# Intro to Astrodynamics

# Introduction to Astrodynamics: Gravitational Fields, Potential and Kinetic Energy, and the Vis-Viva Equation

# Kinematics versus Dynamics


Kepler

• Up to now we have mostly dealt with orbital motions from a kinematics point of view … I.e. Kepler's laws Were used simply as descriptors of orbital motion

• Kepler's laws are a reasonable approximation of the motions of a small body orbiting around a much larger body in a 2-body universe

… but there are no Physics (I.e. Isaac Newton) Involved

• Kepler derived his laws of planetary motion by Empirical observation only.

# Summary: Kepler's Laws

- **Kepler's First Law:** *In a two body universe, orbit of a satellite is a conic section with the Earth centered at one of the focii*

$$\bar{r} = a \frac{1-e^2}{1 + e\cos(\nu)} \bar{i}_r$$

$$\bar{i}_r = \cos(\nu)\,\bar{i} + \sin(\nu)\,\bar{j}$$

# Kepler's Laws (cont'd)

## • Parameters of the Orbit

$$\frac{r_{max} + r_{min}}{2} = a$$

$$\frac{r_{max} - r_{min}}{r_{max} + r_{min}} = e$$

# Kepler's Laws (cont'd)

- **Kepler's Second Law:** *In a two body universe, radius vector from the Earth to the satellite sweeps out equal areas in equal times*

$$\frac{A_{\nu_1}}{a^2} = \frac{A_{\nu_o}}{a^2} + \left[\pi\sqrt{1 - e^2}\right] \times \left[\frac{t_1 - t_0}{T}\right]$$

$$r^2\,\omega = \frac{2\left[a^2\,\pi\sqrt{1 - e^2}\right]}{T} \equiv \mathrm{I}$$

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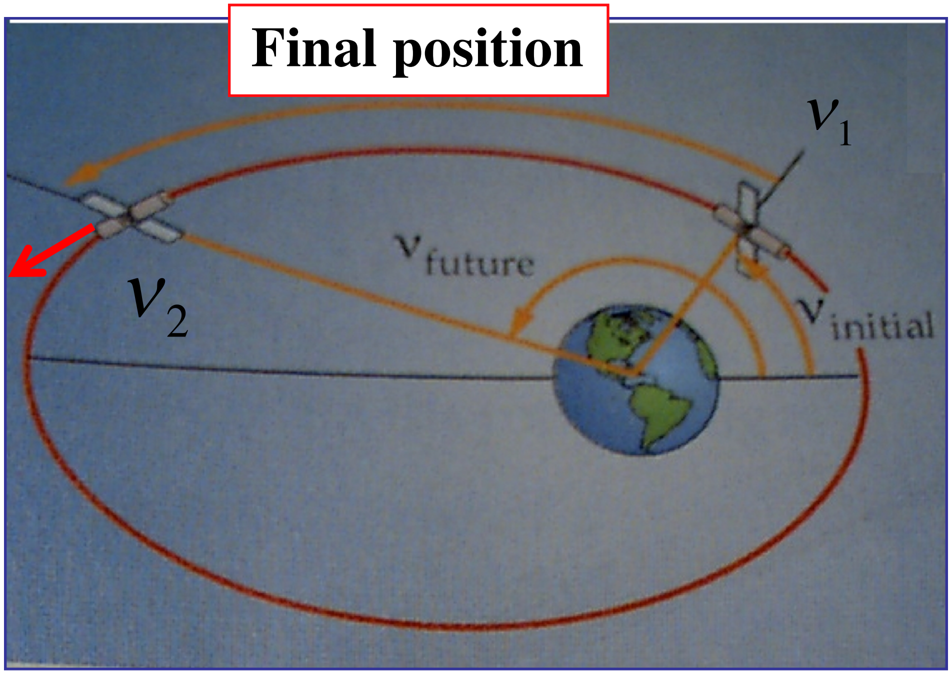
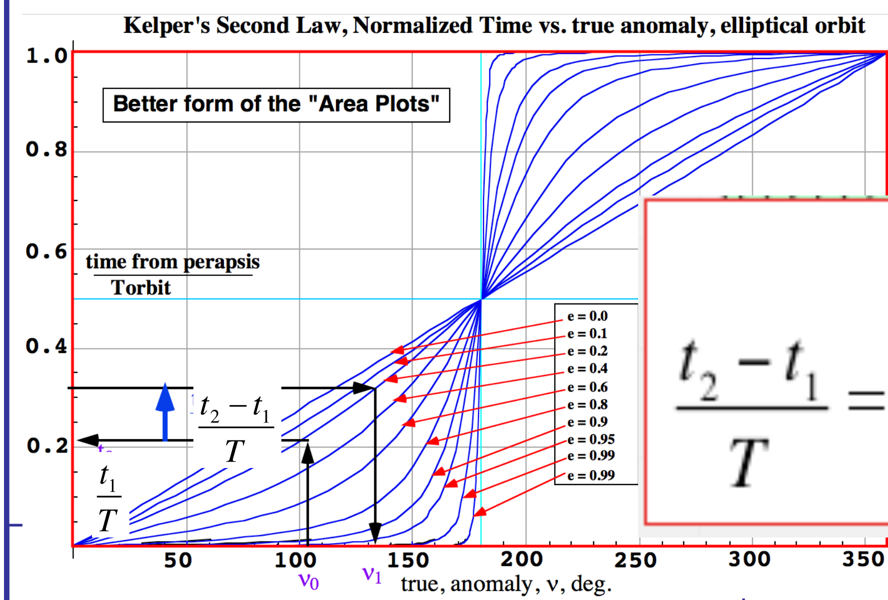
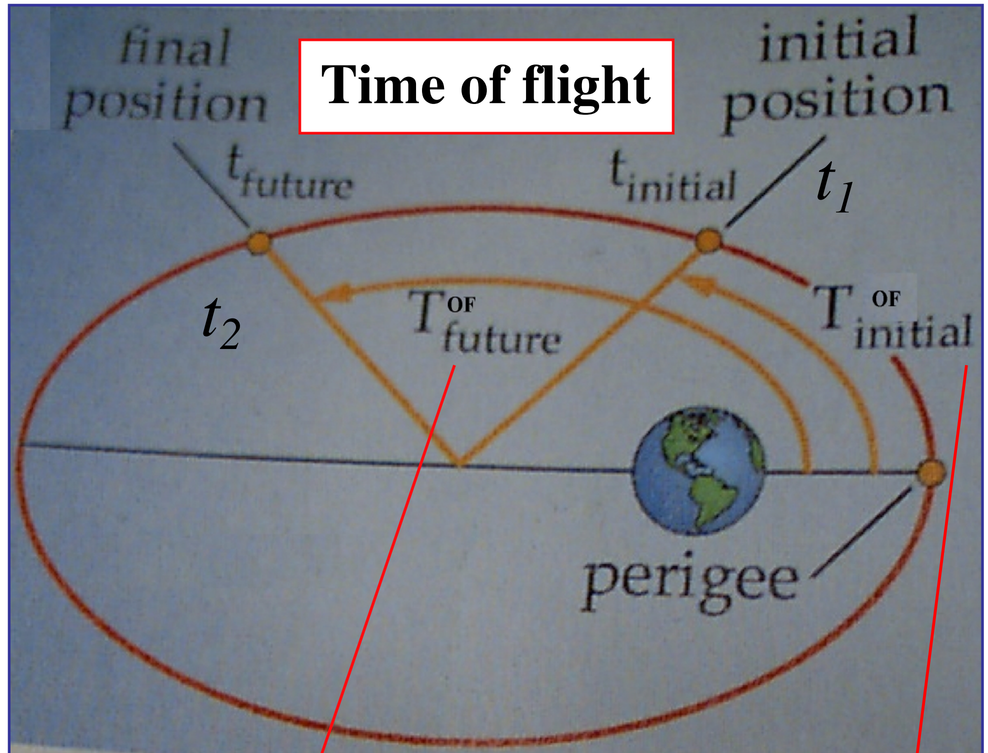
Mechanical & Aerospace Engineering

# Time of Flight




## Propogation of Orbital Position




$$\frac{t_2 - t_1}{T} = \frac{\int_0^{\nu_2}\left[\frac{1-e^2}{1+e\cos(\nu)}\right]^2 d\nu - \int_0^{\nu_1}\left[\frac{1-e^2}{1+e\cos(\nu)}\right]^2 d\nu}{2\pi\sqrt{[1-e^2]}}$$

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# Kepler's Laws (cont'd)

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## Kepler's third law

Kepler's second law

$$\mu \equiv \frac{l^2}{a[1-e^2]} = \frac{\left[\frac{2\left[a^2 \pi \sqrt{1-e^2}\right]}{T}\right]^2}{a[1-e^2]} = \frac{\frac{4a^4\pi^2[1-e^2]}{T^2}}{a[1-e^2]} =$$

$$\frac{\frac{4a^4\pi^2[1-e^2]}{T^2}}{a[1-e^2]} = \frac{4a^3\pi^2}{T^2}$$

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

- $\mu \equiv \frac{l^2}{a[1-e^2]}$ = constant = $\frac{4a^3\pi^2}{T^2}$

- **Kepler's Third Law:** *In a two body universe, square of the period of any object revolving about the sun (Earth) is in the same ratio as the cube of its mean distance*

# Kepler's Laws (cont'd)

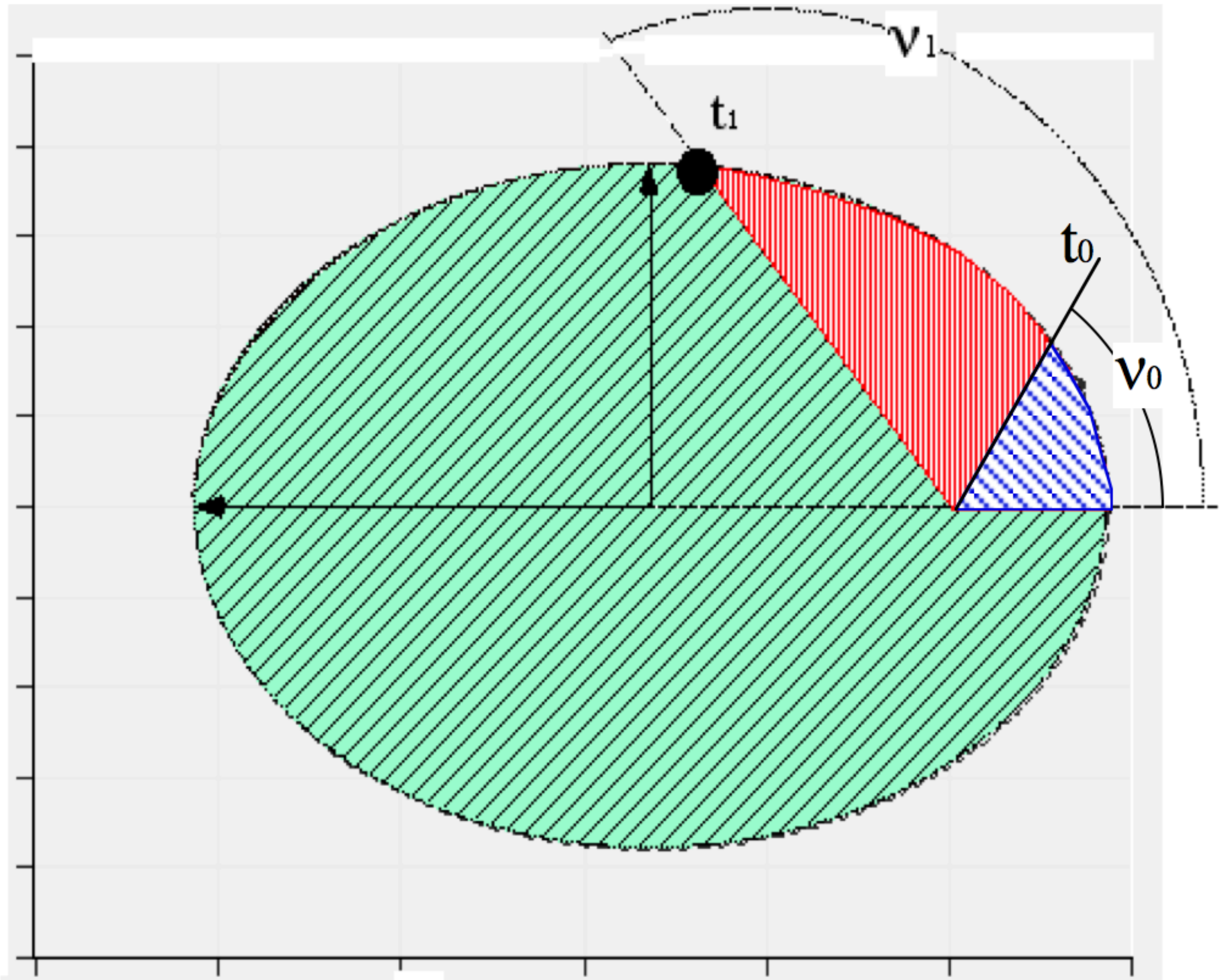
- **Kepler's Third Law:** *In a two body universe, square of the period of any object revolving about the Earth is in the same ratio as the cube of its mean distance*

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

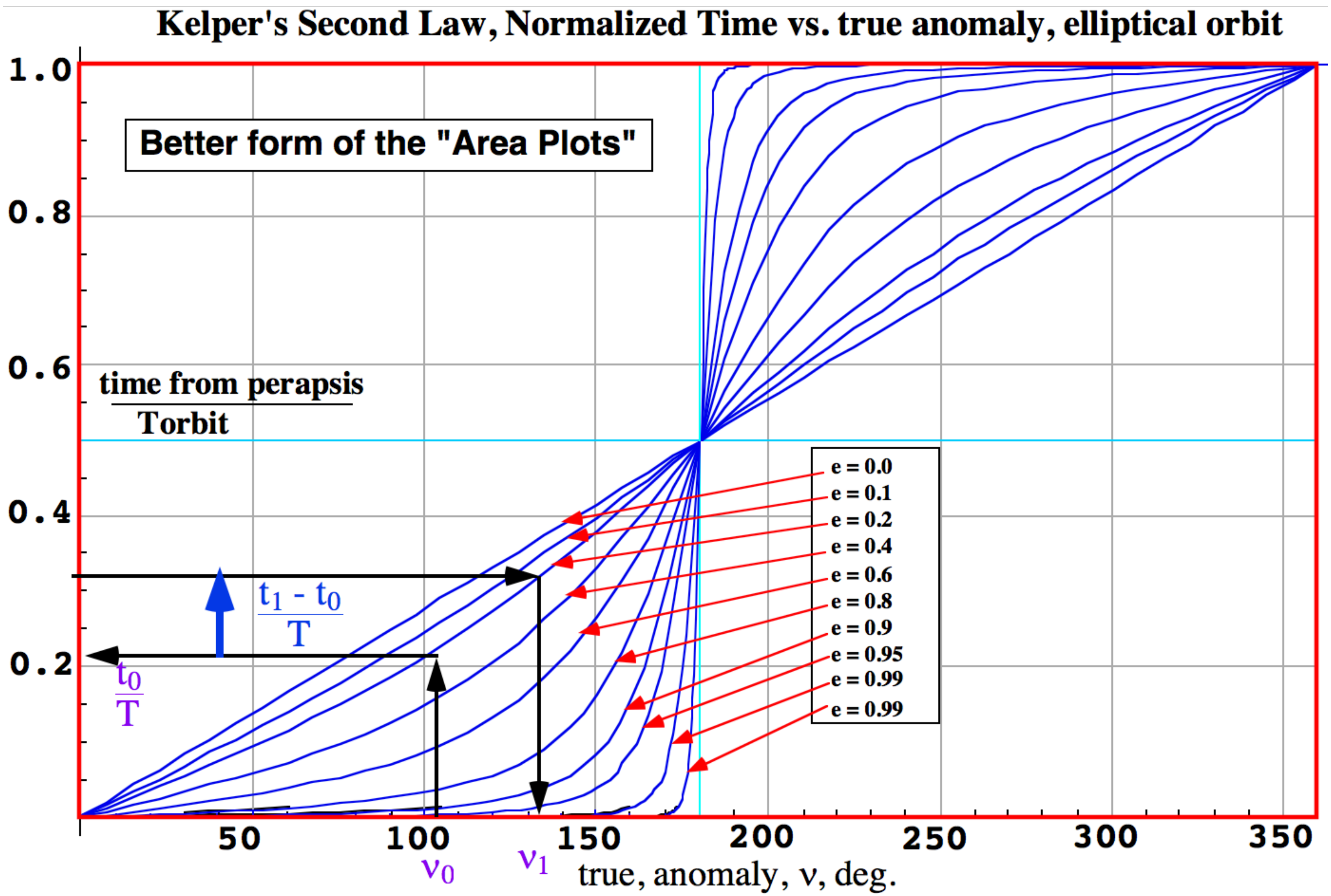
Haven't really proven this! Yet.

$$\mu = G M$$

# Time of Flight Graphs (cont'd)



# Propagation of Orbital Position



# Velocity Vector, Elliptical Orbit

$$\bar{V} = \frac{d}{dt}\bar{r} = \frac{d}{dt}\left[r(\nu)\right]\bar{i}_r + r(\nu)\,\omega\,\bar{i}_\nu$$

$$\frac{d}{dt}\left[r(\nu)\right] = r(\nu)\,\omega\,\frac{\left[e\sin(\nu)\right]}{\left[1 + e\cos(\nu)\right]}$$

$$\bar{V} = r(\nu)\,\omega\left[\frac{\left[e\sin(\nu)\right]}{\left[1 + e\cos(\nu)\right]}\bar{i}_r + \bar{i}_\nu\right]$$

# Angular Velocity of Spacecraft (cont'd)

$$| = \sqrt{\mu\, a\left[1 - e^2\right]} = r_p^2\, \omega \quad \Rightarrow \quad \omega = \frac{\sqrt{\mu\, a\left[1 - e^2\right]}}{r_p^2}$$

Kepler's Second Law

Circle:

$$\omega = \frac{\sqrt{\mu\, a\left[1 - 0\right]}}{a^2} = \frac{\sqrt{\mu}}{a^{3/2}}$$

Ellipse:

$$\omega = \frac{\sqrt{\mu\, a\left[1 - e^2\right]}}{\left[a\left[1 - e^2\right] / \left[1 + e\,\mathrm{Cos}(v)\right]\right]^2} = \frac{\sqrt{\mu}}{\left[a\left[1 - e^2\right]\right]^{3/2}}\left[1 + e\,\mathrm{Cos}(v)\right]^2$$

# Kepler's third law(corollary)

- **In the process of demonstrating Kepler's third law, we have also indirectly demonstrated that, for an elliptical orbit the orbital speed (Magnitude of the Velocity Vector) is**

$$\left|\overline{V}\right|^2 = \frac{l^2}{a\left[1 - e^2\right]}\left[\frac{2}{r} - \frac{1}{a}\right] = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

# Isaac Newton


• Sir Isaac Newton used his new calculus and laws of motion and gravitation to show that Kepler was right.

• One day in 1682 he came up to his friend, Edmund Halley, and casually mentioned to him that he'd proved that, with a $1/r^2$ force law like gravity, planets orbit the sun in the shapes of conic sections.

• This undoubtedly took Halley aback, as Newton had just revealed to him the nature of the Universe (at least the Universe as it was known then).

# Newton …

•Halley then pressed Newton to publish his findings, but he realized that he'd forgotten the proof.

• After struggling to remember how he had proved the t heorem, he published his work and it later appeared in full form in his classic work: *Philosophiae Naturalis Principia Mathematica* -- commonly known as the Principia -- published in1687.

• OK … let walk down Newton's path to enlightenment!

# Postscript: Magnitude of the Velocity vector

## But what is μ ?

• To Understand that ... we'll first have to show that the orbital dynamics are a result of the balance between kenetic and potential energy

• **The Energy Equation**

$$-\frac{\mu}{2a} = \frac{|V|^2}{2} - \frac{\mu}{r}$$

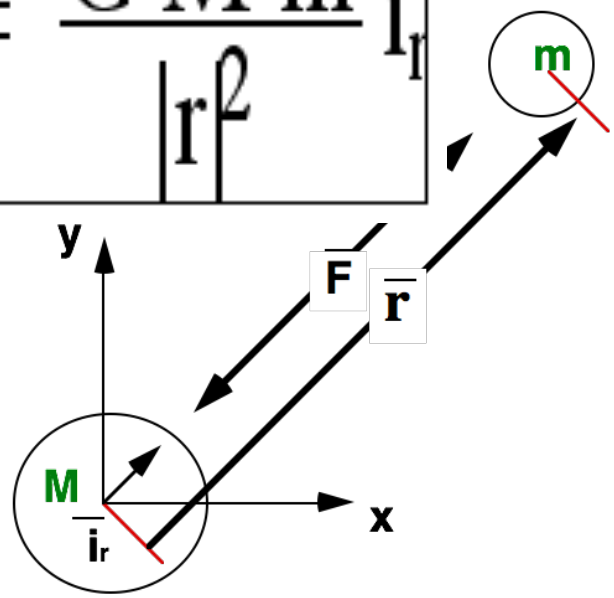
| Total Specific Energy | Specific Kinetic Energy | Specific Potential Energy |
|---|---|---|

# Gravitational Physics

- Now by introducing a bit of *"gravitational physics"* we can unify the entire mathematical analysis

$$\bar{F}_{Mm} = \frac{G\,M\,m}{|r|^2}\,\bar{i}_r$$

"Inverse-square" law "potential" field




Isaac Newton, (1642-1727)

You've seen it before …..

# Gravitational Physics

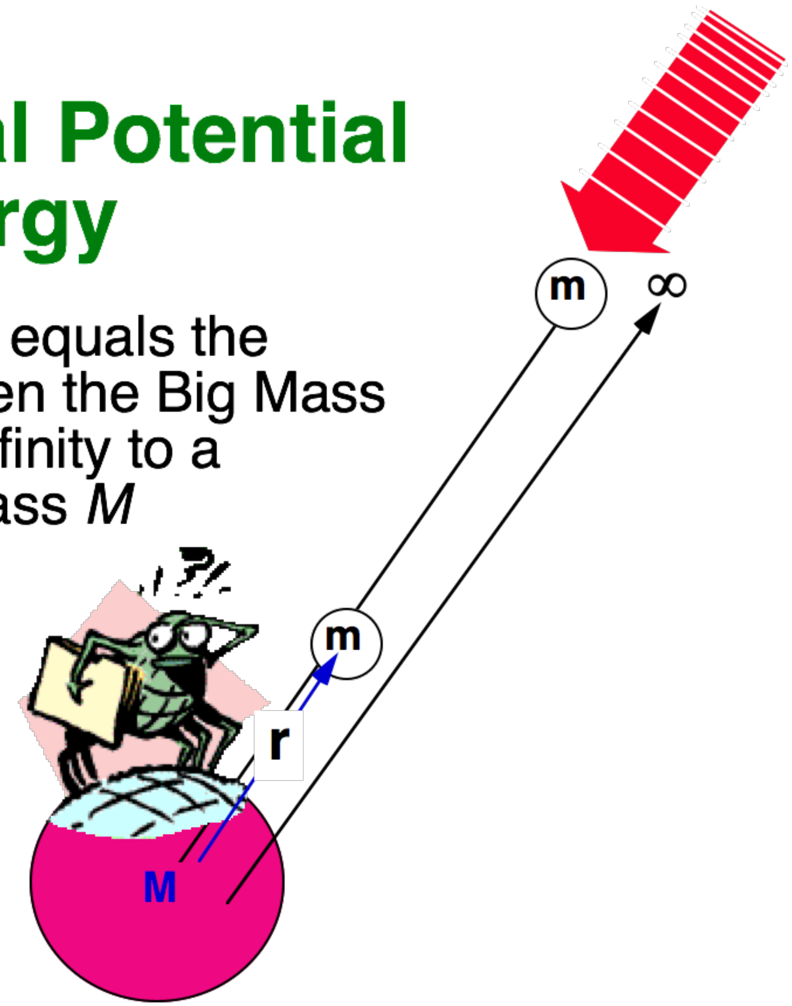
(cont'd)

- Constant *G* appearing in Newton's law of gravitation, known as the *universal gravitational constant*.

- Numerical value of *G*

$$G = 6.672 \times 10^{-11} \frac{\text{Nt-m}^2}{\text{kg}^2} = 3.325 \times 10^{-11} \frac{\text{lbf-ft}^2}{\text{lbm}^2}$$

# Gravitational Potential Energy




- **G***ravitational potential energy* equals the amount of energy released when the Big Mass M pulls the small mass *m* at infinity to a location *r* in the vicinity of a mass *M*

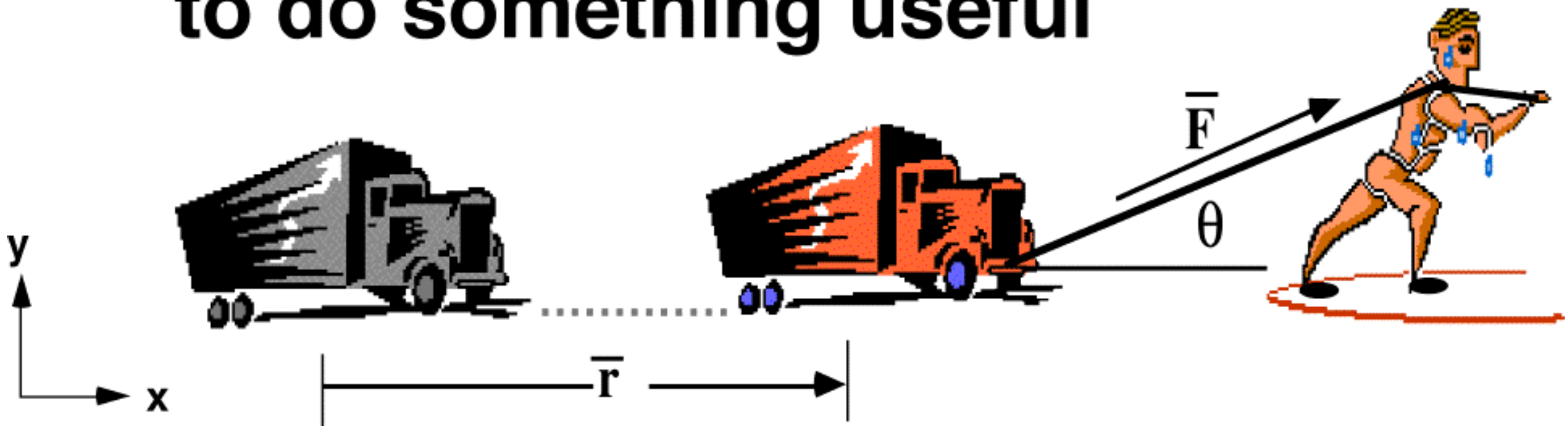
- **Energy of position**

$$P_{E_{grav}} \equiv E_{released} = \int_{\infty}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} =$$

$$\int_{\infty}^{\mathbf{r}} \frac{G\,M\,m}{r^2}\,dr = -\,G\,M\,m\left[\frac{1}{\mathbf{r}} - \frac{1}{\infty}\right] = \boxed{-\,\frac{G\,M\,m}{\mathbf{r}}}$$

# "Work and Potential Energy"

- *Work* can be loosely defined as the ability of an applied force to do something useful



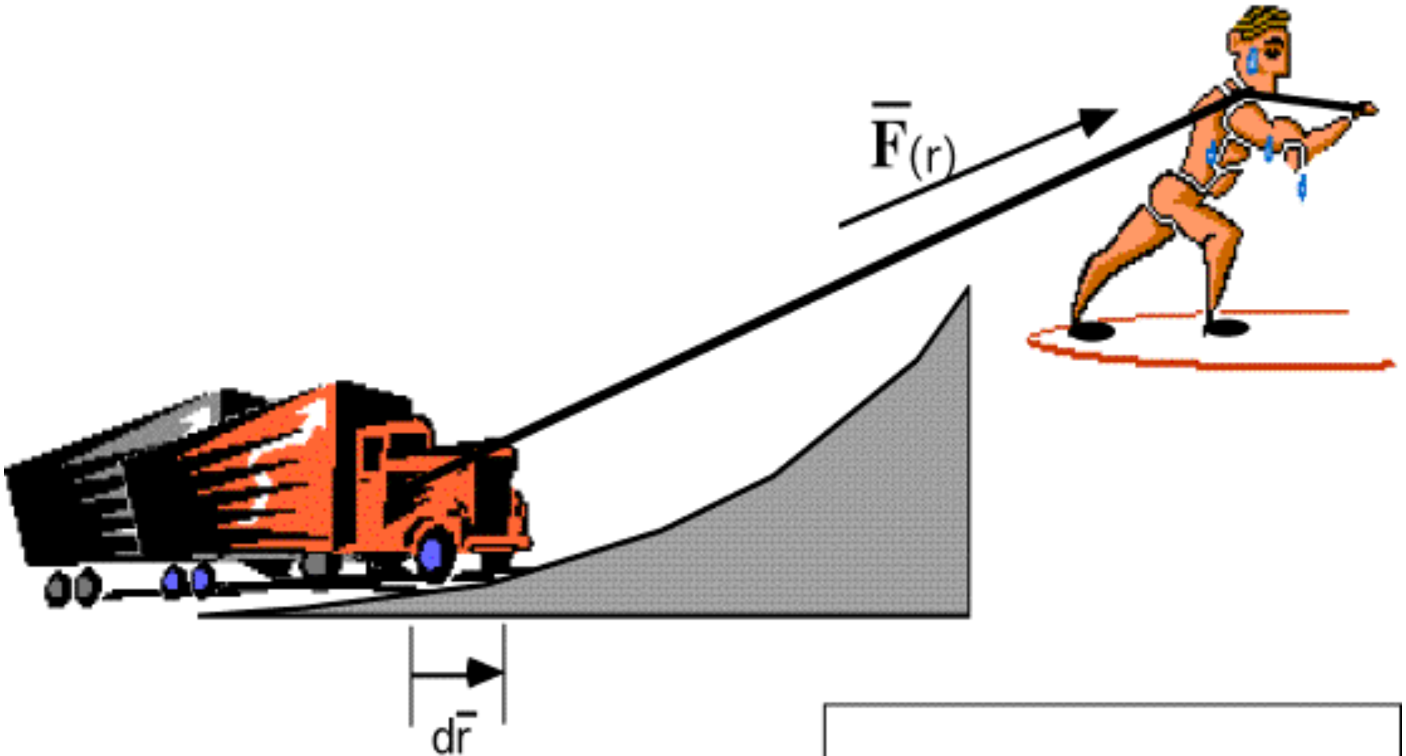
# Mechanical "Work"

- = Inner product of the applied Force vector and the change in the position vector caused by the applied force

$$W = \left|\bar{F}\right| \left|\bar{r}\right| \cos\left[\theta\right]$$

$$W = \bar{F} \cdot \bar{r}$$

# What if $\overline{F}$ is a function of $\overline{r}$?




$$dW = \left[\overline{F(r)} \cdot d\,\overline{r}\right] \Rightarrow W = \int_0^{|\overline{r}|} \overline{F(r)} \cdot d\,\overline{r}$$

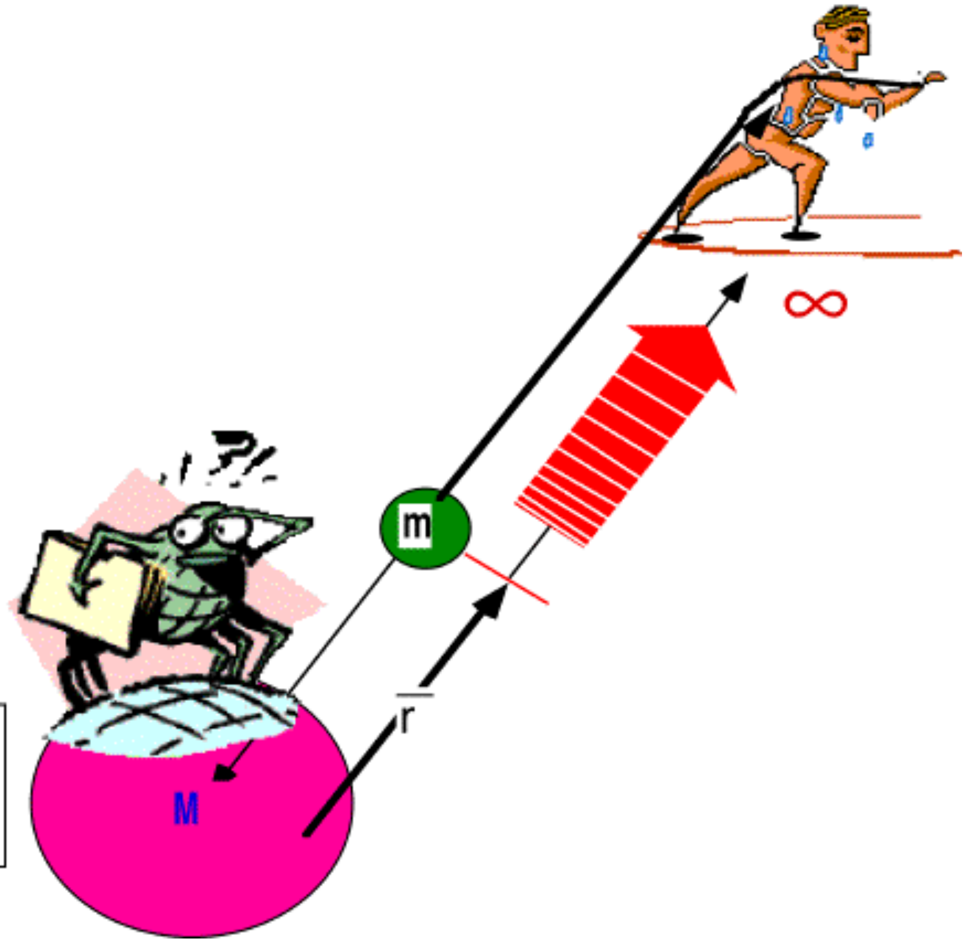
# Work Against a Gravity Field

- Work Done to Pull a mass *m*, against a gravity field caused by mass *M*, from from radius $\bar{r}$ to $\infty$

$$\Rightarrow W_{\substack{\text{performed}\\ \text{against M}}} = \int_r^\infty \mathbf{F} \cdot dr =$$

$$\int_r^\infty \frac{G\,M\,m}{r^2}\, dr =$$

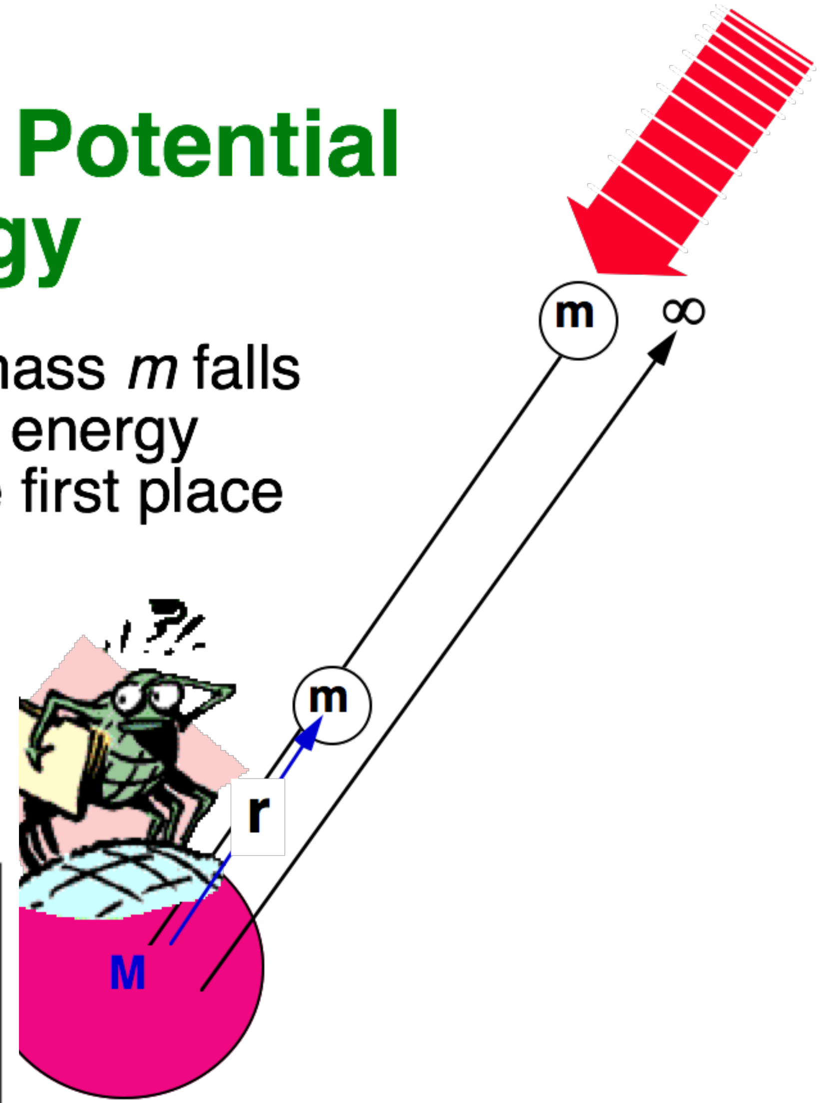
$$- G\,M\,m\left[\frac{1}{\infty} - \frac{1}{r}\right] = \boxed{\frac{G\,M\,m}{\mathbf{r}}}$$



# Gravitational Potential Energy

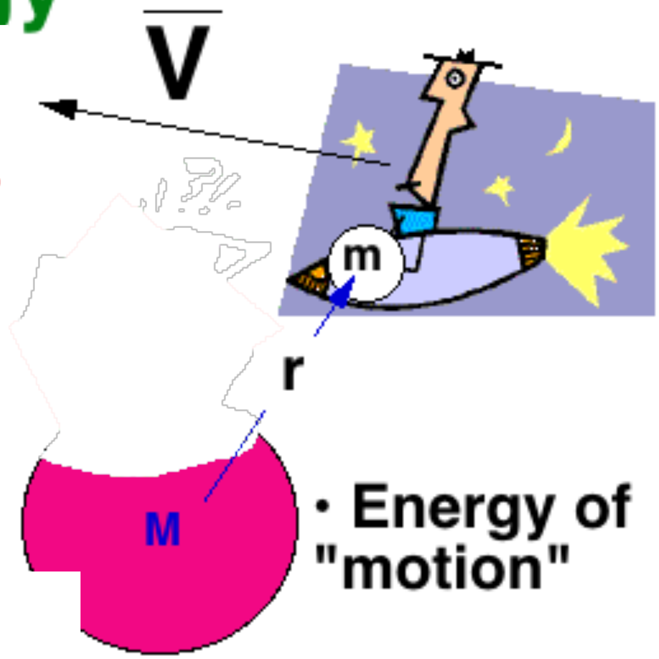
- *Energy released* when the small mass *m* falls back towards *M* is the same as the energy required to move *m* to *inifinty* in the first place

$$P_{E_{grav}} \equiv E_{released} = -W_{\substack{performed \\ against\ M}} = \boxed{-\frac{G\,M\,m}{r}}$$



# Kinetic Energy

Newton's Second Law $\bar{F} = m\,\bar{a} = m\dfrac{d\bar{V}}{dt}$




$$E_{kinetic} \equiv W_{\substack{\text{performed}\\ \text{to accelerate m}}} = \int_0^{\mathbf{V}} \bar{\mathbf{F}} \cdot d\bar{r}$$

- Energy of "motion"

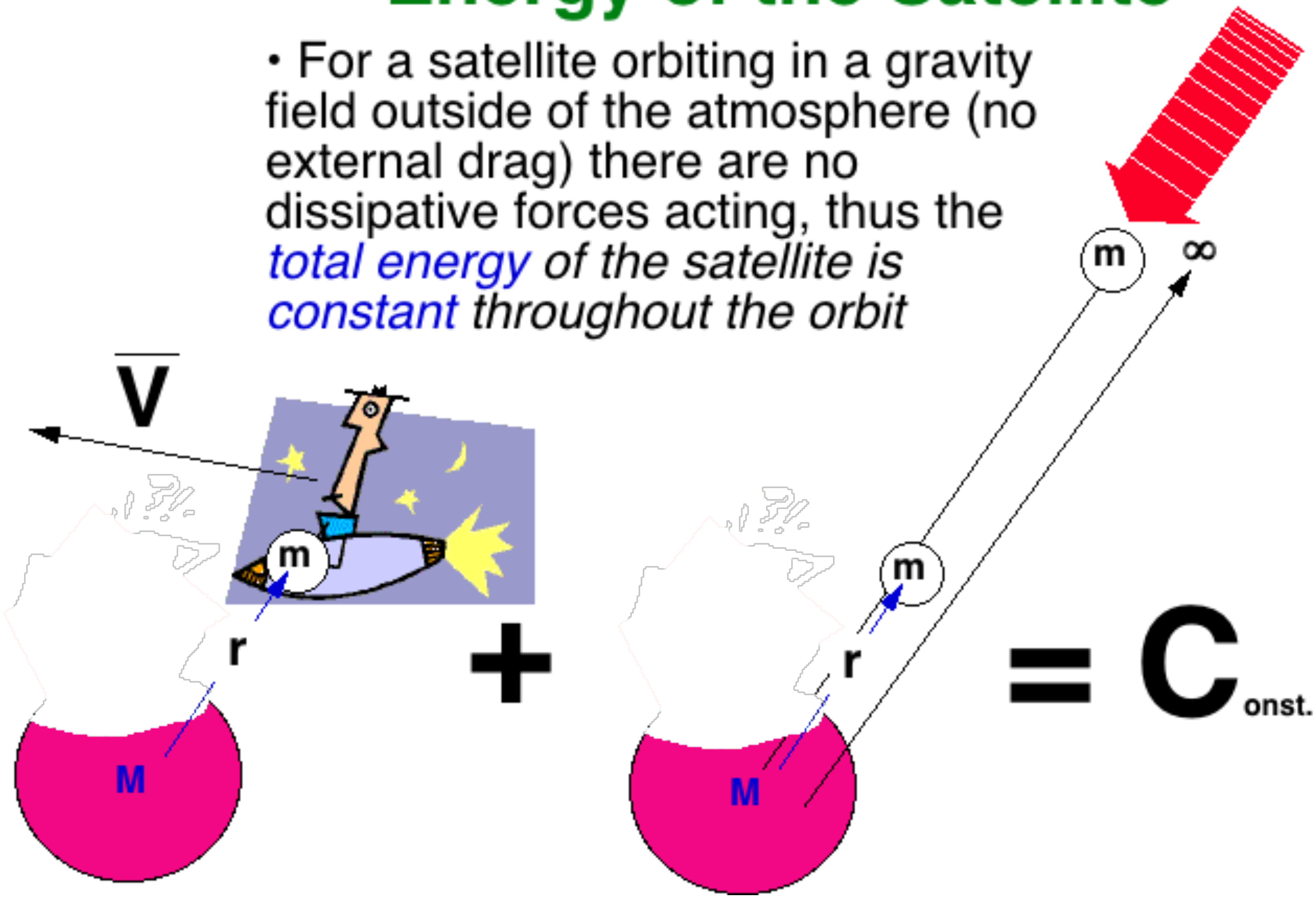
$$\int_0^{\mathbf{V}} m\frac{d\bar{V}}{dt}\cdot d\bar{r} = \int_0^{\mathbf{V}} m\frac{d\bar{r}}{dt}\cdot d\bar{V} \Rightarrow \frac{d\bar{r}}{dt} = \bar{V}$$

- **K***inetic energy* = work required to accelerate mass *m* *initially at rest* to final speed *V*

$$E_{kinetic} = \int_0^{\mathbf{V}} m\,V\cdot dV = \boxed{\frac{1}{2}\,m\,V^2}$$

# Total (Mechanical) Energy of the Satellite

- For a satellite orbiting in a gravity field outside of the atmosphere (no external drag) there are no dissipative forces acting, thus the *total energy of the satellite is constant throughout the orbit*



# Specific Energy

- Specific Energy ~ energy divided by the mass

$$\frac{E_T}{m} \equiv \varepsilon_T = \frac{1}{m}\left[\frac{m\,V^2}{2} - \frac{G\,M\,m}{r}\right] = \text{constant}$$

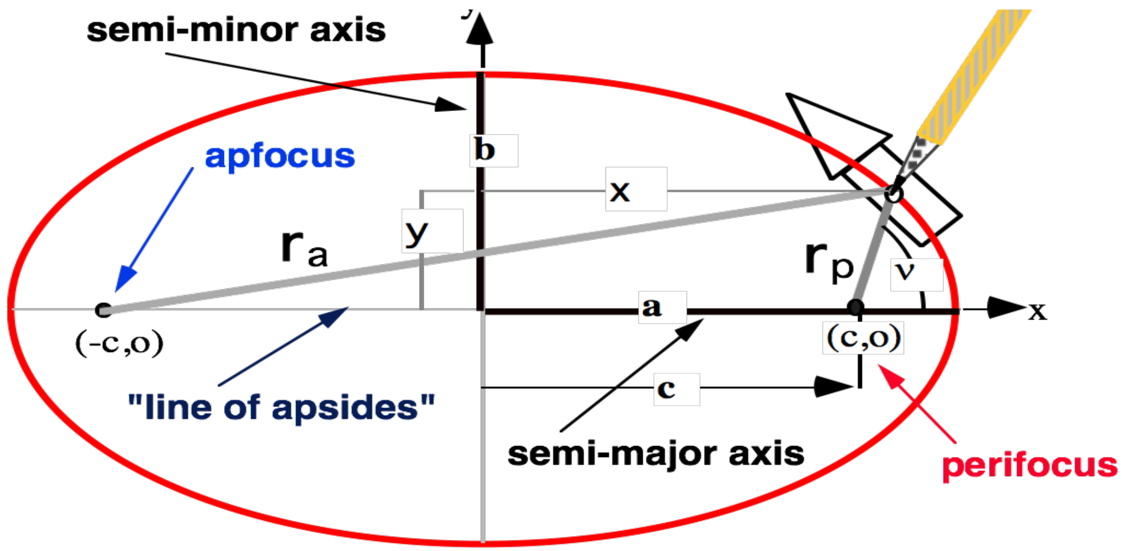
$$\Downarrow$$

$$\varepsilon_T = \left[\frac{V^2}{2} - \frac{\mu}{r}\right]$$

$\mu \equiv G\,M \Rightarrow$ planetary gravitational parameter

# Total Specific Energy

- *First Calculate Radius of Curvature of Ellipse*

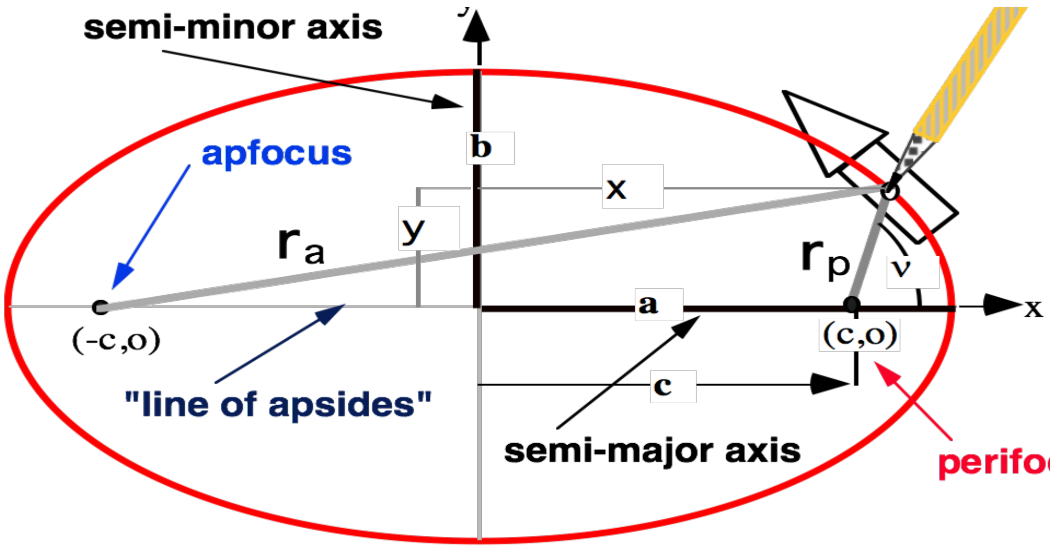



$$R_c = \frac{\left(r_a \cdot r_p\right)^{3/2}}{a^2\sqrt{1-e^2}} \rightarrow r_a = a \cdot \frac{\left(1-e^2\right)}{1+e\cdot\cos(\nu)} \rightarrow r_a + r_p = 2\cdot a$$

$$\rightarrow r_a = 2\cdot a - r_p \rightarrow \boxed{R_c = \frac{\left[\left(2\cdot a - r_p\right)\cdot r_p\right]^{3/2}}{a^2\sqrt{1-e^2}}}$$

# Total Specific Energy (2)

- ***Next Calculate Radius of Curvature of Ellipse at Perigee***



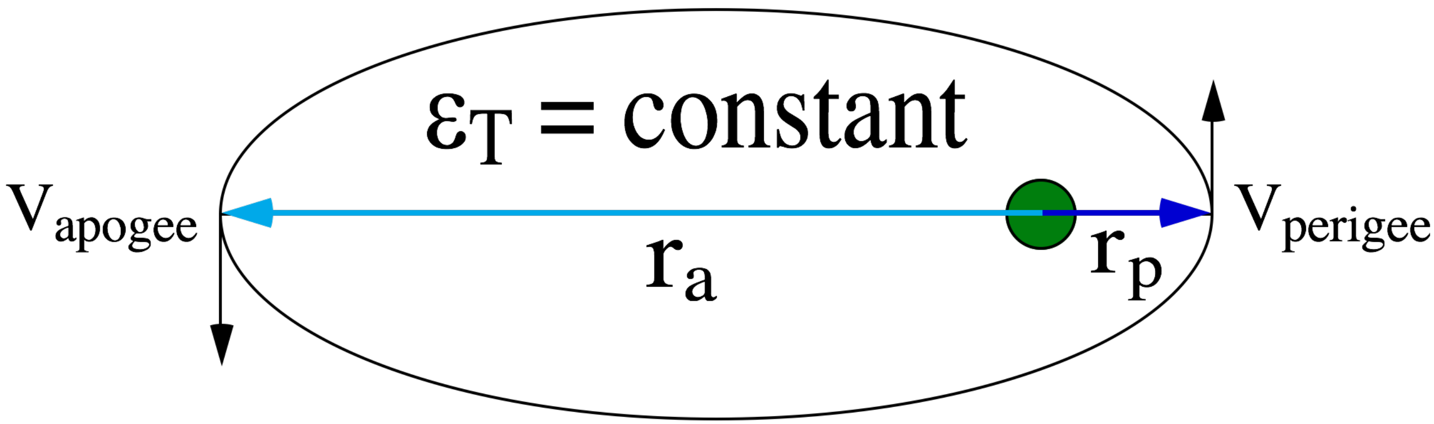

$$R_{c_{perigee}} = \frac{\left[\left(2\cdot a - r_{perigee}\right)\cdot r_{perigee}\right]^{3/2}}{a^2\sqrt{1-e^2}} = \frac{\left[\left(2\cdot a - a\cdot(1-e)\right)\cdot a\cdot(1-e)\right]^{3/2}}{a^2\sqrt{1-e^2}} = \frac{\left[a^2\cdot\left(1-e^2\right)\right]^{3/2}}{a^2\sqrt{1-e^2}} = a\left(1-e^2\right)$$

# Total Specific Energy (3)

$$\varepsilon_T = \left[\frac{V^2}{2} - \frac{\mu}{r}\right]$$

$\varepsilon_T$ constant everywhere in orbit

- Now ***look at perigee condition force balance***




$$\dot{r}_{perigee} = 0 \rightarrow \text{Centrifugal force} = \text{Gravitational force @ Apogee}$$

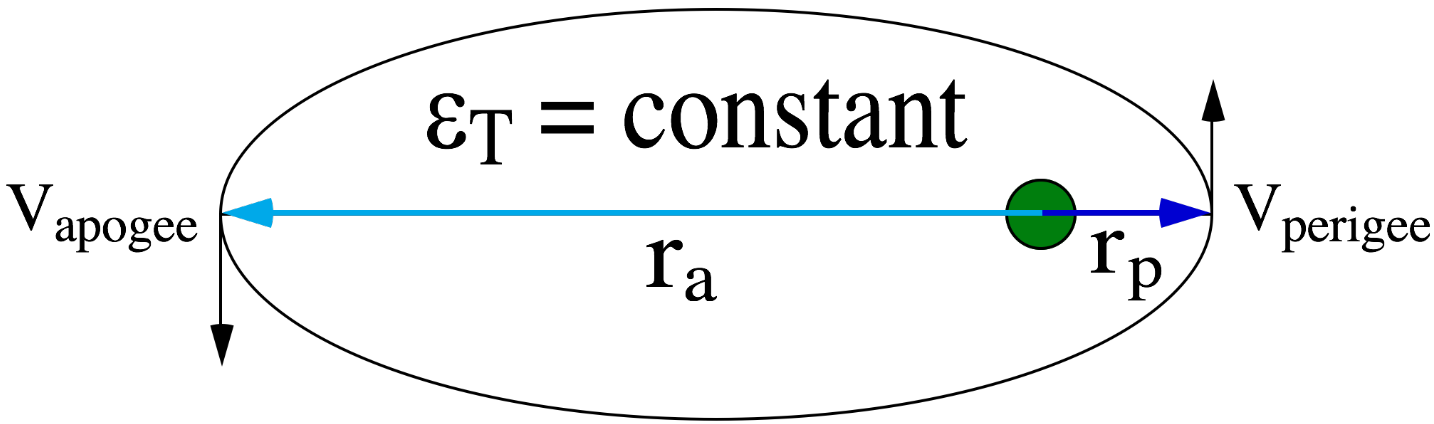
$$\frac{V_{perigee}{}^2}{R_{c_{perigee}}} = \frac{\mu}{r^2{}_{perigee}} \rightarrow V_{perigee}{}^2 = \frac{\mu \cdot R_{c_{perigee}}}{r^2{}_{perigee}} \rightarrow r_{perigee} = a \cdot (1 - e)$$

# Total Specific Energy (4)

$$\varepsilon_T = \left[\frac{V^2}{2} - \frac{\mu}{r}\right]$$

$\varepsilon_T$ constant everywhere in orbit

- Now ***look at perigee condition force balance***




$$\dot{r}_{perigee} = 0 \rightarrow \text{Centrifugal force} = \text{Gravitational force @ Apogee}$$

$$V_{perigee}{}^2 = \frac{\mu \cdot a\left(1-e^2\right)}{a^2 \cdot (1-e)^2} = \frac{\mu \cdot \left(1-e^2\right)}{a \cdot (1-e)^2} = \frac{\mu \cdot (1+e)(1-e)}{a \cdot (1-e)^2} = \frac{\mu \cdot (1+e)}{a \cdot (1-e)}$$

# Total Specific Energy (5)

$$\varepsilon_T = \left[\frac{V^2}{2} - \frac{\mu}{r}\right]$$

$\varepsilon_T$ constant everywhere in orbit

… ***at perigee conditions*** $\quad V_{perigee}^{\ 2} = \frac{\mu \cdot (1+e)}{a \cdot (1-e)}$

- ***Sub Into Energy Equation***

*Energy Equation*

$$\frac{V_{perigee}^{\ 2}}{2} - \frac{\mu}{r_{perigee}} = \varepsilon = \frac{\mu \cdot (1+e)}{2 \cdot a \cdot (1-e)} - \frac{\mu}{a \cdot (1-e)} = \frac{\mu}{2a} \cdot \left(\frac{(1+e)}{(1-e)} - \frac{2}{(1-e)}\right) =$$

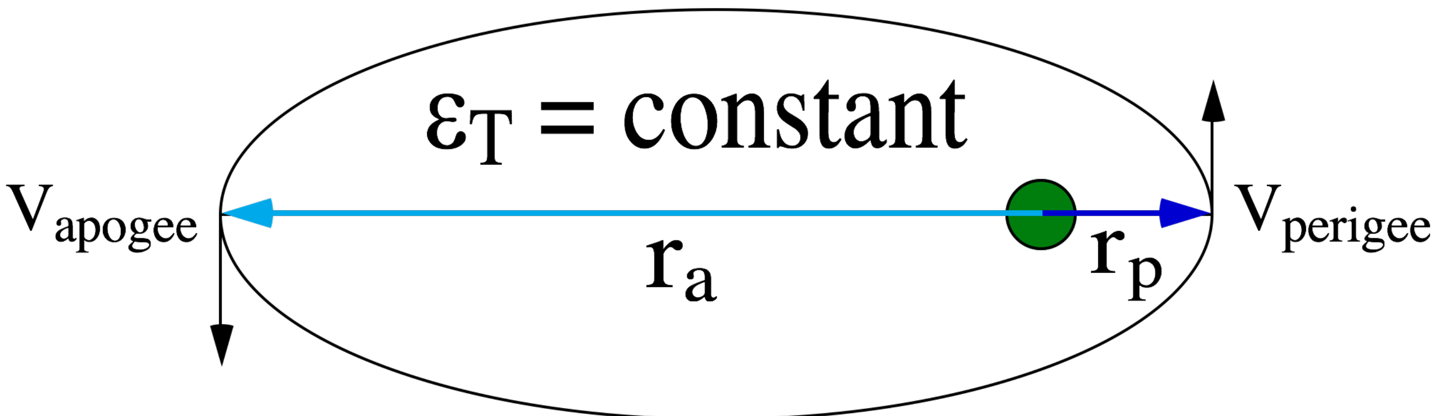
$$\frac{\mu}{2a} \cdot \left(\frac{1+e-2}{1-e}\right) = \frac{\mu}{2a} \cdot \left(\frac{e-1}{1-e}\right) = -\frac{\mu}{2a}$$

# Total Specific Energy (6)

$$\varepsilon_T = \left[\frac{V^2}{2} - \frac{\mu}{r}\right]$$

$\varepsilon_T$ constant everywhere in orbit

- Similarly @ ***Apogee condition force balance***




$$\dot{r}_{apogee} = 0 \rightarrow \text{Centrifugal force} = \text{Gravitational force @ Apogee}$$

$$\rightarrow r_{apogee} = a \cdot (1+e)$$

$$V_{apogee}^{\ 2} = \frac{\mu \cdot R_{c_{apogee}}}{r^2_{\ apogee}} = \frac{\mu \cdot (1+e)(1-e)}{a \cdot (1+e)^2} = \frac{\mu \cdot (1-e)}{a \cdot (1+e)}$$

# Total Specific Energy (5)

$$\varepsilon_T = \left[\frac{V^2}{2} - \frac{\mu}{r}\right]$$

$\varepsilon_T$ constant everywhere in orbit

… ***at perigee conditions*** $V_{apogee}{}^2 = \frac{\mu \cdot (1-e)}{a \cdot (1+e)}$

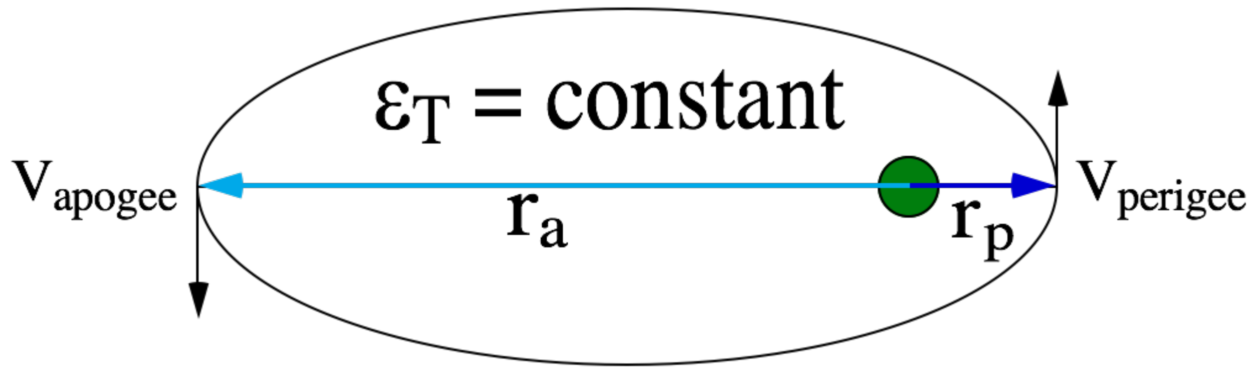
- ***Sub Into Energy Equation***

*Energy Equation*

$$\frac{V_{apogee}{}^2}{2} - \frac{\mu}{r_{apogee}} = \varepsilon = \frac{\mu \cdot (1-e)}{a \cdot (1+e)} - \frac{\mu}{a \cdot (1+e)} = \frac{\mu}{2a} \cdot \left(\frac{1-e-2}{1+e}\right) = -\frac{\mu}{2a}$$

$...Q.E.D$

# Total Specific Energy (6)

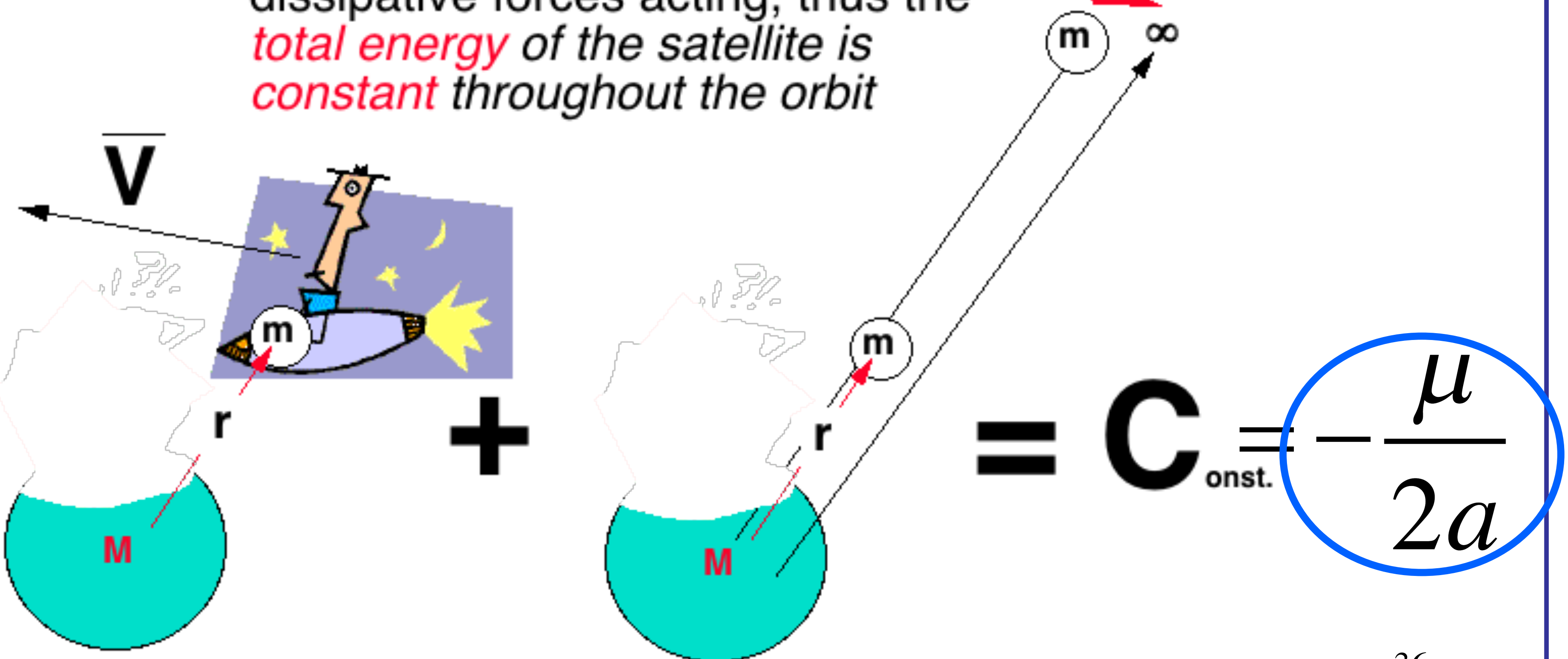



$$\varepsilon_T = \left[\underset{\text{kinetic energy}}{\frac{V^2}{2}} - \underset{\text{potential energy}}{\frac{\mu}{r}}\right]_{perigee} = \left[\underset{\text{kinetic energy}}{\frac{V^2}{2}} - \underset{\text{potential energy}}{\frac{\mu}{r}}\right]_{anywhere} = \underset{\text{total energy}}{-\frac{\mu}{2a}}$$

# Orbital Energy Review

## Total (Mechanical) Energy of the Satellite

- For a satellite orbiting in a gravity field outside of the atmosphere (no external drag) there are no dissipative forces acting, thus the *total energy of the satellite is constant throughout the orbit*



# Orbital Energy

- To Understand that ... we'll first have to show that the orbital dynamics are a result of the balance between kenetic and potential energy
  - Specifically

$$-\frac{\mu}{2a} = \frac{\left|\overline{V}\right|^2}{2} - \frac{\mu}{r}$$

| Total Specific Energy | Specific Kinetic Energy | Specific Potential Energy |
|---|---|---|

# Total Specific Energy (concluded)


- Solving for V, the *elliptical orbit velocity magnitude* is:

$$V = \sqrt{\mu \left[ \frac{2}{r} - \frac{1}{a} \right]}$$

- Newton referred to this equation as the "vis-viva" equation

.... literally translated ... "it's alive"

- Extremely important relationship shows that orbital speed is inversely proportional to square root of the orbital radius

# Orbital Energy

*eps = 0*

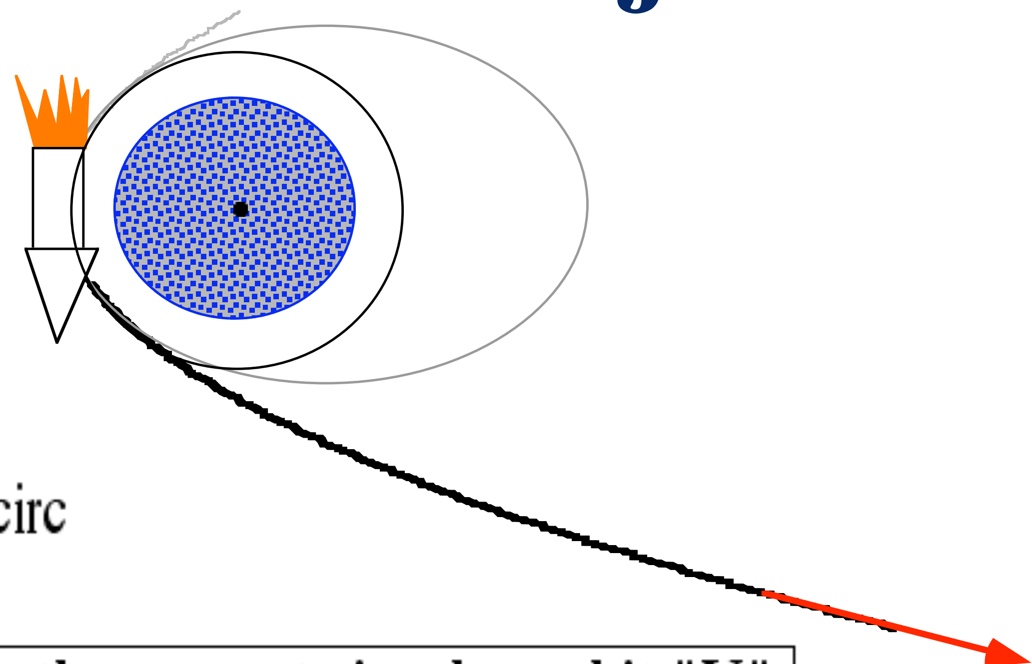
- To Understand that ... we'll first have to show that the orbital dynamics are a result of the balance between kenetic and potential energy
  - Specifically

*eps < 0*

$$-\frac{\mu}{2a} = \frac{\left|\bar{V}\right|^2}{2} - \frac{\mu}{r}$$

| Total Specific Energy | Specific Kinetic Energy | Specific Potential Energy |
|---|---|---|

# Parabolic Trajectories:



$$\Delta V_{esc} = \left[\sqrt{2} - 1\right] V_{circ}$$

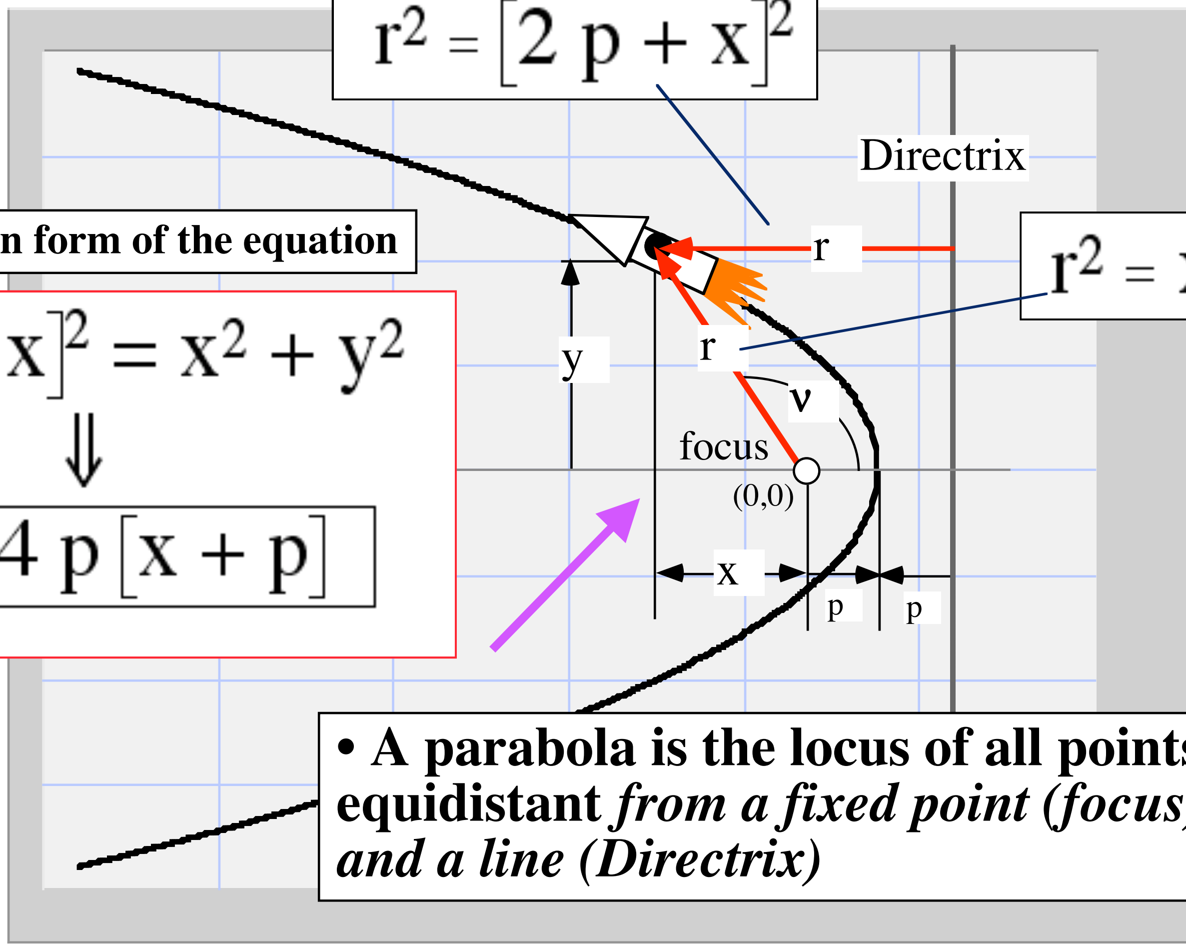
• If we increase the current circular orbit "V" by a factor of $\sqrt{2}$; then the velocity becomes too great for the planet to contain the orbit

• Satellite *escapes* the planet on a parabolic trajectory

# What is a Parabola?

$$r^2 = [2p + x]^2$$

$$r^2 = x^2 + y^2$$




- **Cartesian form of the equation**

$$[2p + x]^2 = x^2 + y^2$$

$$\Downarrow$$

$$y^2 = 4p[x + p]$$

- **A parabola is the locus of all points equidistant *from a fixed point (focus) and a line (Directrix)***

# Postscript:Escape Velocity

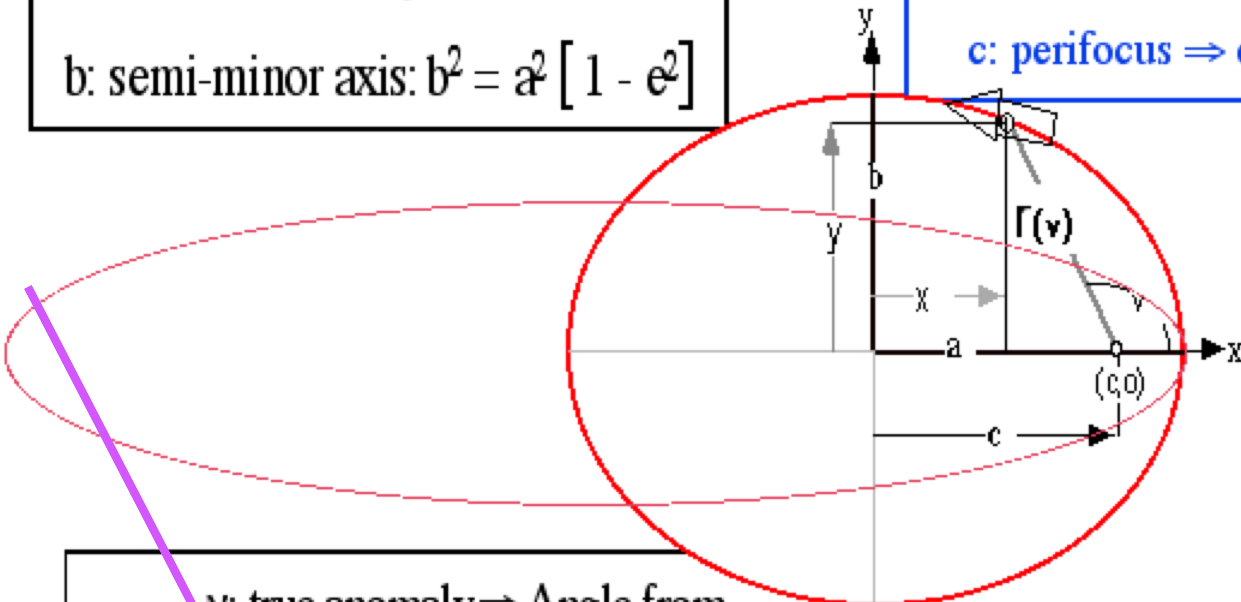
**What happens when a->∞ in an elliptical orbit?**

a: semi-major axis:

b: semi-minor axis: $b^2 = a^2\left[1 - e^2\right]$

e: orbital eccentricity ⇒ $e = \sqrt{1 - \left[\frac{b}{a}\right]^2}$

c: perifocus ⇒ $c = a\sqrt{1 - \left[\frac{b}{a}\right]^2} = a\,e$




$a \rightarrow \infty$

$e \rightarrow 1$

$r(\nu) = \frac{a\left[1 - e^2\right]}{\left[1 + e\cos(\nu)\right]}$ ?

ν: true anomaly ⇒ Angle from perapsis to satellite

(ν): orbital radius ⇒ $r(\nu) = \frac{a\left[1 - e^2\right]}{\left[1 + e\cos(\nu)\right]}$

# Postscript II: Escape Velocity (cont'd)

**What happens when a->∞ in an elliptical orbit?**

$$R_{a\to\infty} = \left[ \frac{\dfrac{a\left[1 - e^2\right]}{1 + e\cos\left[\nu_i\right]}}{\lim \begin{matrix} a \to \infty \\ e \to 1 \end{matrix}} \right] \text{ "indeterminant"}$$

- **But** $a\left[1 - e^2\right] = \underline{a\left[1 - e\right]}\left[1 + e\right] = \underline{R_{perigee}}\left[1 + e\right]$

$$R_{a\to\infty} = \left[ \frac{\dfrac{a\left[1 - e^2\right]}{1 + e\cos\left[\nu_i\right]}}{\lim \begin{matrix} a \to \infty \\ e \to 1 \end{matrix}} \right] = \frac{2\,R_{perigee}}{1 + \cos\left[\nu_i\right]}$$

# Postscript:Escape Velocity (cont'd)

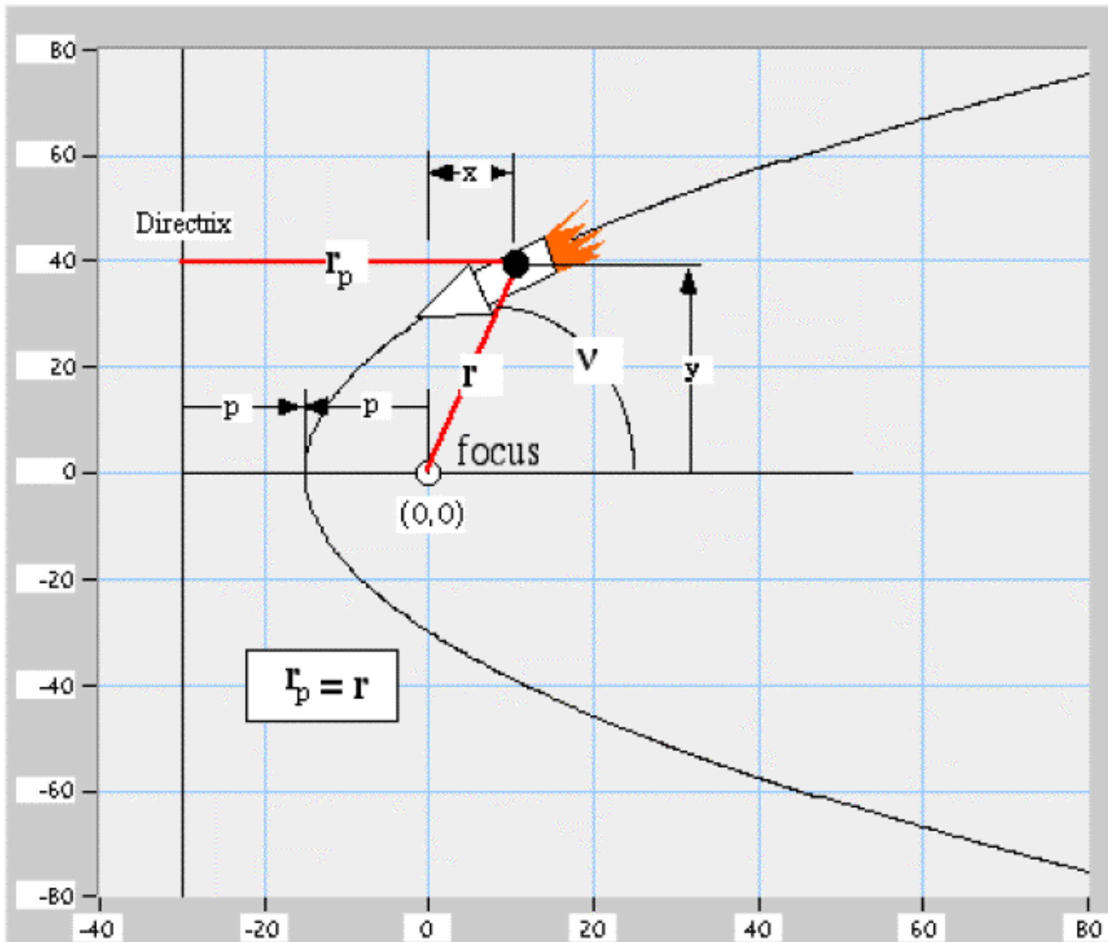
- a->∞ implies an "open" parabolic trajectory

$$R_{a \to \infty} = \frac{2\, R_{perigee}}{1 - \cos\left[\nu_i\right]}$$

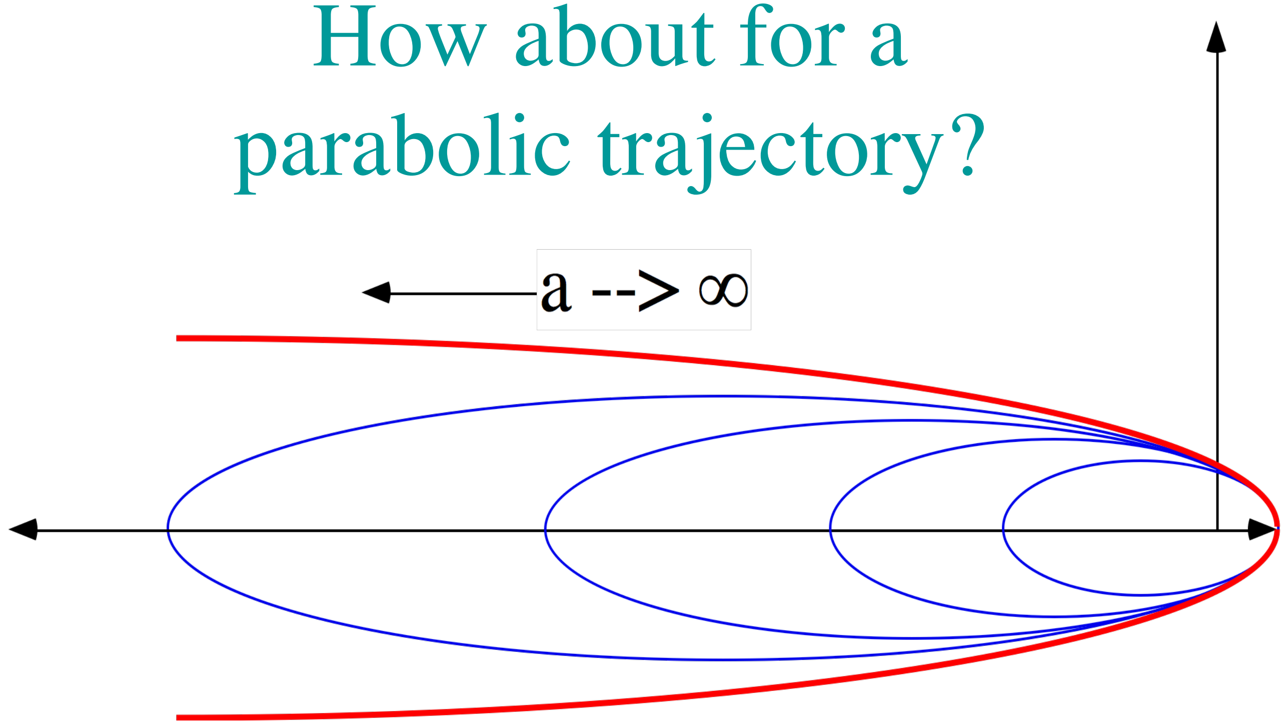
$$\left|\overline{V}\right|^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

0

$$\left|\overline{V}_{esc}\right| = \sqrt{\frac{2\,\mu}{r}}$$



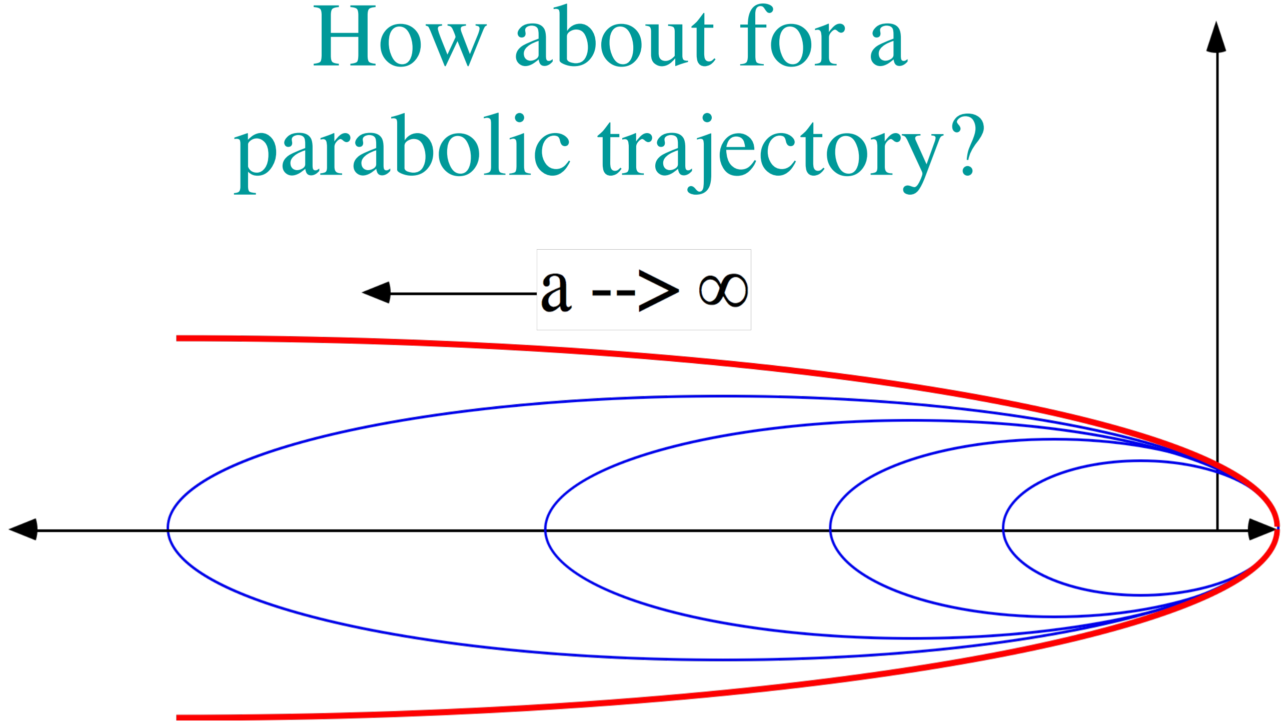
# How about for a parabolic trajectory?




kinetic energy | potential energy | total energy

$$\lim_{a\to\infty}\left\{\varepsilon_T = \left[\frac{V^2}{2} - \frac{\mu}{r}\right] = -\frac{\mu}{2\,a}\right\} = 0$$

# How about for a parabolic trajectory?




- Orbital Energy is with regard to an escape trajectory!

- *Circular, Elliptical Orbit* → $\varepsilon_T < 0$
- *Parabolic (Escape) Trajectory* → $\varepsilon_T = 0$
- *Hyperbolic Trajectory* → $\varepsilon_T >= 0$

***See Appendix 2.3.2 for Hyperbolic Trajectory Proof***

# Vis-Viva Equation for All the Conic-Sections

Circle: $\boxed{r = a} \Rightarrow \quad V = \sqrt{\mu\left[\frac{2}{a} - \frac{1}{a}\right]} = \sqrt{\frac{\mu}{a}}$

Ellipse: $\boxed{r = \frac{a\left[1 - e^2\right]}{\left[1 + e\cos(\nu)\right]}} \Rightarrow \quad V = \sqrt{\mu\left[\frac{2}{r} - \frac{1}{a}\right]}$

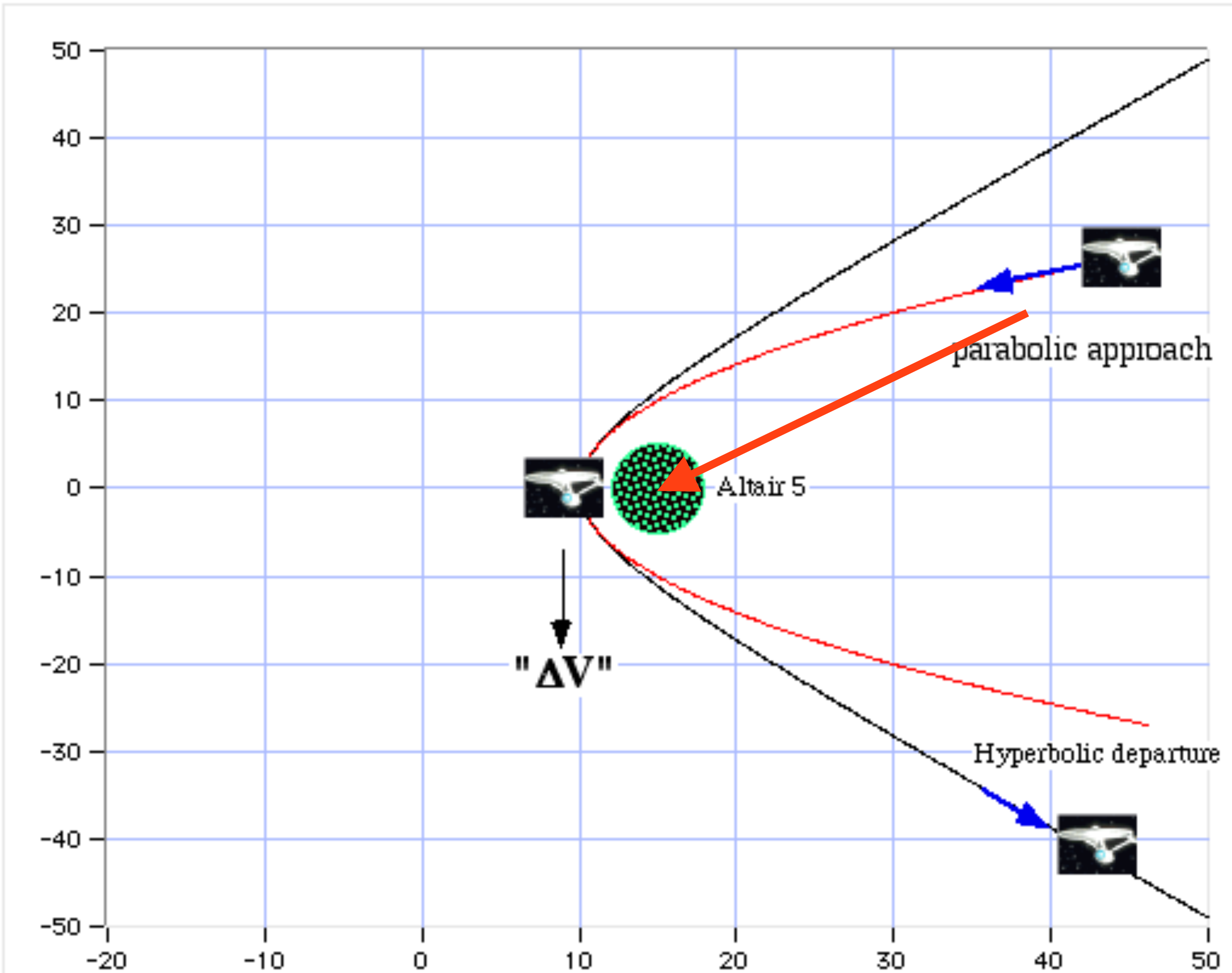
Parabola: $\boxed{r = \frac{2p}{\left[1 + \cos(\nu)\right]}} \Rightarrow \quad V = \sqrt{\mu\left[\frac{2}{r} - \frac{1}{\infty}\right]} = \sqrt{\frac{2\mu}{r}}$

Hyperbola: $\boxed{r = \frac{a\left[e_{hyp}^2 - 1\right]}{\left[1 + e_{hyp}\cos(\nu)\right]}} \Rightarrow \quad V = \sqrt{\mu\left[\frac{2}{r} + \frac{1}{a}\right]}$

***See Appendix 2.3.2 for Hyperbolic Trajectory Proof***

# Homework 3

## Parabolic and Hyperbolic Trajectories



# Homework

## Parabolic and Hyperbolic Trajectories (cont'd)

- ***United Federation of Planets*** **starship** ***Excelsior*** **approaches** ***Klingon*** **outpost** ***Altair 5*** **on a covert retaliatory bombing mission**

- **A cloaking device uses enormous energy &** ***Warp drive*** **is non-operational with the cloak engaged**

- **All maneuvering must be done on** ***impulse power*** **alone**

- **The** ***Excelsior*** **uses a gravity assisted** ***parabolic*** **approach trajectory to** ***Altair 5*** **in order to save on waning impulse power and insure a stealthy approach**

UtahState UNIVERSITY

# Homework

Mechanical & Aerospace Engineering

## Parabolic and Hyperbolic Trajectories (cont'd)

- **After dropping photo-torpedos, Captain Checkov wants to get out the *sphere of influence* (SOI) of*Altair 5* as fast as possible without being spotted**

- **The *Excelsio*r has enough impulse power left for *one big burn* before, having to recharge the *dilithium crystals***

- **The best way to "get out of town fast" is to fire impulse engines at closest approach to Altair 5 -- taking advantage of the gravity assist to give the highest approach speed without using impulse power and then use impulse power to depart on a hyperbolic trajectory at angle of 45 degrees**

- **What is the "*Delta-V*" required to depart on a *Hyperbolic* trajectory with an asymptotic departure angle of 45 degrees**

# Homework:

## Parabolic and Hyperbolic Trajectories (cont'd)

- **Hint 1: For a Parabolic trajectory**

**r is measured from the parabolic *focus* to the location of the *Excelsior***

- **Hint 2: For a Hyperbolic trajectory**

**r is measured from the *right (perifocus) focus* to the location of the *Excelsior***

# Homework:

## Parabolic and Hyperbolic Trajectories (concluded)

- **Hint 3: For a Parabolic to Hyperbolic trajectory transfer**

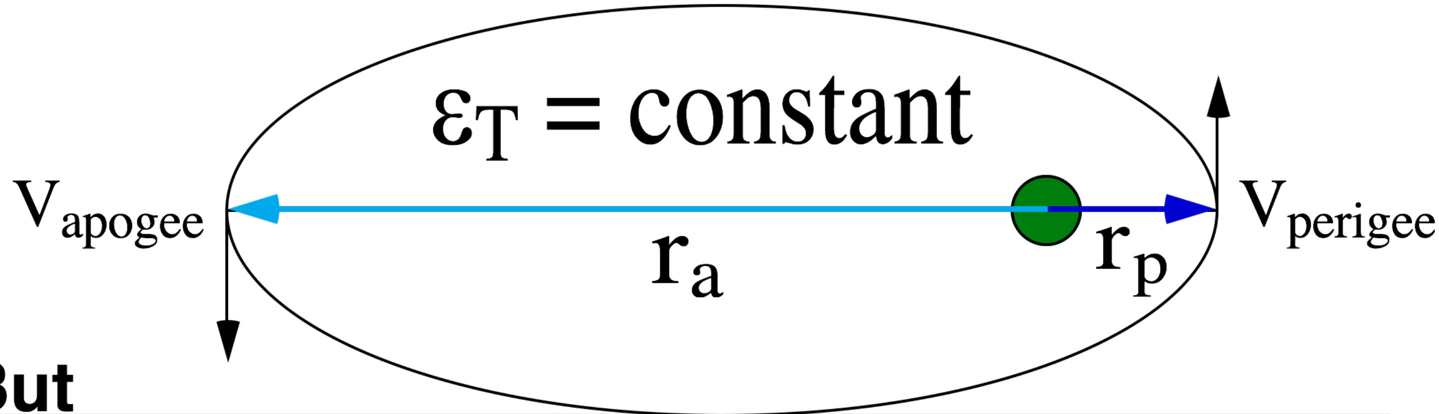
$$"\Delta V" = V_h - V_p = V_p\left[\frac{V_h}{V_p} - 1\right]$$

- **Hint 4: At closest apprach, the distance from the *parabolic focus* to the *Excelsior* must equal the distance from the *Hyperbolic right focus* to the *Excelsior***

- **Your answer should be expressed in terms $\mu$ and $r_{min}$ (closest approach distance)**

# *Appendix 2.3: Total Specific Orbital Energy Alternate Derivation*

# Total Specific Energy (cont'd)




- **But**

Kepler's Second Law:

$r^2 \omega = $ constant

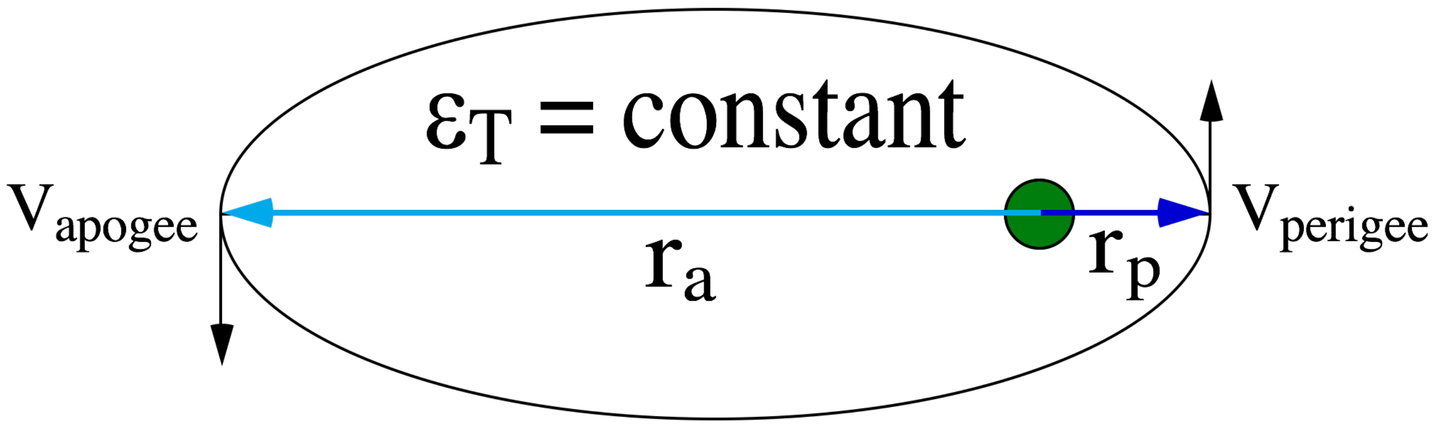
angular momentum

$$\Rightarrow r_a^2 \omega_a = r_p^2 \omega_p \Rightarrow \boxed{\omega_a = \frac{r_p^2}{r_a^2} \omega_p}$$

- **Substituting in for $\omega_a$ and rearranging**

$$2\mu \left[ \frac{r_p - r_a}{r_a r_p} \right] = \left[ r_a \frac{r_p^2}{r_a^2} \omega_p + r_p \omega_p \right] \left[ r_a \frac{r_p^2}{r_a^2} \omega_p - r_p \omega_p \right]$$

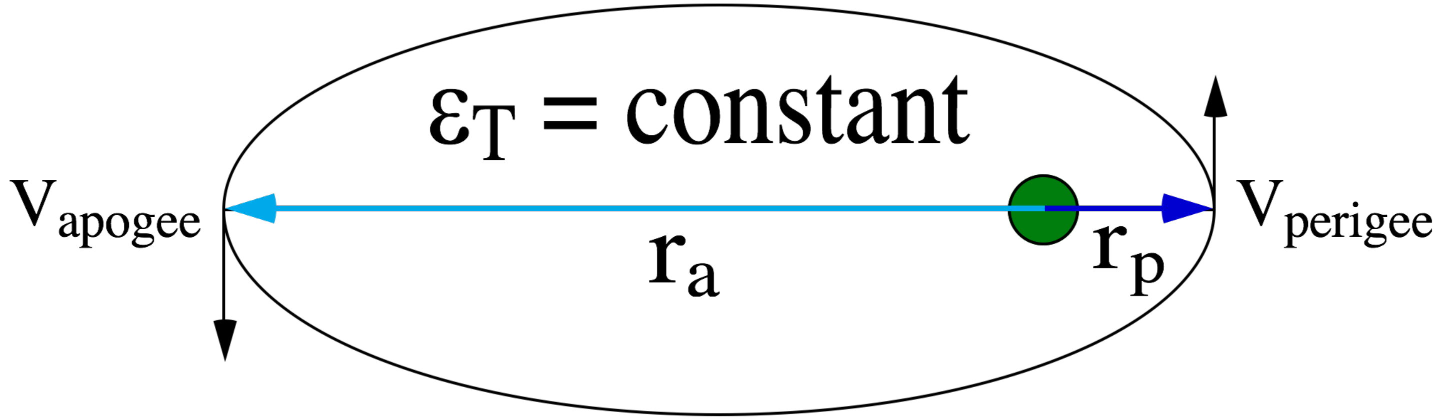
# Total Specific Energy (cont'd)




- Solving for $\omega_p^2$

$$\omega_p^2 = 2\,\mu \left[\frac{r_a}{r_p}\right]^2 \frac{1}{\left[r_p + r_a\right]} \left[\frac{1}{r_a\, r_p}\right]$$

# Total Specific Energy (cont'd)




- **But .....** $r_a = a\,[1+e]$ $\quad r_p = a\,[1-e]$

$$\Downarrow$$

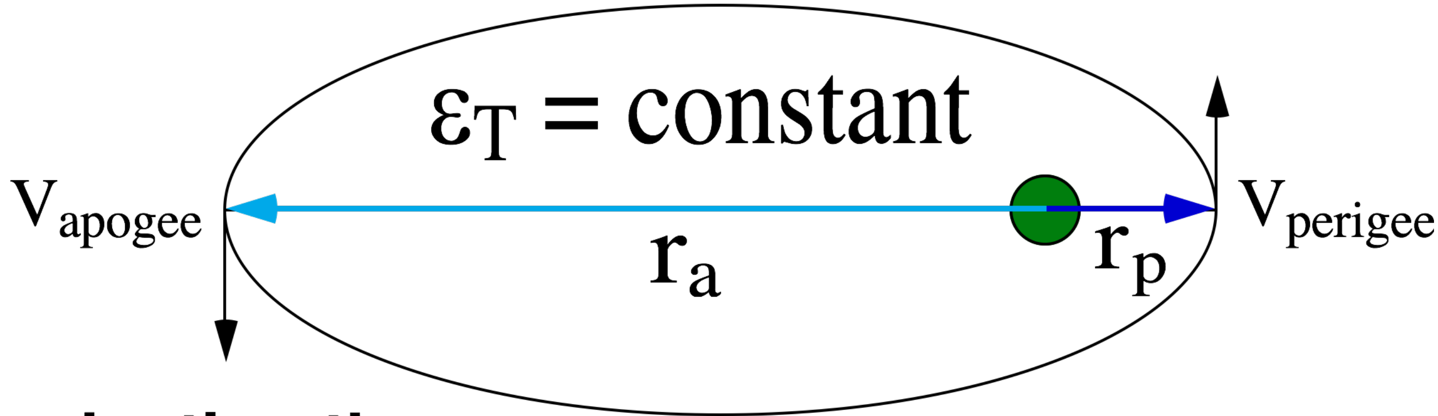
$$\boxed{r_p + r_a = a\,[1-e] + a\,[1+e] = 2\,a}$$

$$\Downarrow$$

$$\boxed{r_a\, r_p = a\,[1-e]\,a\,[1+e] = a^2\,[1-e^2]}$$

$$\omega_p^2 = \frac{\mu}{a^3}\,\frac{[1+e]^2}{[1-e]^2\,[1-e^2]}$$

# Total Specific Energy (cont'd)



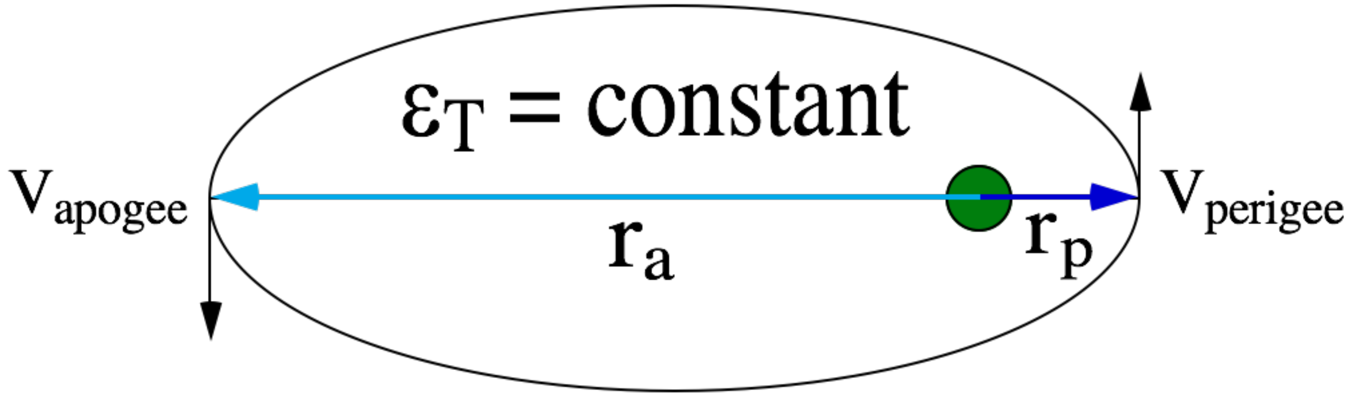

- **Re-evaluating the total specific energy.....**

$$r_p = a\left[1-e\right]$$

$$\omega_p^2 = \frac{\mu}{a^3}\frac{\left[1+e\right]^2}{\left[1-e\right]^2\left[1-e^2\right]}$$

$$\varepsilon_T = \left[\frac{\left[r_p^2\,\omega_p^2\right]}{2} - \frac{\mu}{r_p}\right]$$

# Total Specific Energy (cont'd)




- .... and ..... the result is

$$\varepsilon_T = \left[\frac{V^2}{2} - \frac{\mu}{r}\right]_{perigee} = \left[\frac{\frac{\mu}{a^3}\frac{[1+e]^2}{[1-e]^2[1-e^2]}a^2\,[1-e]^2}{2} - \frac{\mu}{a\,[1-e]}\right] =$$

$$\frac{\mu}{2\,a}\left[\frac{[1+e]^2}{[1-e^2]} - \frac{2}{[1-e]}\right] = \frac{-\mu}{2\,a}\left[\left[\frac{1-e^2}{1-e^2}\right]\right] = \boxed{-\frac{\mu}{2\,a}}$$

# *Appendix 2.3.2: Total Specific Orbital Energy for Hyperbolic Trajectory*

# How about for a hyperbolic trajectory?

- Conservation of Energy and Angular Momentum still hold …. So ….

$$\varepsilon_T = \left[ \underset{\text{kinetic energy}}{\frac{V^2}{2}} \quad \underset{\text{potential energy}}{-\frac{\mu}{r}} \right]_{perigee}$$

….. and ….

$$V_{perigee} = r_p \, \omega_p$$

# Hyperbolic Energy

- Recall, from the "First Law" derivation

$$\omega\, r = \frac{\mu}{|\mathbf{h}|}\left[1 + B\cos(\nu)\right] \Rightarrow B = \frac{|\mathbf{h}|^2}{\mu}\frac{1}{r_p} - 1$$

$$\Downarrow$$

$$\omega\, r = \frac{\mu}{|\mathbf{h}|}\left[1 + \left(\frac{|\mathbf{h}|^2}{\mu}\frac{1}{r_p} - 1\right)\cos(\nu)\right]$$

# Hyperbolic Energy (continued)

- At Perigee, $\nu$=0

$$\omega_p \, r_p = \frac{\mu}{|\boldsymbol{l}|}\left[1 + \left(\frac{|\boldsymbol{l}|^2}{\mu}\frac{1}{r_p} - 1\right)\right] =$$

$$\frac{\mu}{|\boldsymbol{l}|}\left[\frac{|\boldsymbol{l}|^2}{\mu}\frac{1}{r_p}\right] = \left[\frac{|\boldsymbol{l}|}{r_p}\right]$$

# Hyperbolic Energy (continued)

- Substituting into energy equation

$$\varepsilon_T^{(hyp)} = \underbrace{\frac{1}{2}\left[\frac{|\mathbf{l}|}{r_p}\right]^2}_{\text{kinetic energy}} - \underbrace{\frac{\mu}{r_p}}_{\text{potential energy}}$$

And for a hyperbola

$$r_p = \frac{a\left[e_{hyp}^2 - 1\right]}{\left[1 + e_{hyp}\cos(0)\right]} = \frac{a\left[e_{hyp} + 1\right]\left[e_{hyp} - 1\right]}{\left[1 + e_{hyp}\right]} = a\left[e_{hyp} - 1\right]$$

# Hyperbolic Energy (continued)

- Substituting into energy equation

kinetic energy — potential energy

$$\varepsilon_T^{(hyp)} = \frac{1}{2}\left[\frac{\left|\boldsymbol{l}\right|}{a\left[e_{hyp} - 1\right]}\right]^2 - \frac{\mu}{a\left[e_{hyp} - 1\right]}$$

# Hyperbolic Energy (continued)

- But from the General Form for the Conic section

$$r(\nu) = \frac{\frac{|\boldsymbol{l}|^2}{\mu}}{\left[1 + B\cos(\nu)\right]} \Rightarrow$$

$$B = \frac{|\boldsymbol{l}|^2}{\mu}\frac{1}{r_p} - 1 \equiv e_{\text{hyp}}$$

# Hyperbolic Energy (continued)

Evaluating at perigee

$$r_p^{(hyp)} = a\left[ e_{hyp} - 1 \right] = \frac{\dfrac{\left|\boldsymbol{I}\right|^2}{\mu}}{\left[1 + e_{hyp} \cos(0)\right]}$$

$$\Downarrow$$

$$\left|\boldsymbol{I}\right|^2 = \mu a \left[ e_{hyp}^2 - 1 \right]$$

# Hyperbolic Energy (continued)

- Substituting into the Energy equation

$$\varepsilon_T^{(hyp)} = \underbrace{\frac{1}{2}\left[\frac{|\boldsymbol{I}|}{a\left[e_{hyp} - 1\right]}\right]^2}_{\text{kinetic energy}} - \underbrace{\frac{\mu}{a\left[e_{hyp} - 1\right]}}_{\text{potential energy}}$$

$$|\boldsymbol{I}|^2 = \mu a\left[e_{hyp}^2 - 1\right]$$

# Hyperbolic Energy (concluded)

$$\varepsilon_T^{(hyp)} = \frac{1}{2}\frac{\mu a\left[e_{hyp}^2 - 1\right]}{a^2\left[e_{hyp} - 1\right]^2} - \frac{\mu}{a\left[e_{hyp} - 1\right]} =$$

$$\frac{1}{2}\frac{\mu a\left[e_{hyp} + 1\right]\left[e_{hyp} - 1\right]}{a^2\left[e_{hyp} - 1\right]^2} - \frac{\mu}{a\left[e_{hyp} - 1\right]} =$$

$$\frac{1}{2}\frac{\mu\left[e_{hyp}+1\right]}{a\left[e_{hyp}-1\right]} - \frac{\mu}{a\left[e_{hyp}-1\right]} = \frac{\mu\left[\frac{e_{hyp}}{2} - \frac{1}{2}\right]}{a\left[e_{hyp}-1\right]} = \frac{\mu}{2a}$$