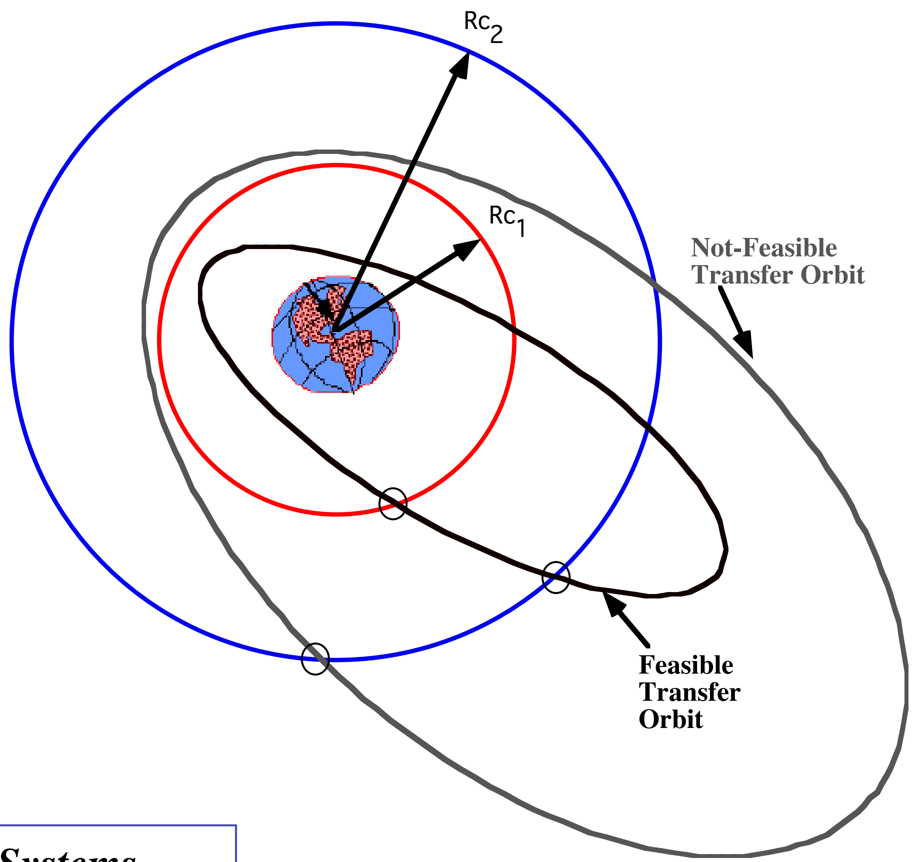
Utah State University

Mechanical & Aerospace Engineering

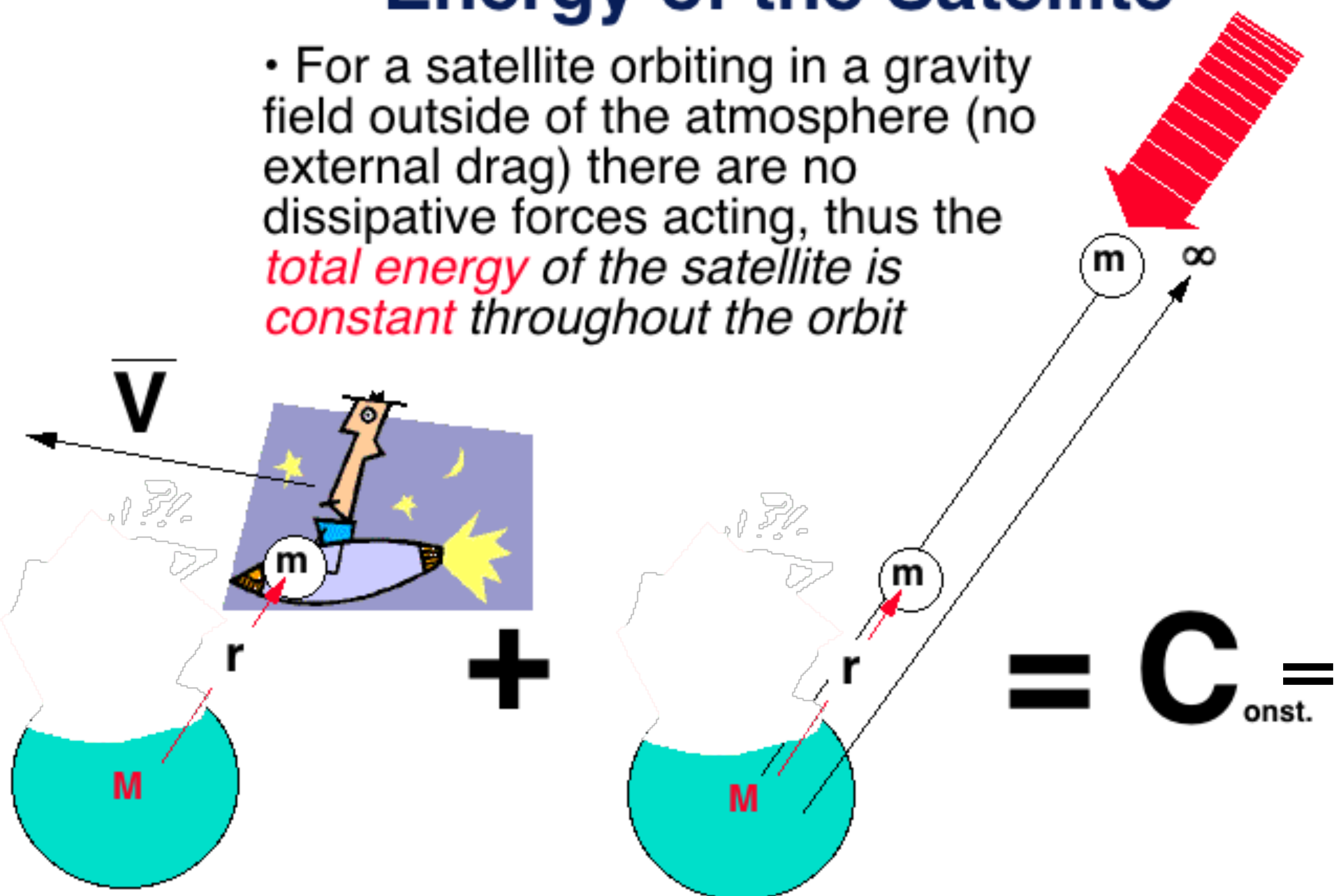
Section 2.4: Applications of the Vis-Viva Equation: The Hohmann Transfer



Orbital Energy Review

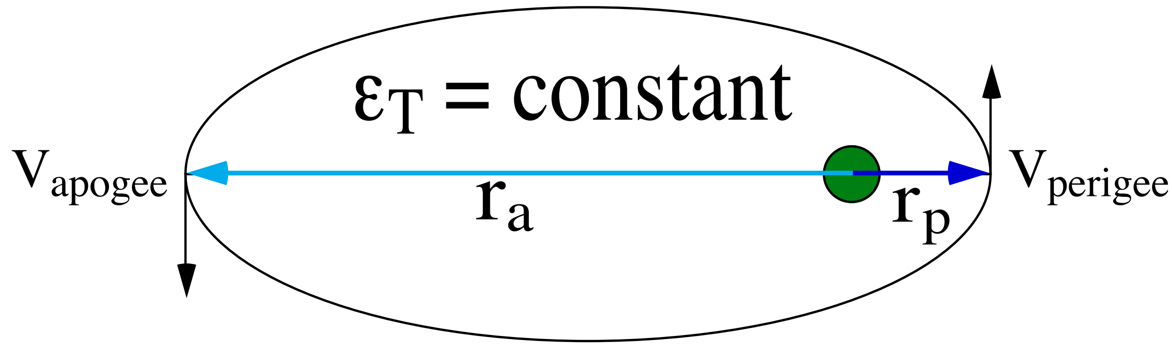
Total (Mechanical) Energy of the Satellite

• For a satellite orbiting in a gravity field outside of the atmosphere (no external drag) there are no dissipative forces acting, thus the *total energy of the satellite is constant throughout the orbit*



$$= -\frac{\mu}{2a}$$

Total Specific Energy (6)



$$\varepsilon_T = \left[\underset{\text{kinetic energy}}{\frac{V^2}{2}} \underset{\text{potential energy}}{- \frac{\mu}{r}}\right]_{perigee} = \left[\underset{\text{kinetic energy}}{\frac{V^2}{2}} \underset{\text{potential energy}}{- \frac{\mu}{r}}\right]_{anywhere} = \underset{\text{total energy}}{- \frac{\mu}{2a}}$$

Total Specific Energy (concluded)

- Solving for V, the *elliptical orbit velocity magnitude* is:

$$V = \sqrt{\mu \left[\frac{2}{r} - \frac{1}{a} \right]}$$

- Newton referred to this equation as the "vis-viva" equation

.... literally translated ... "it's alive"

- Extremely important relationship shows that orbital speed is inversely proportional to square root of the orbital radius

Vis-Viva Equation for All the Conic-Sections

Circle: $\boxed{r = a}$ $\Rightarrow$ $V = \sqrt{\mu\left[\frac{2}{a} - \frac{1}{a}\right]} = \sqrt{\frac{\mu}{a}}$

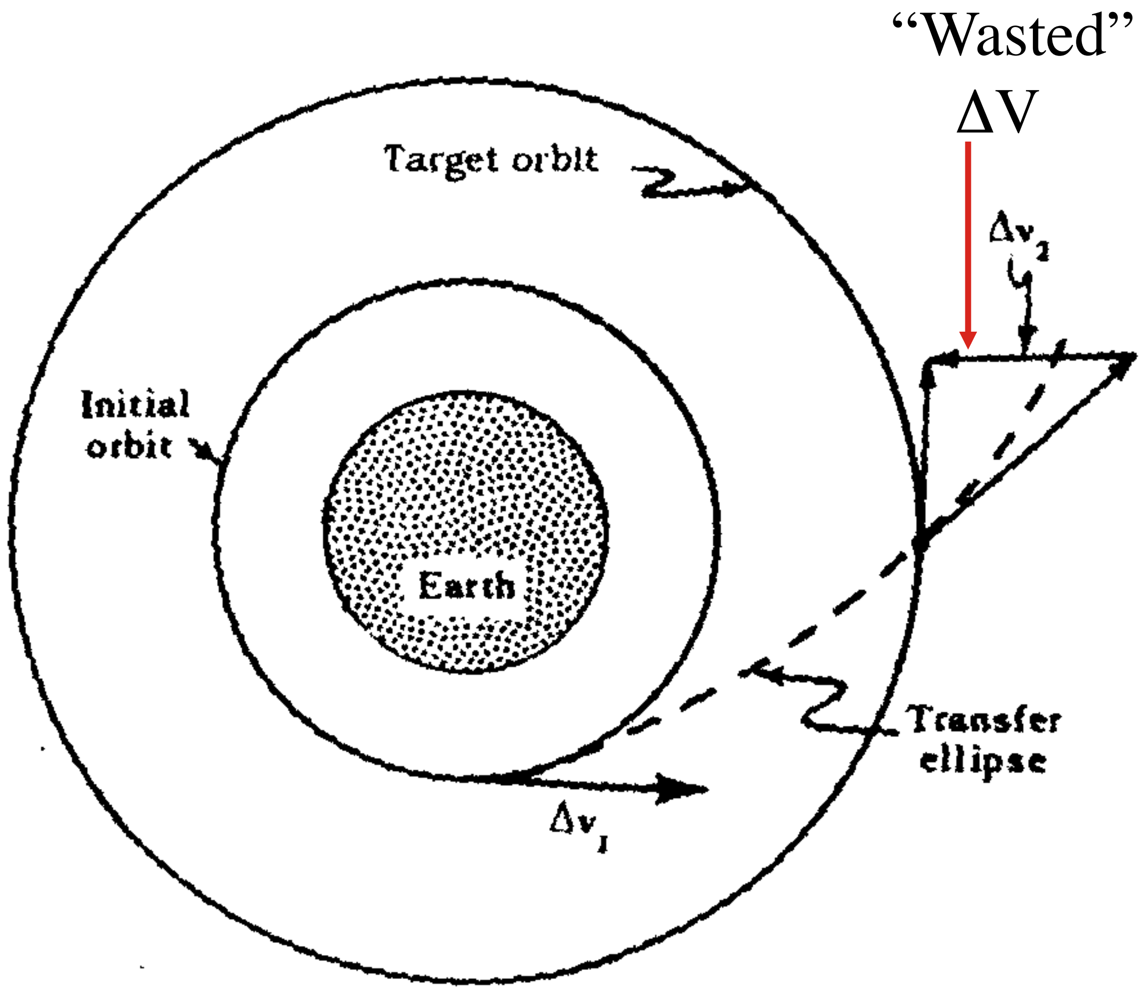
Ellipse: $\boxed{r = \frac{a\left[1 - e^2\right]}{\left[1 + e\ \cos(\nu)\right]}}$ $\Rightarrow$ $V = \sqrt{\mu\left[\frac{2}{r} - \frac{1}{a}\right]}$

Parabola: $\boxed{r = \frac{2p}{\left[1 + \cos(\nu)\right]}}$ $\Rightarrow$ $V = \sqrt{\mu\left[\frac{2}{r} - \frac{1}{\infty}\right]} = \sqrt{\frac{2\mu}{r}}$

Hyperbola: $\boxed{r = \frac{a\left[e_{hyp}^2 - 1\right]}{\left[1 + e_{hyp}\cos(\nu)\right]}}$ $\Rightarrow$ $V = \sqrt{\mu\left[\frac{2}{r} + \frac{1}{a}\right]}$

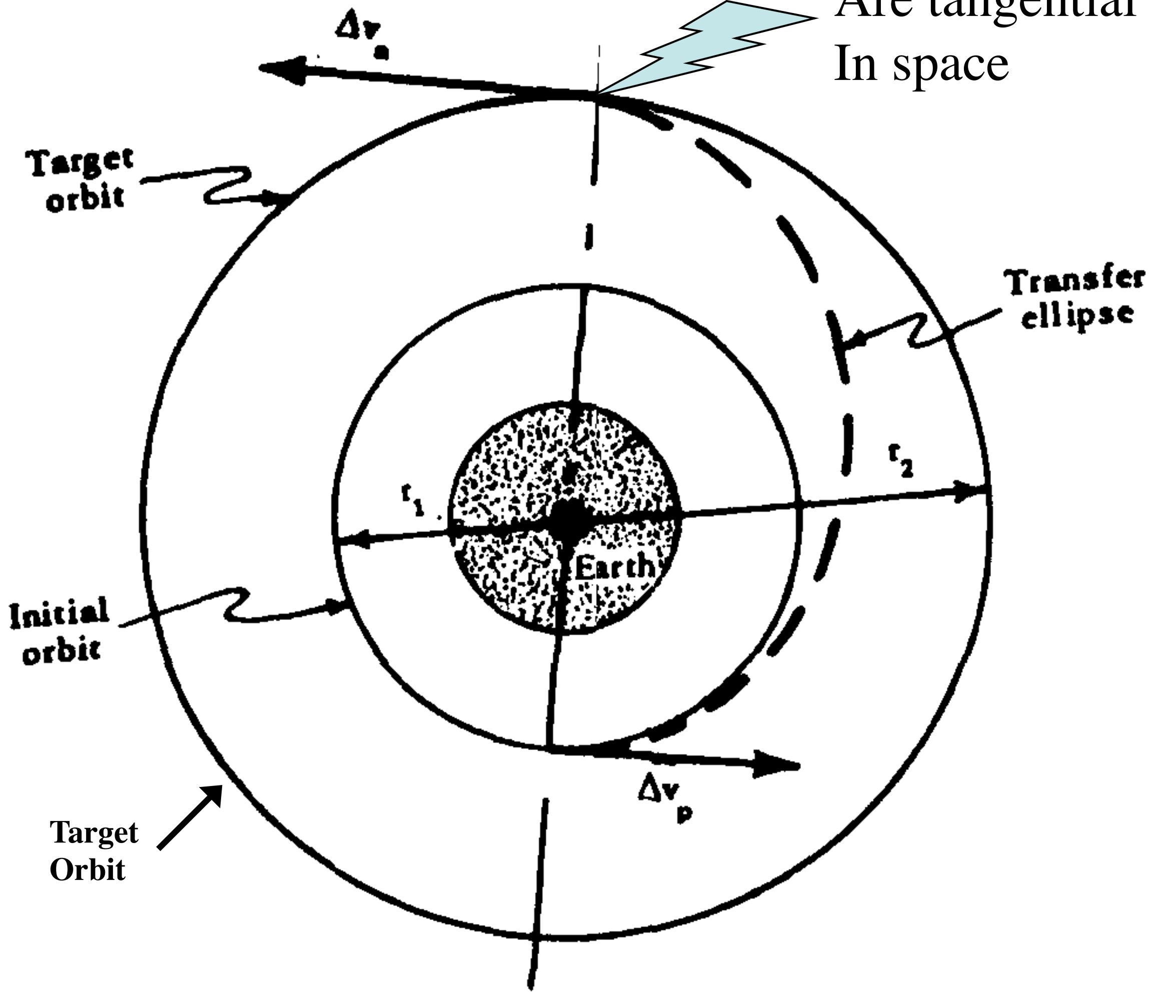
See Appendix 2.3.2 for Hyperbolic Trajectory Proof

Excess-energy transfer orbit



Optimal-energy transfer Orbit

Two curves
Are tangential
In space

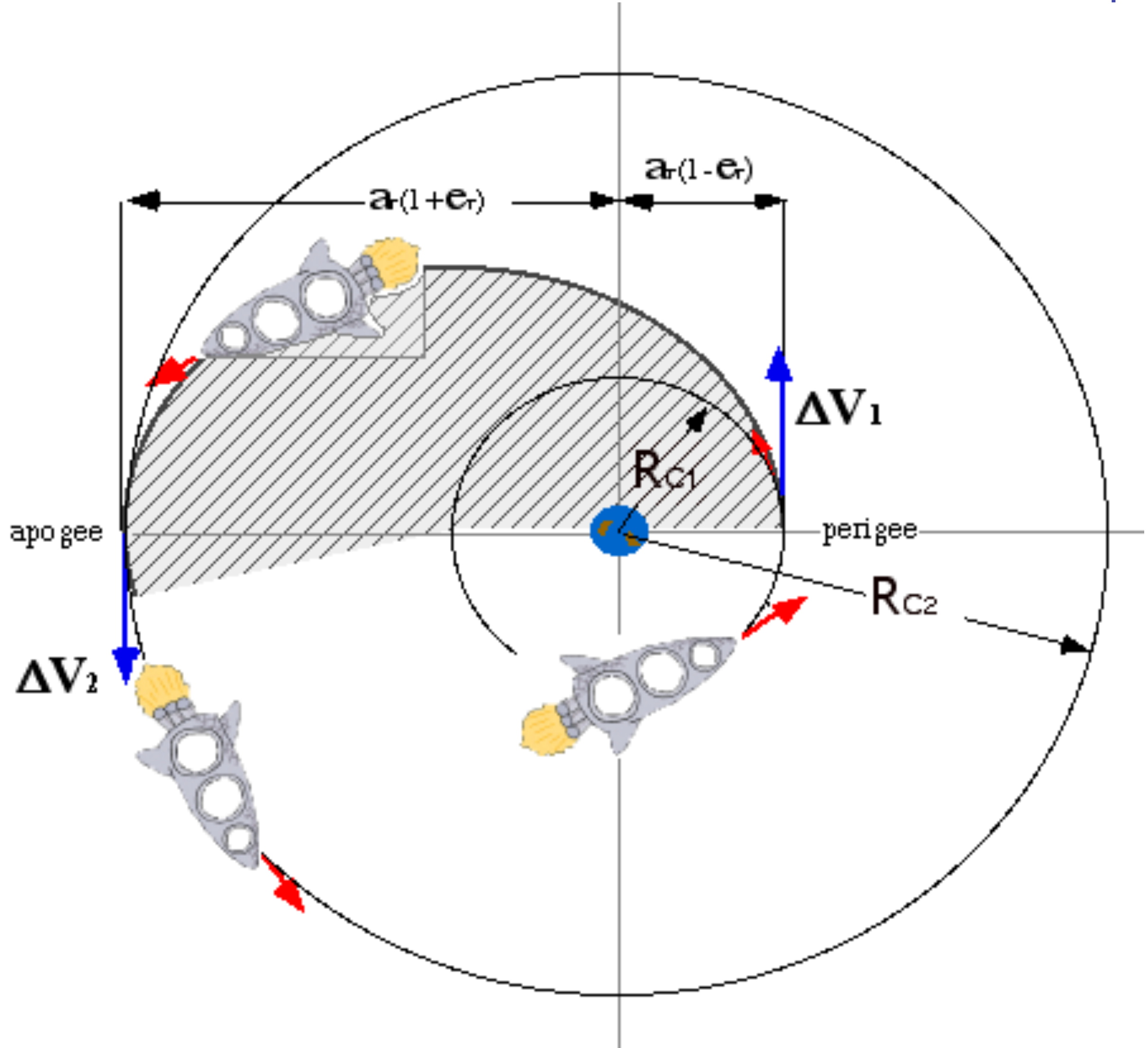


… The Hohmann Transfer

- We want our "transfer-orbit" to be an ellipse with parameters

$$a_T = \frac{\left[R_{c_2} + R_{c_1}\right]}{2}$$

$$e_T = \frac{R_{c_2} - R_{c_1}}{\left[R_{c_2} + R_{c_1}\right]}$$



The Hohmann transfer

- Most fuel efficient method
  - All velocity changes are tangential
  - (change velocity magnitude but not direction)

- Between circular or (aligned elliptical orbits)

- Takes longer than other less efficient transfers

- Tangential elliptical transfer orbit

- (example: Geosynchronous Transfer Orbit GTO)

Hohmann Transfer Steps

- **0 : Calculate transfer orbit semi-major axis & eccentricity**
- 1: Calculate circular velocity of parking orbit
- 2: Calculate perigee velocity of transfer orbit
- 3: Determine perigee delta V
- 4: Calculate apogee velocity of transfer orbit
- 5: Calculate circular velocity of final orbit
- 6: Determine apogee delta V
- 7: Determine total delta V

Hohmann Transfer: Calculating the ΔV's

- Applying the Vis-Viva equation for *Transfer orbit*

$$V_{T_1} = \sqrt{\mu\left[\frac{2}{R_{c_1}} - \frac{1}{a_T}\right]} = \sqrt{\mu\left[\frac{2}{R_{c_1}} - \frac{2}{\left[R_{c_2} + R_{c_1}\right]}\right]}$$

$$\sqrt{2\mu\left[\frac{1}{R_{c_1}} - \frac{1}{\left[R_{c_2} + R_{c_1}\right]}\right]} =$$

$$\sqrt{\frac{\mu}{R_{c_1}}}\sqrt{2\left[1 - \frac{R_{c_1}}{\left[R_{c_2} + R_{c_1}\right]}\right]} = \sqrt{\frac{\mu}{R_{c_1}}}\sqrt{2\left[1 - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]}$$

Hohmann Transfer: Calculating the ΔV's

- Velocity (magnitude) change required to go from starting orbit to transfer orbit is:

$$\text{Initial Orbit: } V_{c_1} = \sqrt{\frac{\mu}{R_{c_1}}}$$

$$\Delta V_1 = V_{T_1} - V_{c_1} = \sqrt{\frac{\mu}{R_{c_1}}}\sqrt{2\left[1 - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]} - \sqrt{\frac{\mu}{R_{c_1}}} =$$

$$\Downarrow$$

$$\frac{\Delta V_1}{\sqrt{\frac{\mu}{R_{c_1}}}} = \sqrt{2\left[1 - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]} - 1$$

Hohmann Transfer: Calculating the ΔV's

- **Again applying the Vis-Viva equation, but this time at R_{c_2}**

  - Velocity at transfer Orbit Intercept

$$V_{T_2} = \sqrt{\mu\left[\frac{2}{R_{c_2}} - \frac{1}{a_T}\right]} = \sqrt{\mu\left[\frac{2}{R_{c_2}} - \frac{1}{\frac{\left[R_{c_2} + R_{c_1}\right]}{2}}\right]} =$$

$$\sqrt{\frac{2\mu}{R_{c_1}}\left[\frac{R_{c_1}}{R_{c_2}} - \frac{R_{c_1}}{\left[R_{c_2} + R_{c_1}\right]}\right]} = \sqrt{\frac{\mu}{R_{c_1}}}\sqrt{2\left[\frac{1}{\left[\frac{R_{c_2}}{R_{c_1}}\right]} - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]}$$

Hohmann Transfer: Calculating the ΔV's

- ΔV change required to go from transfer orbit to final orbit is:

$$\text{Final Orbit: } V_{c_2} = \sqrt{\frac{\mu}{R_{c_2}}}$$

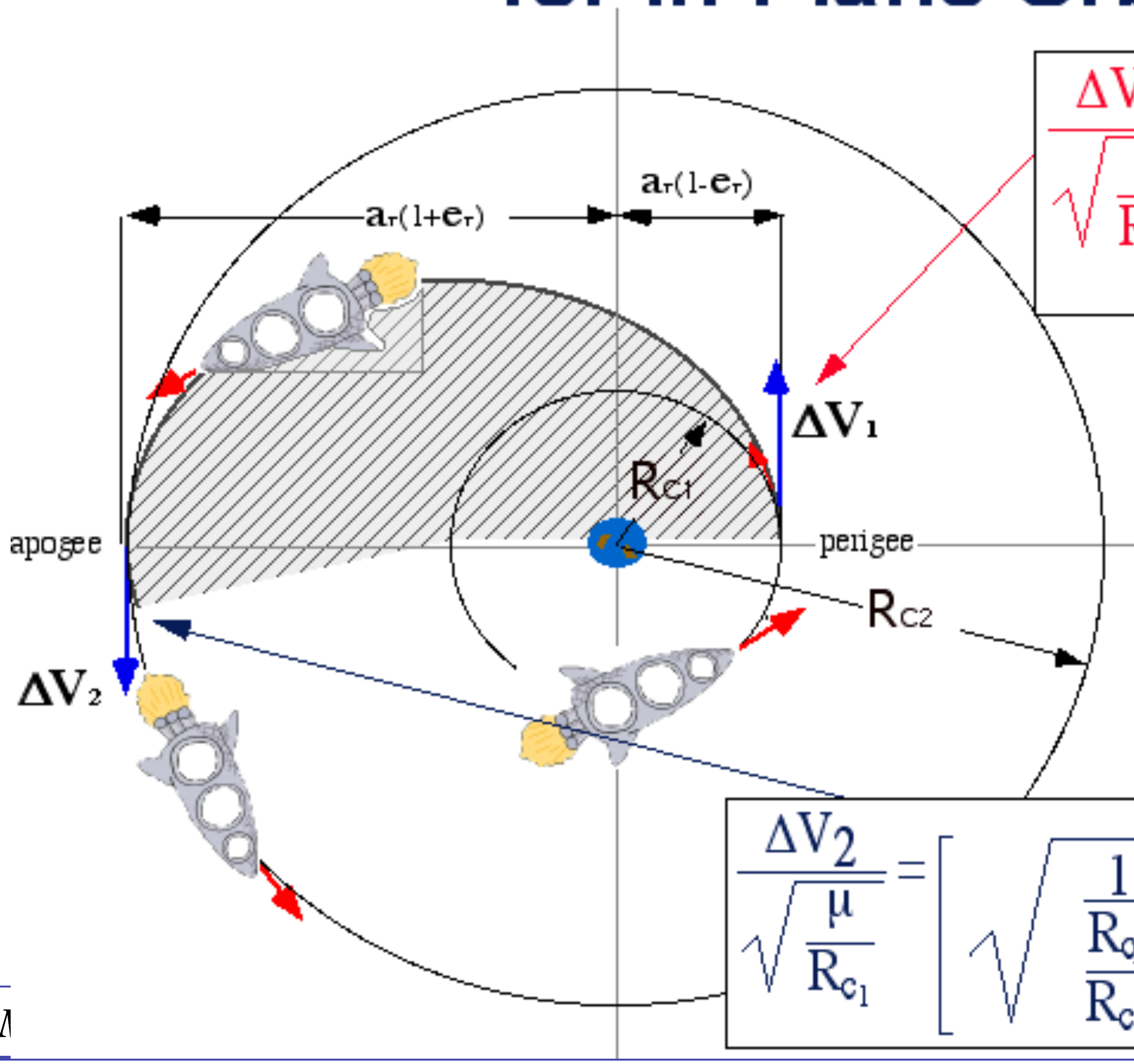
$$\Delta V_2 = V_{c_1} - V_{T_2} = \sqrt{\frac{\mu}{R_{c_2}}} - \sqrt{\frac{\mu}{R_{c_1}}}\sqrt{2\left[\frac{1}{\left[\frac{R_{c_2}}{R_{c_1}}\right]} - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]} =$$

$$\sqrt{\frac{\mu}{R_{c_1}}}\left[\sqrt{\frac{1}{\frac{R_{c_2}}{R_{c_1}}}} - \sqrt{2\left[\frac{1}{\left[\frac{R_{c_2}}{R_{c_1}}\right]} - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]}\right]$$

$$\Downarrow$$

$$\frac{\Delta V_2}{\sqrt{\frac{\mu}{R_{c_1}}}} = \left[\sqrt{\frac{1}{\frac{R_{c_2}}{R_{c_1}}}} - \sqrt{2\left[\frac{1}{\left[\frac{R_{c_2}}{R_{c_1}}\right]} - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]}\right]$$

Total Delta-Vee Required for in-Plane Orbit Transfer



$$\frac{\Delta V_1}{\sqrt{\frac{\mu}{R_{c_1}}}} = \sqrt{2\left[1 - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]} - 1$$

- **Again remember ΔV implies FUEL !**

$$\frac{\Delta V_2}{\sqrt{\frac{\mu}{R_{c_1}}}} = \left[\sqrt{\frac{1}{\frac{R_{c_2}}{R_{c_1}}}} - \sqrt{2\left[\frac{1}{\left[\frac{R_{c_2}}{R_{c_1}}\right]} - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]}\right]$$

Hohmann Transfer: Total ΔV

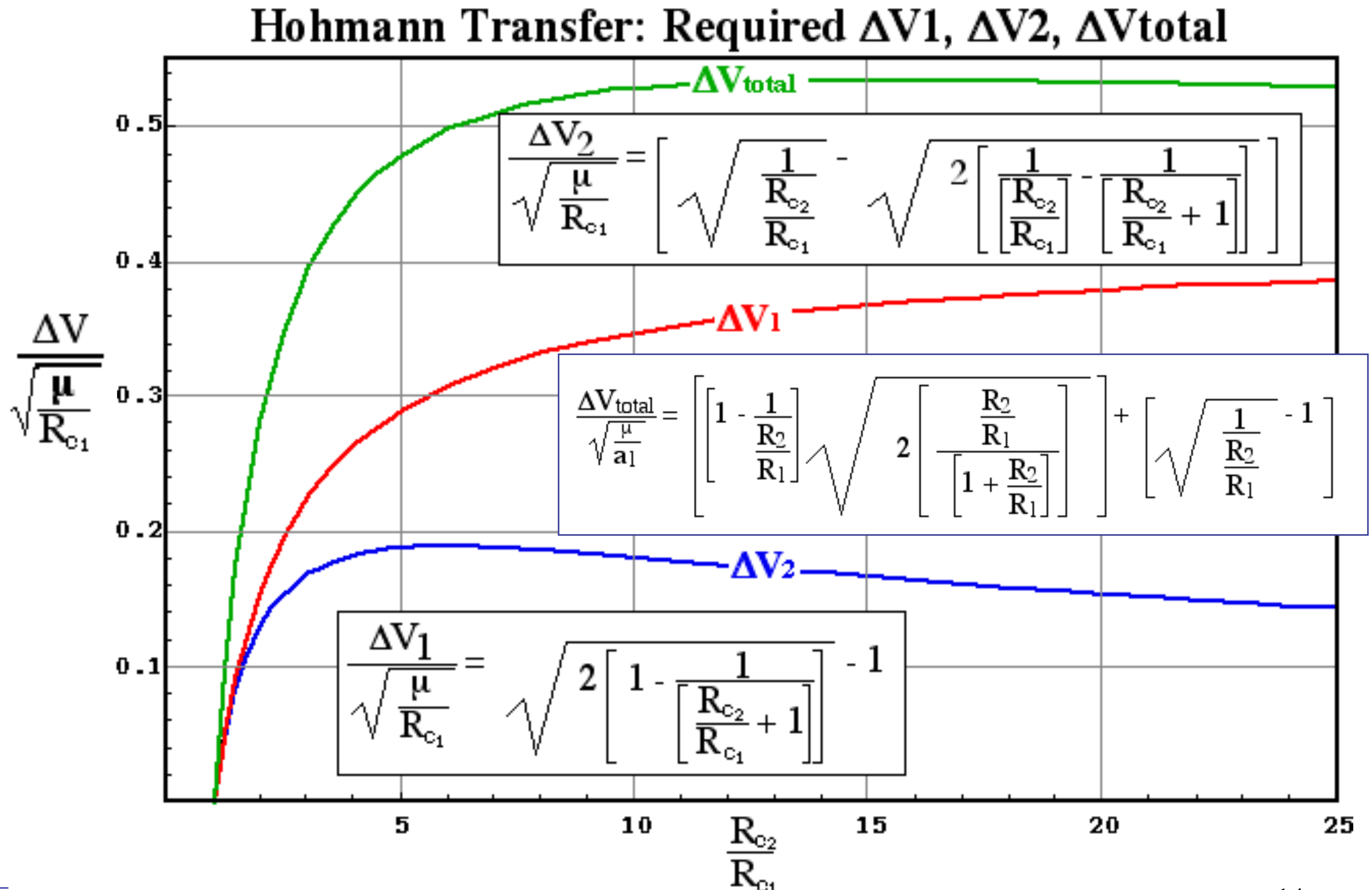
$$\frac{\Delta V_1}{\sqrt{\frac{\mu}{a_1}}} = \left[\sqrt{2\left[1 - \frac{1}{\left[1 + \frac{R_2}{R_1}\right]}\right]}\right] = \sqrt{2\left[\frac{\frac{R_{c_2}}{R_{c_1}}}{\frac{R_{c_2}}{R_{c_1}} + 1}\right]} - 1$$

-1

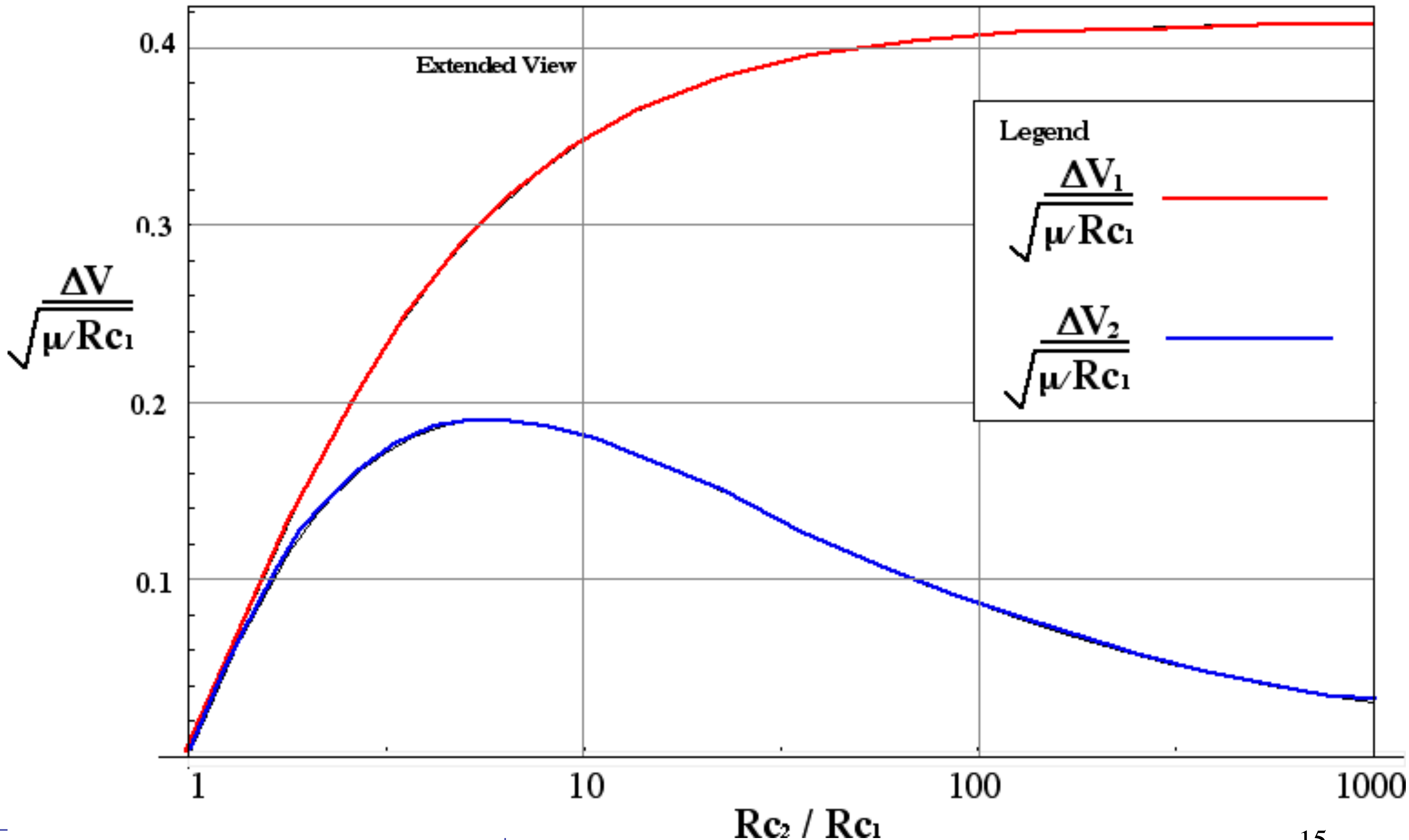
$$\frac{\Delta V_2}{\sqrt{\frac{\mu}{a_1}}} = \left[\sqrt{\frac{1}{\frac{R_2}{R_1}}}\right] - \sqrt{2\left[\frac{1}{\frac{R_2}{R_1}} - \frac{1}{\left[1 + \frac{R_2}{R_1}\right]}\right]} = \left[\sqrt{\frac{1}{\frac{R_2}{R_1}}}\right] - \left[\frac{1}{\frac{R_2}{R_1}}\sqrt{2\left[\frac{\frac{R_2}{R_1}}{\left[1 + \frac{R_2}{R_1}\right]}\right]}\right]$$

$$\frac{\Delta V_{total}}{\sqrt{\frac{\mu}{a_1}}} = \left[\left[1 - \frac{1}{\frac{R_2}{R_1}}\right]\sqrt{2\left[\frac{\frac{R_2}{R_1}}{\left[1 + \frac{R_2}{R_1}\right]}\right]}\right] + \left[\sqrt{\frac{1}{\frac{R_2}{R_1}}} - 1\right]$$

Hohmann Transfer Problem … Solved!



Delta-Vee Plots (Extended View)



Example:
ΔV required for Hohmann Transfer from LEO to GEO

First Compute Radius of Geo Orbit

- Kepler's Third law

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$$T_{geo} = 24 \text{ hrs} \times 3600 \frac{\text{sec}}{\text{hr}} = 86400 \text{ sec}$$

$$a_{geo} = \left[\frac{\sqrt{\mu}\, T_{geo}}{2\pi}\right]^{2/3} =$$

$$\left[\frac{\sqrt{3.986 \times 10^5 \frac{\text{km}^3}{\text{sec}^2}} \times 86400 \text{ sec}}{2\pi}\right]^{2/3} = \boxed{42241 \text{ km}}$$

Example: ΔV required for Hohmann Transfer from LEO to GEO (cont'd)

- Compute Orbit ratio

$$\mathcal{R} \equiv \frac{R_{c_2}}{R_{c_1}} = \frac{42241 \text{ km}}{[160+6371] \text{ km}} = 6.47$$

- Compute Normalized ΔV

$$\frac{\Delta V_{total}}{\sqrt{\frac{\mu}{R_{c_1}}}} = \left[1 - \frac{1}{6.47}\right]\sqrt{2\left[\frac{6.47}{1+6.47}\right]} + \sqrt{\frac{1}{6.47}} - 1 = 0.5059$$

Example:
ΔV required for Hohmann Transfer from LEO to GEO (cont'd)

- Compute Initial Orbit Velocity

$$V_1 = \sqrt{\frac{\mu}{R_{c_1}}} = \sqrt{\frac{3.986 \times 10^5 \frac{km^3}{sec^2}}{[160+6371]\,km}} = 7.812 \frac{km}{sec}$$

- Compute Required ΔV

$$\Delta V_{total} = 7.8123 \times 0.5059 = 3.952 \frac{km}{sec}$$

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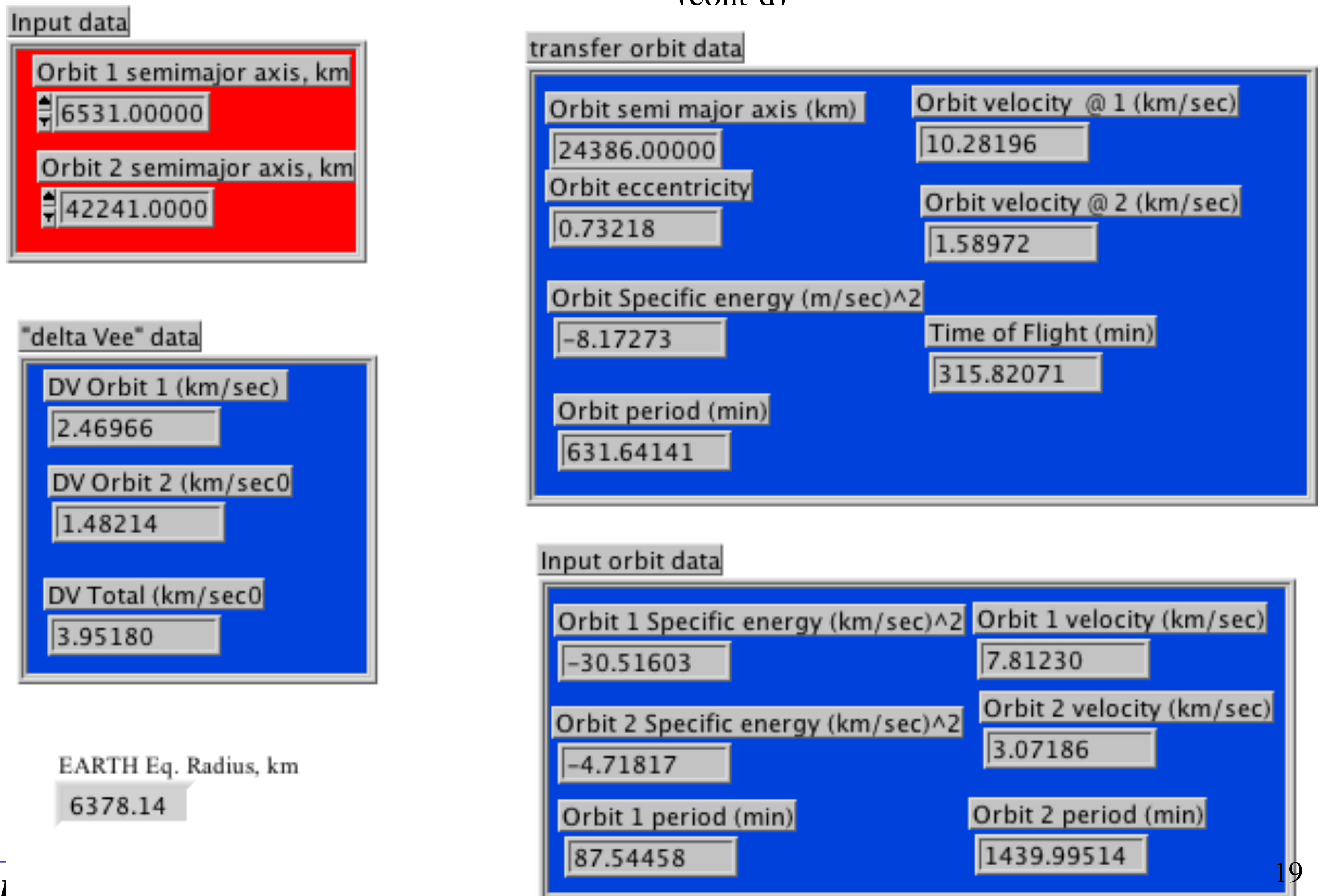
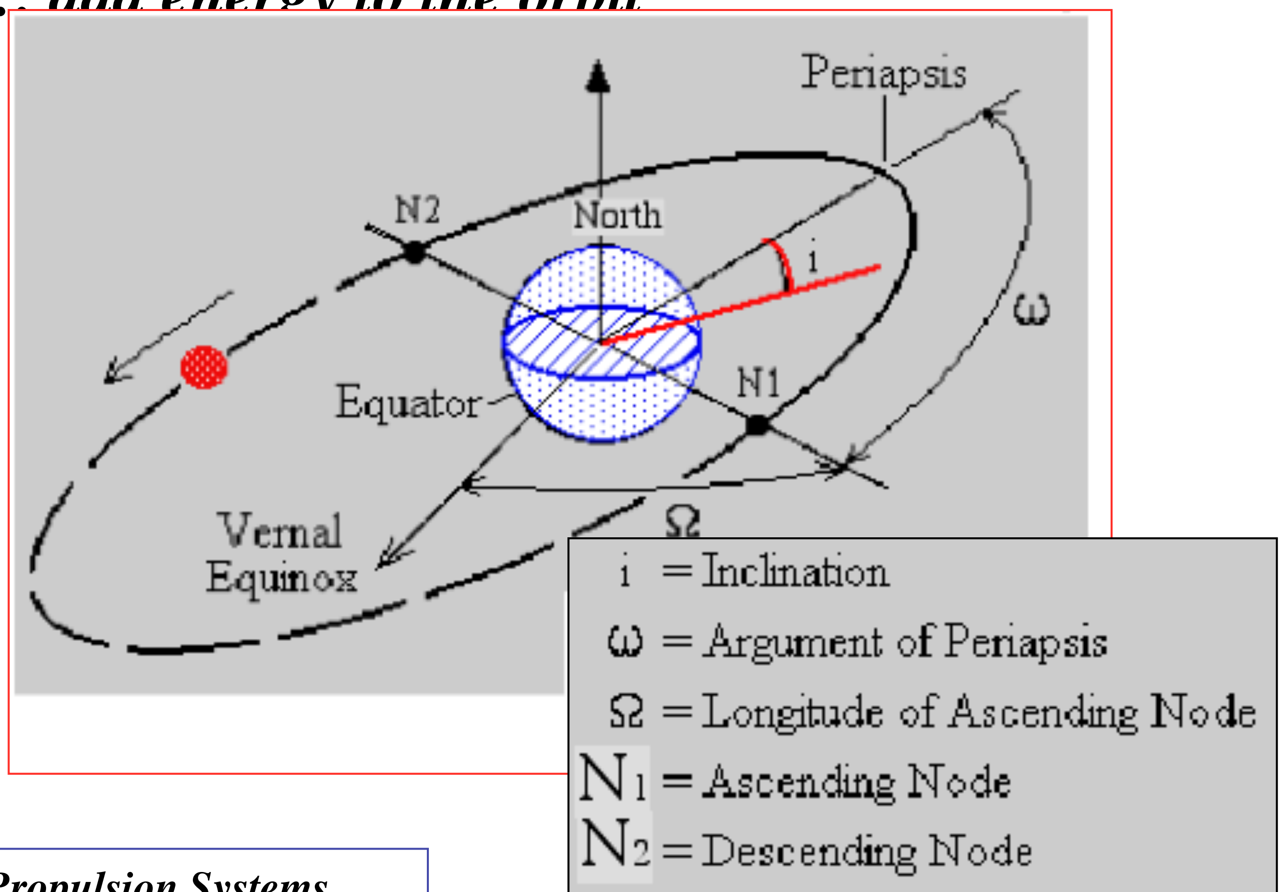
Example:
ΔV required for Hohmann Transfer from LEO to GEO
(cont'd)



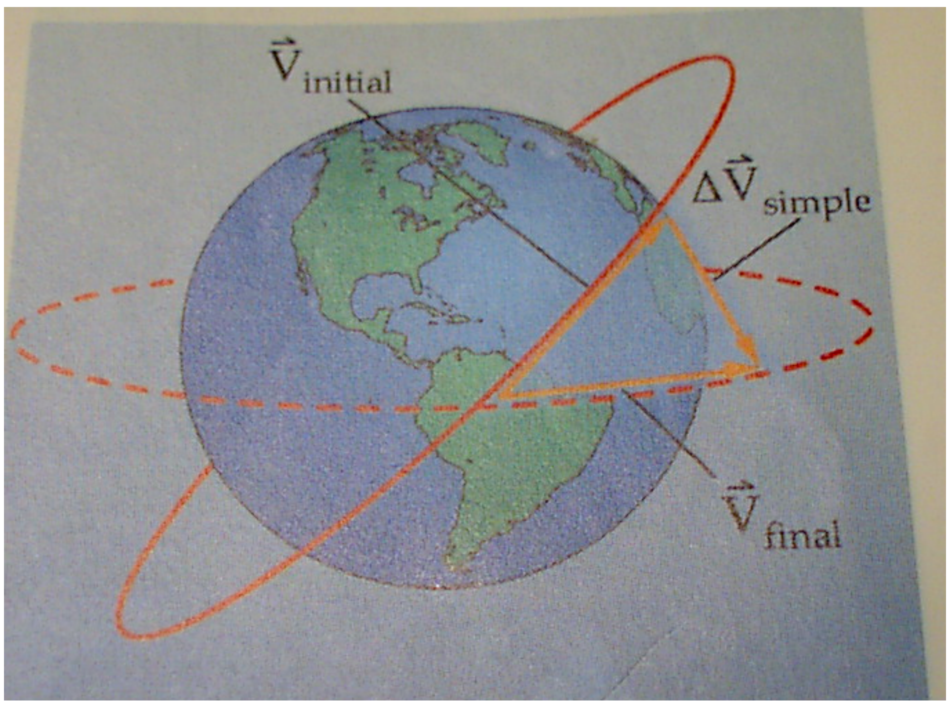
Orbital Plane Changes (1)

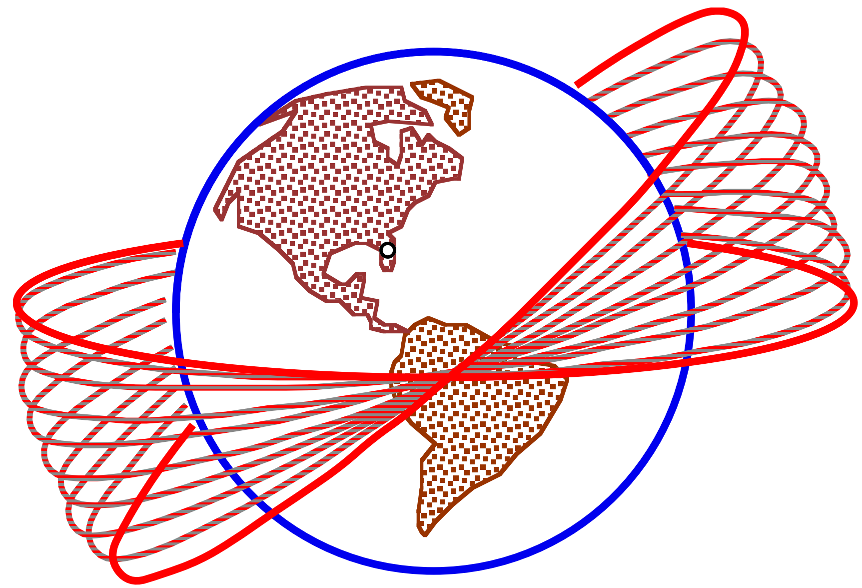
• ***Once Launch Systems burns out … .. And payload is placed in orbit … … ignoring the small effects of … drag and gravity perturbations … your orbit inclination is fixed unless You … add energy to the orbit***



Orbital Plane Changes (2)

- Launch Puts you into a fixed orbit inclination
- What does it cost ***(ΔV)*** to change orbit planes?





• You can only change planes When the planes at your orbits cross

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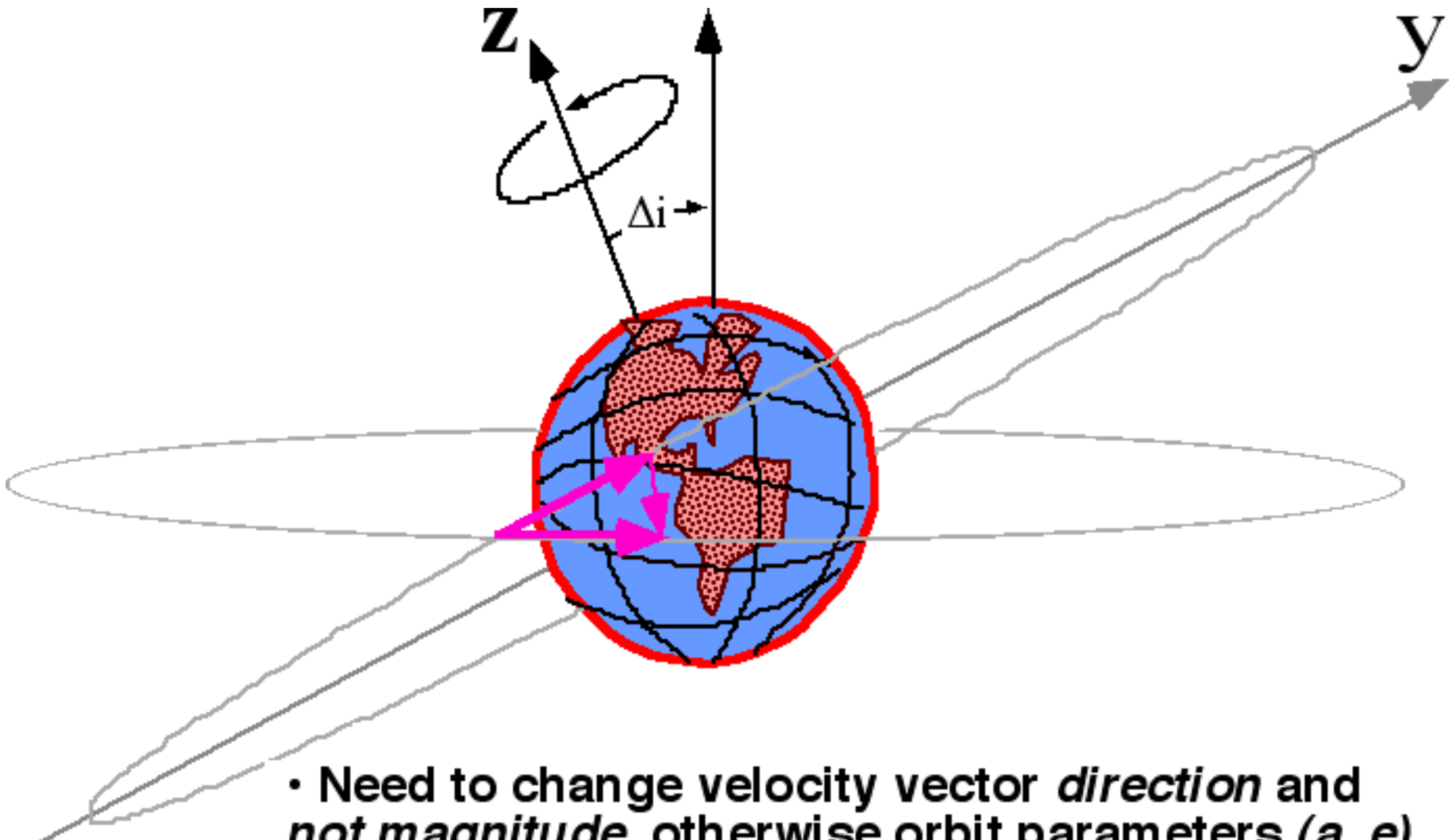
Simple Plane Change

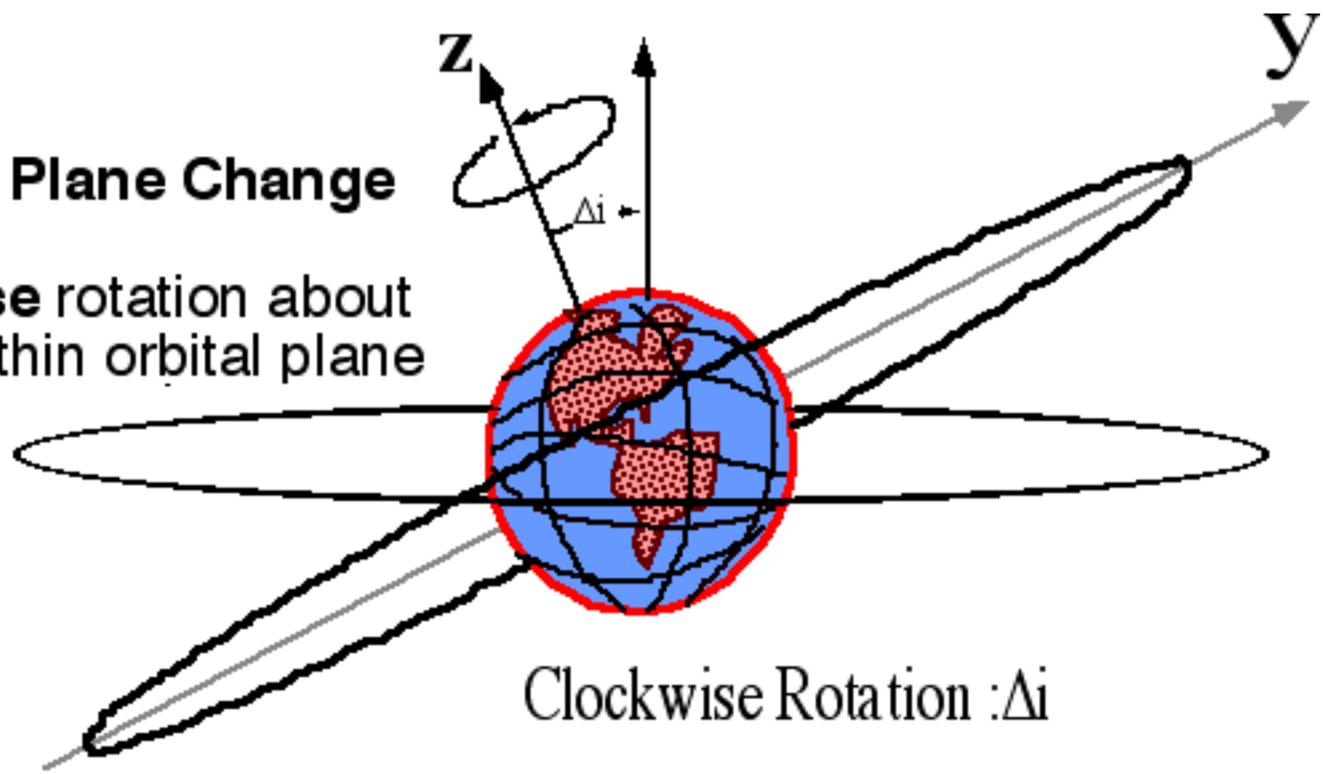


- Need to change velocity vector *direction* and *not magnitude*, otherwise orbit parameters *(a, e)* will change

Simple Plane Change (cont'd)

- **Simple Plane Change**

clockwise rotation about X axis within orbital plane



Clockwise Rotation :Δi

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\Delta i) & -\sin(\Delta i) \\ 0 & \sin(\Delta i) & \cos(\Delta i) \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_1$$

"Norm preserving"

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Simple Plane Change (cont'd)

- "ΔV" - Vector ... includes burn directionality

$$\begin{bmatrix} \Delta V_x \\ \Delta V_y \\ \Delta V_z \end{bmatrix} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_2 - \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_1 =$$

rotation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\Delta i) & -\sin(\Delta i) \\ 0 & \sin(\Delta i) & \cos(\Delta i) \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_1 - \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_1 =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\Delta i) & -\sin(\Delta i) \\ 0 & \sin(\Delta i) & \cos(\Delta i) \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_1 - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_1 =$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(\Delta i) - 1 & -\sin(\Delta i) \\ 0 & \sin(\Delta i) & \cos(\Delta i) - 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

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Simple Plane Change (cont'd)

- ΔV can be expressed in polar form also

$$\begin{bmatrix} \Delta V_r \\ \Delta V_v \\ \Delta V_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(\Delta i) - 1 & -\sin(\Delta i) \\ 0 & \sin(\Delta i) & \cos(\Delta i) - 1 \end{bmatrix} \begin{bmatrix} V_r \\ V_v \\ 0 \end{bmatrix}$$

Simple Plane Change (cont'd)

- Evaluate magnitude of ΔV

$$|\Delta V|^2 = \Delta V^T \Delta V =$$

$$\begin{bmatrix} V_r \\ V_v \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(\Delta i) - 1 & \sin(\Delta i) \\ 0 & -\sin(\Delta i) & \cos(\Delta i) - 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(\Delta i) - 1 & -\sin(\Delta i) \\ 0 & \sin(\Delta i) & \cos(\Delta i) - 1 \end{bmatrix} \begin{bmatrix} V_r \\ V_v \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} V_r \\ V_v \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & [\cos(\Delta i) - 1]^2 + \sin^2(\Delta i) & 0 \\ 0 & 0 & [\cos(\Delta i) - 1]^2 + \sin^2(\Delta i) \end{bmatrix} \begin{bmatrix} V_r \\ V_v \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} V_r \\ V_v \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ \left[[\cos(\Delta i) - 1]^2 + \sin^2(\Delta i)\right] V_v \\ 0 \end{bmatrix} = \left[[\cos(\Delta i) - 1]^2 + \sin^2(\Delta i)\right] V^2{}_v$$

Simple Plane Change (cont'd)

$$|\Delta V|^2 = \left[\left[\cos(\Delta i) - 1\right]^2 + \sin^2(\Delta i)\right] V^2{}_v =$$

$$\Downarrow$$

$$\left[\cos(\Delta i) - 1\right]^2 + \sin^2(\Delta i) = \cos^2(\Delta i) - 2\cos(\Delta i) + 1 + \sin^2(\Delta i) =$$

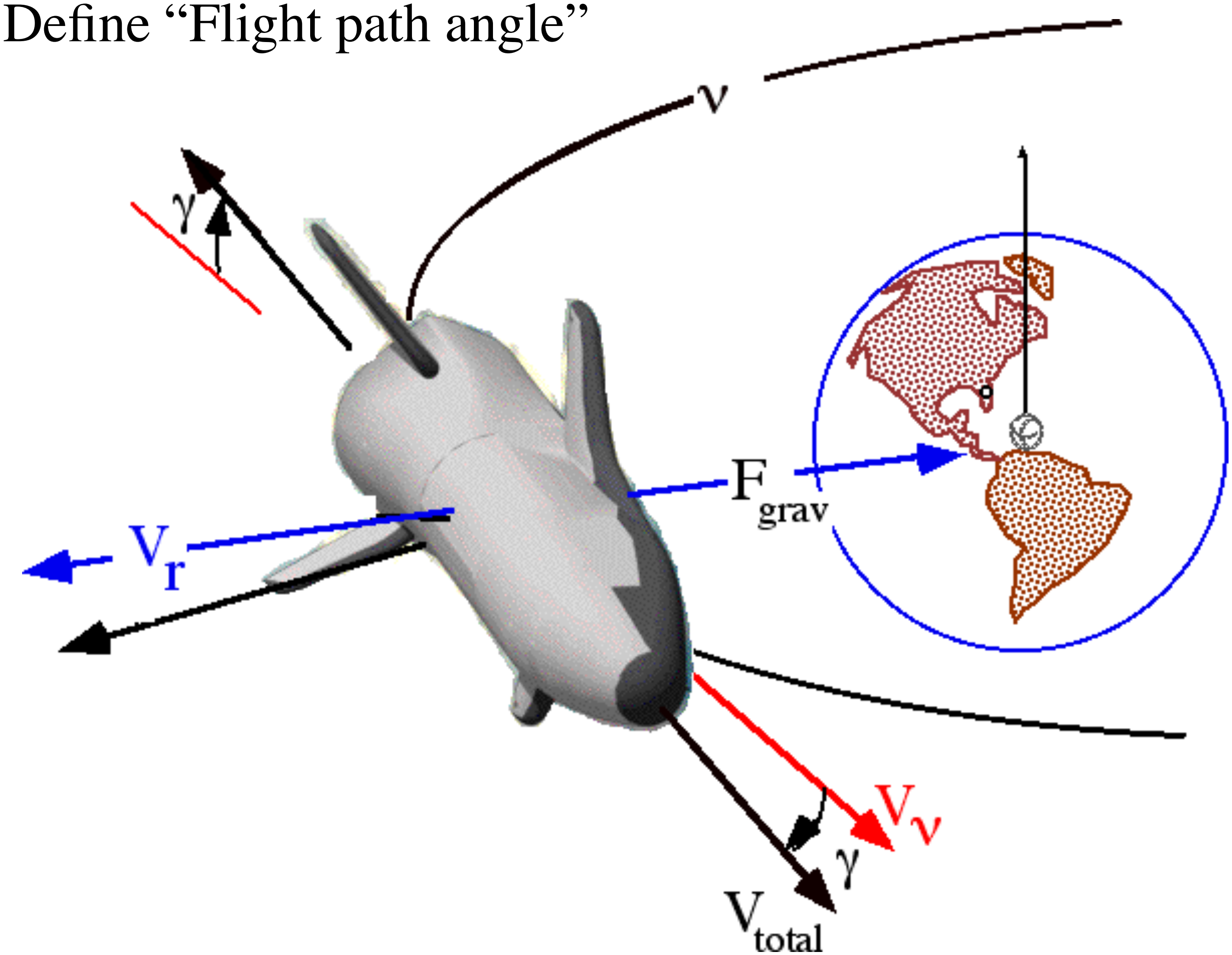
$$2\left[1 - \cos(\Delta i)\right] = 4\frac{\left[1 - \cos(\Delta i)\right]}{2} = 4\sin^2\left(\frac{\Delta i}{2}\right)$$

$$|\Delta V|_{\text{simple plane change}} = 2\sin\left(\frac{\Delta i}{2}\right) V_v$$

Why no V_r component?

Simple Plane Change (cont'd)

- Define "Flight path angle"



Simple Plane Change (cont'd)

- Define "Flight path angle"

$$\left|\Delta V\right|_{\substack{\text{simple}\\\text{plane}\\\text{change}}} = 2\sin\left(\frac{\Delta i}{2}\right)V_\nu$$

$$\gamma \equiv \tan^{-1}\left[\frac{V_r}{V_\nu}\right] \Rightarrow \left|\Delta V\right|_{\substack{\text{simple}\\\text{plane}\\\text{change}}} = 2\left|V\right|\cos(\gamma)\sin\left(\frac{\Delta i}{2}\right)$$

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Simple Plane Change (cont'd)

- In-Plane Velocity Vector

$$\bar{V} = r(\nu)\ \omega \left[\frac{[e \sin(\nu)]}{[1 + e\cos(\nu)]} \bar{i}_r + \bar{i}_\nu \right]$$

$$\Downarrow$$

$$\omega = \frac{\sqrt{\mu\, a\, [1 - e^2]}}{r^2}$$

$$\begin{bmatrix} V_r \\ V_\nu \\ V_z \end{bmatrix} = \sqrt{\frac{\mu}{a[1 - e^2]}} \begin{bmatrix} e \sin(\nu) \\ 1 + e\cos(\nu) \\ 0 \end{bmatrix}$$

perifocal

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Simple Plane Change (cont'd)

$$\left|\Delta V\right|_{\substack{\text{simple}\\\text{plane}\\\text{change}}} = 2\sin\left(\frac{\Delta i}{2}\right)\sqrt{\frac{\mu}{a\left[1-e^2\right]}}\left[1 + e\cos\left(\nu\right)\right]$$

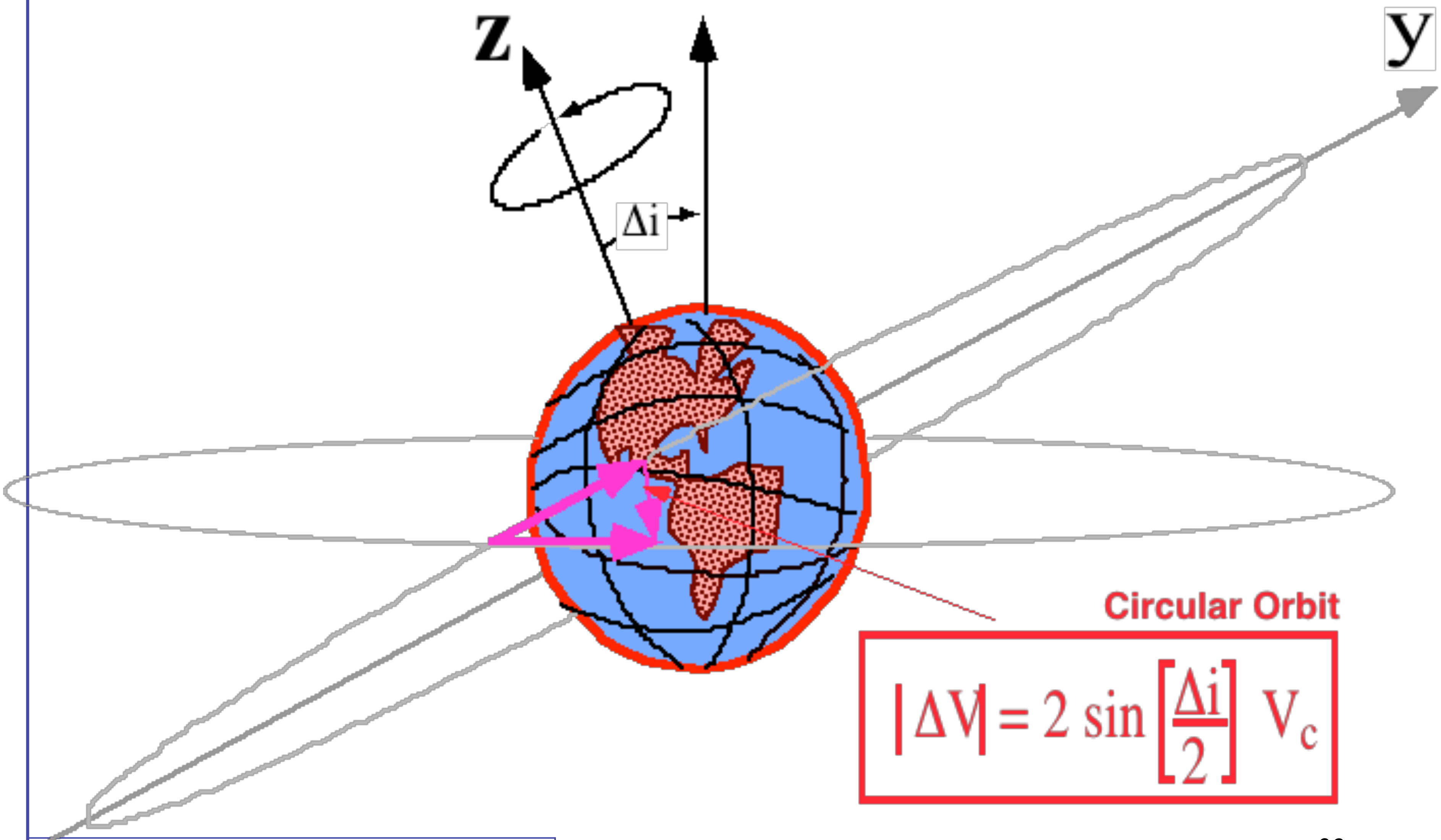
$$\Downarrow$$

for e-> 0

$$\left[\left|\Delta V\right|_{\substack{\text{simple}\\\text{plane}\\\text{change}}}\right]_c = 2\sin\left(\frac{\Delta i}{2}\right)\sqrt{\frac{\mu}{a}} = 2\sin\left(\frac{\Delta i}{2}\right)V_c$$

Circular orbit

Simple Plane Change (Circular orbit)



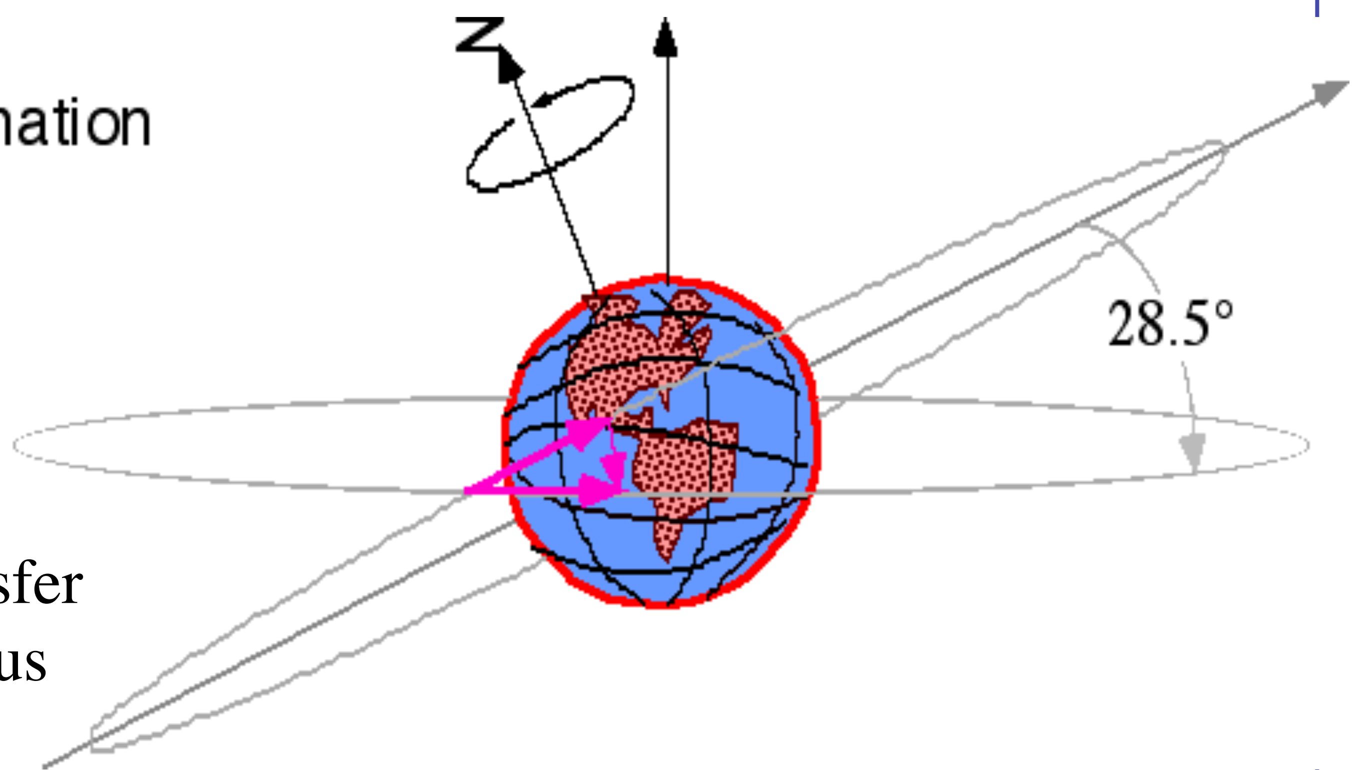
$$|\Delta V| = 2 \sin\left[\frac{\Delta i}{2}\right] V_c$$

Example Geo-synchronous Transfer

i) Direct launch to 28.5 deg inclination LEO (160 km)

ii) Plane change to equatorial orbit

iii) Hohmann transfer To geo-synchronous orbit



Calculate required ΔV

Orbital Speed

- Compute Required Orbital Speed for 160 km LEO

$$V_{LEO} = \left[\sqrt{\frac{\mu}{r_{LEO}}}\right] =$$

$$\sqrt{\frac{3.986 \times 10^{14} \dfrac{\text{kg-m}^3}{\text{kg sec}^2}}{[160 \text{ km} + 6371 \text{ km}] \times 1000 \dfrac{\text{m}}{\text{km}}}} = 7812.2 \frac{\text{m}}{\text{sec}}$$

Gravitational Specific Energy

- Equivalent specific energy required to lift unit mass to Orbital altitude

$$\Delta V_{gravity} = \sqrt{2 \frac{\mu\, h}{r_e\,(r_e + h)}} =$$

$$\sqrt{2 \frac{3.986 \times 10^5 \frac{km^3}{sec^2} \times 160\ km}{6371\ km\,([160\ km + 6371\ km])}} =$$

$$1.7508\ \frac{km}{sec} = 1750.8\ \frac{m}{sec}$$

V_{boost} Due to Earth's Rotation

- Compute Earth Rotational velocity at 28.5° (KSC) latitude

$$V_{rot_{Earth}} = \omega_{Earth} \times r_{Earth} \times \cos[Lat] =$$

$$\left[0.000072921\,\frac{radians}{sec}\right] \times \left[6373.23\text{ km} \times 1000\,\frac{m}{km}\right] \times \cos\left[\frac{28.5\,\pi}{180}\,radians\right] = 408.4\,\frac{m}{sec}$$

- For Launch from the cape in to a 28.5° inclination orbit

$$\Delta V_{total}^{(ksc)} = \sqrt{\left[7812.2\,\frac{m}{sec} - 404.8\,\frac{m}{sec}\right]^2 + \left[1750.8\,\frac{m}{sec}\right]^2} =$$

$$7611.5\,\frac{m}{sec}$$

Plane Change ΔV

- Delta V required for 28.5° plane change

$$|\Delta V| = 2 \sin\left[\frac{\Delta i}{2}\right] V_c =$$

$$|\Delta V| = 2 \sin\left[\frac{28.5}{2} \times \frac{\pi}{180}\right] \times 7812.3 \frac{m}{sec} = 3846.05$$

Total Delta V required to Reach Equatorial Leo Orbit from KSC

$$\text{Total } \Delta V \text{ required:} \quad [3846.1 + 7611.5] =$$

$$11457.6 \ \frac{\text{m}}{\text{sec}} \approx 51\% \text{ more } \Delta V!$$

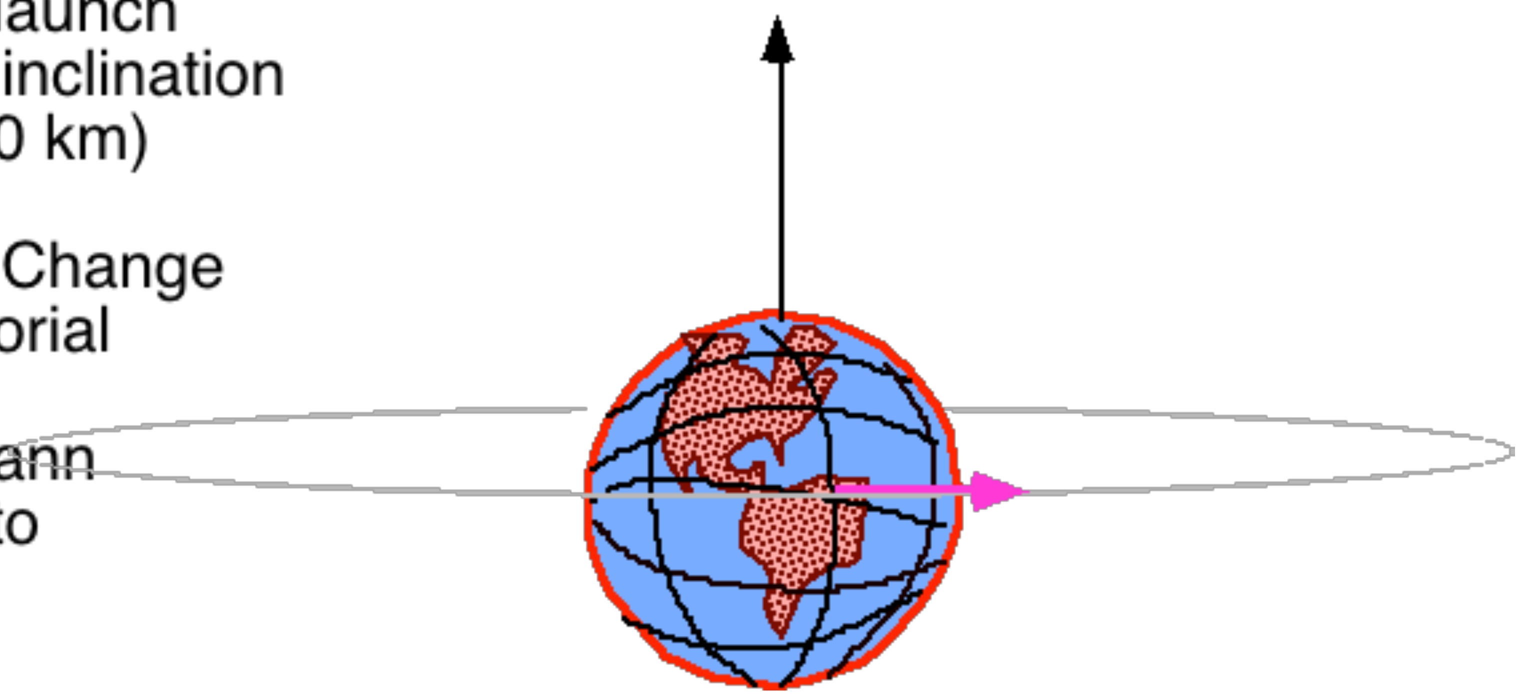
than what is required just to obtain orbit!

How About Transfer to Geo-Stationary Orbit

i) Direct launch
to 0 deg inclination
LEO (160 km)

ii) Plane Change
to Equatorial

ii) Hohmann
transfer to
GEO



Hohmann transfer

- Delta V for Hohmann transfer to GEO

$$R_{c_2}/R_{c_1} = \frac{42164.2 \text{ km}}{[160 + 6371] \text{ km}} = 6.456$$

$$V_1 = \sqrt{\mu/R_{c_1}} \sqrt{2\left[1 - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]} = 2467.02 \frac{\text{m}}{\text{sec}}$$

$$\Delta V_2 = \sqrt{\frac{\mu}{R_{c_1}}} \sqrt{\frac{1}{\frac{R_{c_2}}{R_{c_1}}}} - \sqrt{2\left[\frac{1}{\left[\frac{R_{c_2}}{R_{c_1}}\right]} - \frac{1}{\left[\frac{R_{c_2}}{R_{c_1}} + 1\right]}\right]} = 1481.41 \frac{\text{m}}{\text{sec}}$$

KSC Launch example: Option 1

- **28.5° North Latitude Launch Minimum Total Delta V to GEO**

$$\Delta V_{total} = \Delta V_{launch} + \Delta V_{\substack{Plane \\ Change}} + \Delta V_{1_{hohmann}} + \Delta V_{2_{hohmann}} =$$

$$[7611.5 + 3846.1 + 2467.0 + 1481.4]\ \frac{m}{sec} = 15405.9\ \frac{m}{sec}$$

- **THAT'S almost twice orbital velocity ... How do we ever get there?**

OK, lets try another approach

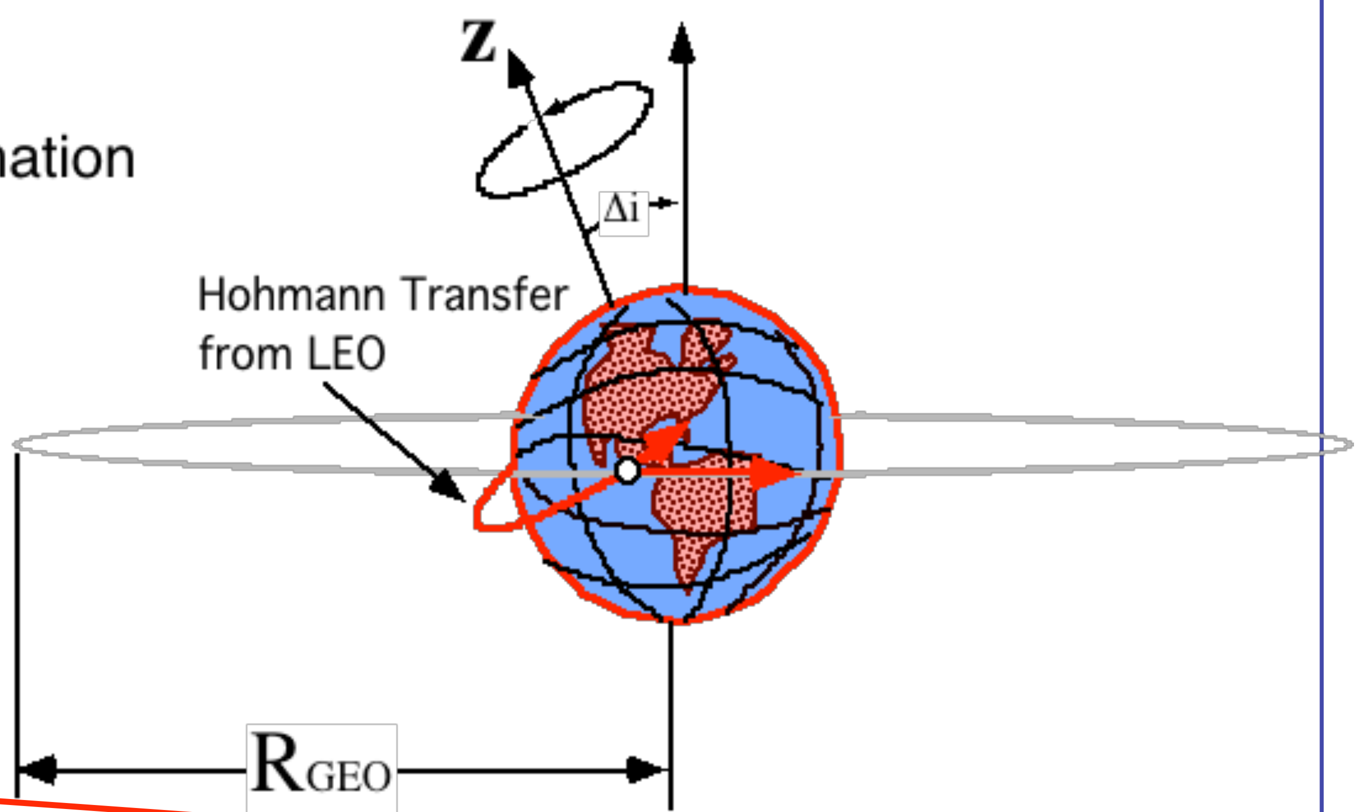
- **Do the math!**

Option 2:

i) Direct launch
to 28.5 deg inclination
LEO (160 km)

ii) Hohmann
transfer to
circular orbit
r_c = 42164.2 km

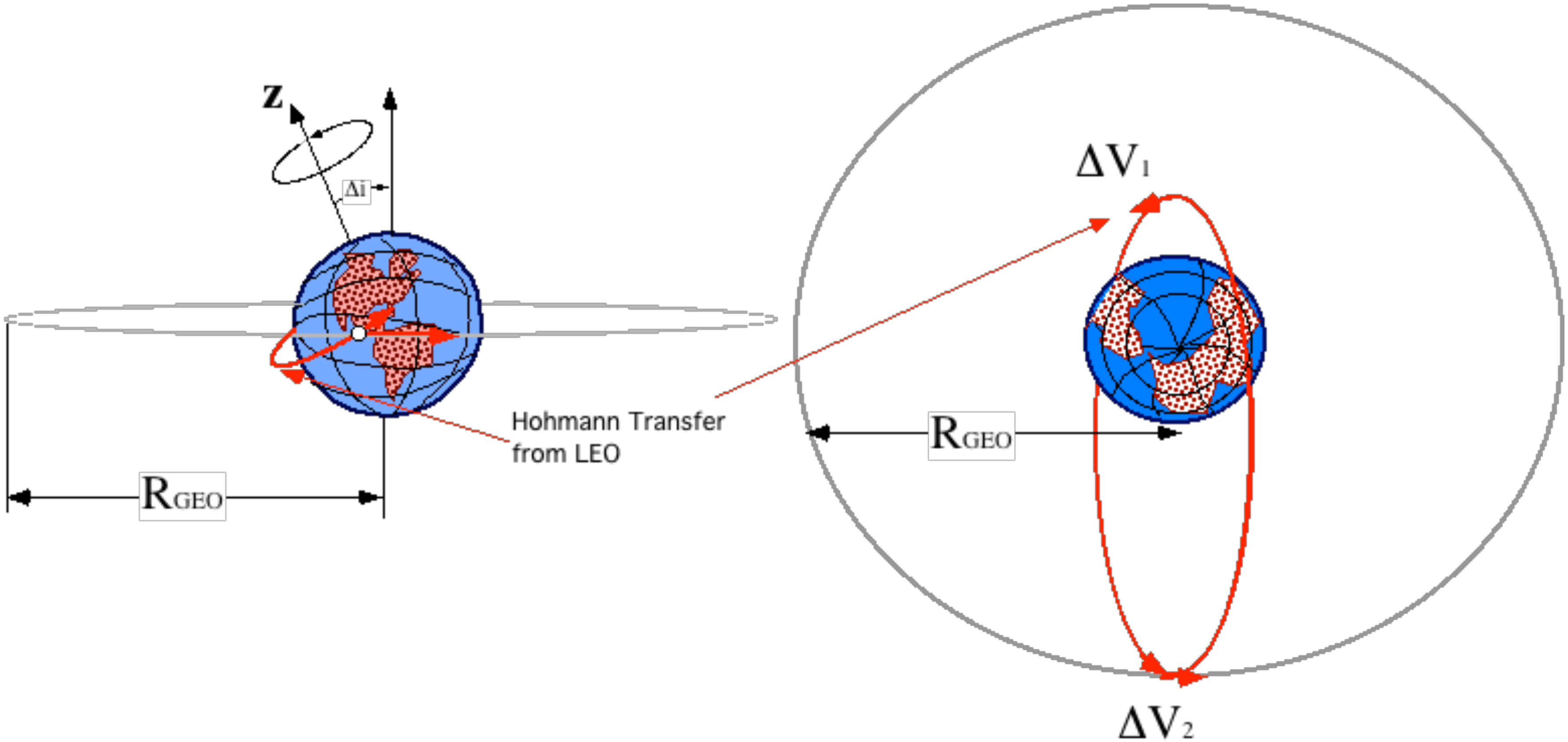
iii) Plane change
to equatorial
orbit



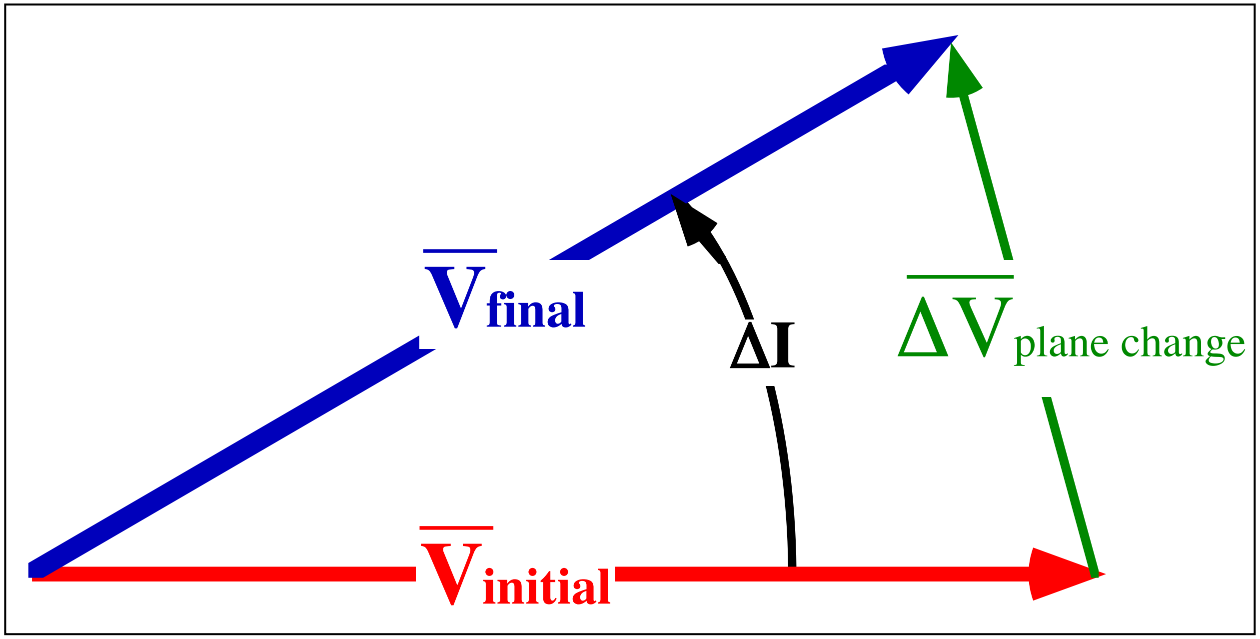
"where V is smaller"

$$\left|\Delta V\right|_{\substack{\text{simple}\\\text{plane}\\\text{change}}} = 2\left|V\right| \cos\left(\gamma\right) \sin\left(\frac{\Delta i}{2}\right)$$

Option 2



Higher the orbit, Less ΔV Required for Change of Planes



Less ΔV Required for Change of Planes at Apogee than any Other Position within Orbit

i) elliptical orbit

$$\left|\Delta V\right|_{\substack{\text{simple}\\\text{plane}\\\text{change}}} = 2\sin\left(\frac{\Delta i}{2}\right)\sqrt{\frac{\mu}{a\left[1-e^2\right]}}\left[1 + e\cos\left(\nu\right)\right]$$

ii) Minimum Value when $\nu = \pi$

Plane Change at Apogee

$$\left[\left|\Delta V\right|_{\substack{\text{simple}\\\text{plane}\\\text{change}}}\right]_{\text{apogee}} =$$

$$2\left|V\right|\sqrt{\frac{\mu}{a\left[1-e^2\right]}}\left[1 + e\cos(\pi)\right] =$$

$$2\left|V\right|\sqrt{\frac{\mu\left[1-e\right]\left[1-e\right]}{a\left[1+e\right]\left[1-e\right]}} = 2\left|V\right|\sqrt{\frac{\mu\left[1-e\right]}{a\left[1+e\right]}}$$

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KSC Launch Example OPT 2: Continued

- Delta V required for 28.5 plane change at transfer orbit Apogee

$$\left|\Delta V\right|_{apogee}^{\substack{plane\\change}} = 2\sin\left[\frac{\Delta i}{2}\right]\sqrt{\frac{\mu}{a}\frac{[1-e]}{[1+e]}} =$$

$$\left|\Delta V\right| = 2\sin\left[\frac{28.5}{2}\times\frac{\pi}{180}\right]\times\sqrt{\frac{3.986\times10^{5}\frac{km^{3}}{sec^{2}}}{24347.6\ km}\times\frac{1-0.73176}{1.73176}}\times 1000\frac{m}{km} =$$

$$783.96\ \frac{m}{sec}$$

KSC Launch Option 2

$$\Delta V_{total} = \Delta V_{launch} + \Delta V_{1_{hohmann}} + \Delta V_{\substack{Plane \\ Change}} + \Delta V_{2_{hohmann}} =$$

$$[7611.5 + 2467.0 + 783.96 + 1481.4]\frac{m}{sec} = 12343.9\ \frac{m}{sec}$$

$$\begin{matrix}\text{Energy savings} \\ \text{Compared to} \\ \text{plane transfer in LEO}\end{matrix} = 100\% \times \frac{15405.9.0 - 12343.9}{15405.9} = 19.9\%$$

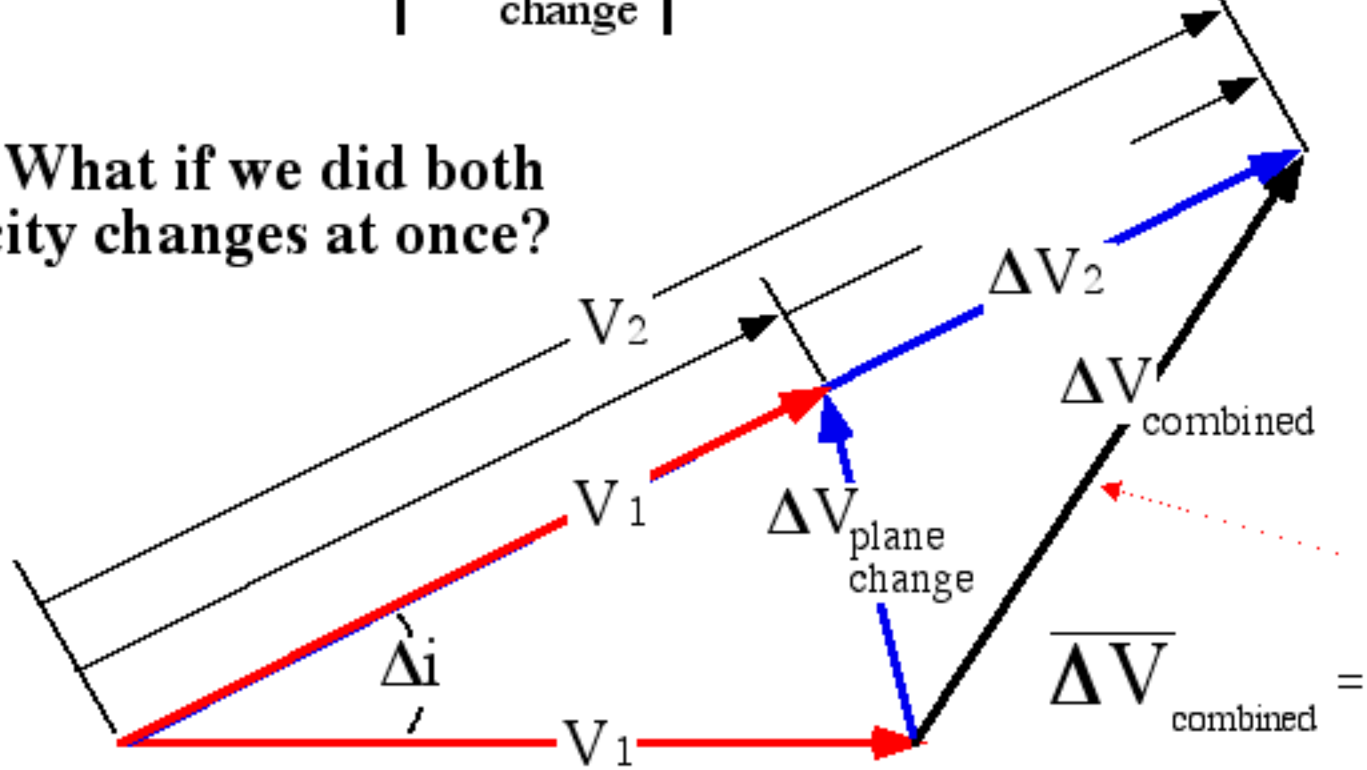
- **Better …. But can we still do better than this?**

Combined Plane Change

- Re examine the transfer at GEO

$$|\Delta V_{total}| = |\Delta V_{\substack{plane\\change}}| + |\Delta V_2|$$ ← This is what WE did

- But, What if we did both velocity changes at once?



$$\overline{\Delta V}_{combined} = \overline{V}_2 - \overline{V}_1$$

$$|\Delta V_{combined}| < |\Delta V_{\substack{plane\\change}}| + |\Delta V_2|$$

Combined Plane Change (cont'd)

$$|\Delta V|^2_{combined} = \Delta V^T \Delta V = [V_2 - V_1]^T [V_2 - V_1] =$$

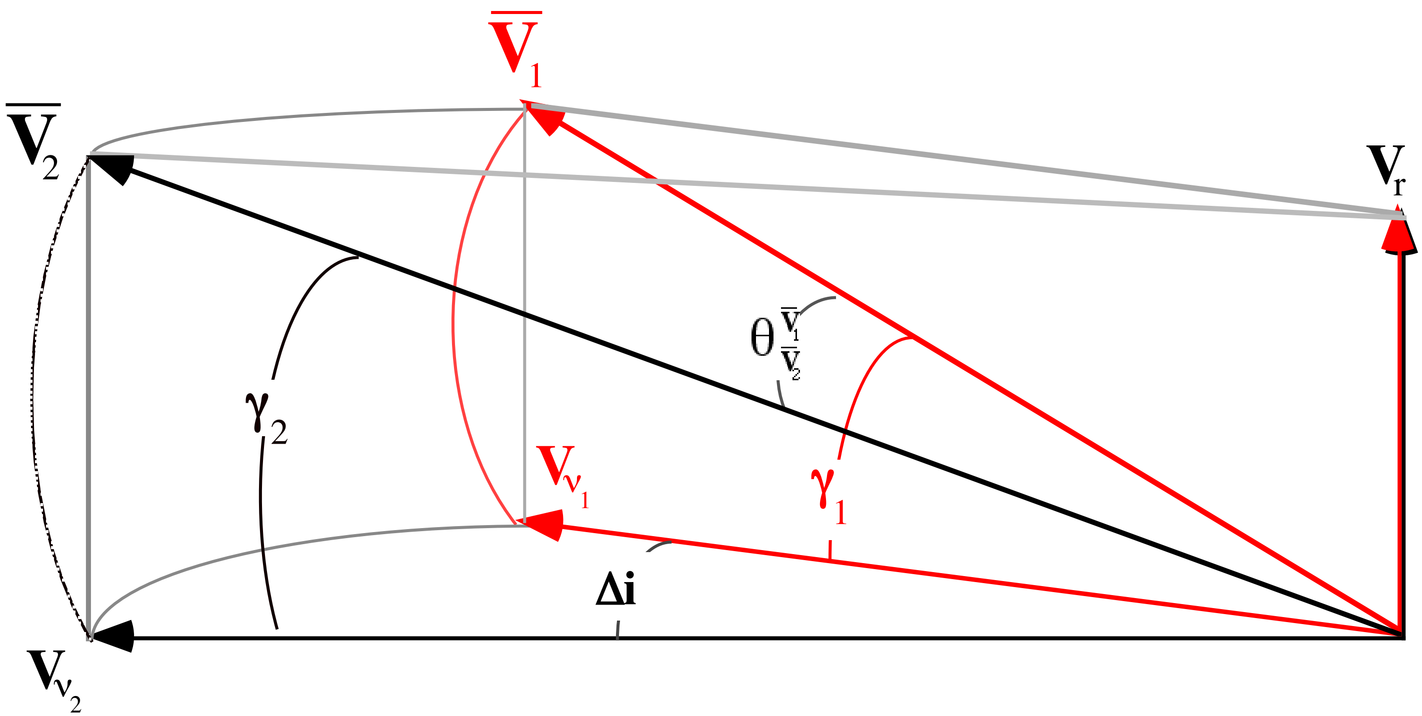
$$V_2^T V_2 - V_2^T V_1 - V_1^T V_2 + V_1^T V_1 =$$

$$|V_2|^2 + |V_1|^2 - 2 V_2^T V_1$$

$$V_2^T V_1 = |V_2||V_1| \cos(\theta_{\substack{V_2 \\ V_1}})$$

Angle between vectors

Combined Plane Change (cont'd)



$$\cos\left(\theta_{\overline{V}_1}^{\overline{V}_2}\right) =$$

$$\cos(\gamma_2)\cos(\gamma_1)\cos(\Delta i)$$

Combined Plane Change (cont'd)

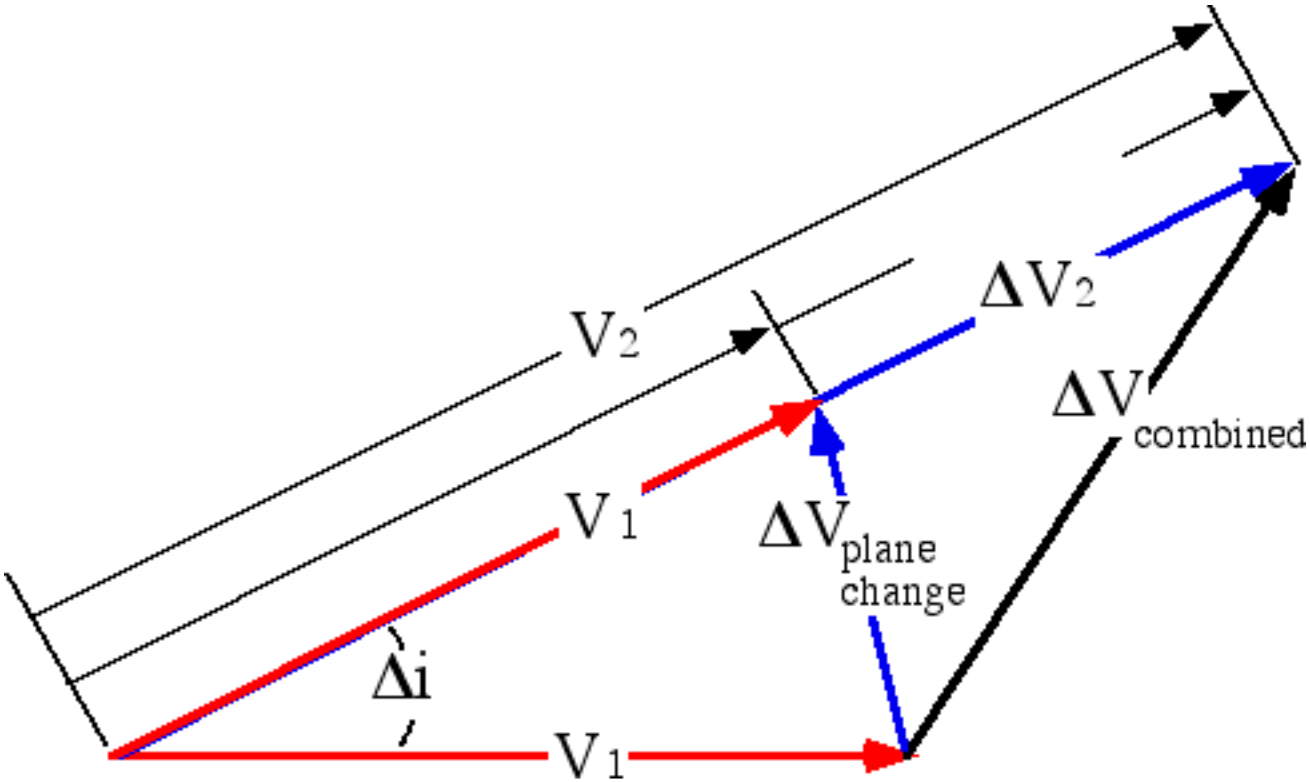
$$\cos\left(\theta_{V_1}^{V_2}\right) = \cos(\gamma_2)\cos(\gamma_1)\cos(\Delta i)$$

$$\Downarrow$$

$$V_2^T V_1 = \left|V_2\cos(\gamma_2)\right|\left|V_1\cos(\gamma_1)\right|\cos(\Delta i)$$

$$\Delta V_{combined} = \sqrt{V_2^2 + V_1^2 - 2\left|V_2\cos(\gamma_2)\right|\left|V_1\cos(\gamma_1)\right|\cos(\Delta i)}$$

Combined Plane Change (cont'd)



$$\Delta V_{combined} = \sqrt{V_2^2 + V_1^2 - 2\,|V_2 \cos(\gamma_2)|\,|V_1 \cos(\gamma_1)| \cos(\Delta i)}$$

Re-Visit the GEO Transfer from KSC

• Orbital velocity magnitude at GEO (V_2)

Option 3:

i) Direct launch to 28.5 deg inclination LEO (160 km)

ii) Hohmann transfer to circular orbit $r_c = 42164.2$ km

iii) Combined Plane change to Circular GEO orbit

$$V_{GEO} = \left[\sqrt{\frac{\mu}{r_{GEO}}}\right] =$$

$$\left[\sqrt{\frac{3.986 \times 10^{14} \dfrac{\text{kg-m}^3}{\text{kg sec}^2}}{[42164.2 \text{ km}] \times 1000 \dfrac{\text{m}}{\text{km}}}}\right] = 3074.7 \frac{\text{m}}{\text{sec}}$$

Re-Visit the GEO Transfer from KSC (cont'd)

- Orbital velocity magnitude of transfer orbit at Apogee (V_1)

$$V_T^{(apogee)} = \sqrt{\frac{2\mu}{a_T\left[1+e_T\right]} - \frac{\mu}{a_T}} =$$

$$\sqrt{\frac{2\mu - \mu\left[1+e_T\right]}{a_T\left[1+e_T\right]}} = \sqrt{\frac{\mu\left[1-e_T\right]}{a_T\left[1+e_T\right]}} =$$

$$\sqrt{\frac{3.986 \times 10^5 \frac{km^3}{sec^2} \times \left[1-0.73176\right]}{24347.6\ km \times \left[1+0.73176\right]}} = 1593.3\ \frac{m}{sec}$$

Re-Visit the GEO Transfer from KSC (cont'd)

- Compute ΔV combined

$$\left|\Delta V\right|_{combined} = \sqrt{\left|V_2\right|^2 + \left|V_1\right|^2 - 2\left|V_2\right|\left|V_1\right|\cos\left[\Delta i\right]} =$$

$$\sqrt{\left|3074.7\,\frac{m}{sec}\right|^2 + \left(1593.3\,\frac{m}{sec}\right)^2 - 2\left|3074.7\,\frac{m}{sec}\right|\left(1593.3\,\frac{m}{sec}\right)\cos\left[28.5^\circ \times \frac{\pi}{180}\right]} =$$

$$1793.5\,\frac{m}{sec}$$

Re-Visit the GEO Transfer from KSC (concluded)

- Minimum Total Delta V GEO

$$\Delta V_{total} = \Delta V_{launch} + \Delta V_{1_{hohmann}} + \Delta V_{combined} =$$

$$[7611.5 + 2467.0 + 1793.5] \frac{m}{sec} = 11872 \frac{m}{sec}$$

savings Compared to plane transfer in LEO

$$= 100\% \times \frac{15405.9.0 - 11872}{11872} = 23.1\%$$

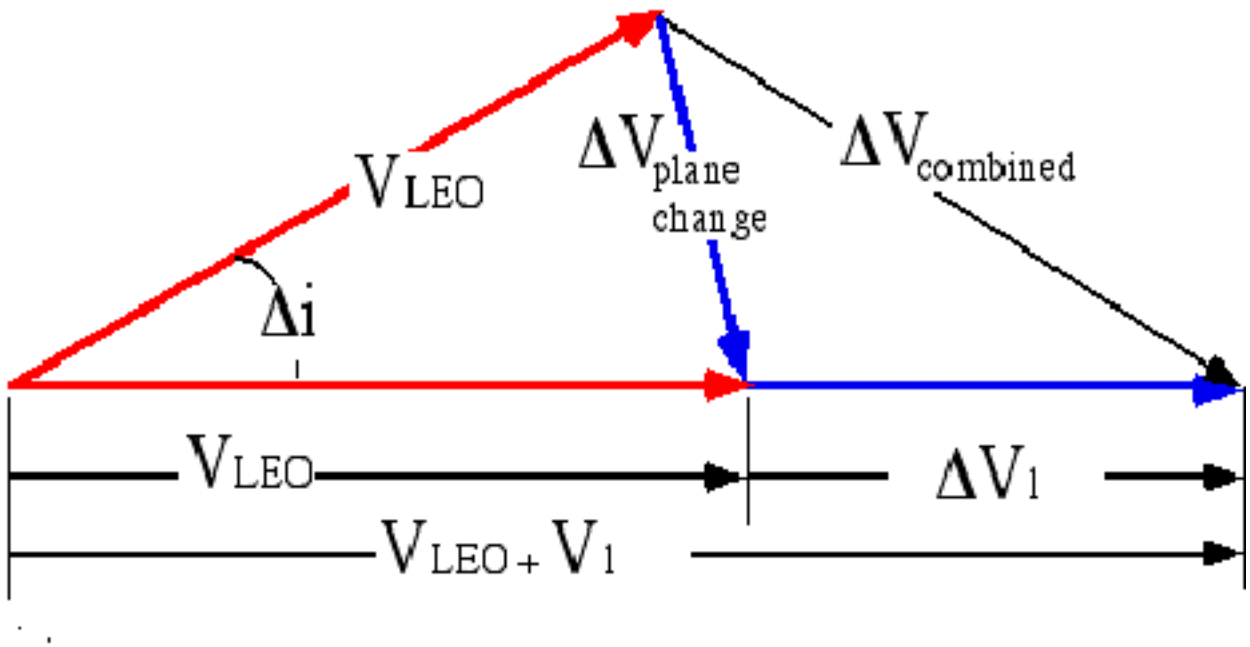
- Compute the Minimum Total ΔV Required for Launch, Combined plane change & Hohmann Transfer at LEO, and then Circularize Orbit at GEO

Option 4:

i) Direct launch to 28.5 deg LEO (160 km)

ii) Combined Plane Change (28.5°) Hohmann transfer
r_c = 42164.2 km

iii) Circularize Orbit



Option 4

- Compute Earth Rotational velocity at 28.5° (KSC) latitude

$$V_{rot_{Earth}} = \omega_{Earth} \times r_{Earth} \times \cos[\mathrm{Lat}] =$$

$$\left[0.000072921\ \frac{\mathrm{radians}}{\mathrm{sec}}\right] \times \left[6373.23\ \mathrm{km} \times 1000\ \frac{\mathrm{m}}{\mathrm{km}}\right] \times \cos\left[\frac{28.5\pi}{180}\ \mathrm{radians}\right] = 408.4\ \frac{\mathrm{m}}{\mathrm{sec}}$$

- For Launch from the cape in to a 28.5° inclination orbit

$$\Delta V_{total}^{(ksc)} = \sqrt{\left[7812.2\ \frac{\mathrm{m}}{\mathrm{sec}} - 404.8\ \frac{\mathrm{m}}{\mathrm{sec}}\right]^2 + \left[1750.8\ \frac{\mathrm{m}}{\mathrm{sec}}\right]^2} =$$

$$7611.5\ \frac{\mathrm{m}}{\mathrm{sec}}$$

KSC Launch Example Option 4: Continued

- Compute Transfer Orbit Parameters

$$e_T = \frac{R_{c_2} - R_{c_1}}{\left[R_{c_2} + R_{c_1}\right]} = \frac{\left[42164.2 - \left[6371 + 160\right]\right] \text{km}}{\left[42164.2 + \left[6371 + 160\right]\right] \text{km}} = 0.73176$$

$$a_T = \frac{R_{c_2} + R_{c_1}}{2} = \frac{\left[42164.2 + \left[6371 + 160\right]\right] \text{km}}{2} = 24347.6 \text{ km}$$

Compute Velocity at Transfer Orbit Perigee

$$V_T^{(perigee)} = \sqrt{\frac{2\mu}{a_T[1-e_T]} - \frac{\mu}{a_T}} =$$

$$\sqrt{\frac{2\mu - \mu[1-e_T]}{a_T[1-e_T]}} = \sqrt{\frac{\mu[1+e_T]}{a_T[1-e_T]}} =$$

$$\sqrt{\frac{3.986 \times 10^5 \frac{km^3}{sec^2} \times [1+0.73176]}{24347.6\ km\ km \times [1-0.73176]}} = 10280.7\ \frac{m}{sec}$$

KSC Launch Example Option 4: Continued

- Compute ΔV combined (28.5° plane change)

$$\left|\Delta V\right|_{combined} = \sqrt{\left|V_T^{(perigee)}\right|^2 + \left|V_{LEO}\right|^2 - 2\left|V_T^{(perigee)}\right|\left|V_{LEO}\right|\cos[\Delta i]} =$$

$$\sqrt{\left|10280.7\ \frac{m}{sec}\right|^2 + \left(7812.3\ \frac{m}{sec}\right)^2 - 2\left|10280.7\ \frac{m}{sec}\right|\left(7812.3\ \frac{m}{sec}\right)\cos\left[28.5°\times\frac{\pi}{180}\right]} =$$

$$5055.57\ \frac{m}{sec}$$

KSC Launch Example Option 4: Continued

- Compute ΔV_2 for Circularized Orbit at GEO

$$R_{c_2}/R_{c_1} = \frac{42164.2\ \text{km}}{[160 + 6371]\ \text{km}} = 6.456$$

$$\Delta V_2 = \frac{\sqrt{\frac{\mu}{R_{c_1}}}\sqrt{[R_{c_1}/R_{c_2}]}}{\left[\sqrt{2\frac{1}{R_{c_2}/R_{c_1}+1}}+1\right]}\left[\frac{R_{c_2}/R_{c_1} - 1}{[R_{c_2}/R_{c_1}+1]}\right] = 1481.41\ \frac{\text{m}}{\text{sec}}$$

KSC Launch Example Option 4: Concluded

- Minimum Total Delta V to GEO

$$\Delta V_{total} = \Delta V_{launch} + \Delta V_{combined} + \Delta V_{2_{hohmann}} =$$

$$\left[7611.5 - 5055.57 + 1481.41\right]\frac{m}{sec} = 14148.6\ \frac{m}{sec}$$

$$\text{Energy cost Compared to equatorial launch} = 100\% \times \frac{14148.6 - 11503.1}{11503.1} = 23.0\%$$

Cost to get to GEO from Equatorial Launch

$$V_{rot_{Earth}} = \omega_{Earth} \times r_{Earth} \times \cos[Lat] =$$

$$\left[0.000072921 \frac{\text{radians}}{\text{sec}}\right] \times \left[6371 \text{ km} \times 1000 \frac{\text{m}}{\text{km}}\right] \times \cos[0 \text{ radians}] = 465.1 \frac{\text{m}}{\text{sec}}$$

$$\Delta V_{total}^{(equator)} = \sqrt{\left[[7812.2 - 465.1] \frac{\text{m}}{\text{sec}}\right]^2 + [\varepsilon\ 1750.8]^2} =$$

$$\boxed{7552.8 \frac{\text{m}}{\text{sec}}}$$

Cost to get to GEO from Equatorial Launch

$$\Delta V_{total} = \Delta V_{launch} + \Delta V_{1_{hohmann}} + \Delta V_{2_{hohmann}} =$$

$$\left[7552.8 + 2467.0 + 1481.4\right]\frac{\text{m}}{\text{sec}} = 111501.2\frac{\text{m}}{\text{sec}}$$

$$\begin{matrix}\text{savings} \\ \text{Compared to} \\ \text{best Launch Option} \\ \text{from KSC}\end{matrix} = 100\% \times \frac{11872.0 - 11501.2}{11872.00} = 3.1\%$$

- That's the bottom Line … get a satellite to GEO-stationary Orbit from KSC costs you >3% ΔV Compared to Equatorial LAUNCH

Summary: Launch to GEO Options ΔV Table

Option	Description	ΔV_{total}		ΔV_{orbit}	% ΔV_{cost} Total	% ΔV_{cost} Orbit
Sea launch	Direct launch Hohmann transfer	**11503.1**	m/sec	3948.4 m/sec	0.0%	0.0%
KSC Opt. 1:	28.5° Launch, 28.5° Plane Change, Hohman ΔV_1, ΔV_2	**15406.0**	m/sec	7794.5 m/sec	34.5%	97.4%
KSC Opt. 2:	28.5° Launch, Hohman ΔV_1, ΔV_2, 28.5° Plane Change	**12344.0**	m/sec	4732.4 m/sec	7.4%	19.9%
KSC Opt. 3:	28.5° Launch, Hohman ΔV_1, Combined 28.5° Plane Change + ΔV_2	**11872.1**	m/sec	4242.7 m/sec	3.25%	7.5%
KSC Opt. 4:	28.5° Launch, Combined 28.5° Plane Change + ΔV_1, Hohman ΔV_2	**14148.6**	m/sec	6537.0 m/sec	23.4%	65.6%