Orbital Mechanics:

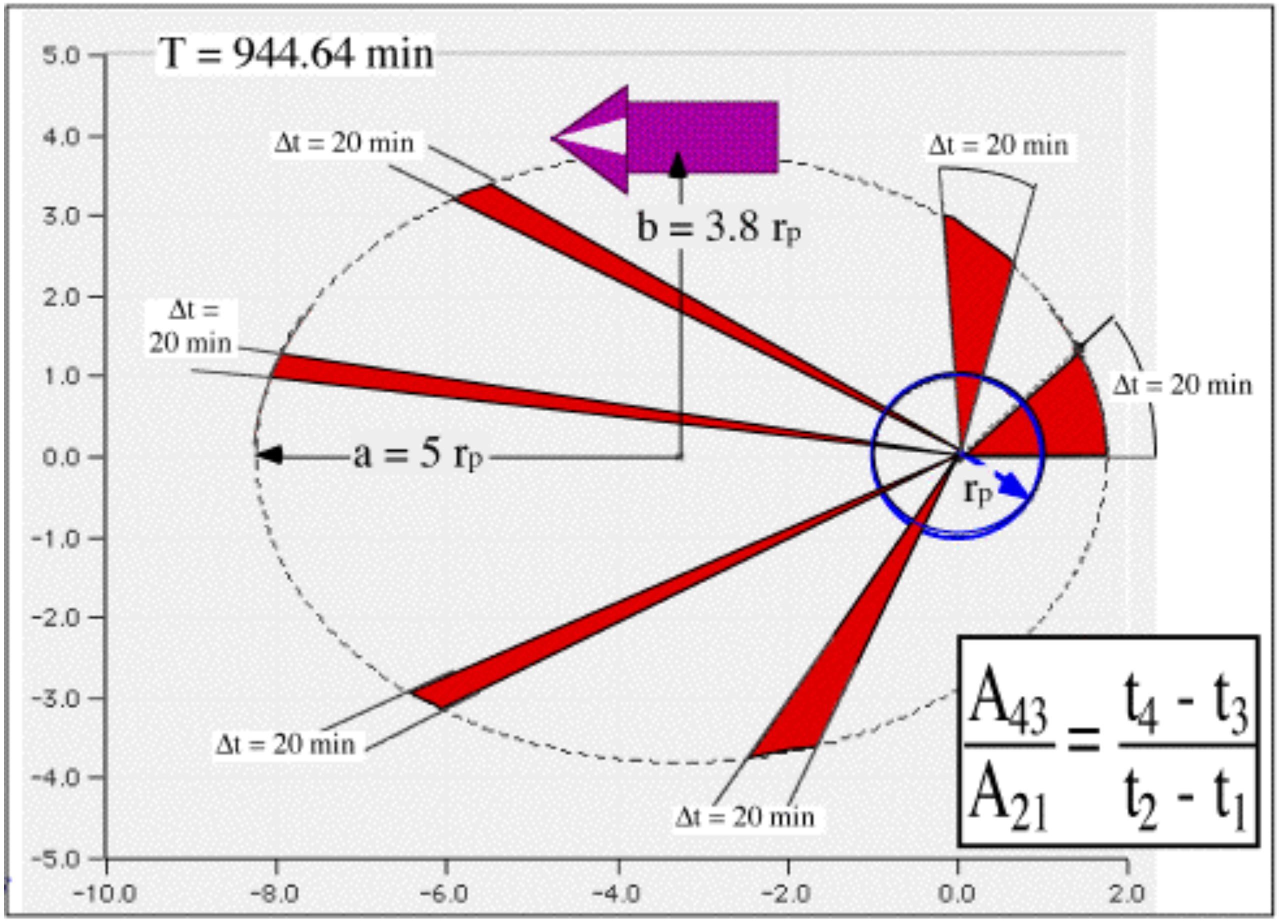
Conservation of Angular Momentum and the In-plane Velocity vector

Kepler's Second & Third Laws

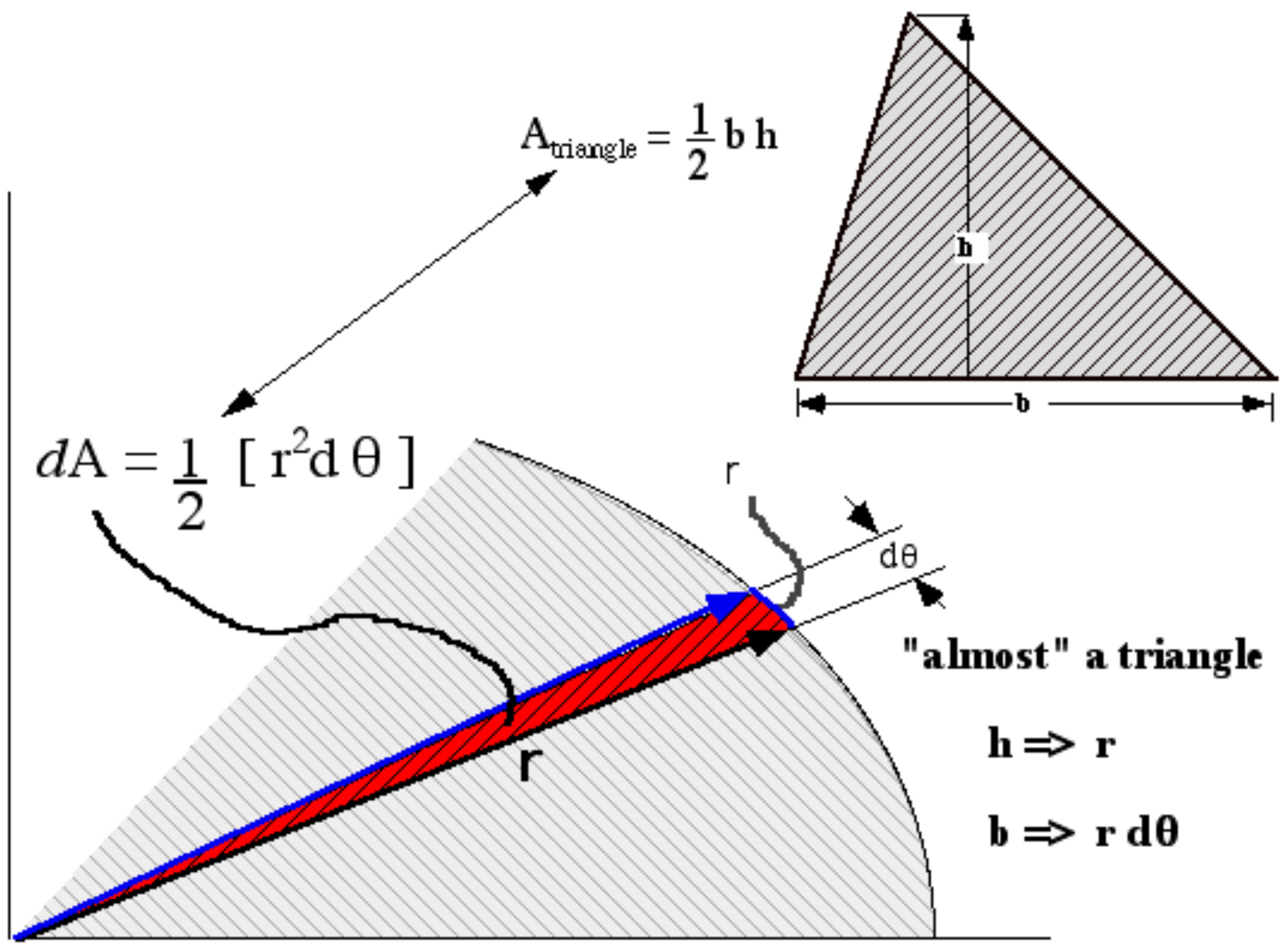
Sutton and Biblarz: Chapter 4

Kepler's Second Law

Kepler's Second Law: *In a two body universe, radius vector from the sun (Earth) to the planet (satellite) sweeps out equal areas in equal times*

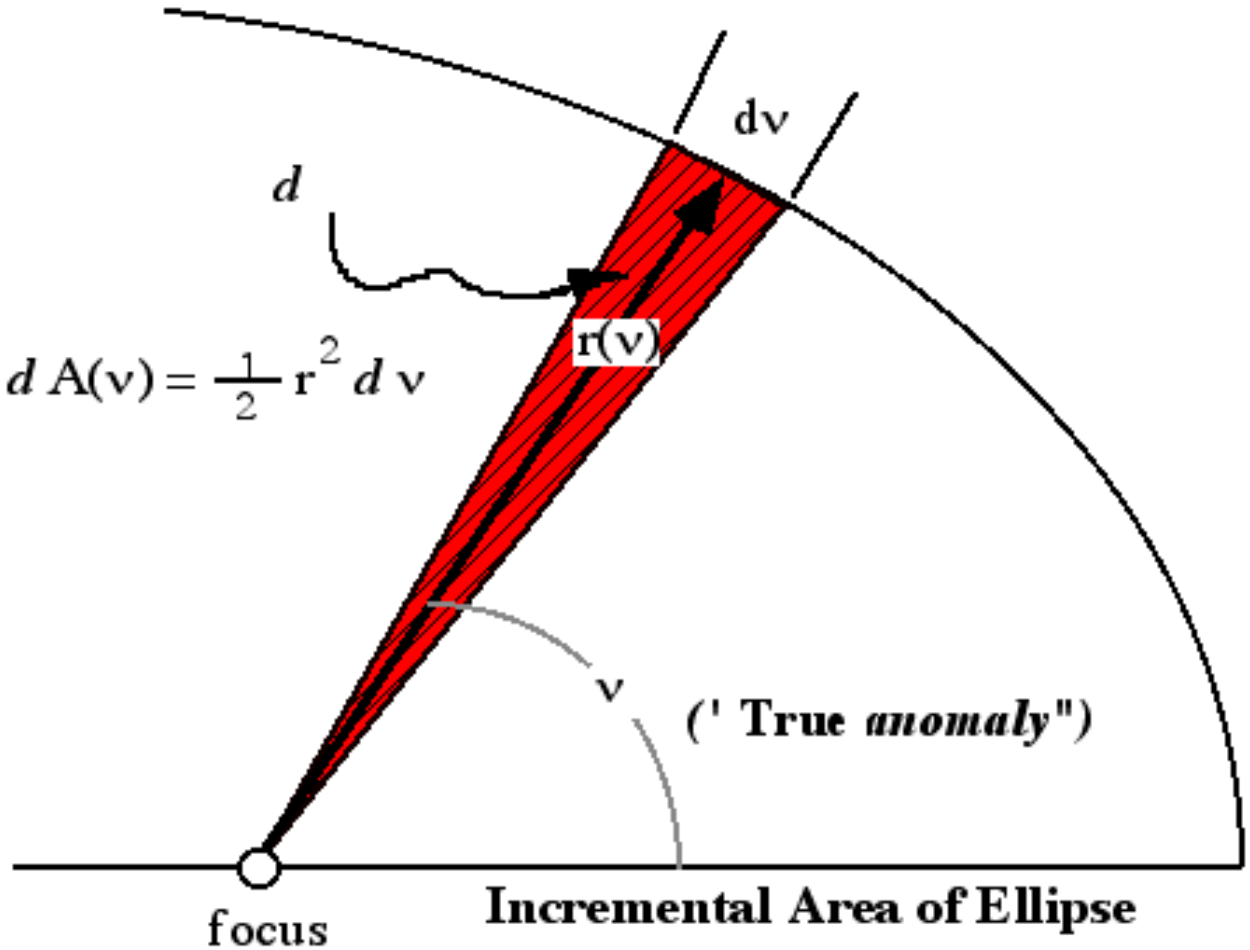


Incremental Area Swept Out by Radius vector



Incremental Area Swept out by an arc

Area Swept out by an Elliptical Arc



Incremental Area of Ellipse

Area Swept out by an Elliptical Arc (cont'd)

$$A_{\substack{\text{ellip.}\\ \text{arc}}} = \int_{\nu_0}^{\nu_1} \left[\frac{1}{2} r(\nu)^2 \, d\nu \right] =$$

$$A_{\substack{\text{ellip.}\\ \text{arc}}} = \int_{\nu_0}^{\nu_1} \left[\frac{1}{2} \left[\frac{a[1 - e^2]}{[1 + e \cos(\nu)]} \right]^2 d\nu \right] =$$

$$\boxed{\frac{1}{2}[a[1 - e^2]]^2 \int_{\nu_0}^{\nu_1} \left[\frac{1}{[1 + e \cos(\nu)]^2} d\nu \right]}$$

"very difficult" integral

Area Swept out by an Elliptical Arc (concluded)

$$A_{\substack{\text{ellip.}\\ \text{arc}}} = \int_{\nu_0}^{\nu_1} \left[\frac{1}{2} r(\nu)^2 \, d\nu \right] =$$

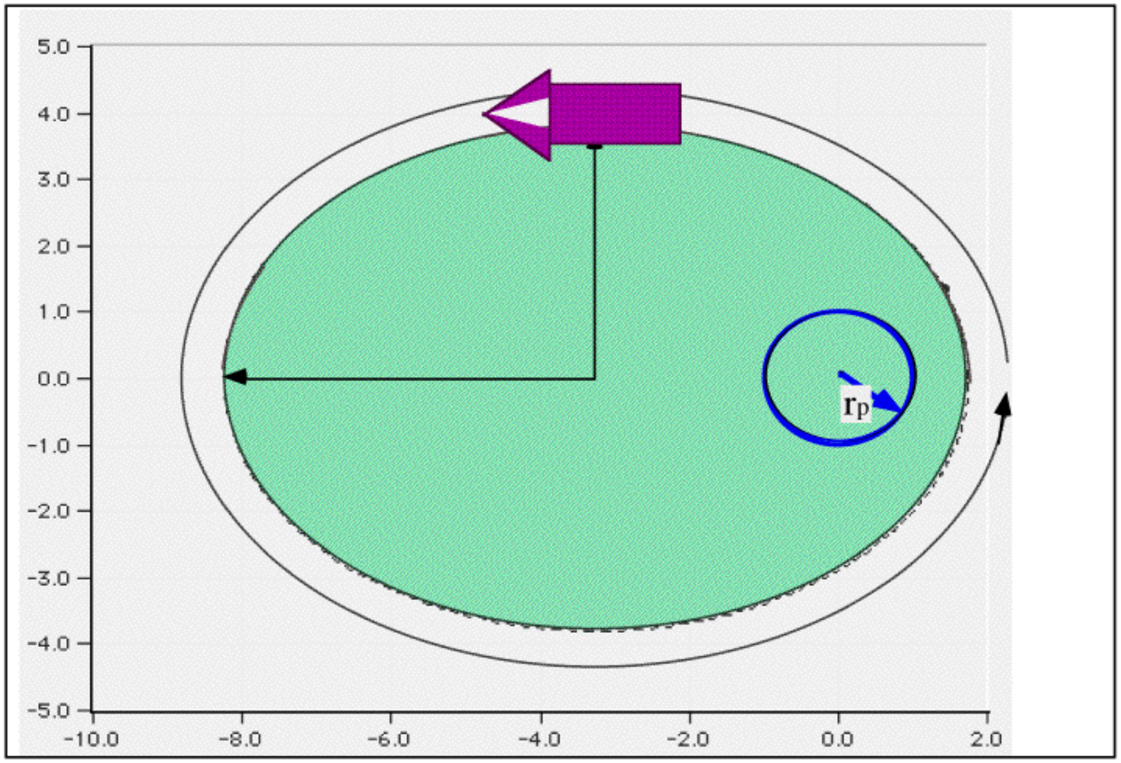
Ouch!

$$\frac{1}{2}\left[a\left[1 - e^2\right]\right]^2 \left[\frac{e\sqrt{e^2-1}\,\mathrm{Sin}(\nu_1) - 2F_1 - 2e\,\mathrm{Cos}(\nu_1)F_1}{(e^2-1)^{3/2}\left[1 + e\mathrm{Cos}(\nu_1)\right]} - \frac{e\sqrt{e^2-1}\,\mathrm{Sin}(\nu_0) - 2F_0 - 2e\,\mathrm{Cos}(\nu_0)F_0}{(e^2-1)^{3/2}\left[1 + e\mathrm{Cos}(\nu_0)\right]} \right]$$

$$\Downarrow$$

$$\boxed{F_1 = \mathrm{Tanh}^{-1}\left[\frac{(e-1)\,\mathrm{Tan}\left[\frac{\nu_1}{2}\right]}{\sqrt{e^2-1}} \right]} \qquad \boxed{F_0 = \mathrm{Tanh}^{-1}\left[\frac{(e-1)\,\mathrm{Tan}\left[\frac{\nu_0}{2}\right]}{\sqrt{e^2-1}} \right]}$$

Total Area of an Elliptical Orbit

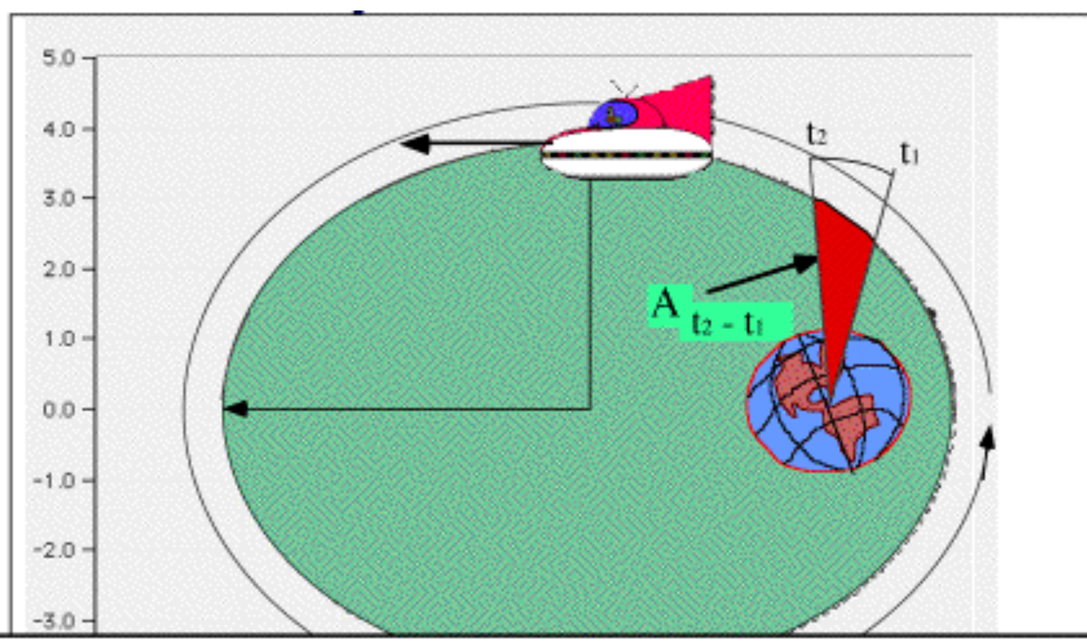


- But … the Total area integral Is a "Pretty Nice" form

$$A_{\substack{\text{ellipse}\\\text{total}}} = \int_0^{2\pi} \left[\frac{1}{2} r(\nu)^2 \, d\nu \right] = \boxed{a^2 \pi \sqrt{1 - e^2}}$$

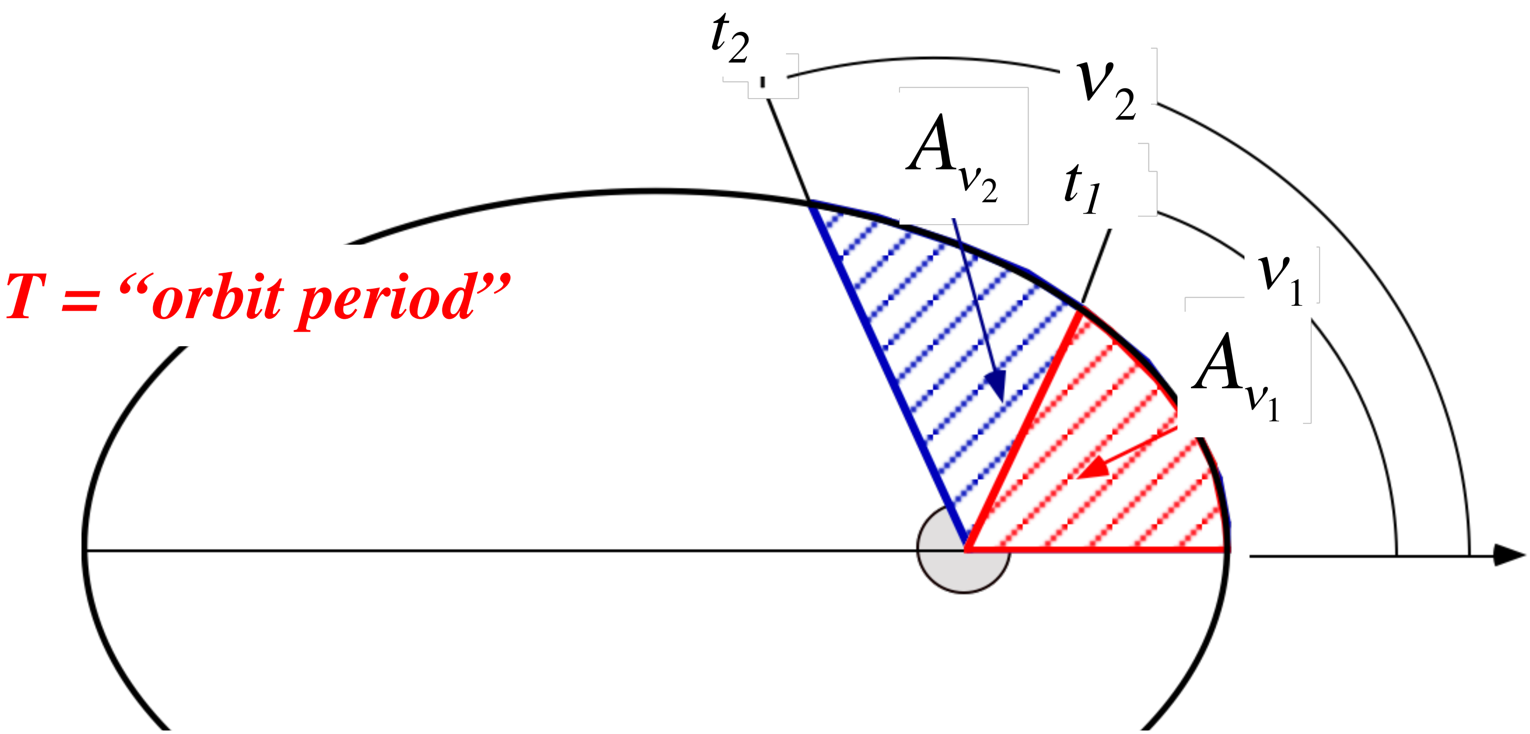
Mathematical Representation of Kepler's Second Law

T -->
Orbital period



$$A_{t_2 - t_1} = A_{\substack{\text{ellipse}\\ \text{total}}} \frac{t_2 - t_1}{T} \Rightarrow \boxed{A_{t_2 - t_1} = \left[a^2 \pi \sqrt{1 - e^2} \right] \frac{t_2 - t_1}{T}}$$

Area Swept from Perapsis



$$A_{\nu_2} = A_{\nu_1} + a^2 \cdot \pi \cdot \sqrt{1-e^2} \cdot \frac{t_2 - t_1}{T}$$

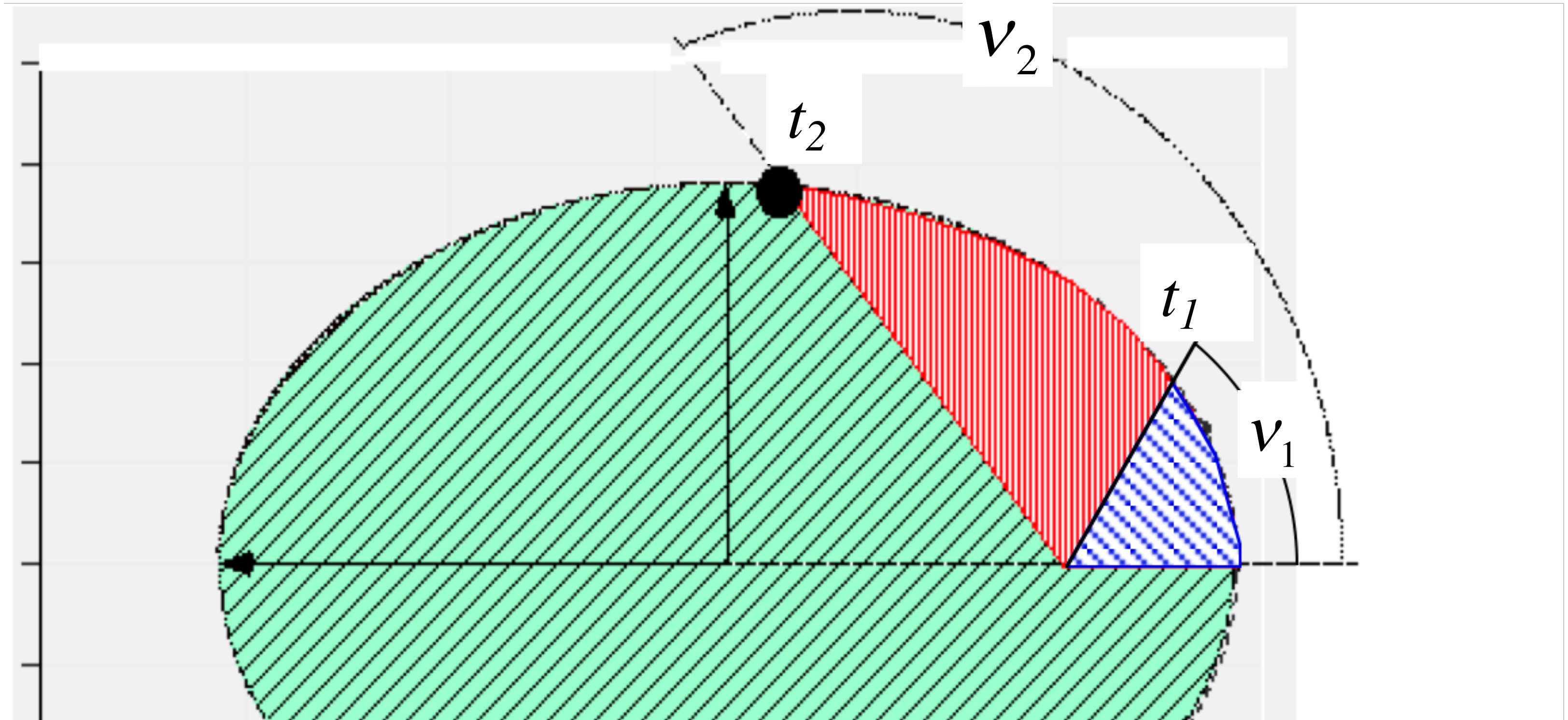
Time-Of-Flight Graphs

$$\frac{A_\nu}{a^2} = \cancelto{0}{\frac{A_0}{a^2}} + \pi\sqrt{1 - e^2}\left[\frac{t-0}{T}\right]$$

$$\Downarrow$$

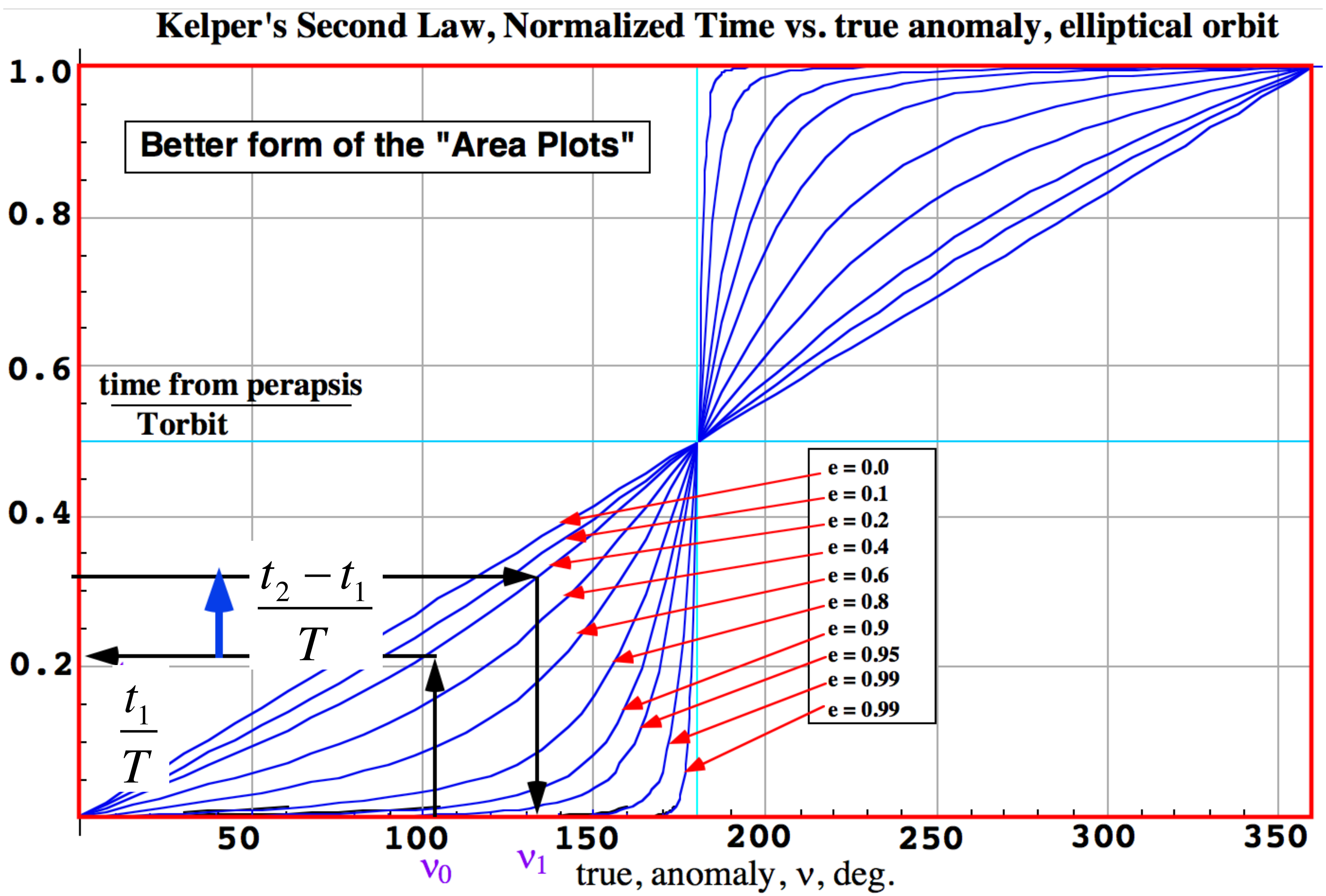
$$\frac{t_{perapsis}}{T} = \frac{A_\nu}{a^2\pi\sqrt{1 - e^2}} = \frac{\frac{1}{2}\int_0^{\nu_1}\left[\frac{1 - e^2}{1 + e\cos(\nu)}\right]^2 d\nu}{\pi\sqrt{\left[1 - e^2\right]}}$$

Time of Flight Graphs (cont'd)



$$\frac{t_2 - t_1}{T} = \frac{\displaystyle\int_0^{\nu_2} \left[\frac{1 - e^2}{1 + e\cos(\nu)}\right]^2 d\nu - \int_0^{\nu_1} \left[\frac{1 - e^2}{1 + e\cos(\nu)}\right]^2 d\nu}{2\pi\sqrt{\left[1 - e^2\right]}}$$

Propogation of Orbital Position



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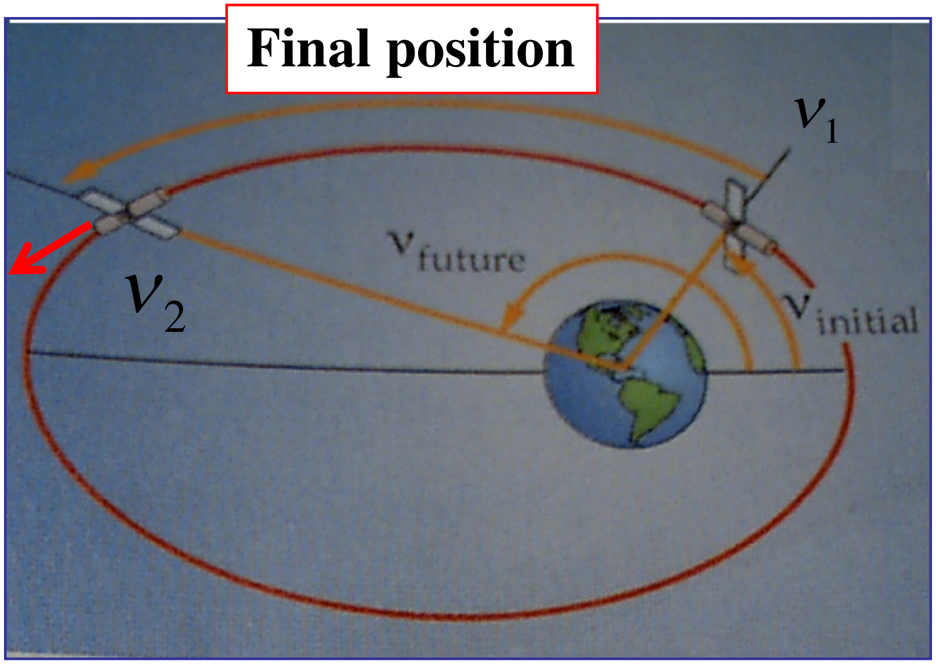
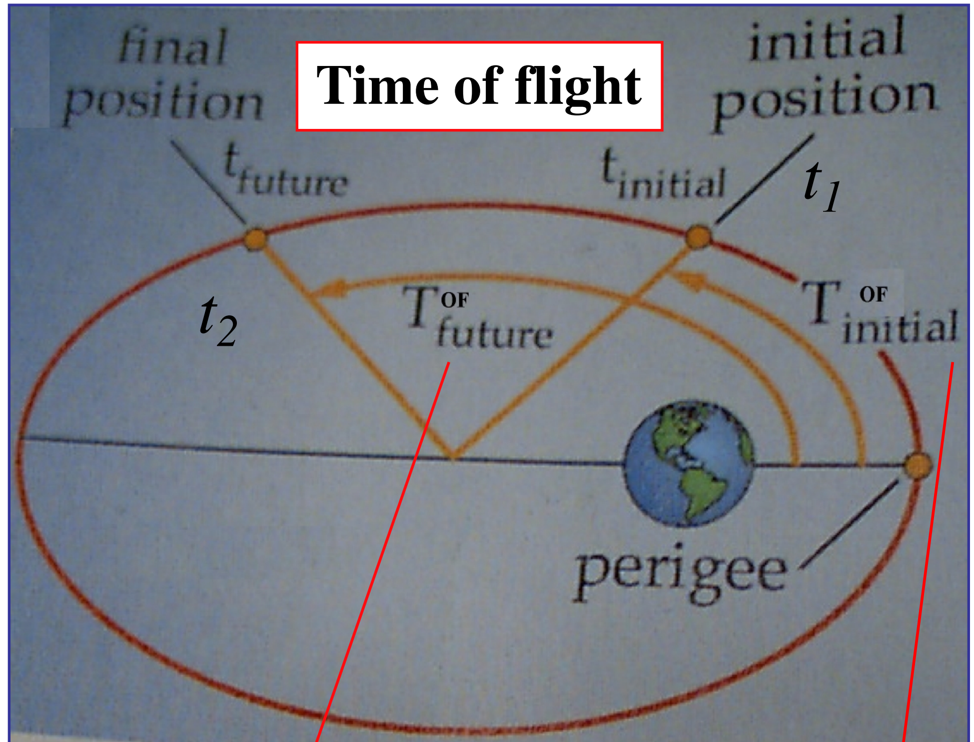
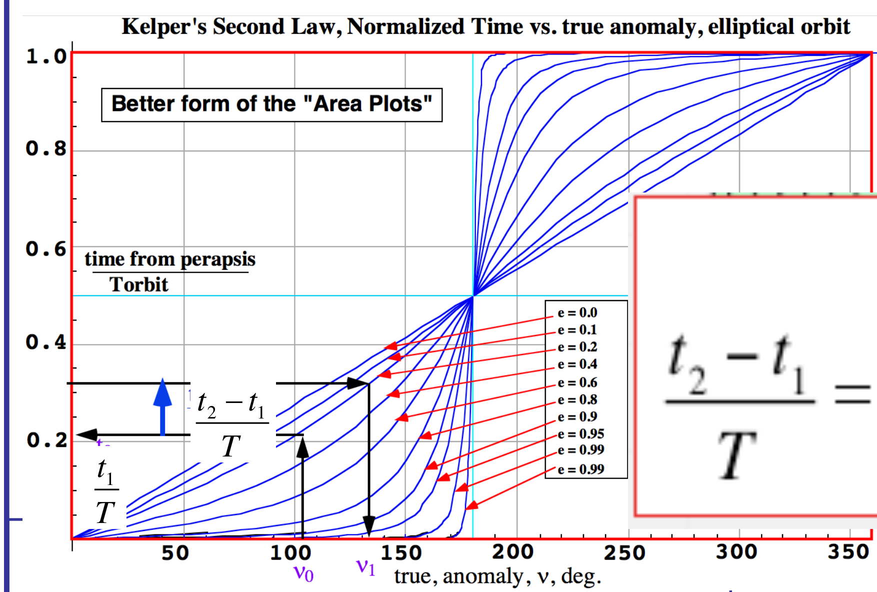
Time of Flight





Propogation of Orbital Position



$$\frac{t_2 - t_1}{T} = \frac{\int_0^{\nu_2}\left[\frac{1-e^2}{1+e\cos(\nu)}\right]^2 d\nu - \int_0^{\nu_1}\left[\frac{1-e^2}{1+e\cos(\nu)}\right]^2 d\nu}{2\pi\sqrt{[1-e^2]}}$$

Orbit Propagation … Kepler's Equation

$$\frac{t_2 - t_1}{T} = \frac{\int_0^{\nu_2} \left[\frac{1 - e^2}{1 + e\cos(\nu)}\right]^2 d\nu - \int_0^{\nu_1} \left[\frac{1 - e^2}{1 + e\cos(\nu)}\right]^2 d\nu}{2\pi\sqrt{\left[1 - e^2\right]}}$$

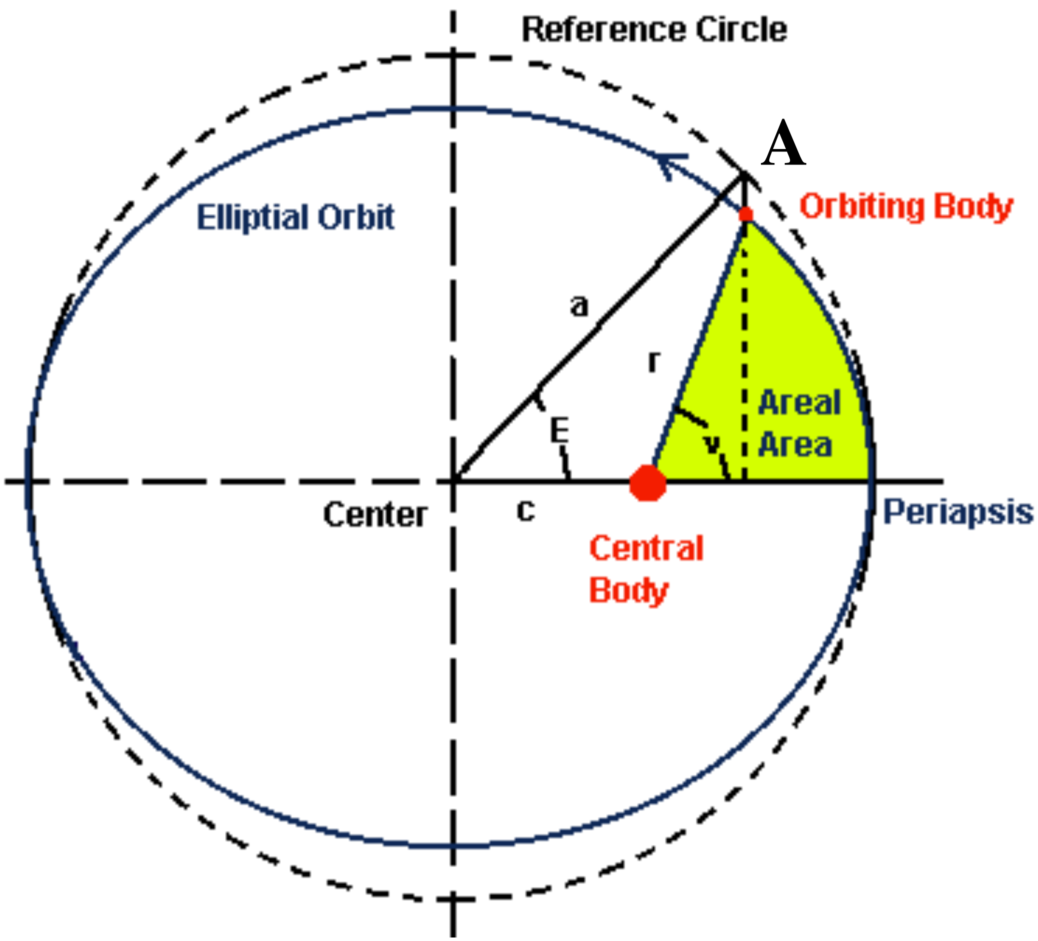
Solving this numerical integral for time of flight … given initial And final positions is "doable"

$$i.e.\ \ given ... \{\nu_1, \nu_2, T\} \rightarrow solve\ for\ transit\ time \rightarrow \{t_2 - t_1\}$$

… but the inverse problem is numerically unstable ….
and while TOF charts are good for illustrative purposes …
they are impractical for orbit propagation calculations …

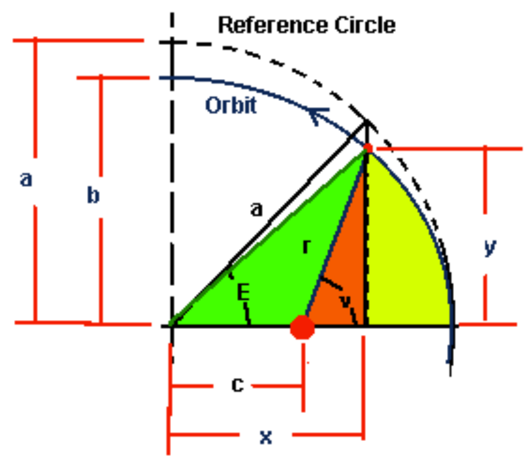
Fortunately …. 17th century mathematicians developed a better way

True and Eccentric Anomaly

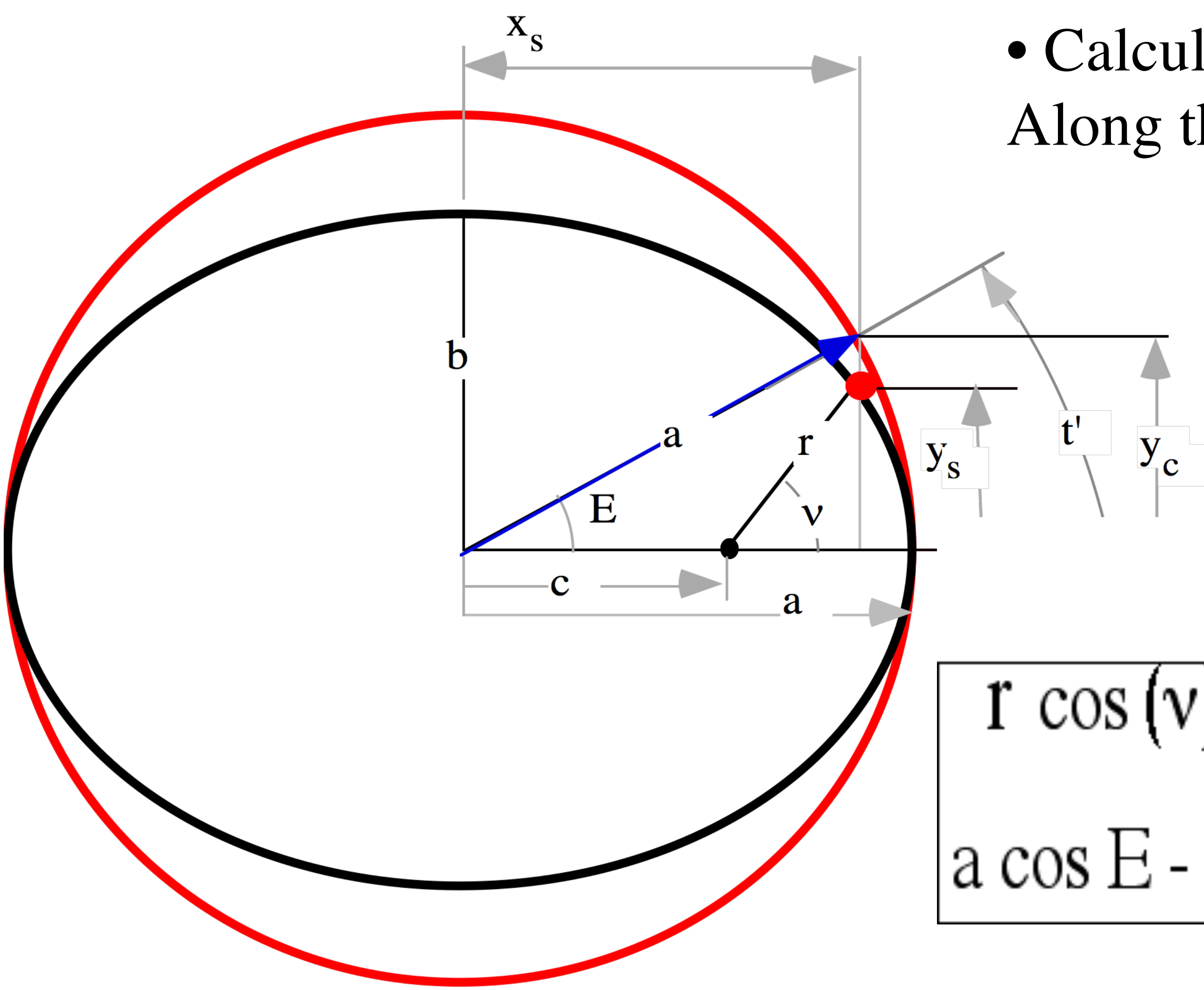


- **Define the angle obtained by drawing an auxiliary circle of around the ellipse, and drawing a line perpendicular to the semi-major axis and intersecting it at Point A**

- **The angle defined by the center radius "a" is called the "eccentric anomaly" E**



True and Eccentric Anomaly



• Calculating the projections Along the semi major axis

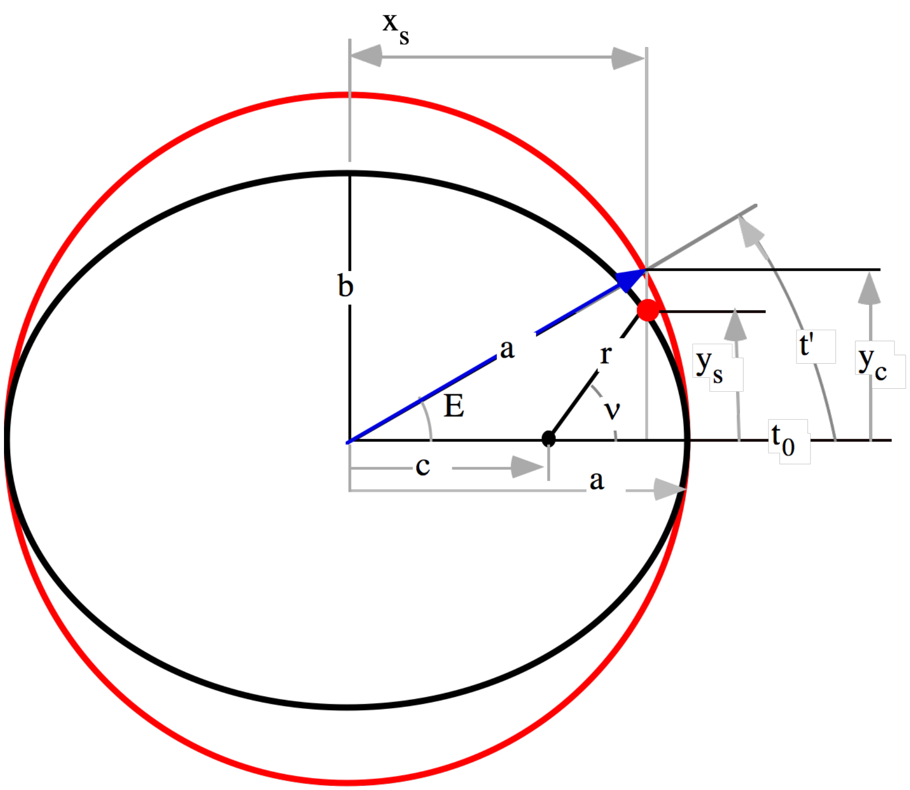
$$\boxed{c = a\,e}$$

$$r\cos(\nu) = a\cos E - c = a\cos E - a\,e = a[\cos E - e]$$

M

Anomaly: True and Eccentric

True and Eccentric Anomaly (cont'd)



Anomaly: True and Eccentric

$$r = a\left[1 - e\cos E\right]$$

- **Relationship of *r* (distance from planet to satellite) with respect to *E* (Eccentric anomaly) with *a*, *e* as parameters**

True and Eccentric Anomaly (cont'd)

- **Solving for *E* gives**

$$\left[1 - e\cos(E)\right]\cos(\nu) = \cos(E) - e \Rightarrow$$

$$\cos(\nu) + e = \cos(E) + e\cos(E)\cos(\nu) \Rightarrow$$

$$\boxed{\cos(E) = \frac{\cos(\nu) + e}{\left[1 + e\cos(\nu)\right]}}$$

True and Eccentric Anomaly (cont'd)

- **Thus we get the relationships**

$$\cos(\nu) = \frac{\cos(E) - e}{[1 - e\cos(E)]}$$

$$\cos(E) = \frac{\cos(\nu) + e}{[1 + e\cos(\nu)]}$$

- **These equations can be unified into a single equation by performing additional geometry**

True and Eccentric Anomaly (concluded)

After some REALLY Messy algebra …

- Or the final expression for the "true Anomaly" In terms of the "eccentric anomaly"

$$\tan\left[\frac{\nu}{2}\right] = \sqrt{\frac{1+e}{1-e}}\,\tan\left[\frac{E}{2}\right]$$

- Not even CLOSE! to finished yet .....

- OK, lets hang onto this ... we'll come back to it later

FINALLY ... (WHEW!) "KEPLER'S EQUATION"

And after some even Messier algebra the "area integral" reduces to

$$2\pi\left[\frac{t - t_0}{T}\right] = M_{t-0} = \left\{E - e\sin(E)\right\}$$

"M" – Mean Anomaly → Normalized Orbit Period Fraction

- **Where ..... t_0 is the time of perapsis passage and**

$$\tan\left[\frac{\nu}{2}\right] = \sqrt{\frac{1+e}{1-e}}\tan\left[\frac{E}{2}\right]$$

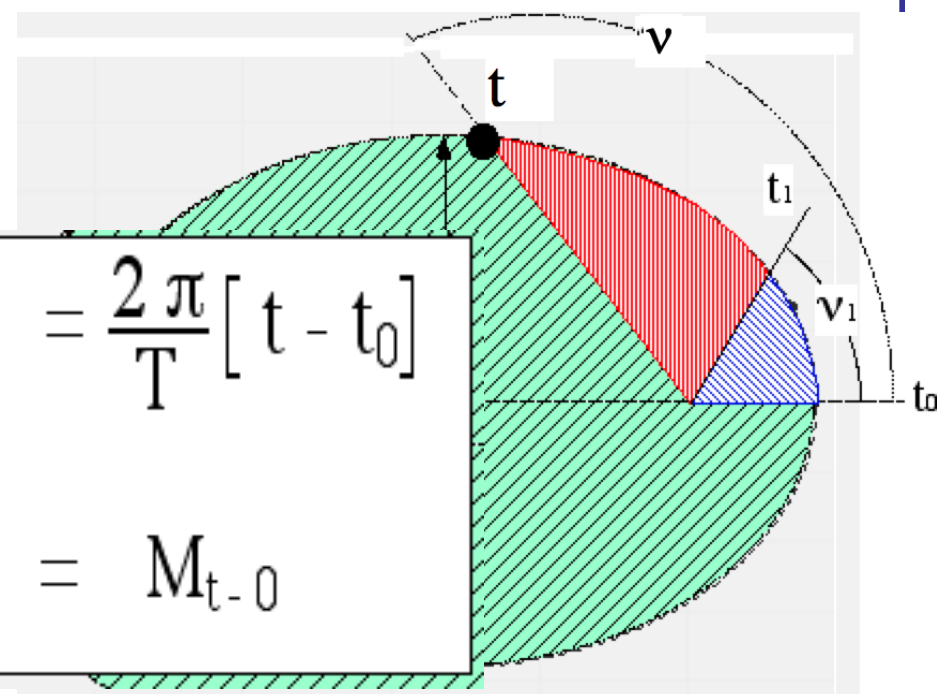
Mean anomaly

$$M_t \equiv 2\pi\frac{t_{\text{from perapsis}}}{T}$$

Using Mean Anomaly to Propagate the Orbital Position

- **Kepler's Equation defines the swept area from perapsis to the current position ...**

- **Adapted to an arbitrary starting point by performing following transformation**



$$\frac{2\pi}{T}\left[t - t_1\right] + \frac{2\pi}{T}\left[t_1 - t_0\right] = \frac{2\pi}{T}\left[t - t_0\right]$$

$$\frac{2\pi}{T}\left[t - t_1\right] + M_{t_1 - 0} = M_{t - 0}$$

Using Mean Anomaly to Propagate the Orbital Position
(cont'd)

- **Substituting into Kepler's equation**

$$M_{t-0} = \left\{ E - e \sin(E) \right\} = \frac{2\pi}{T}\left[t - t_1\right] + M_{t_1-0}$$

- **Accounting for the fact that t may be large enough that multiple orbits may have passed during the time from t_1 to t**

$$M_{t-0} = \left\{ E - e \sin(E) \right\} = \frac{2\pi}{T}\left[t - t_1 - kT\right] + M_{t_1-0}$$

where $k = \mathrm{INT}\left[\frac{t - t_1}{T}\right]$ (integer number of periods elapsed since t_1)

Using Mean Anomaly to Propagate the Orbital Position
(cont'd)

- The time increment term can be further simplified by noting that

$$\frac{2\pi}{T}\left[t - t_1 - kT\right] = \frac{2\pi}{T}\left[t - t_1 - \mathrm{INT}\left[\frac{t - t_1}{T}\right] \times T\right] =$$

$$\frac{2\pi}{T}\underbrace{\mathrm{Modulus}\left[\left(t - t_1\right) , T\right]}_{\text{"Remainder function"}}$$

$$\boxed{\text{i.e. } \mathrm{Modulus}[31 , 7] = 31 - \mathrm{int}[31/7] \times 7 = 3}$$

Using Mean Anomaly to Propagate the Orbital Position

(concluded)

- **And the working form of Kepler's equation results**

$$\boxed{\{E - e\ \sin(E)\} = \frac{2\pi}{T}\mathrm{Modulus}[(t - t_1), T] + M_{t_1 - 0}}$$

Kepler

Kepler's Revenge!

The Propagation Algorithm

- Given the orbit parameters *(a, e)* and an Initial Starting Time and Position

$$t_1 \, , \nu_1$$

1) Compute *Starting Eccentric Anomaly (work in radians!)*

$$E_1 = 2\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\left[\frac{\nu_1}{2}\right]\right]$$

Propagation Algorithm
(continued)

2) Now Compute *Current Mean Anomaly (work in radians!)*

$$M_{t_1 - 0} = \left\{ E_1 - e \sin\left(E_1\right) \right\}$$

Propagation Algorithm

(continued)

3) Compute the orbit Period

$$T = 2\pi \frac{a^{3/2}}{\sqrt{\mu}}$$

Kepler's third law

We still need to derive this!

4) and the *future mean anomaly* (at time t)

$$M_{t-0} = \frac{2\pi}{T} \mathrm{Modulus}\left[\left(t - t_1\right), T\right] + M_{t_1 - 0}$$

Propagation Algorithm

(continued)

5) Now Solve Kepler's Equation for the *New Eccentric Anomaly*

$$M_{t-0} = \left\{ E_t - e \sin\left(E_t\right) \right\}$$

……… Use your Newton Solver! ……………

True Mean Anomaly

Iteration index

$$E^{(j+1)} = E^{(j)} + \frac{M - \overbrace{\left[E^{(j)} - e \sin\left(E^{(j)}\right)\right]}^{\text{Current estimate of mean anomaly}}}{\underbrace{1 - e \cos\left(E^{(j)}\right)}_{\text{Steepest descent}}}$$

Current estimate

Refined estimate

Propagation Algorithm
(concluded)

6) Compute the *NEW true anomaly*

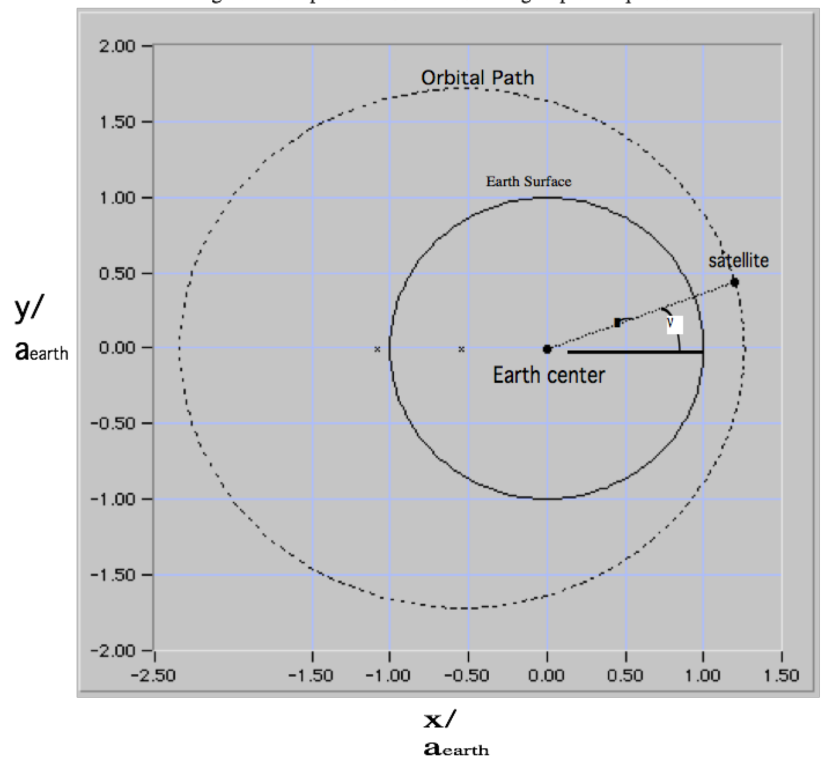
$$\nu_t = 2\tan^{-1}\left[\sqrt{\frac{1+e}{1-e}}\tan\left[\frac{E_t}{2}\right]\right]$$

7) Finally compute the new radius vector

$$r_t = \frac{a\left[1 - e^2\right]}{1 + e\cos\left(\nu_t\right)}$$

- **At this point you have propagated the orbit for exactly One time point**

Figure 5: Sample Orbit Calculation using Kepler's Equation



Kepler's Second Law (Alternate form)
What is the physical Interpretation?

Reconsider the “Swept Area” Integral

$$\boxed{A_{\nu_2} = A_{\nu_1} + a^2 \cdot \pi \cdot \sqrt{1-e^2} \cdot \frac{t_2 - t_1}{T}}$$

Let’s look at this integral in differential form …

Kepler's Second Law (Alternate form) What is the physical Interpretation?

$$\left\{ \begin{array}{l} \text{Let: } [\, t_2 \Rightarrow t_1] \rightarrow t_2 - t_1 = dt \\ \text{Then: } A_{t_2 - t_1} = d\,A(t) \end{array} \right\} \Rightarrow d\,A(t) = \left[\, a^2 \pi \sqrt{1 - e^2} \,\right] \frac{dt}{T}$$

- **But** $\boxed{dA(t) = \frac{1}{2} r^2\, d\nu}$

and

$$\boxed{\frac{d\,A(t)}{dt} = \frac{\left[\, a^2 \pi \sqrt{1 - e^2} \,\right]}{T} = \frac{\frac{1}{2} r^2\, d\nu}{dt} = \frac{1}{2} r^2 \frac{d\nu}{dt}}$$

Mathematical Representation of Kepler's Second Law (continued)

$$2 \times \frac{d\,A(t)}{d\,t} \qquad 2 \times \frac{Total\ Area}{Period} \qquad Specific\ Angular\ momentum$$

$$r^2 \frac{d\nu}{d\,t} = \frac{2\left[\,a^2 \pi \sqrt{1 - e^2}\,\right]}{T} \equiv l$$

Prove This!

"Constant"

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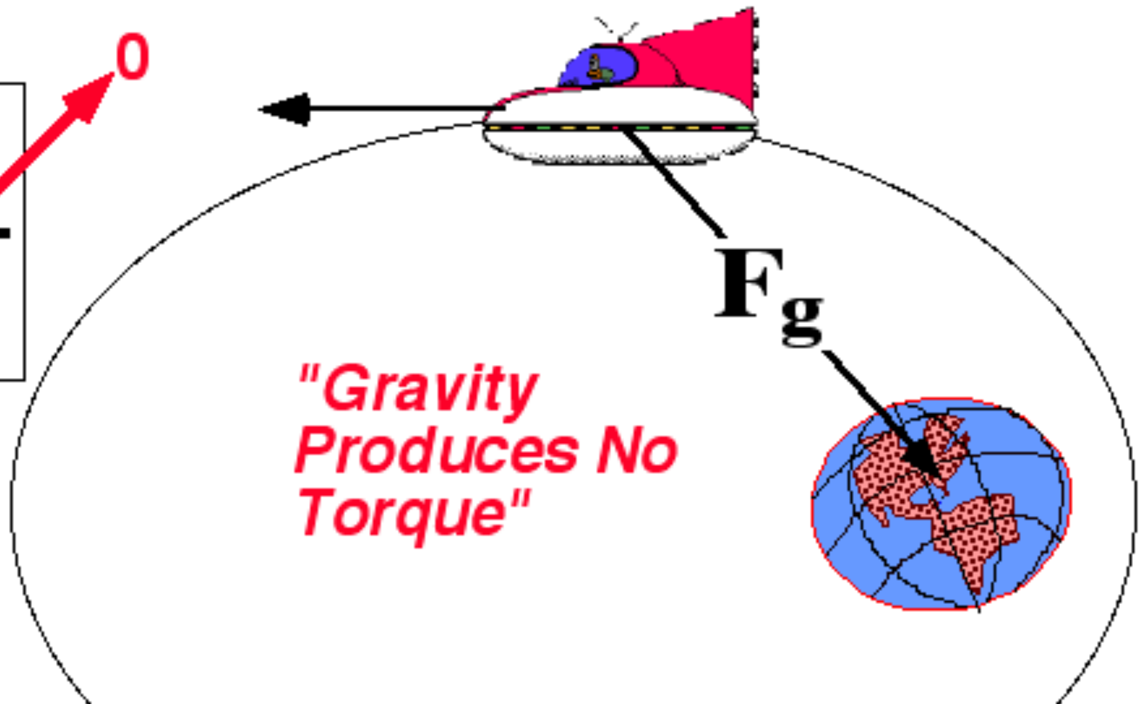
Torque Acting on Orbiting Space Craft

"rate of change of angular momentum"

$$\tau = \cancelto{0}{\frac{d\mathbf{L}}{dt}}$$

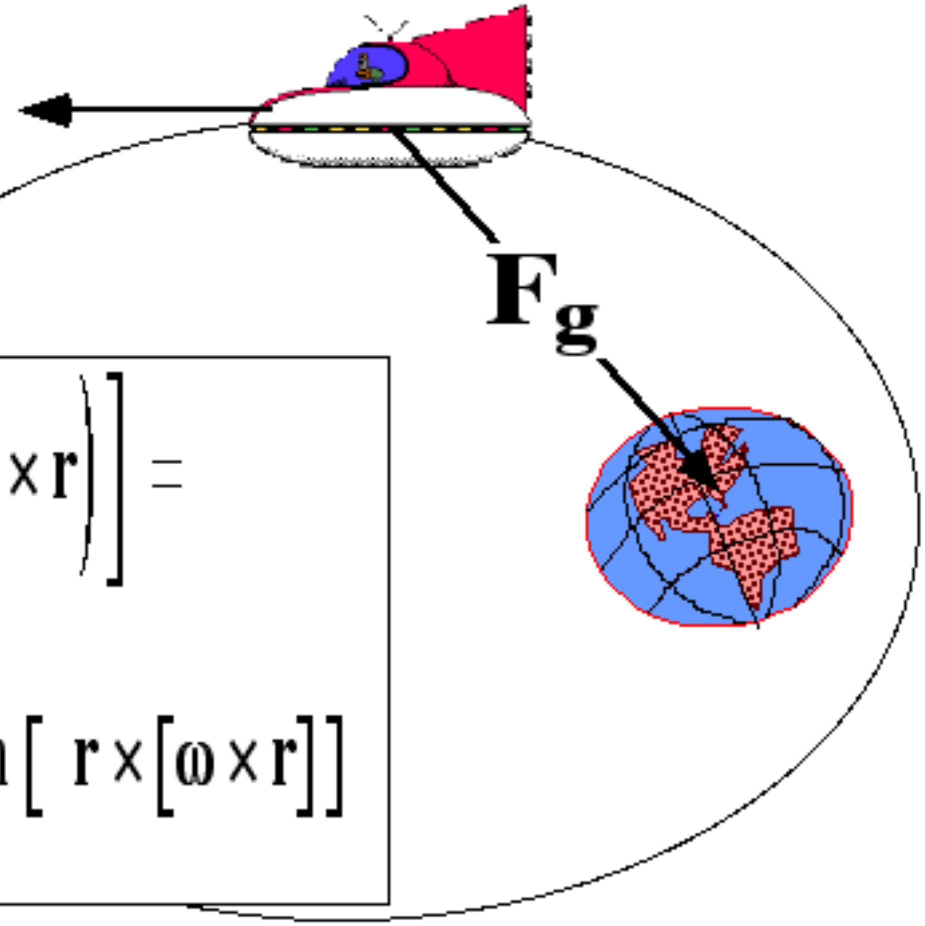


$$\bar{\tau} = \bar{\mathbf{r}} \times \bar{\mathbf{F}}_{\mathbf{g}} = \bar{\mathbf{r}} \times \frac{(\mathrm{m\ M})}{\mathrm{r}^2}\,\bar{i}_r = \frac{(\mathrm{m\ M})}{\mathrm{r}^2}\,\bar{\mathbf{r}} \times \bar{i}_r = 0$$

Angular Momentum of An Orbiting Spacecraft

$$\mathbf{L} = m\left(\mathbf{r} \times \mathbf{V}\right)$$

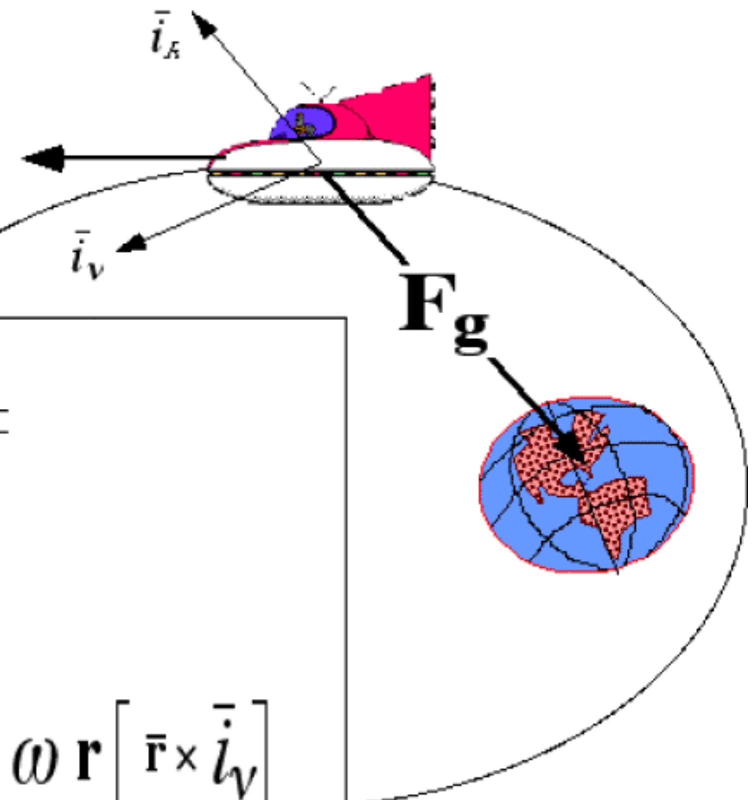
$$\mathbf{V} = \frac{\partial r}{\partial t}\mathbf{i}_r + \boldsymbol{\omega} \times \mathbf{r}$$



$$\mathbf{L} = m\left[\mathbf{r} \times \left(\frac{\partial r}{\partial t}\mathbf{i}_r + \boldsymbol{\omega} \times \mathbf{r}\right)\right] =$$

$$m\left[\frac{\partial r}{\partial t}\mathbf{r} \times \mathbf{i}_r + \mathbf{r} \times \left[\boldsymbol{\omega} \times \mathbf{r}\right]\right] = m\left[\ \mathbf{r} \times \left[\boldsymbol{\omega} \times \mathbf{r}\right]\right]$$

Angular Momentum of An Orbiting Spacecraft (cont'd)



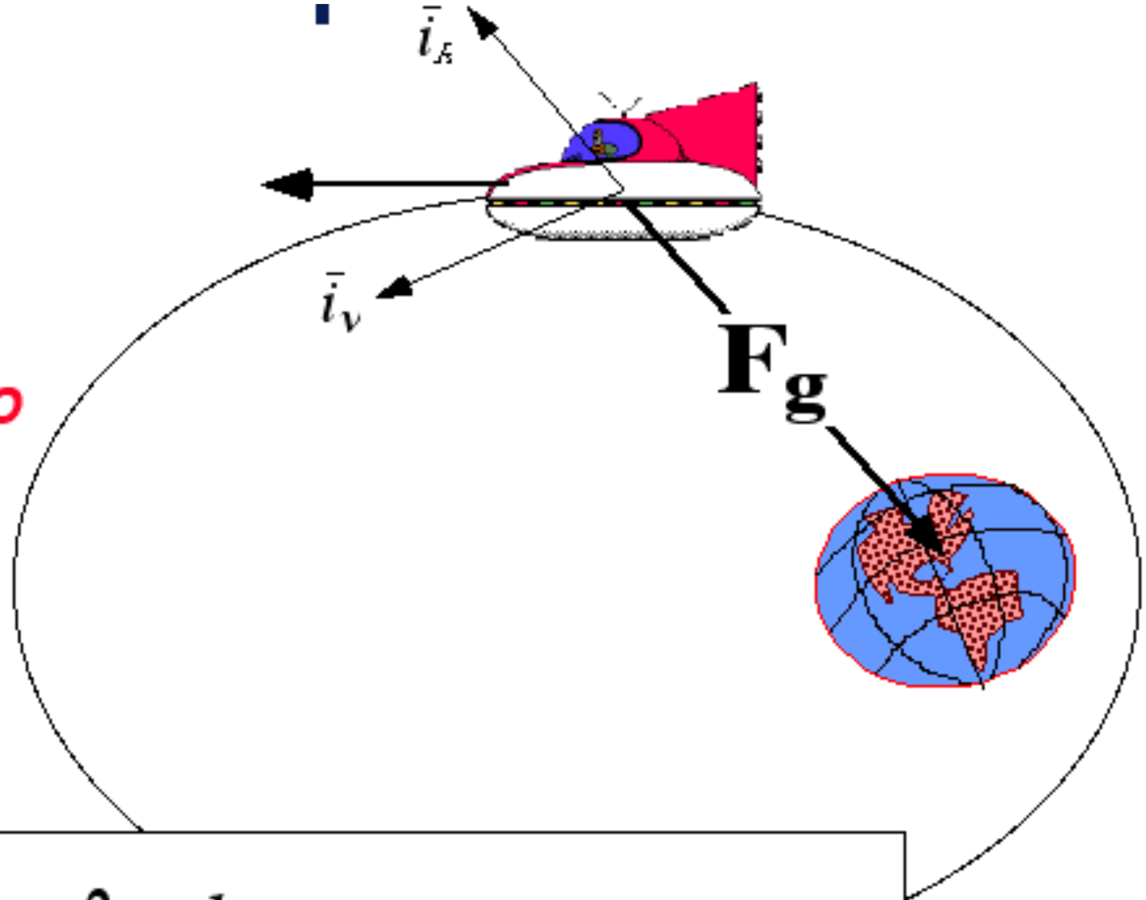
$$\frac{\bar{L}}{m} = \bar{r} \times \left[\frac{\partial \bar{r}}{\partial t} + \left[\bar{\omega} \times \bar{r} \right] \right] =$$

$$\bar{r} \times \left[\frac{\partial \mathbf{r}}{\partial t} \bar{i}_R \right] + \bar{r} \times \begin{bmatrix} \bar{i}_R & \bar{i}_V & \bar{i}_k \\ 0 & 0 & \omega \\ \mathbf{r} & 0 & 0 \end{bmatrix} = \omega\, \mathbf{r} \left[\bar{r} \times \bar{i}_V \right]$$

Angular Momentum of An Orbiting Spacecraft (cont'd)

$$\tau = \frac{d\mathbf{L}}{dt} = 0$$

"Gravity Produces No Torque"



$$\frac{\bar{\mathbf{L}}}{\mathbf{m}} \equiv \bar{l} = \omega\,\mathbf{r}^2\,\bar{i}_k \Rightarrow \omega\,\mathbf{r}^2 = l \text{ (specific angular momentum)}$$

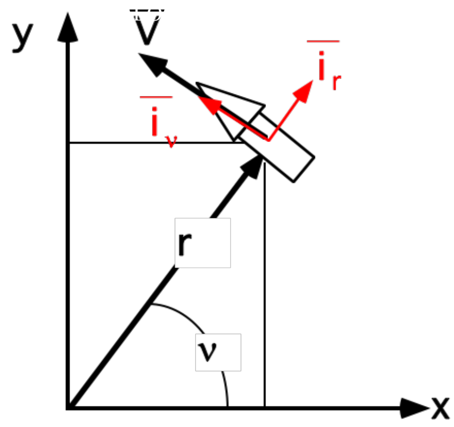
"Constant"

Alternate Statement of of Kepler's Second law:

$$\frac{\bar{\mathbf{L}}}{\mathbf{m}} \equiv \bar{l} = \omega\, \mathbf{r}^2\, \bar{i}_k \Rightarrow \underbrace{\omega\, \mathbf{r}^2 = l \,(\textit{specific angular momentum})}$$

"The angular momentum of an orbiting object is constant"

The velocity vector:



- for a while we have Only worried about position

… now revisit velocity

Apply the "Chain Rule"

$$\overline{V}_s = \frac{d}{dt}\left[\overline{R}_s\right] = \frac{d}{dt}\left[r\,\overline{i}_r\right] = \frac{d}{dt}\left[r\right]\overline{i}_r + r\,\frac{d}{dt}\left[\,\overline{i}_r\right] =$$

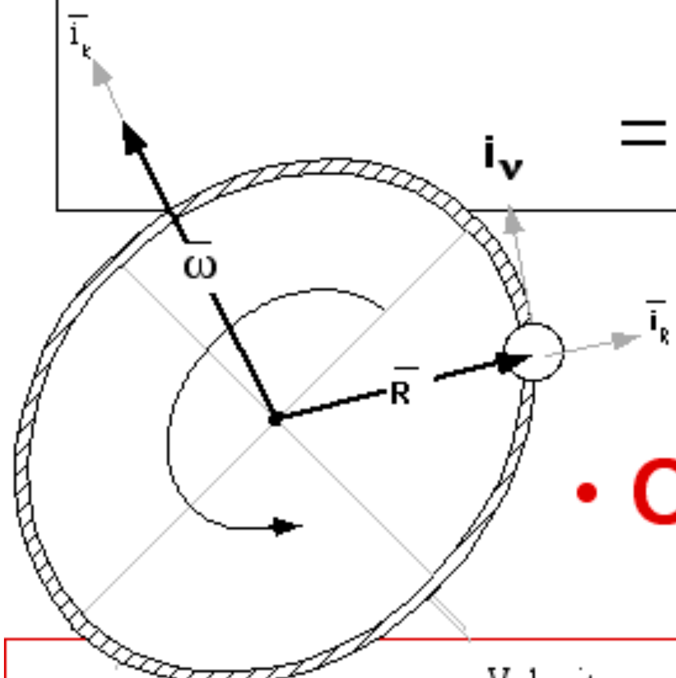
$$= \dot{r}\,\overline{i}_r + \left[r\right]\frac{d}{dt}\left[\,\overline{i}\cos\left[\nu\right] + \overline{j}\sin\left[\nu\right]\right] =$$

$$\dot{r}\,\overline{i}_r + r\left[-\overline{i}\sin\left[\nu\right] + \overline{j}\cos\left[\nu\right]\right]\dot{\nu}$$

The velocity vector:

$$\overline{V}_s = \frac{d}{dt}\left[\overline{R}_s\right] = \dot{r}\,\overline{i}_r + r\left[-\overline{i}\sin[\nu] + \overline{j}\cos[\nu]\right]\dot{\nu}$$

$$= \dot{r}\,\overline{i}_r + r\,\dot{\nu}\,\overline{i}_\nu$$



- Or In more general terms

Velocity relative to external world		Velocity relative to center of coordinates		Angular velocity of *spacecraft*	cross product	Position vector relative to center of coordinates
$V_{external}$	$=$	$\frac{\partial \overline{R}}{\partial t}$	$+$	$\overline{\omega}$	$\times$	$\overline{R}$

The velocity vector: (cont'd)

$$\overline{V}_s = \frac{d}{dt}\left[\overline{R}_s\right] = \dot{r}\,\overline{i}_r + r\left[-\overline{i}\,\sin[\nu] + \overline{j}\,\cos[\nu]\right]\dot{\nu}$$

$$= \dot{r}\,\overline{i}_r + r\,\dot{\nu}\,\overline{i}_\nu$$

$$\omega = \frac{\partial \nu}{\partial t}$$

$$\overline{\mathbf{V}} = \frac{\partial \overline{\mathbf{R}}}{\partial t} + \left[\overline{\boldsymbol{\omega}} \times \overline{\mathbf{R}}\right] =$$

$$\frac{\partial \mathbf{r}}{\partial t}\overline{i}_r + \left|\begin{bmatrix} \overline{i}_r & \overline{i}_\nu & \overline{i}_k \\ 0 & 0 & \omega \\ \mathbf{r} & 0 & 0 \end{bmatrix}\right| = \frac{\partial \mathbf{r}}{\partial t}\overline{i}_r + \omega\,\mathbf{r}\,\overline{i}_\nu$$

Velocity Vector, Elliptical orbit

$$\overline{V} = \frac{\partial r}{\partial t}\bar{i}_r + \omega\, r\, \bar{i}_\nu$$

- The *polar form* of the *ellipse equation*

$$r = \frac{a\left[1 - e^2\right]}{\left[1 + e\ \cos(\nu)\right]}$$

$$\dot{r} \neq 0$$

Velocity Vector, Elliptical Orbit

$$\overline{V} = \frac{d}{dt}\overline{r} = \frac{d}{dt}\left[r(\nu)\right]\overline{i}_r + r(\nu)\,\omega\,\overline{i}_\nu$$

$$\overline{r} = \left[\frac{a\left[1 - e^2\right]}{1 + e\cos(\nu)}\right]$$

$$\frac{d}{dt}\left[r(\nu)\right] = \frac{d}{dt}\left[\left[\frac{a\left[1 - e^2\right]}{1 + e\cos(\nu)}\right]\right] = \frac{-a\left[1 - e^2\right]}{\left[1 + e\cos(\nu)\right]^2}\left[-e\sin(\nu)\right]\frac{d\nu}{dt} =$$

$$\frac{a\left[1 - e^2\right]}{\left[1 + e\cos(\nu)\right]}\frac{\left[e\sin(\nu)\right]}{\left[1 + e\cos(\nu)\right]}\omega = \boxed{r(\nu)\,\omega\frac{\left[e\sin(\nu)\right]}{\left[1 + e\cos(\nu)\right]}}$$

Velocity Vector, Elliptical Orbit
(concluded)

$$\bar{V} = \frac{d}{dt}\bar{r} = \frac{d}{dt}\left[r(\nu)\right]\bar{i}_r + r(\nu)\,\omega\,\bar{i}_\nu$$

$$\frac{d}{dt}\left[r(\nu)\right] = r(\nu)\,\omega\,\frac{\left[e\sin(\nu)\right]}{\left[1 + e\cos(\nu)\right]}$$

$$\bar{V} = r(\nu)\,\omega\left[\frac{\left[e\sin(\nu)\right]}{\left[1 + e\cos(\nu)\right]}\bar{i}_r + \bar{i}_\nu\right]$$

Angular Velocity of Spacecraft

$$r^2\omega = \frac{2\, a^2 \pi \sqrt{1 - e^2}}{T} = \boldsymbol{I}$$

Kepler's Second Law

$$\frac{\boldsymbol{I}^2}{a\left[1 - e^2\right]} = \mu \Rightarrow \boldsymbol{I} = \sqrt{\mu\, a\left[1 - e^2\right]}$$

$$\Downarrow$$

$$\omega = \frac{\sqrt{\mu\, a\left[1 - e^2\right]}}{r^2}$$

Later we'll show
This is the same

$$\mu = G \cdot M_{\oplus}$$

Kepler's third law

Kepler's second law

Let

$$\mu \equiv \frac{I^2}{a\left[1-e^2\right]} = \frac{\left[\frac{2\left[a^2 \pi \sqrt{1-e^2}\right]}{T}\right]^2}{a\left[1-e^2\right]} = \frac{\frac{4a^4 \pi^2 \left[1-e^2\right]}{T^2}}{a\left[1-e^2\right]} =$$

$$\frac{\frac{4a^4 \pi^2 \left[1-e^2\right]}{T^2}}{a\left[1-e^2\right]} = \boxed{\frac{4a^3 \pi^2}{T^2}}$$

Later we'll show
This is the same

$$\mu = G \cdot M_{\oplus}$$

- $\mu \equiv \dfrac{I^2}{a\left[1-e^2\right]}$ = constant = $\boxed{\dfrac{4a^3 \pi^2}{T^2}}$

$$\boxed{T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}}$$

- **Kepler's Third Law:** *In a two body universe, square of the period of any object revolving about the sun (Earth) is in the same ratio as the cube of its mean distance*

Angular Velocity of Spacecraft (cont'd)

$$l = \sqrt{\mu\, a\left[1 - e^2\right]} = r_p^2\, \omega \Rightarrow \boxed{\omega = \frac{\sqrt{\mu\, a\left[1 - e^2\right]}}{r_p^2}}$$

Kepler's Second Law

Circle:

$$\omega = \frac{\sqrt{\mu\, a\left[1 - 0\right]}}{a^2} = \frac{\sqrt{\mu}}{a^{3/2}}$$

Ellipse:

$$\omega = \frac{\sqrt{\mu\, a\left[1 - e^2\right]}}{\left[a\left[1 - e^2\right] / \left[1 + e\, \mathrm{Cos}(\nu)\right]\right]^2} = \frac{\sqrt{\mu}}{\left[a\left[1 - e^2\right]\right]^{3/2}}\left[1 + e\, \mathrm{Cos}(\nu)\right]^2$$

Angular Velocity of Spacecraft (cont'd)

Circle:

$$\omega\, T = \frac{\sqrt{\mu}}{a^{3/2}} \times \frac{2\,\pi\, a^{3/2}}{\sqrt{\mu}} = 2\pi$$

Constant!

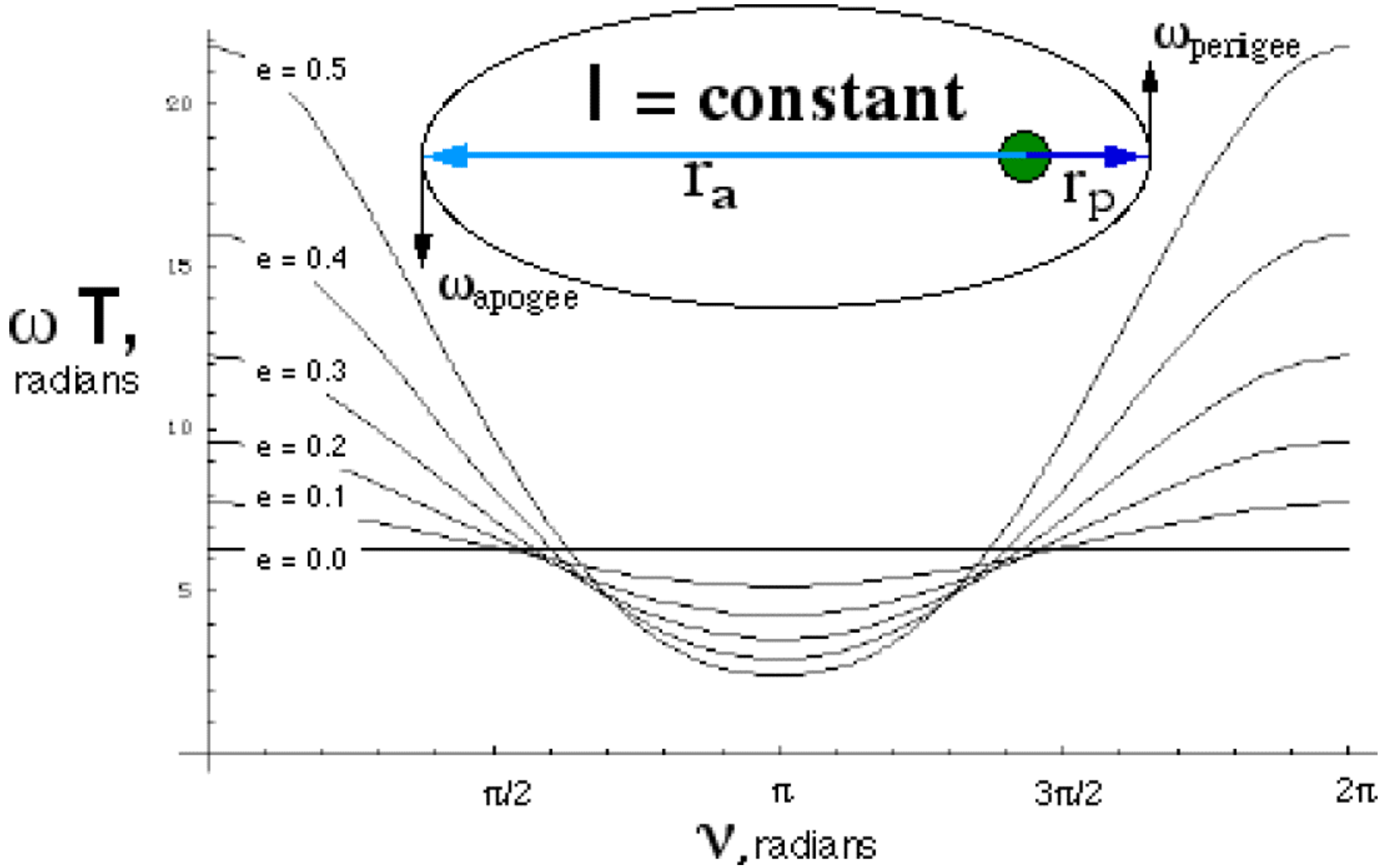
Ellipse:

$$\omega\, T = \frac{\sqrt{\mu}}{[\,a[1 - e^2]\,]^{3/2}}[1 + e\,\mathrm{Cos}(\nu)\,]^2 \times \frac{2\,\pi\, a^{3/2}}{\sqrt{\mu}} =$$

$$2\,\pi\,\frac{[1 + e\,\mathrm{Cos}(\nu)\,]^2}{[\;[1 - e^2]\,]^{3/2}}$$

Not ! Constant

Elliptical Orbit, Normalized Angular Velocity



Orbital Speed -- Magnitude of the Velocity Vector (cont'd)

- Taking the Magnitude of the Velocity Vector to get Orbital Speed

$$\boxed{\bar{V}_p = r_{p(\nu)}\,\omega\left[\frac{[e\sin(\nu)]}{[1+e\cos(\nu)]}\,\bar{i}_r + \bar{i}_\nu\right]}$$

$$\boxed{\left|\bar{V}_p\right|^2 = \left[r_{p(\nu)}\,\omega\right]^2\left[\left[\frac{[e\sin(\nu)]}{[1+e\cos(\nu)]}\right]^2 + 1\right]}$$

Orbital Speed -- Elliptical Orbit (cont'd)

- But from Kelper's second law (angular momentum form)

$$\boxed{r^2\,\omega = \frac{2\left[\,a^2\,\pi\,\sqrt{1-e^2}\,\right]}{T} \equiv \boldsymbol{I}}$$

$$\Downarrow$$

$$[r(\nu)\,\omega]^2 = \left[r(\nu)\,\omega\,\frac{1}{r(\nu)}\right]^2 = \frac{\left[r^2(\nu)\,\omega\right]^2}{r^2(\nu)} = \boxed{\left[\frac{\boldsymbol{I}}{r(\nu)}\right]^2}$$

$$\boxed{\left|\overline{V}\right|^2 = \left[\frac{\mathbf{I}}{r(\nu)}\right]^2 \left[\left[\frac{\left[\,e\,\sin(\nu)\right]}{\left[1+e\cos(\nu)\right]}\right]^2 + 1\right]}$$

Orbital Speed -- Elliptical Orbit (cont'd)

- **Expanding squares and collecting terms**

$$|\bar{V}|^2 = \left[\frac{l}{r}\right]^2 \left[\frac{\left[e^2 \sin^2(\nu) + [1 + e\cos(\nu)]^2\right]}{[1 + e\cos(\nu)]^2}\right]$$

$$\left[\frac{l}{r}\right]^2 \left[\frac{e^2 \sin^2(\nu) + 1 + 2e\cos(\nu) + e^2\cos^2(\nu)}{[1 + e\cos(\nu)]^2}\right] =$$

$$\left[\frac{l}{r}\right]^2 \left[\frac{1 + 2e\cos(\nu) + e^2}{[1 + e\cos(\nu)]^2}\right] = \left[\frac{l}{r}\right]^2 \left[\frac{2 + 2e\cos(\nu) + e^2 - 1}{[1 + e\cos(\nu)]^2}\right] =$$

$$\left[\frac{l}{r}\right]^2 \left[\frac{2[1 + e\cos(\nu)] - (1 - e^2)}{[1 + e\cos(\nu)]^2}\right] = \left[\frac{l}{r}\right]^2 \left[\frac{2}{1 + e\cos(\nu)} - \frac{1 - e^2}{[1 + e\cos(\nu)]^2}\right]$$

Orbital Speed -- Elliptical Orbit (cont'd)

- Substituting in the "radius" equation

$$|\overline{V}|^2 = \left[\frac{l}{r}\right]^2 \left[\frac{2}{1 + e\cos(\nu)} - \frac{1 - e^2}{[1 + e\cos(\nu)]^2}\right]$$

$$\frac{\left[\frac{l}{r}\right]^2}{1 + e\cos(\nu)} \left[2 - \frac{1 - e^2}{[1 + e\cos(\nu)]}\right] =$$

$$\boxed{\frac{r}{a[1 - e^2]} = \frac{1}{[1 + e\cos(\nu)]}}$$

$$\boxed{|\overline{V}|^2 = \left[\frac{l}{r}\right]^2 \frac{r}{a[1 - e^2]}\left[2 - \frac{r}{a}\right]}$$

Orbital Speed -- Elliptical Orbit (cont'd)

- Collecting like terms in r

$$\left|\bar{V}\right|^2 = \left[\frac{\ell}{r}\right]^2 \frac{r}{a\left[1 - e^2\right]}\left[2 - \frac{r}{a}\right] = \left[\frac{\ell^2}{r^2}\right] \frac{r^2}{a\left[1 - e^2\right]}\left[\frac{2}{r} - \frac{1}{a}\right] =$$

$$\boxed{\left|\bar{V}\right|^2 = \frac{\ell^2}{\underline{a\left[1 - e^2\right]}}\left[\frac{2}{r} - \frac{1}{a}\right] = \underline{\mu}\left[\frac{2}{r} - \frac{1}{a}\right]}$$

Later we'll show
This is the same

$$\mu = G \cdot M_{\oplus}$$

Kepler's Second and Third Law Summary

- **Kepler's Second Law:** *In a two body universe, radius vector from the sun (Earth) to the planet (satellite) sweeps out equal areas in equal times*

→ *Derives from constant angular momentum*

- *Swept Area Rule:*

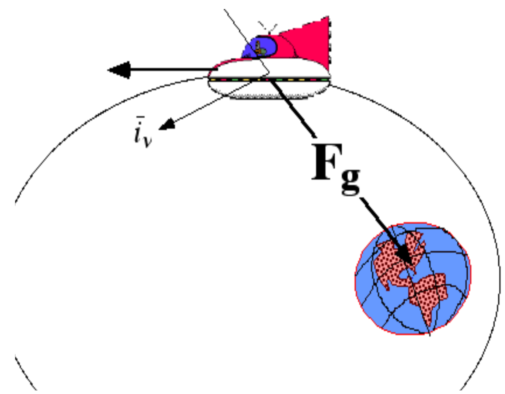
$$\frac{dA(t)}{dt} = \frac{a^2\pi\sqrt{1-e^2}}{T} = \frac{1}{2}r^2\frac{d\nu}{dt} = \frac{\omega \cdot r^2}{2}$$

- *Angular Momentum:*

$$\vec{l} = \frac{\vec{L}}{m} = \omega \cdot r^2 \cdot \vec{i}_\kappa$$

- *Gravitational Torque:*

$$\tau_{grav} = \vec{r} \times \vec{F}_{grav} = 0 \rightarrow \frac{\vec{L}}{m} = const$$



Kepler's Second and Third Law Summary (2)

- **Kepler's Third Law:** *In a two body universe, square of the period of any object revolving about the sun (Earth) is in the same ratio as the cube of its mean distance*

→ *Corollary to Second law*

$$\{l, a, e \to const\} \to define \text{ constant } \mu \equiv \frac{l^2}{a \cdot (1 - e^2)}$$

$$\to \frac{a^2 \pi \sqrt{1-e^2}}{T} = \frac{l}{2} \to T = \frac{2 \cdot a^2 \pi \sqrt{1-e^2}}{l} = \frac{2 \cdot a^2 \pi \sqrt{1-e^2}}{\sqrt{\mu \cdot a \cdot (1-e^2)}} = \left(\frac{2 \cdot \pi}{\sqrt{\mu}}\right) \cdot a^{3/2}$$

$$\boxed{T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}}$$

Kepler's Second and Third Law Summary (3)

- *Angular Velocity* :

$$\omega = \frac{1}{r^2}\frac{2a^2\pi\sqrt{1-e^2}}{T}$$

- *Velocity Vector* :

$$\vec{V} = \begin{bmatrix} r\cdot\omega\cdot\dfrac{e\cdot\sin\nu}{1+e\cdot\cos\nu}\cdot\vec{i}_r \\ r\cdot\omega\cdot\vec{i}_\nu \end{bmatrix} \rightarrow \begin{bmatrix} r = a\cdot\dfrac{1-e^2}{1-e\cdot\cos\nu} \\ \omega = \dfrac{1}{r^2}\dfrac{2a^2\pi\sqrt{1-e^2}}{T} \end{bmatrix}$$

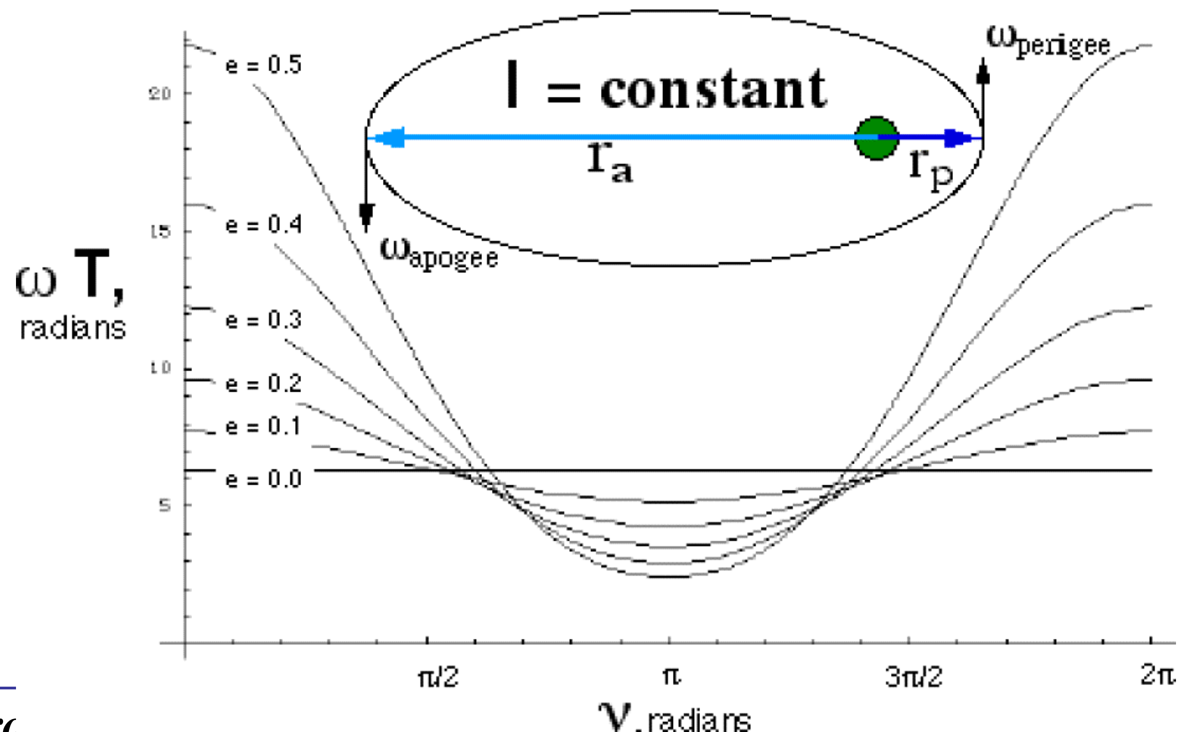
- *Normalized Angular Velocity* :

$$\omega\cdot T = \frac{2a^2\pi\sqrt{1-e^2}}{r^2} = \frac{2a^2\pi\sqrt{1-e^2}}{\left(a\cdot\dfrac{1-e^2}{1-e\cos\nu}\right)^2} = 2\cdot\pi\cdot\frac{\left(1+e\cdot\cos\nu\right)^2}{\left(1-e^2\right)^{3/2}}$$

Kepler’s Second and Third Law Summary (4)

- *Normalized Angular Velocity :*

$$\omega \cdot T = \frac{2a^2\pi\sqrt{1-e^2}}{r^2} = \frac{2a^2\pi\sqrt{1-e^2}}{\left(a \cdot \frac{1-e^2}{1-e\cos\nu}\right)^2} = 2\cdot\pi\cdot\frac{\left(1+e\cdot\cos\nu\right)^2}{\left(1-e^2\right)^{3/2}}$$



• Kepler's Second and Third Law Summary (5)

•*Orbital Speed*

$$\left\|\vec{V}\right\|^2 = r^2 \cdot \omega^2 \cdot \left(\left(\frac{e \cdot \sin\nu}{1 + e \cdot \cos\nu}\right)^2 + 1\right) = \frac{l^2}{a \cdot \left(1 - e^2\right)} \cdot \left(\frac{2}{r} - \frac{1}{a}\right) \equiv \mu \cdot \left(\frac{2}{r} - \frac{1}{a}\right)$$

$$\boxed{\left\|\vec{V}\right\| = \sqrt{\frac{2 \cdot \mu}{r} - \frac{\mu}{a}}}$$

Linear Velocity of Spacecraft

- Just an alternate Form of the Energy Equation

$$\left|\overline{V}\right|^2 = \frac{l^2}{a\left[1 - e^2\right]}\left[\frac{2}{r} - \frac{1}{a}\right] = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

Orbital Energy

- To Understand that ... we'll first have to show that the orbital dynamics are a result of the balance between kenetic and potential energy

- Specifically

$$-\frac{\mu}{2a} = \frac{\left|\vec{V}\right|^2}{2} - \frac{\mu}{r}$$

Total Specific Energy	Specific Kinetic Energy	Specific Potential Energy

- Next Isaac Newton and His Apple!

Isaac Newton, (1642-1727)

Linear Velocity of Spacecraft

$$\mu \equiv G\,M \Rightarrow \text{planetary gravitational parameter}$$

$$\mu_{earth} = G\,M \approx 6.672 \times 10^{-11}\,\frac{\text{Nt-m}^2}{\text{kg}^2} \times 5.974 \times 10^{24}\,\text{kg} =$$

$$3.98565 \times 10^{14}\,\frac{\text{Nt-m}^2}{\text{kg}} = 3.986 \times 10^{14}\,\frac{\text{m}^3}{\text{sec}^2} = 1.4076 \times 10^{16}\,\frac{\text{ft}^3}{\text{sec}^2}$$

$$\mu_{moon} = 4.903 \times 10^{3}\,\frac{\text{m}^3}{\text{sec}^2}$$

$$\mu_{sun} = 1.327 \times 10^{20}\,\frac{\text{m}^3}{\text{sec}^2}$$

$$\mu_{Mars} = 4.269 \times 10^{4}\,\frac{\text{m}^3}{\text{sec}^2}$$

We’ll prove this next!


ANY QUESTIONS? NO?
GOOD!