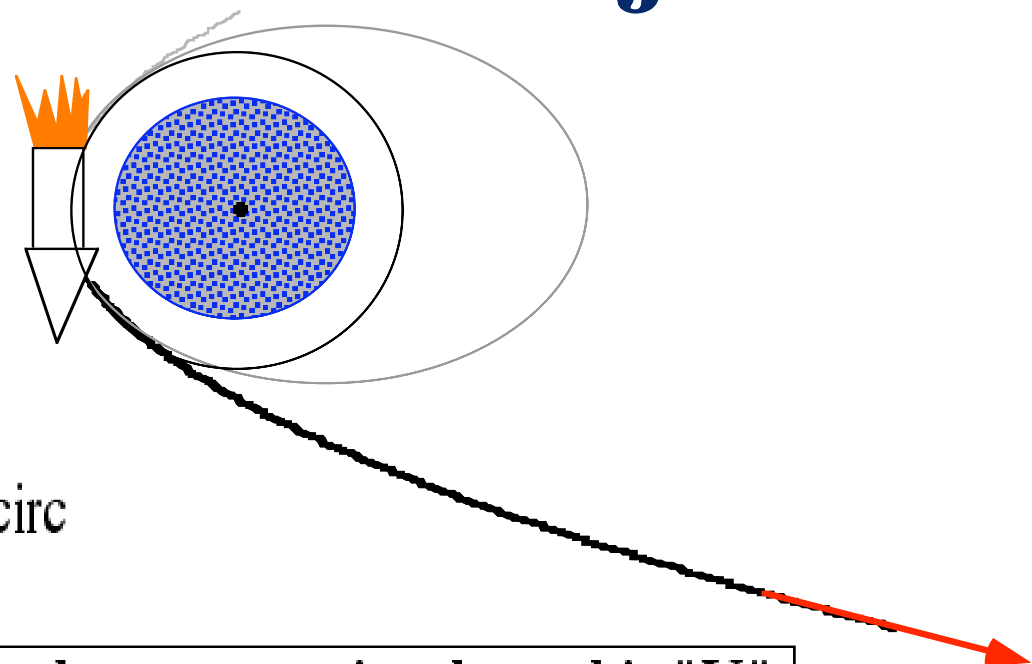
$$\left|\overline{V}\right|^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

$\frac{1}{a} \to 0$

$$\left|\overline{V}_{esc}\right| = \sqrt{\frac{2\,\mu}{r}}$$



Parabolic Trajectories:



$$\Delta V_{esc} = \left[\sqrt{2} - 1\right] V_{circ}$$

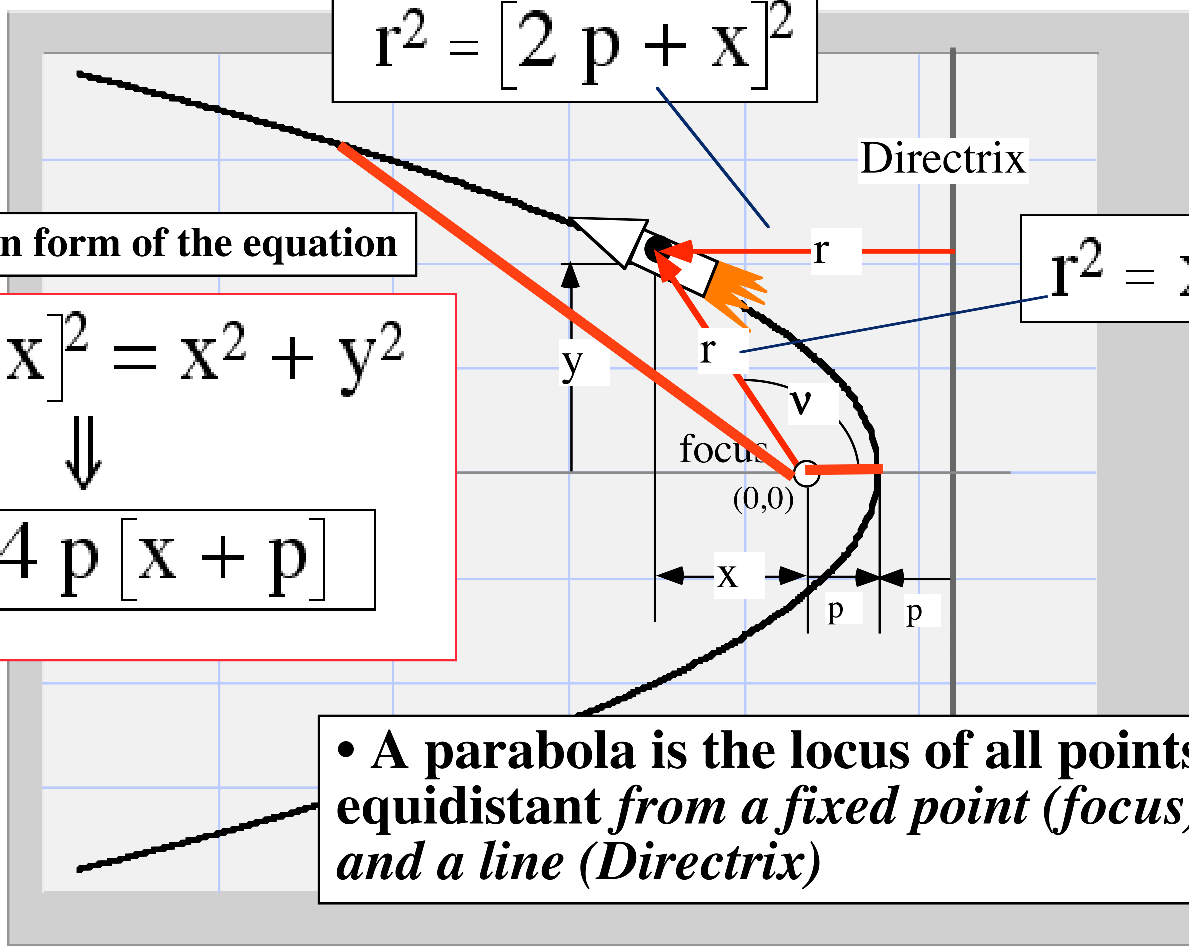
- If we increase the current circular orbit "V" by a factor of $\sqrt{2}$; then the velocity becomes too great for the planet to contain the orbit

- Satellite *escapes* the planet on a parabolic trajectory

What is a Parabola?

$$r^2 = [2\,p + x]^2$$

$$r^2 = x^2 + y^2$$



• **Cartesian form of the equation**

$$[2\,p + x]^2 = x^2 + y^2$$
$$\Downarrow$$
$$y^2 = 4\,p\,[x + p]$$

• **A parabola is the locus of all points equidistant *from a fixed point (focus) and a line (Directrix)***

Parabola Equation

Polar Form

$$y = \mathbf{r}\sin(\nu)$$
$$x = -\mathbf{r}\cos(\nu)$$

$\Rightarrow$

$$\mathbf{y}^2 = 4p\left[p + x\right]$$
$$\Downarrow$$
$$\left[\mathbf{r}\sin(\nu)\right]^2 = 4p\left[p - \mathbf{r}\cos(\nu)\right]$$
$$\Downarrow$$
$$\left[\sin^2(\nu)\right]\mathbf{r}^2 + \left[4p\cos(\nu)\right]\mathbf{r} - 4p^2 = 0$$

Use quadratic formula to solve

Parabola Equation

Polar Form (cont'd)

- Solve for r using quadratic formula

$$a\,\mathbf{r}^2 + b\,\mathbf{r} + c = 0 \Rightarrow \mathbf{r} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\mathbf{r} = \frac{-4p\cos(\nu) \pm \sqrt{16p^2\cos^2(\nu) + 16p^2\sin^2(\nu)}}{2\sin^2(\nu)}$$

- Use some trig identities to simplify

$$\cos^2(\nu) + \sin^2(\nu) = 1$$

$$\sin^2(\nu) = 1 - \cos^2(\nu) = [1 - \cos(\nu)][1 + \cos(\nu)]$$

$$\mathbf{r} = \frac{-2p[\cos(\nu) \pm 1]}{[1 - \cos(\nu)][1 + \cos(\nu)]}$$

Parabola Equation

Polar Form (concluded)

- r must be ≥ 0, therefore pick - sign

$$r = \frac{-2p[\cos(\nu) \pm 1]}{[1-\cos(\nu)][1+\cos(\nu)]} = \frac{-2p[\cos(\nu) - 1]}{[1-\cos(\nu)][1+\cos(\nu)]}$$

$$= \frac{2p}{[1+\cos(\nu)]}$$

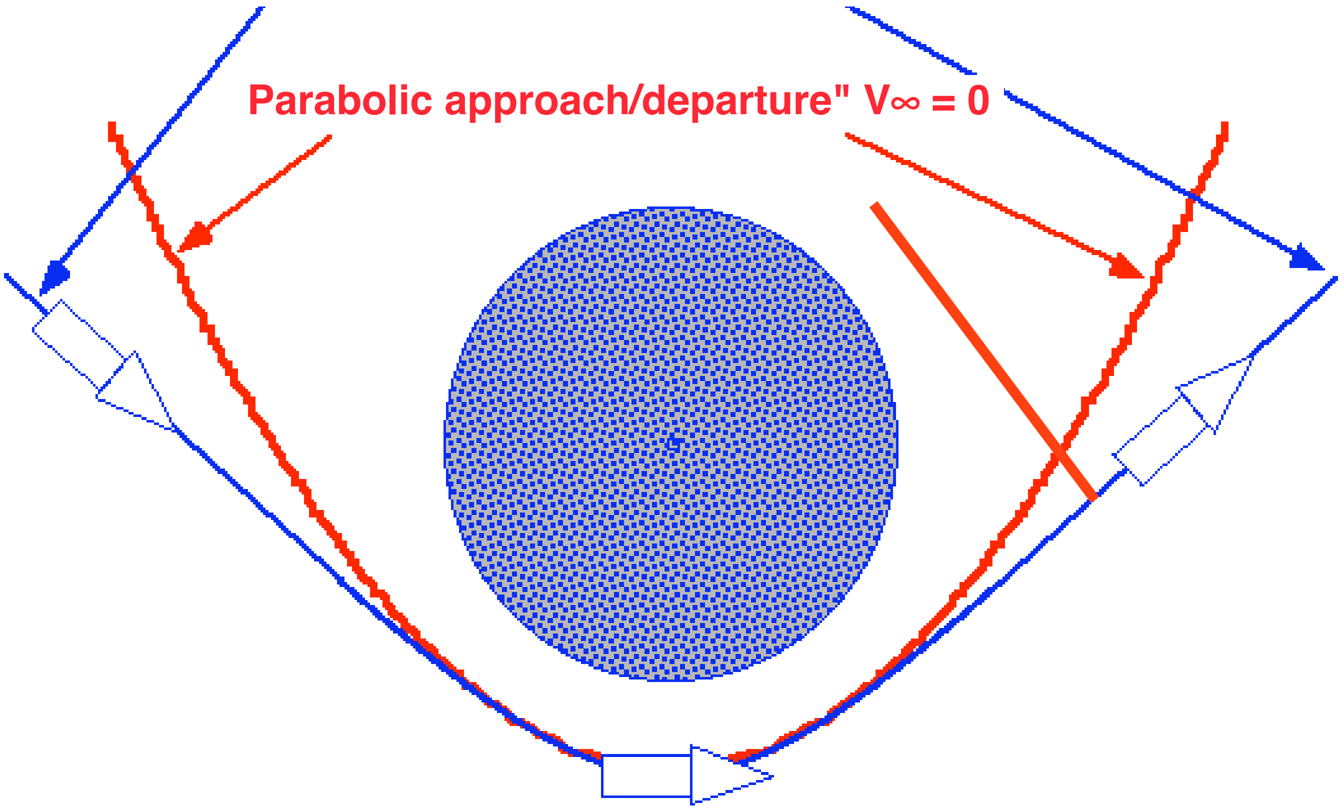
$$\{r_{max} @ \nu = \pi\},\ \frac{2p}{[1+\cos(\pi)]} = \frac{2p}{[1+(-1)]} = \infty$$

$$\{r_{min} @ \nu = 0\},\ \frac{2p}{[1+\cos(0)]} = \frac{2p}{[1+(1)]} = p$$

$$p = r_{min}$$

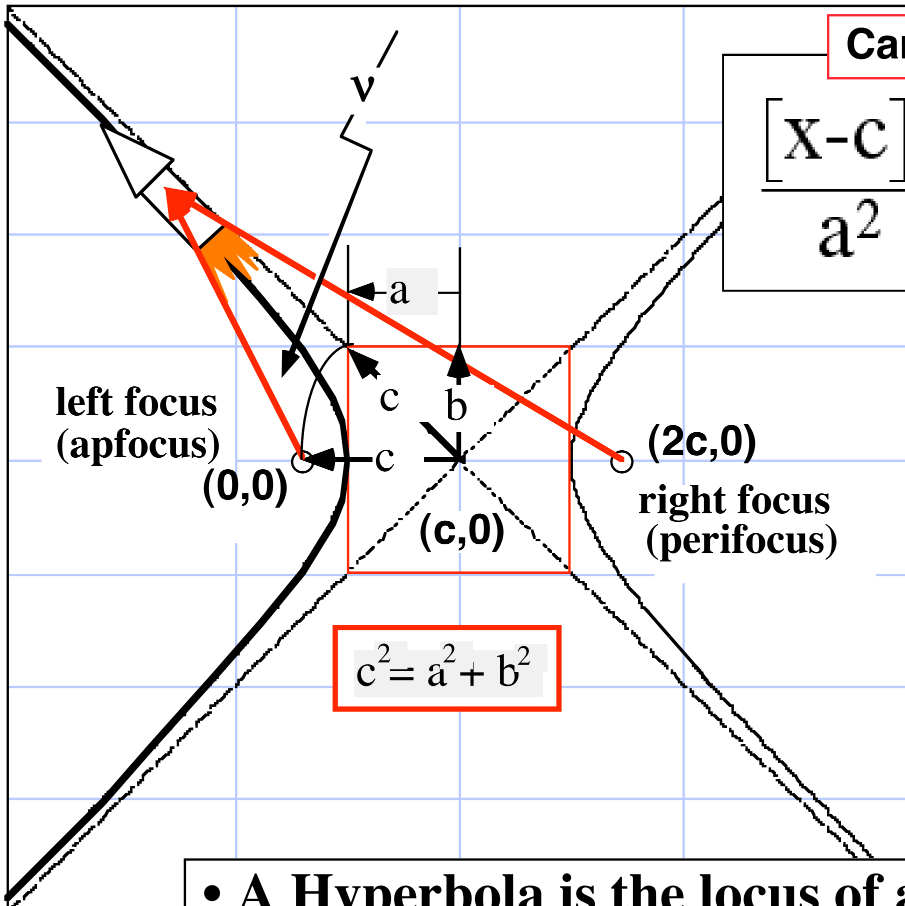
Perigee!

Hyperbolic Trajectories:



• If "V ∞" > 0 , then probe will approach and depart along a hyperbolic trajectory

What is a Hyperbola?



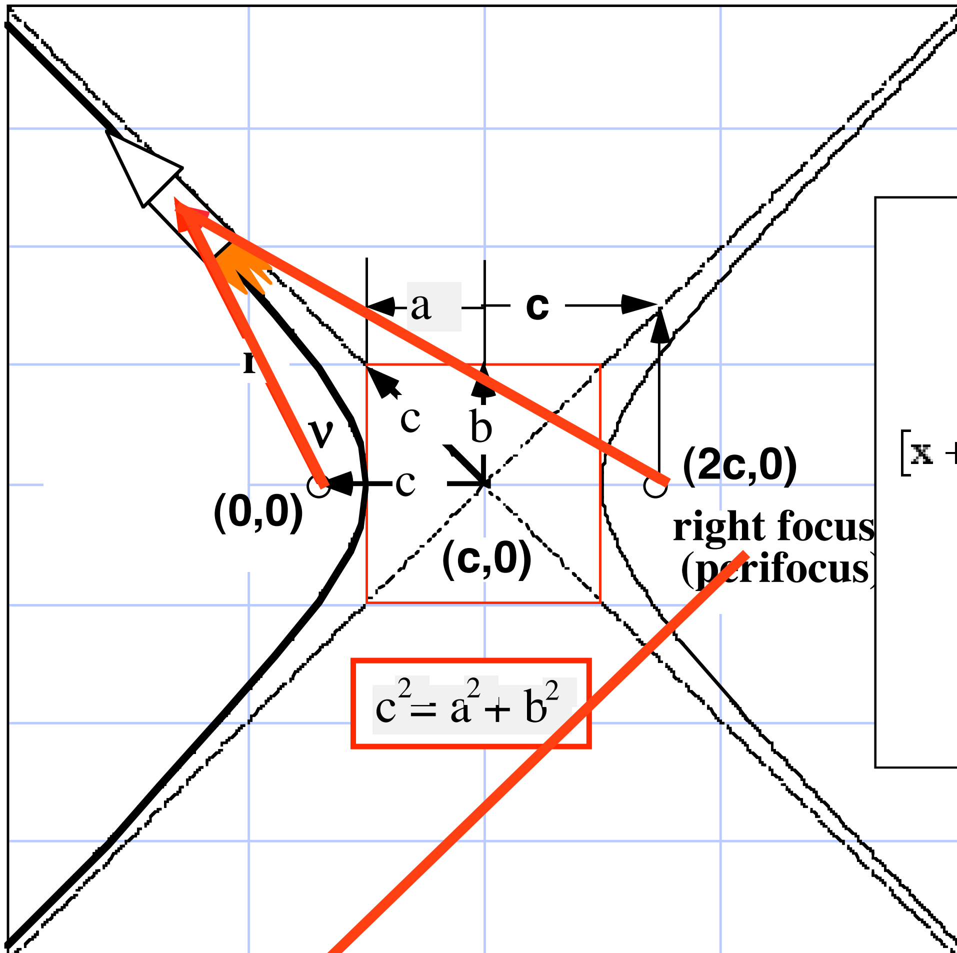
Cartesian Equation

$$\frac{[x-c]^2}{a^2} - \frac{y^2}{b^2} = 1$$

- **A Hyperbola is the locus of all points whose *difference* from two fixed points is constant**

Hyperbolic Equation:

polar form



$$\sqrt{[x + 2c]^2 + y^2} - \sqrt{x^2 + y^2} = 2a$$
$$\Downarrow$$
$$[x + 2c]^2 + y^2 = \left[2a + \sqrt{x^2 + y^2}\right]^2$$
$$\Downarrow$$
$$[x + 2c]^2 + y^2 = 4a^2 + 4a\sqrt{x^2 + y^2} + x^2 + y^2$$
$$\Downarrow$$
$$xc + c^2 = a^2 + a\sqrt{x^2 + y^2}$$
$$\Downarrow$$
$$\boxed{\frac{c}{a}[x + c] - a = \sqrt{x^2 + y^2}}$$

Hyperbolic Equation:

polar form (cont'd)

- From earlier analysis

$$\frac{c}{a}\left[x+c\right]-a=\sqrt{x^2+y^2}=r$$

$$x=-r\cos(\nu)$$

$$\frac{c}{a}\left[c-r\cos(\nu)\right]-a=r$$

$$\Downarrow$$

$$r\left[1+\frac{c}{a}\cos(\nu)\right]=\frac{c^2-a^2}{a}$$

$$\Downarrow$$

$$\frac{c^2-a^2}{a}=\frac{\left[a^2+b^2\right]-a^2}{a}=\frac{b^2}{a}$$

$$\Downarrow$$

$$r=\frac{b^2}{a}\frac{1}{\left[1+\frac{c}{a}\cos(\nu)\right]}$$

Hyperbolic Equation:

polar form (concluded)

- Defining the "*hyperbolic eccentricity*"

$$e_{hyp} = \sqrt{1 + \frac{b^2}{a^2}} \Rightarrow \frac{b^2}{a} = a\left[\ e_{hyp}^2 - 1\right]$$

$$e^2{}_{hyp} = \frac{a^2 + b^2}{b^2} = \frac{c^2}{b^2}$$

$$\Downarrow$$

$$r = \frac{a\left[\ e_{hyp}^2 - 1\right]}{\left[1 + e_{hyp}\cos(\nu)\right]}$$

Hyperbolic Asymptotes

- **What is the behavior of a Hyperbola at "infinity" i.e. along away from earth**

$$\lim_{x,\,y \to \infty}\left[\frac{[x-c]^2}{a^2} - \frac{y^2}{b^2} = 1\right] \Rightarrow \lim_{x,\,y \to \infty}\left[[x-c]^2 - \frac{a^2}{b^2}y^2 - a^2 = 0\right]$$

$$\Downarrow$$

asymptotes

$$\Downarrow$$

$$[x-c]^2 = \frac{a^2}{b^2}y^2 \Rightarrow \boxed{y = \pm\frac{b}{a}[x-c]}$$

Hyperbolic Behavior at ∞

asymptotes

$$\Downarrow$$

$$y = \pm \frac{b}{a}[x-c] \Rightarrow r\sin(\nu) = \pm\frac{b}{a}[-r\cos(\nu) - c]$$

$$\frac{\sin(\nu) + \left[\pm\frac{b}{a}\cos(\nu)\right] = -\pm\frac{b}{a}\frac{c}{\mathbf{r}}}{\lim \mathbf{r} \to \infty} = \sin(\nu_\infty) + \left[\pm\frac{b}{a}\cos(\nu_\infty)\right] = 0$$

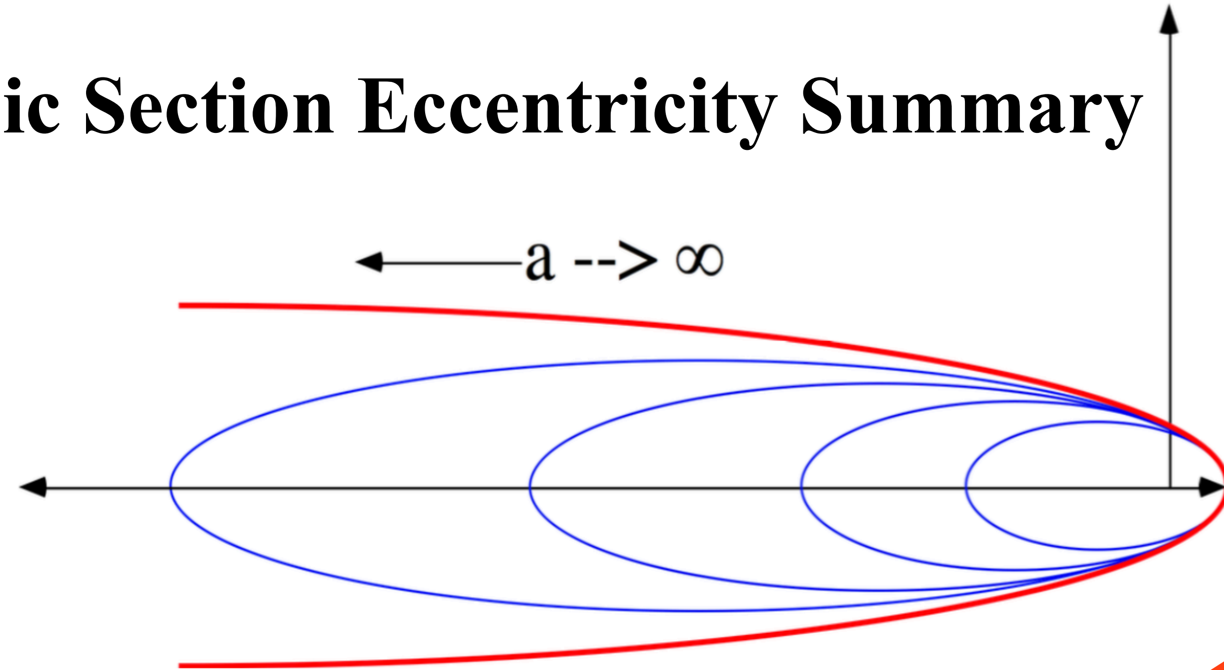
$$\Downarrow$$

$$\tan(\nu_\infty) = \mp\frac{b}{a} = \mp\sqrt{e_{hyp}^2 - 1} \Rightarrow$$

departure: $\nu_\infty = \pi - \tan^{-1}\left[\sqrt{e_{hyp}^2 - 1}\right]$

arrival: $\nu_\infty = \pi + \tan^{-1}\left[\sqrt{e_{hyp}^2 - 1}\right]$

Conic Section Eccentricity Summary



- Orbital Energy is with regard to an escape trajectory!

- *Circular, Elliptical Orbit* → $\varepsilon_T < 1$
- *Parabolic (Escape) Trajectory* → $\varepsilon_T = 1$
- *Hyperbolic Trajectory* → $\varepsilon_T > 1$

See Appendix 2.3.2 for Hyperbolic Trajectory Proof

Vis-Viva Equation for All the Conic-Sections

Circle: $\boxed{r = a} \Rightarrow$ $V = \sqrt{\mu\left[\frac{2}{a} - \frac{1}{a}\right]} = \sqrt{\frac{\mu}{a}}$

Ellipse: $\boxed{r = \frac{a\left[1 - e^2\right]}{\left[1 + e\ \cos(\nu)\right]}} \Rightarrow$ $V = \sqrt{\mu\left[\frac{2}{r} - \frac{1}{a}\right]}$

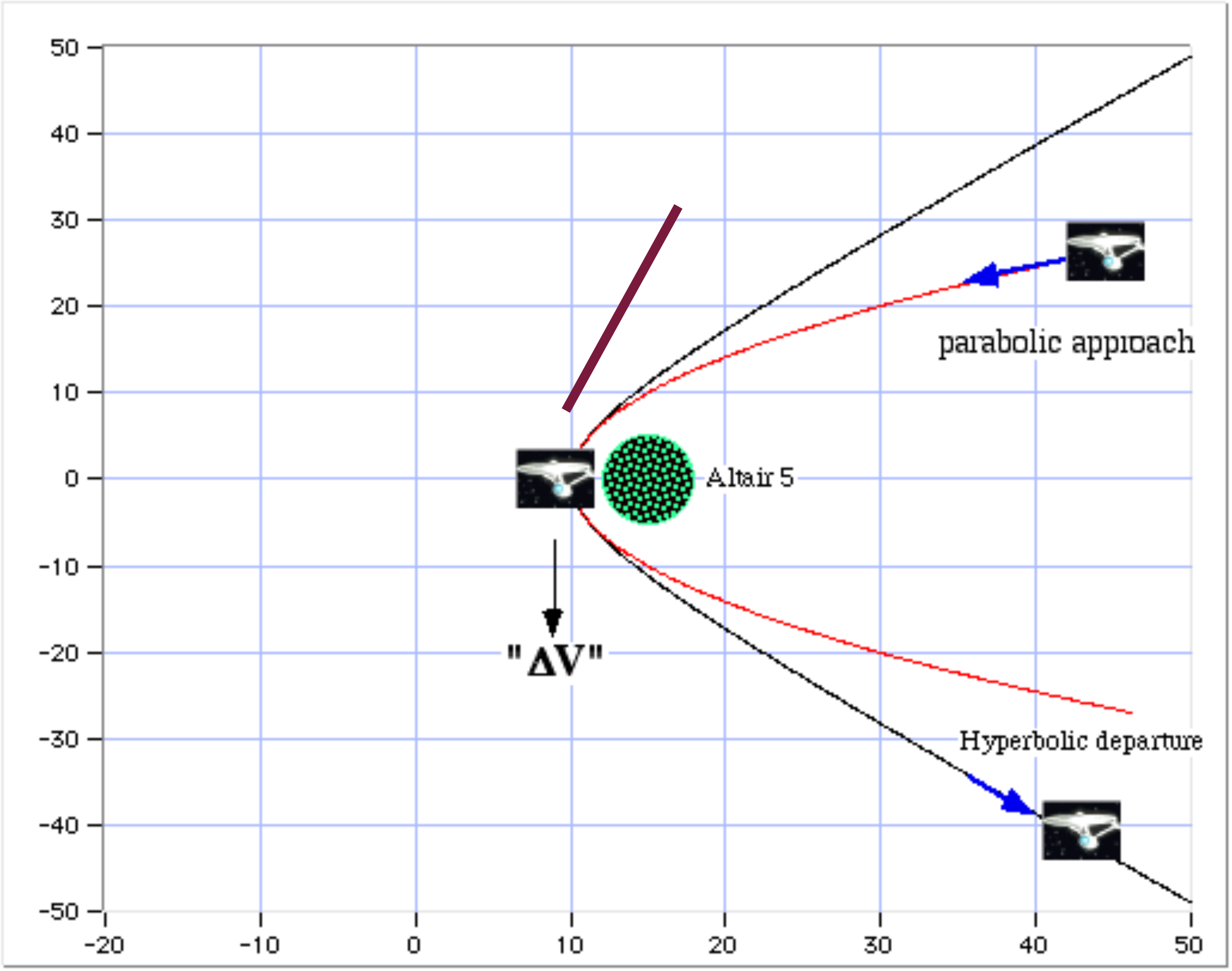
Parabola: $\boxed{r = \frac{2p}{\left[1 + \cos(\nu)\right]}} \Rightarrow$ $V = \sqrt{\mu\left[\frac{2}{r} - \frac{1}{\infty}\right]} = \sqrt{\frac{2\mu}{r}}$

Hyperbola: $\boxed{r = \frac{a\left[e_{hyp}^2 - 1\right]}{\left[1 + e_{hyp}\cos(\nu)\right]}} \Rightarrow$ $V = \sqrt{\mu\left[\frac{2}{r} + \frac{1}{a}\right]}$

See Appendix 2.3.2 for Hyperbolic Trajectory Proof

Homework ~~3~~ 4

Parabolic and Hyperbolic Trajectories



Homework

Parabolic and Hyperbolic Trajectories (cont'd)

- ***United Federation of Planets*** **starship** ***Excelsior*** **approaches** ***Klingon*** **outpost** ***Altair 5*** **on a covert retaliatory bombing mission**

- **A cloaking device uses enormous energy &** ***Warp drive*** **is non-operational with the cloak engaged**

- **All maneuvering must be done on** ***impulse power*** **alone**

- **The** ***Excelsior*** **uses a gravity assisted** ***parabolic*** **approach trajectory to** ***Altair 5*** **in order to save on waning impulse power and insure a stealthy approach**

Homework

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Mechanical & Aerospace Engineering

Parabolic and Hyperbolic Trajectories (cont'd)

- After dropping photo-torpedos, Captain Checkov wants to get out the *sphere of influence* (SOI) of*Altair 5* as fast as possible without being spotted

- The *Excelsio*r has enough impulse power left for *one big burn* before, having to recharge the *dilithium crystals*

- The best way to "get out of town fast" is to fire impulse engines at closest approach to Altair 5 -- taking advantage of the gravity assist to give the highest approach speed without using impulse power and then use impulse power to depart on a hyperbolic trajectory at angle of 45 degrees

- What is the "*Delta-V*" required to depart on a *Hyperbolic* trajectory with an asymptotic departure angle of 45 degrees

Homework:

Parabolic and Hyperbolic Trajectories (cont'd)

- **Hint 1: For a Parabolic trajectory**

r is measured from the parabolic *focus* to the location of the *Excelsior*

- **Hint 2: For a Hyperbolic trajectory**

r is measured from the *right (perifocus) focus* to the location of the *Excelsior*

Homework:

Parabolic and Hyperbolic Trajectories (concluded)

- **Hint 3: For a Parabolic to Hyperbolic trajectory transfer**

$$"\Delta V" = V_h - V_p = V_p \left[\frac{V_h}{V_p} - 1 \right]$$

- **Hint 4: At closest apprach, the distance from the *parabolic focus* to the *Excelsior* must equal the distance from the *Hyperbolic right focus* to the *Excelsior***

- **Your answer should be expressed in terms μ and r_{min} (closest approach distance)**


ANY QUESTIONS? NO?
GOOD!