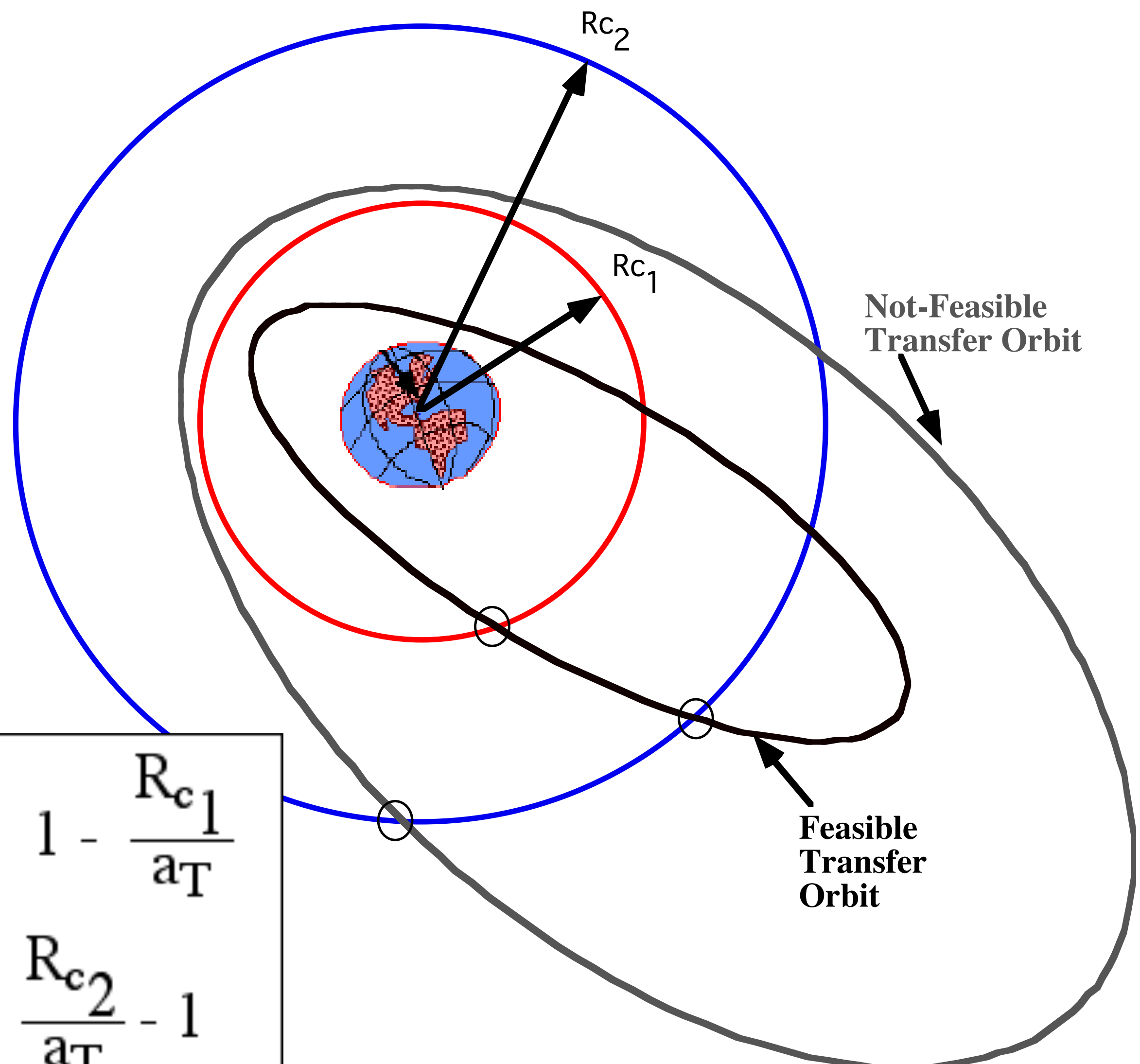
MAE 5540 Feasible In-Plane Transfer Orbits

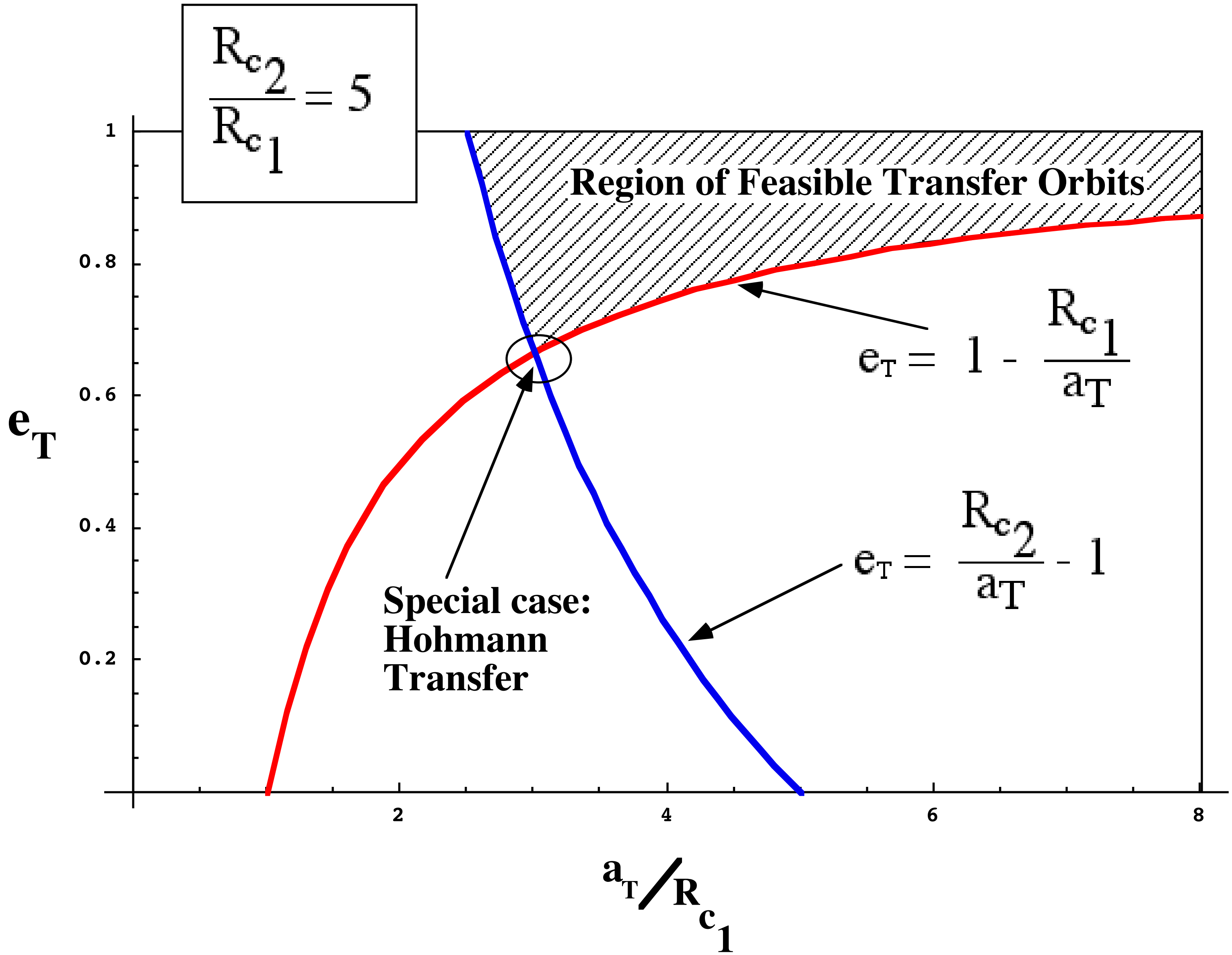
- Transfer Orbit MUST Intersect BOTH Inner and Outer Orbits



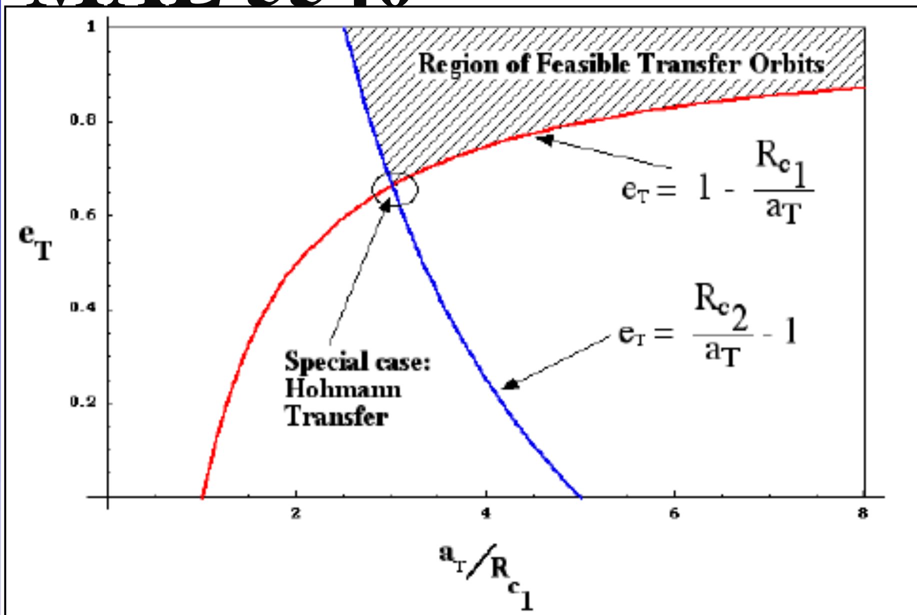
$$a_T\left[1-e_T\right] \leq R_{c_1} \Rightarrow e_T \geq 1 - \frac{R_{c_1}}{a_T}$$

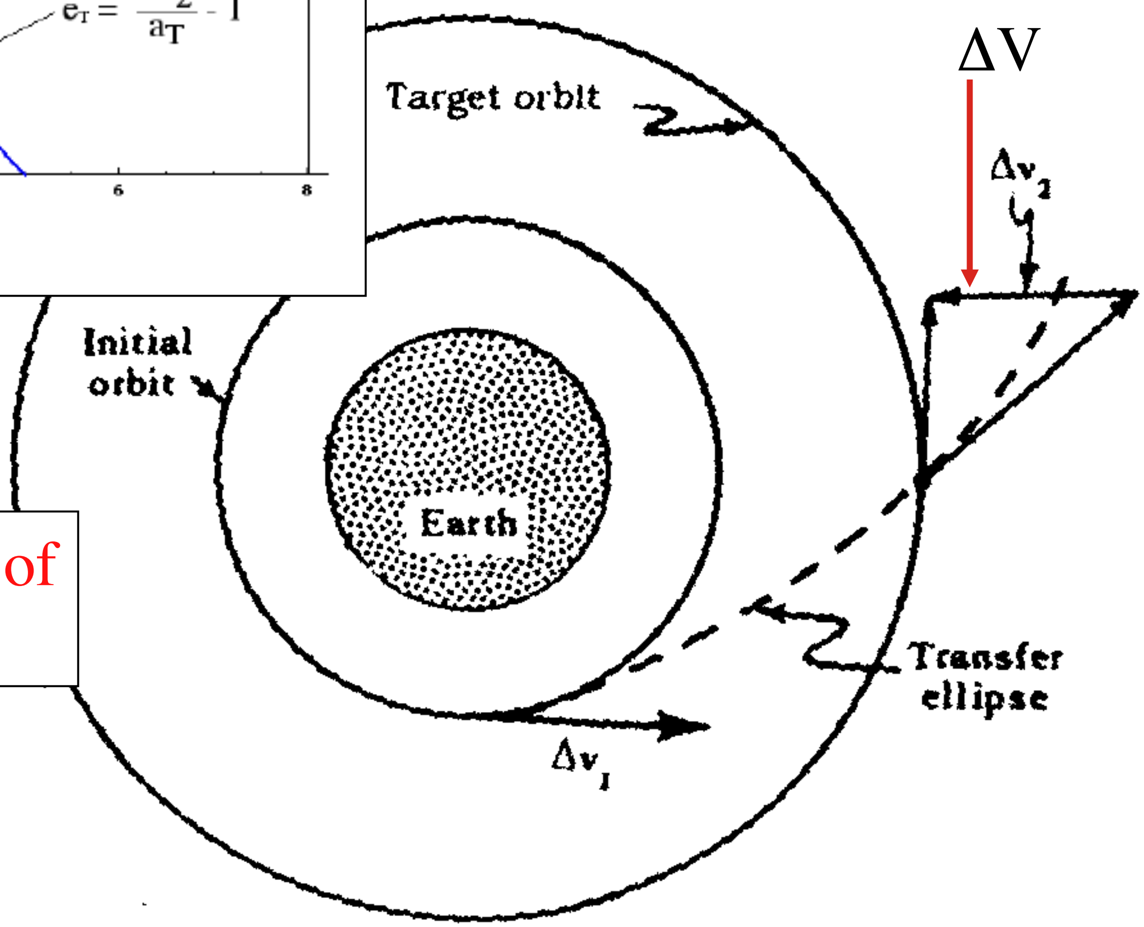
$$a_T\left[1+e_T\right] \geq R_{c_2} \Rightarrow e_T \geq \frac{R_{c_2}}{a_T} - 1$$

MAE 5540 Feasible Transfer Orbits (cont'd)



Excess-energy transfer orbit

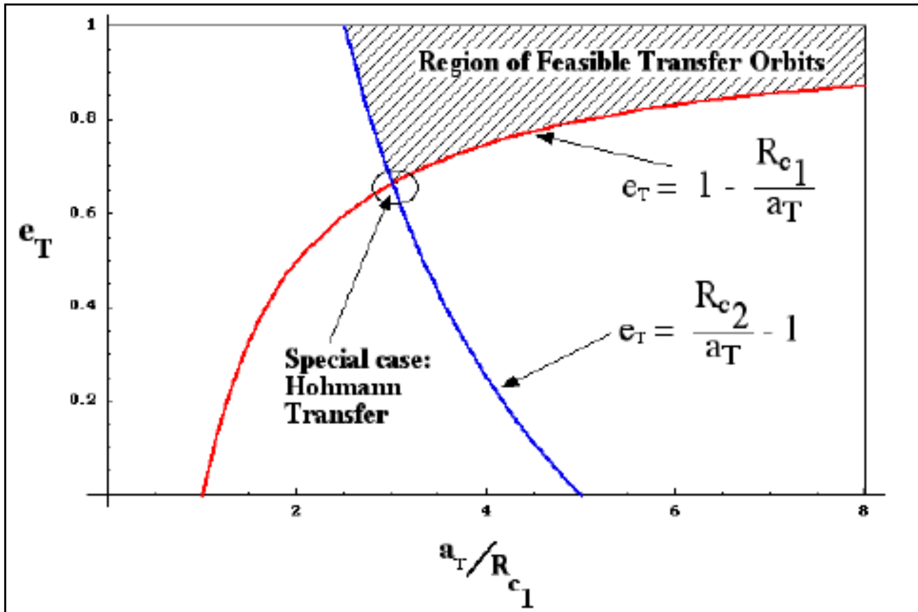




- Shaded region of a_T, e_T phase plot

Mecha

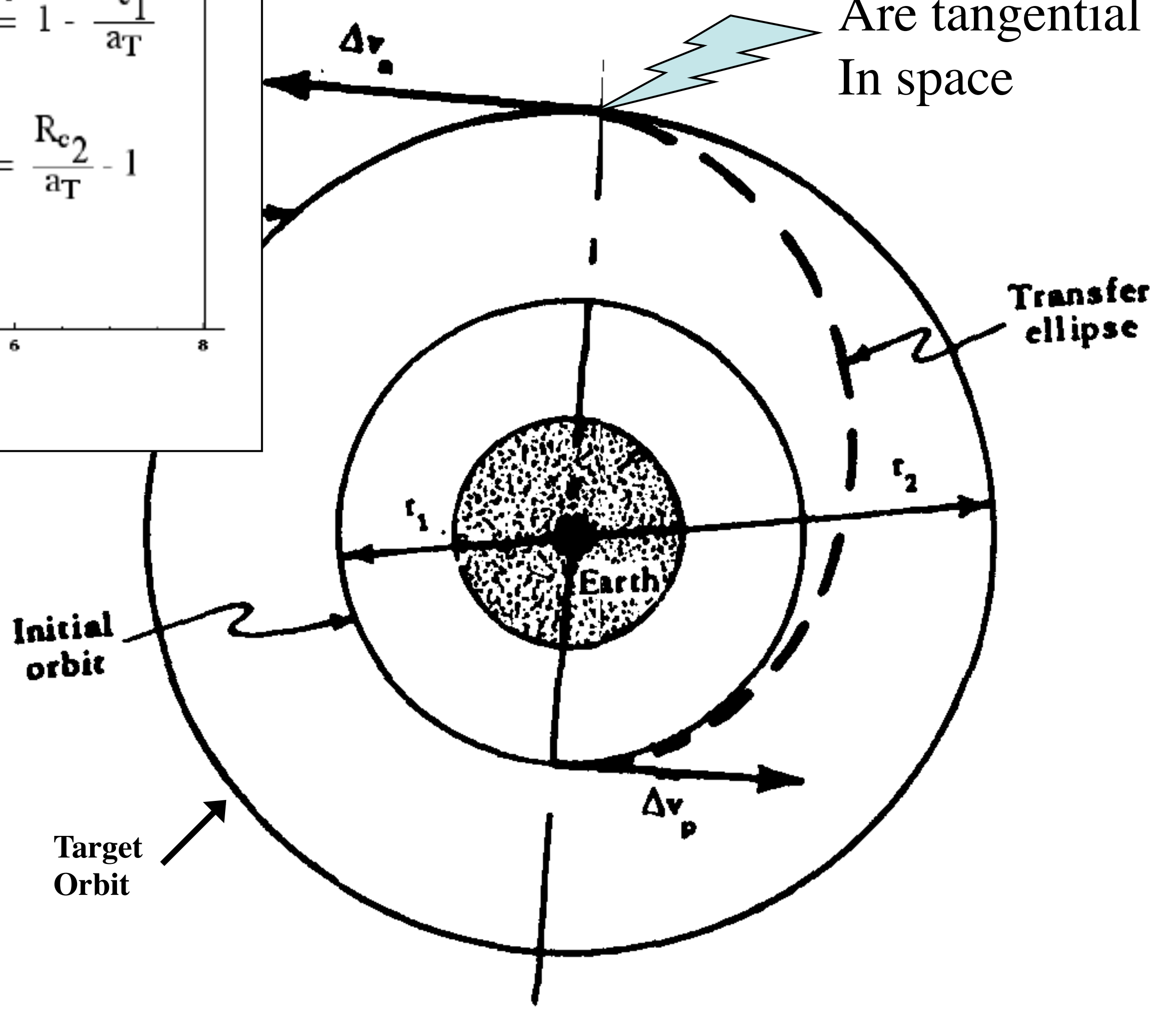
Optimal-energy transfer Orbit



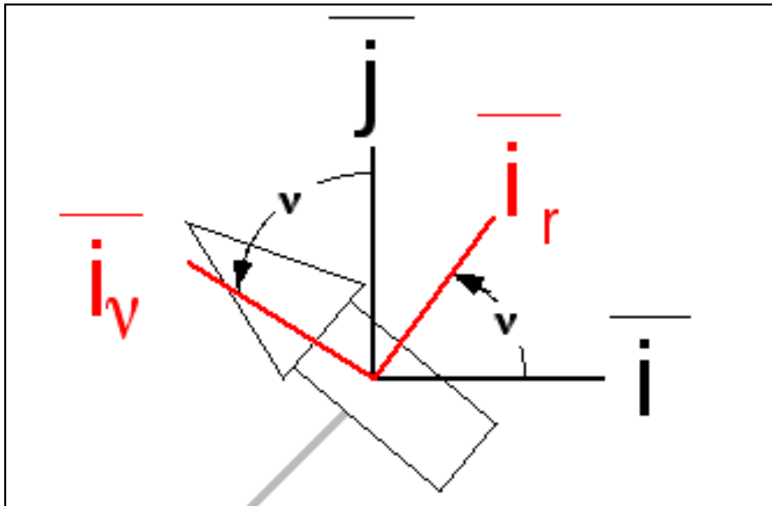
- Intersection of the Curves on the a_T, e_T phase plot

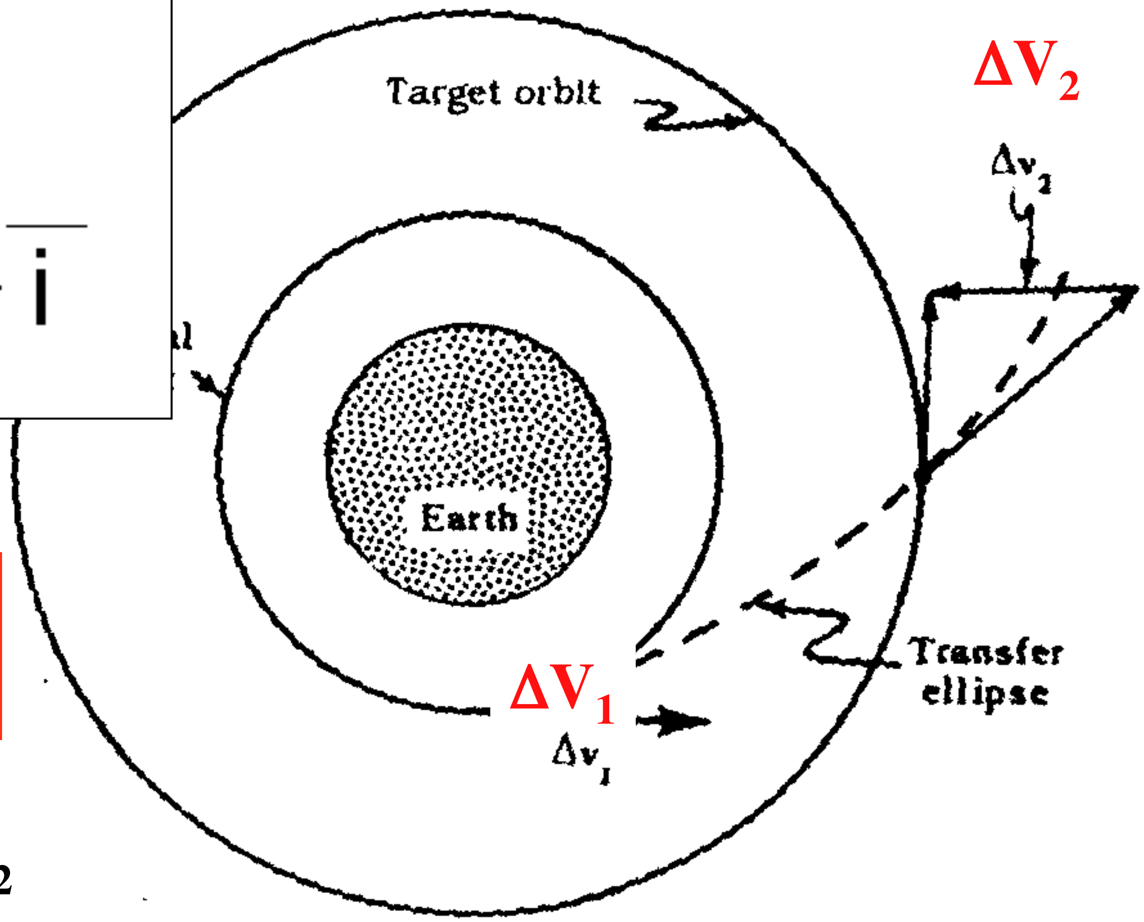
Why, … well it is a Constrained optimization Problem … and the answer Is a bit subtle … but here goes …

Two curves
Are tangential
In space



Why is Hohmann Transfer Optimal For ΔV



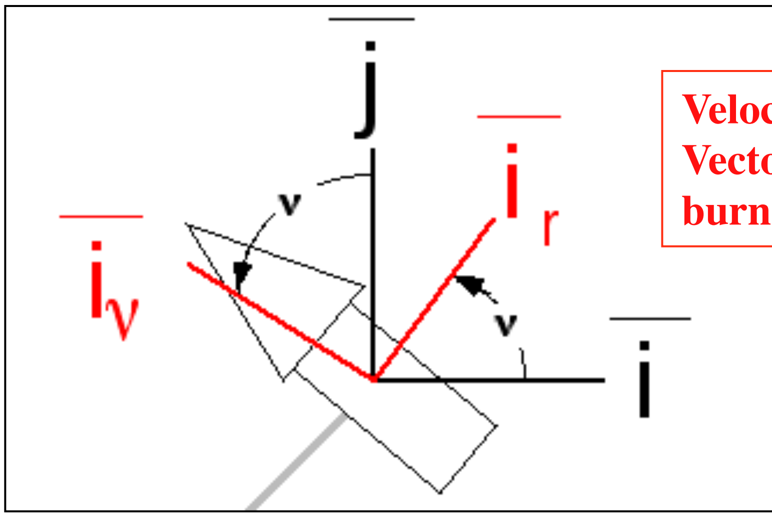


$$\left|\overline{\Delta V_T}\right| = \left|\overline{\Delta V_1}\right| + \left|\overline{\Delta V_2}\right|$$

Burn 1

Burn 2

Why is Hohmann Transfer Optimal For ΔV (cont'd)



Velocity Vector after burn 1

Velocity Of initial Orbit

Velocity Of target Orbit

$$\left|\overline{\Delta V_T}\right| = \left|\overline{\Delta V_1}\right| + \left|\overline{\Delta V_2}\right| \Rightarrow$$

$$\overline{\Delta V_1} = \left(\overline{V}_T\right)_1 - \sqrt{\frac{\mu}{R_{c_1}}}\ \bar{i}_\nu$$

$$\overline{\Delta V_2} = \left(\overline{V}_T\right)_2 - \sqrt{\frac{\mu}{R_{c_2}}}\ \bar{i}_\nu$$

Velocity Vector after burn 2

Why is Hohmann Transfer Optimal For ΔV (cont'd)

$$\left|\overline{\Delta V_T}\right| = \left|\overline{\Delta V_1}\right| + \left|\overline{\Delta V_2}\right| \Rightarrow$$

$$\overline{\Delta V_1} = \left(\overline{V}_T\right)_1 - \sqrt{\frac{\mu}{R_{c_1}}}\,\bar{i}_v$$

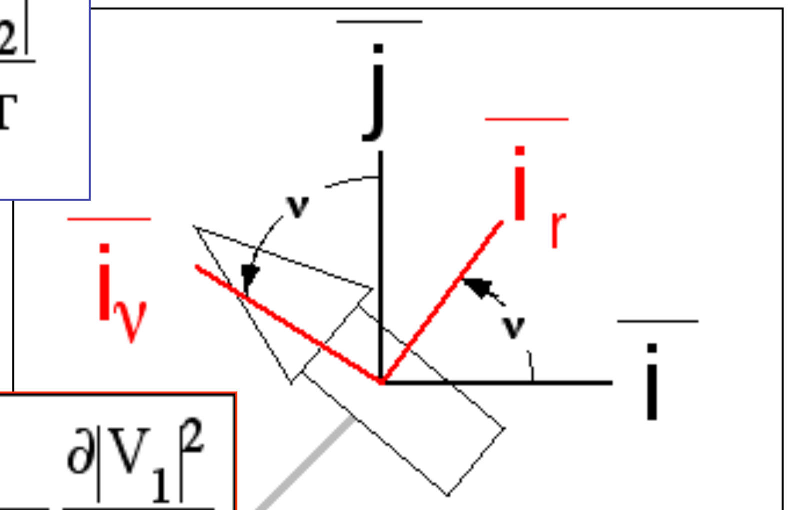
$$\overline{\Delta V_2} = \left(\overline{V}_T\right)_2 - \sqrt{\frac{\mu}{R_{c_2}}}\,\bar{i}_v$$

$$\frac{\partial\left|\Delta V_T\right|}{\partial e_T} = \frac{\partial\left|\Delta V_1\right|}{\partial e_T} + \frac{\partial\left|\Delta V_2\right|}{\partial e_T} = \frac{\partial\left|V_1\right|}{\partial e_T} + \frac{\partial\left|V_2\right|}{\partial e_T}$$

MAE 5540

Why is Hohmann Transfer Optimal For ΔV (cont'd)

$$\frac{\partial |\Delta V_T|}{\partial e_T} = \frac{\partial |\Delta V_1|}{\partial e_T} + \frac{\partial |\Delta V_2|}{\partial e_T} = \frac{\partial |V_1|}{\partial e_T} + \frac{\partial |V_2|}{\partial e_T}$$



$$\frac{\partial |V_1|^2}{\partial e_T} = 2\,|V_1|\,\frac{\partial |V_1|}{\partial e_T} \Rightarrow \boxed{\frac{\partial |V_1|}{\partial e_T} = \frac{1}{2\,|V_1|}\frac{\partial |V_1|^2}{\partial e_T}}$$

$$\frac{\partial |V_2|^2}{\partial e_T} = 2\,|V_2|\,\frac{\partial |V_2|}{\partial e_T} \Rightarrow \boxed{\frac{\partial |V_2|}{\partial e_T} = \frac{1}{2\,|V_2|}\frac{\partial |V_2|^2}{\partial e_T}}$$

Why is Hohmann Transfer Optimal For ΔV (cont'd)

$$\frac{\partial\left[\Delta V_T\right]}{\partial e} = \frac{1}{2}\left[\frac{1}{|V_1|}\frac{\partial\left[|V_1|^2\right]}{\partial e} + \frac{1}{|V_2|}\frac{\partial\left[|V_2|^2\right]}{\partial e}\right]$$

- Recall that from the Kepler's Third law derivation …

$$|\overline{V}|^2 = \frac{l^2}{a\left[1-e^2\right]}\left[\frac{2}{r}-\frac{1}{a}\right]$$

$$\frac{\partial\left[|V|^2\right]}{\partial e} = \frac{\partial\left[\frac{l^2}{a\left[1-e^2\right]}\left[\frac{2}{r}-\frac{1}{a}\right]\right]}{\partial e} =$$

$$2e \times \frac{l^2}{a\left[1-e^2\right]^2}\left[\frac{2}{r}-\frac{1}{a}\right] = \frac{2e}{\left[1-e^2\right]} \times \frac{l^2}{a\left[1-e^2\right]}\left[\frac{2}{r}-\frac{1}{a}\right] =$$

$$\frac{2e}{\left[1-e^2\right]}|V|^2$$

MAE 5540 Why is Hohmann Transfer Optimal For ΔV (cont'd)

$$\frac{\partial[\Delta V_T]}{\partial e} = \frac{1}{2}\left[\frac{1}{|V_1|}\frac{2\,e}{[1-e^2]}|V_1|^2 + \frac{1}{|V_2|}\frac{2\,e}{[1-e^2]}|V_2|^2\right] = \frac{e}{[1-e^2]}\left[|V_1| + |V_2|\right]$$

Or … plugging in the Vis Visa equation

$$\frac{\partial[\Delta V_T]}{\partial e} = \frac{e}{[1-e^2]}\left[\left|\frac{2\,\mu}{R_{c_1}} - \frac{\mu}{a_T}\right| + \left|\frac{2\,\mu}{R_{c_2}} - \frac{\mu}{a_T}\right|\right]$$

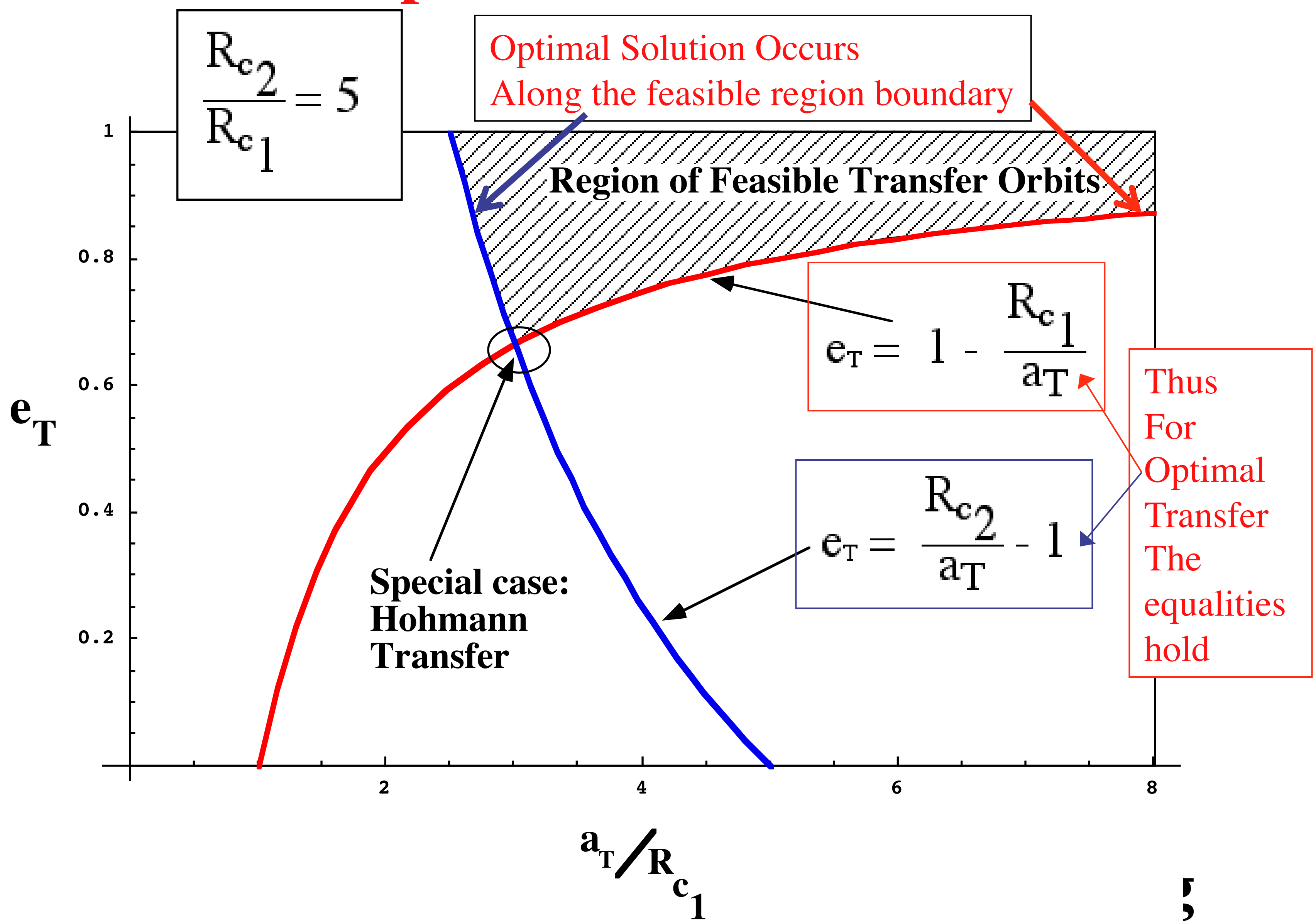
Why is Hohmann Transfer Optimal For ΔV (cont'd)

- Since e < 1 …. $\frac{\partial[\Delta V_T]}{\partial e}$ Is always positive

$$\frac{\partial[\Delta V_T]}{\partial e} = \frac{e}{[1-e^2]}\left[\left|\frac{2\mu}{R_{c_1}} - \frac{\mu}{a_T}\right| + \left|\frac{2\mu}{R_{c_2}} - \frac{\mu}{a_T}\right|\right]$$

- Thus for a given a_T, the minimum ΔV_T occurs at the minimum Allowable eccentricity

Why is Hohmann Transfer Optimal For ΔV (cont'd)



Why is Hohmann Transfer Optimal For ΔV (cont'd)

- Now Look at Orbital Energies

1 $$E_{\substack{Initial\\Orbit}} = \frac{-\mu}{R_{c_1}}$$

2 $$E_{\substack{Transfer\\Orbit}} = \frac{-\mu}{a_T}$$

3 $$E_{\substack{Initial\\Orbit}} = \frac{-\mu}{R_{c_1}}$$

$$\Delta E_{Transfer} = \left| E_{\substack{Transfer\\Orbit}} - E_{\substack{Initial\\Orbit}} \right| + \left| E_{\substack{Final\\Orbit}} - E_{\substack{Transfer\\Orbit}} \right|$$

Why is Hohmann Transfer Optimal For ΔV (cont'd)

- Substituting in from the Vis-Viva equation

$$\Delta E = \left|\left[\frac{-\mu}{2a_T} + \frac{\mu}{2R_{c_1}}\right]\right| + \left|\left[\frac{-\mu}{2R_{c_2}} + \frac{\mu}{2a_T}\right]\right|$$

$$\Delta E = \left|\left[\frac{-\mu(1-e_T)}{2R_{c_1}} + \frac{\mu}{2R_{c_1}}\right]\right| + \left|\left[\frac{-\mu}{2R_{c_2}} + \frac{\mu(1+e_T)}{2R_{c_2}}\right]\right|$$

$$\Delta E = \left[\frac{e_T\,\mu}{2R_{c_1}}\right] + \left[\frac{e_T\,\mu}{2R_{c_2}}\right] = \frac{e_T\,\mu}{2}\left[\frac{R_{c_2} + R_{c_1}}{R_{c_1}\,R_{c_2}}\right]$$

Boundary Value @ Burn 1 Curve

Boundary Value @ Burn 2 Curve

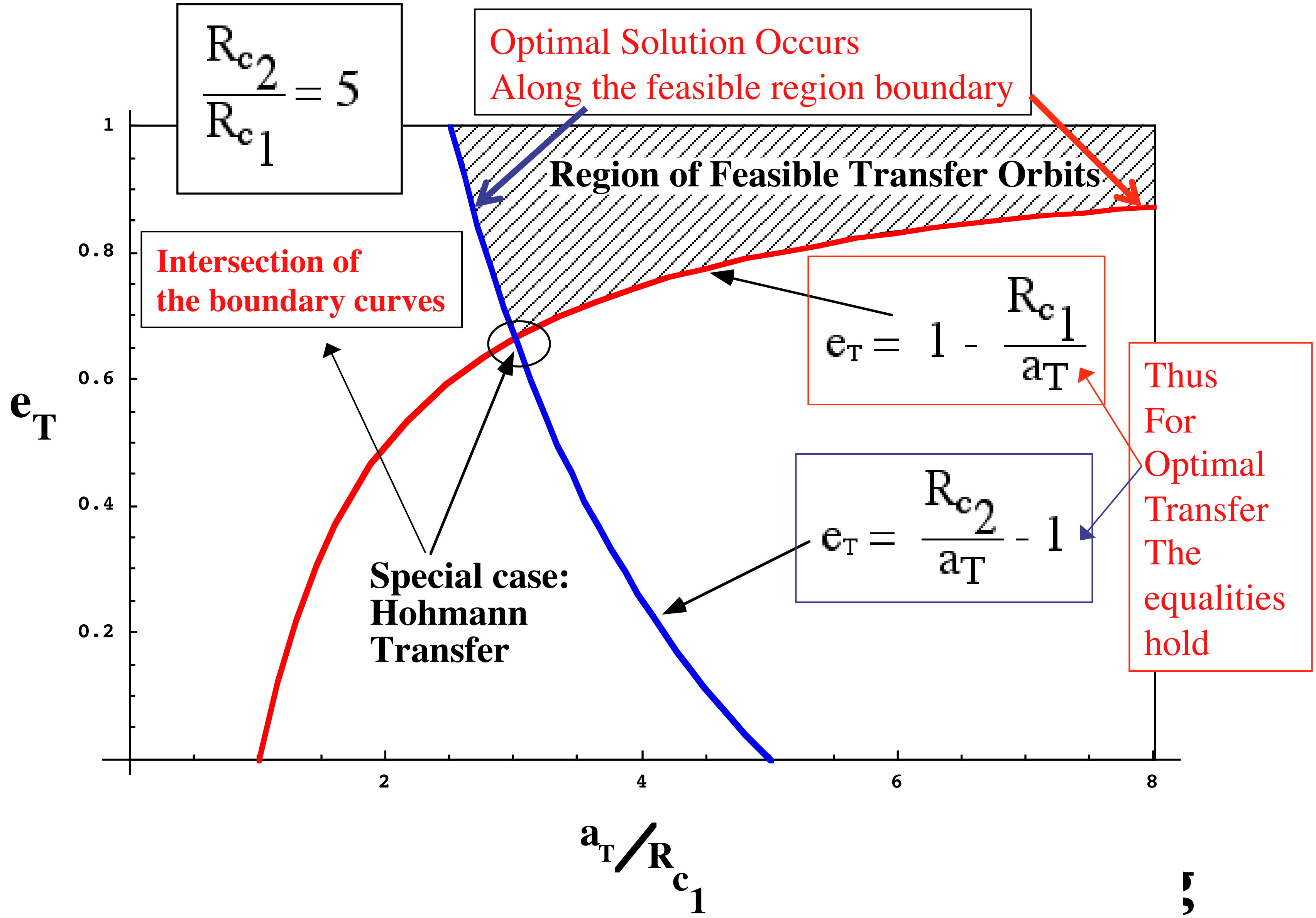
Why is Hohmann Transfer Optimal For ΔV (cont'd)

- Substituting in from the Vis-Viva equation

$$\Delta E_{Transfer} = \frac{e_T\, \mu}{2}\left[\frac{R_{c_2} + R_{c_1}}{R_{c_1}\, R_{c_2}}\right]$$

- Minimum transfer energy will be at minimum Eccentricity value … along the feasibility boundary

Why is Hohmann Transfer Optimal For ΔV (cont'd)



Why is Hohmann Transfer Optimal For ΔV (cont'd)

- Solving for a_T, e_T

$$a_T(1-e_T) = R_{c_1}$$
$$a_T(1+e_T) = R_{c_2}$$
$$\Rightarrow 2a_T = R_{c_1} + R_{c_2}$$

$$a_T = \frac{R_{c_1} + R_{c_2}}{2}$$

Why is Hohmann Transfer Optimal For ΔV (cont'd)

- Solving for a_T, e_T

$$e_T = \frac{R_{c_2}}{a_T} - 1 = \frac{R_{c_2}}{\frac{R_{c_1} + R_{c_2}}{2}} - 1 =$$

$$e_T = \frac{2R_{c_2} -}{R_{c_1} + R_{c_2}} - \frac{R_{c_1} + R_{c_2}}{R_{c_1} + R_{c_2}} = \boxed{\frac{R_{c_2} - R_{c_1}}{R_{c_2} + R_{c_1}}}$$

… The Hohmann Transfer

- We want our "transfer-orbit" to be an ellipse with parameters

$$a_T = \frac{\left[R_{c_2} + R_{c_1}\right]}{2}$$

$$e_T = \frac{R_{c_2} - R_{c_1}}{\left[R_{c_2} + R_{c_1}\right]}$$