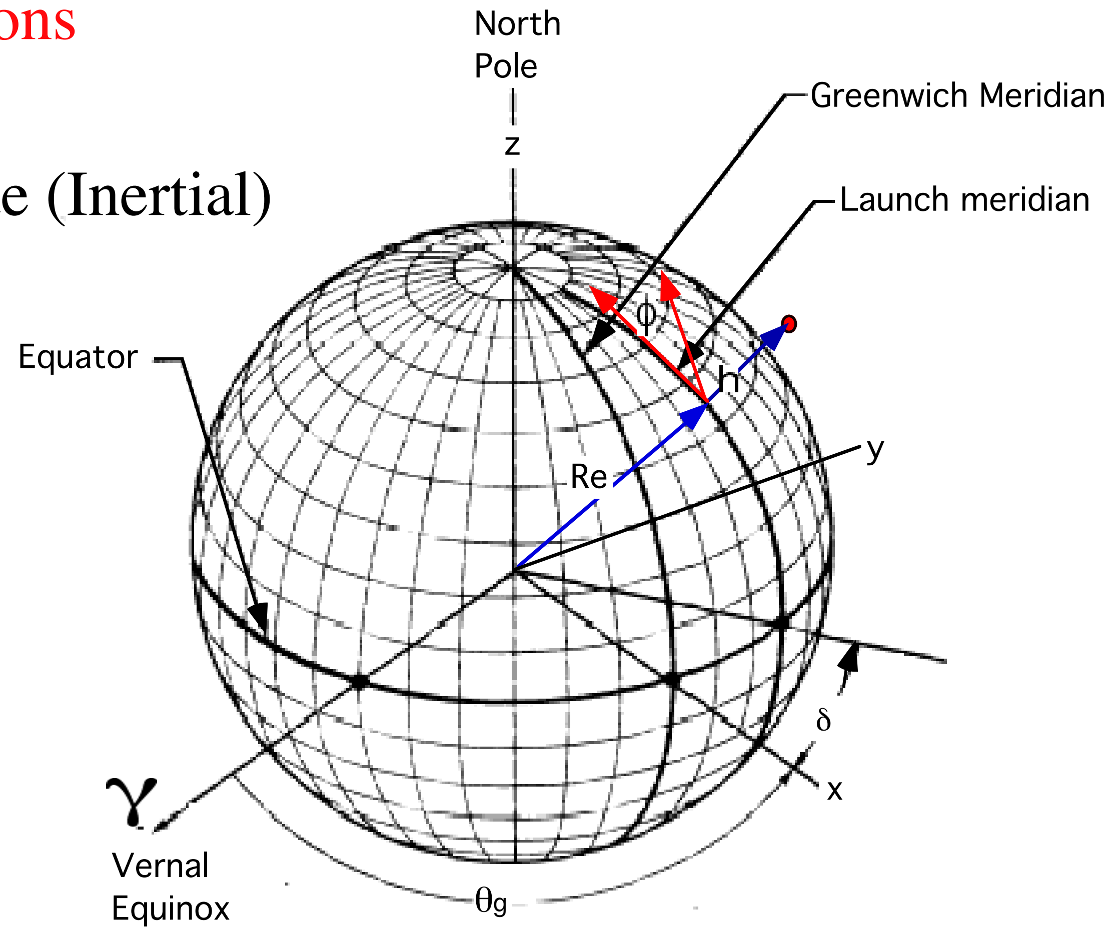
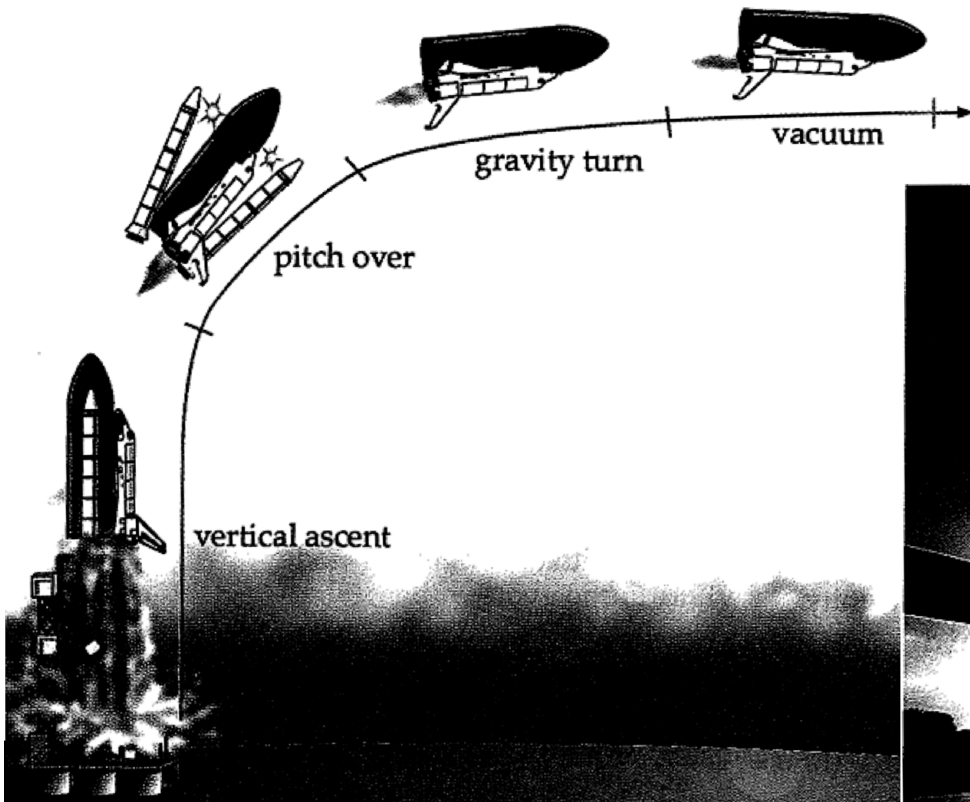
Appendix : Rigorous Derivation of Realizable Launch Inclination

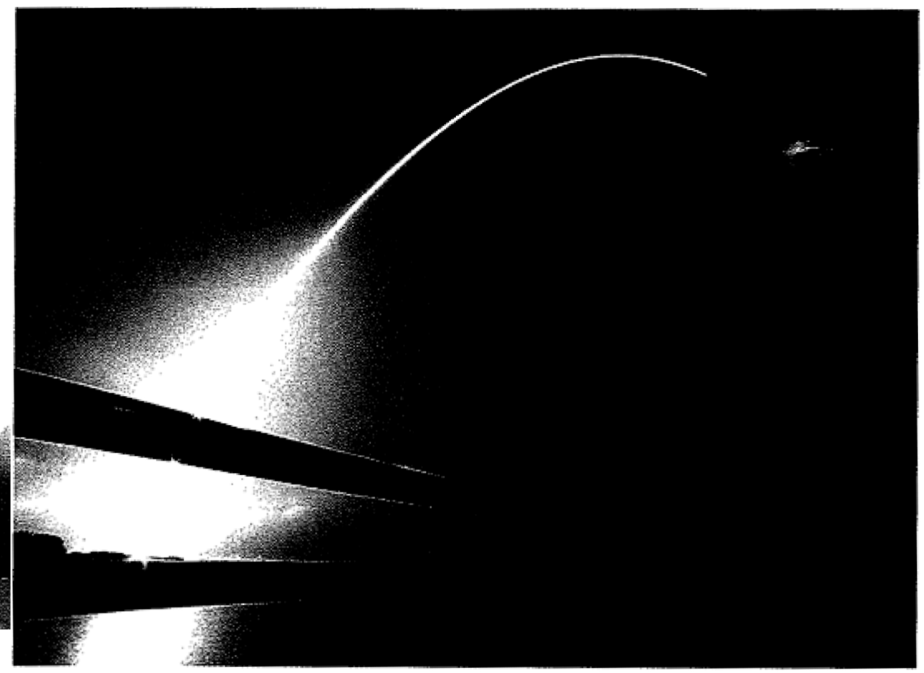
- Launch Initial Conditions
- Position: λ, Latitude
  Ω, Longitude (Inertial)
  h, Altitude



What Happens at Launch?



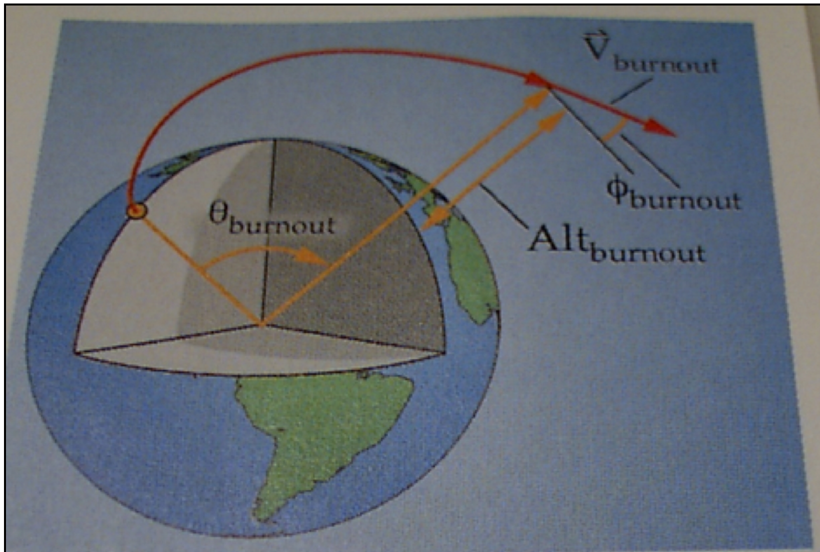
Phases of Launch Vehicle Ascent. During ascent a launch vehicle goes through four phases—vertical ascent, pitch over, gravity turn, and vacuum.



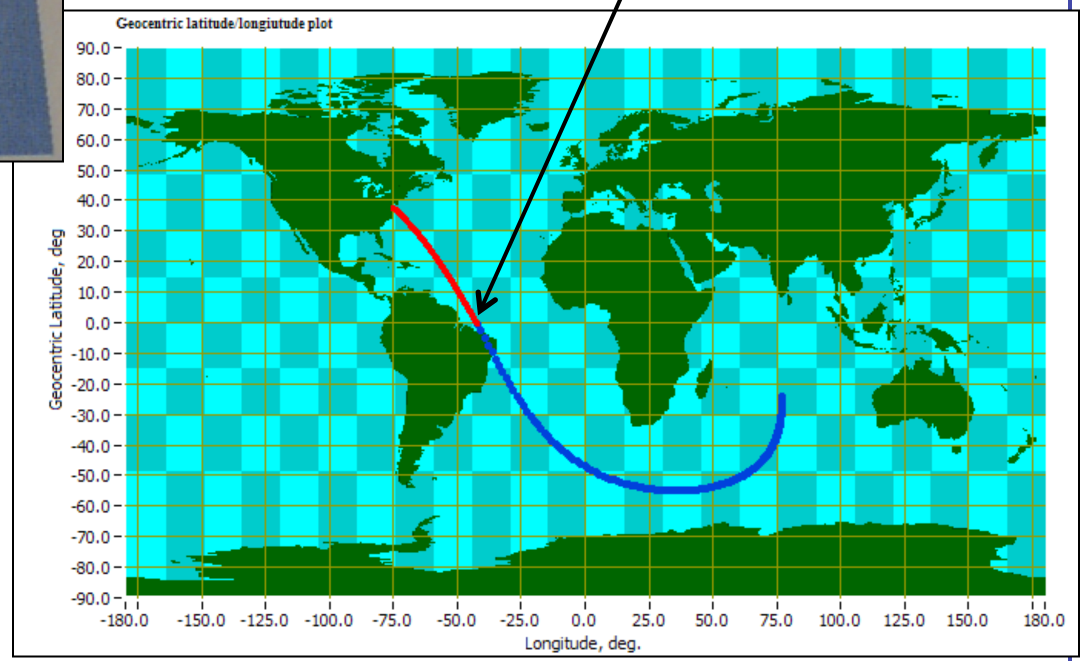
Gravity-turn maneuver of an ascending Delta II rocket with Messenger spacecraft on August 3, 2004.

What Happens at Launch?

- Velocity and Position at Burnout Determine Orbit



Final Stage Burnout



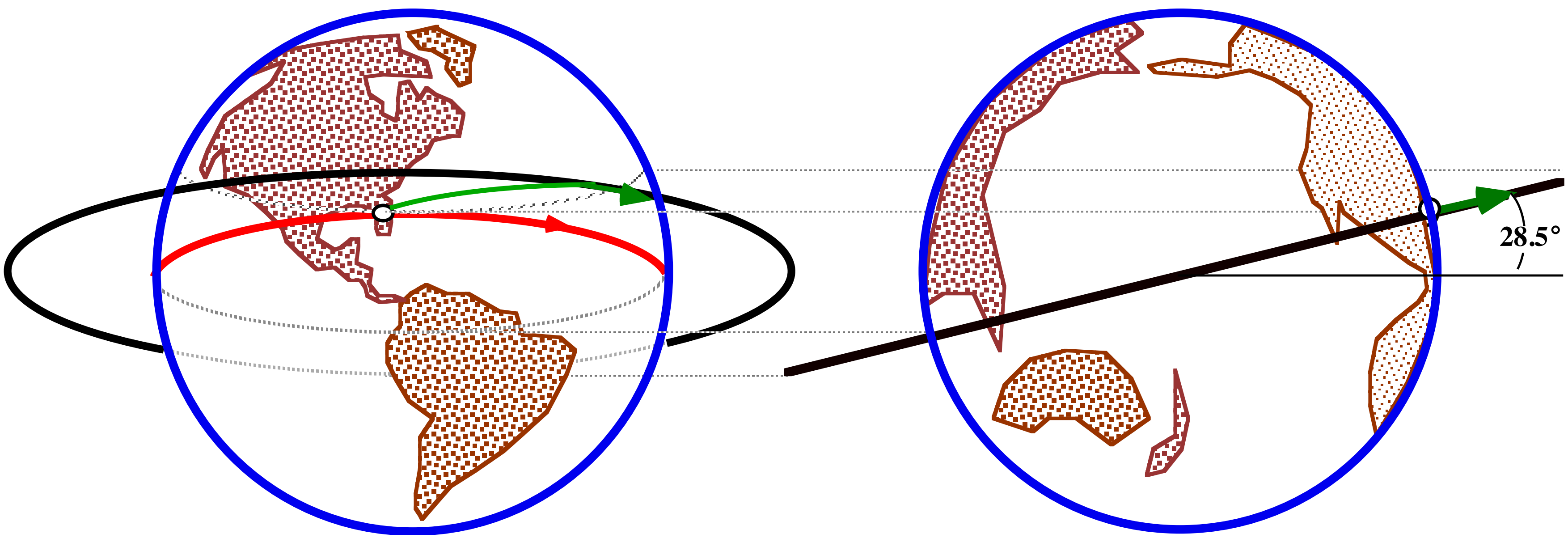
What Happens on Launch?

- Example

Kennedy Space Center (KSC)

Due-East Launch

28.5° Inclination Orbit

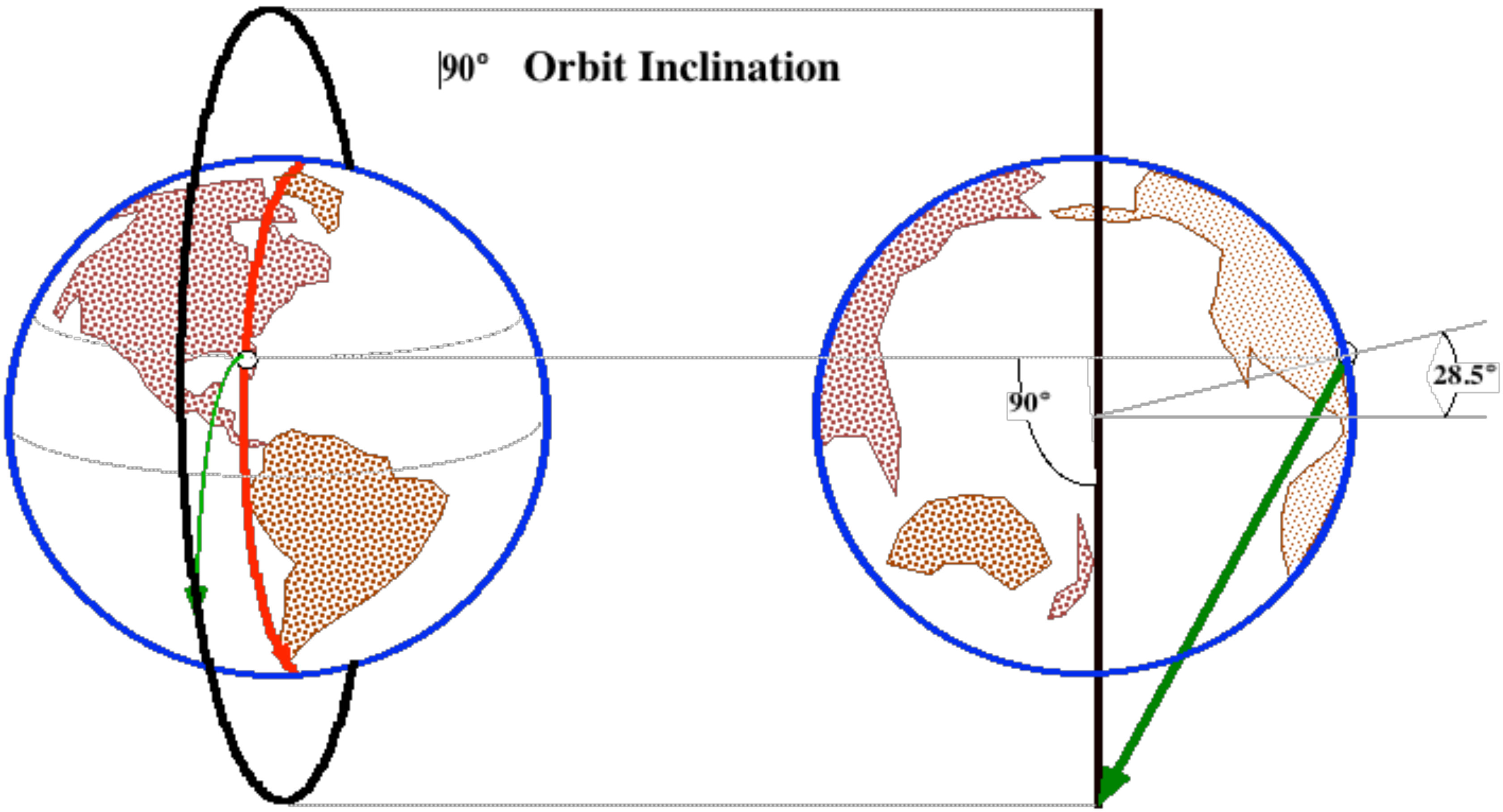


What Happens on Launch? (cont'd)

- Example

Kennedy Space Center (KSC)

Due South (from east)Launch Direction



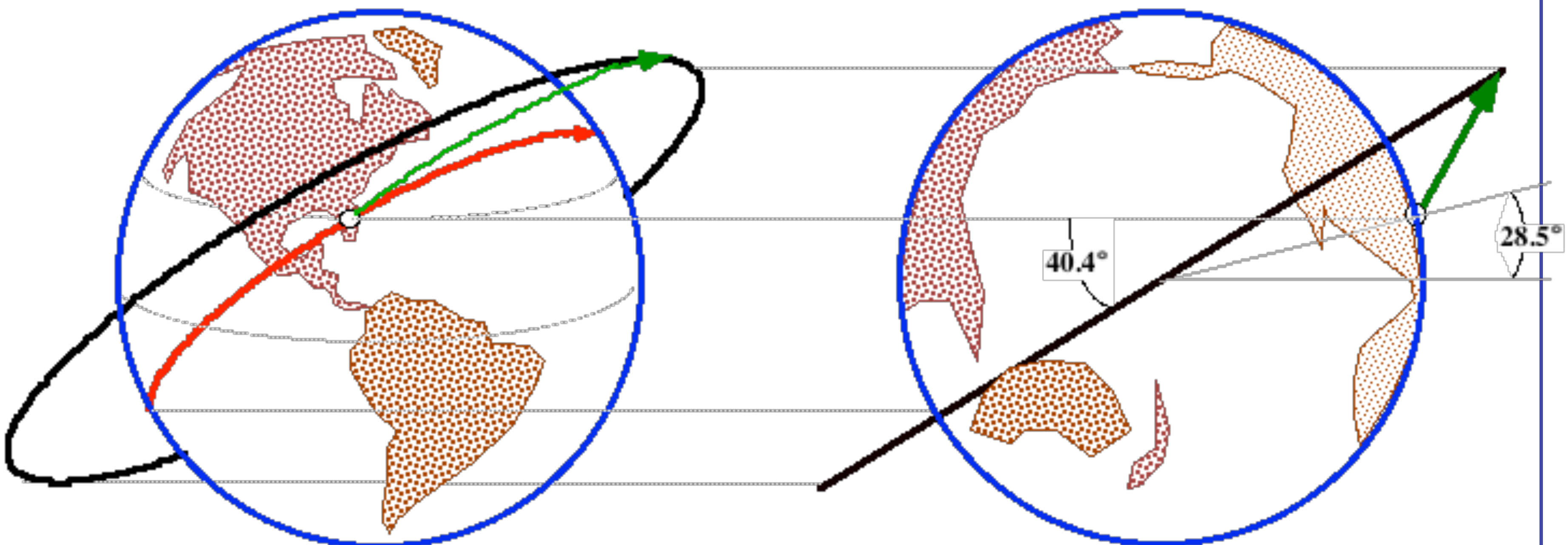
What Happens on Launch? (cont'd)

- Example

Kennedy Space Center (KSC)

30° North (from east)Launch Direction

40.4° Orbit Inclination Huh?



What Happens on Launch? (cont'd)

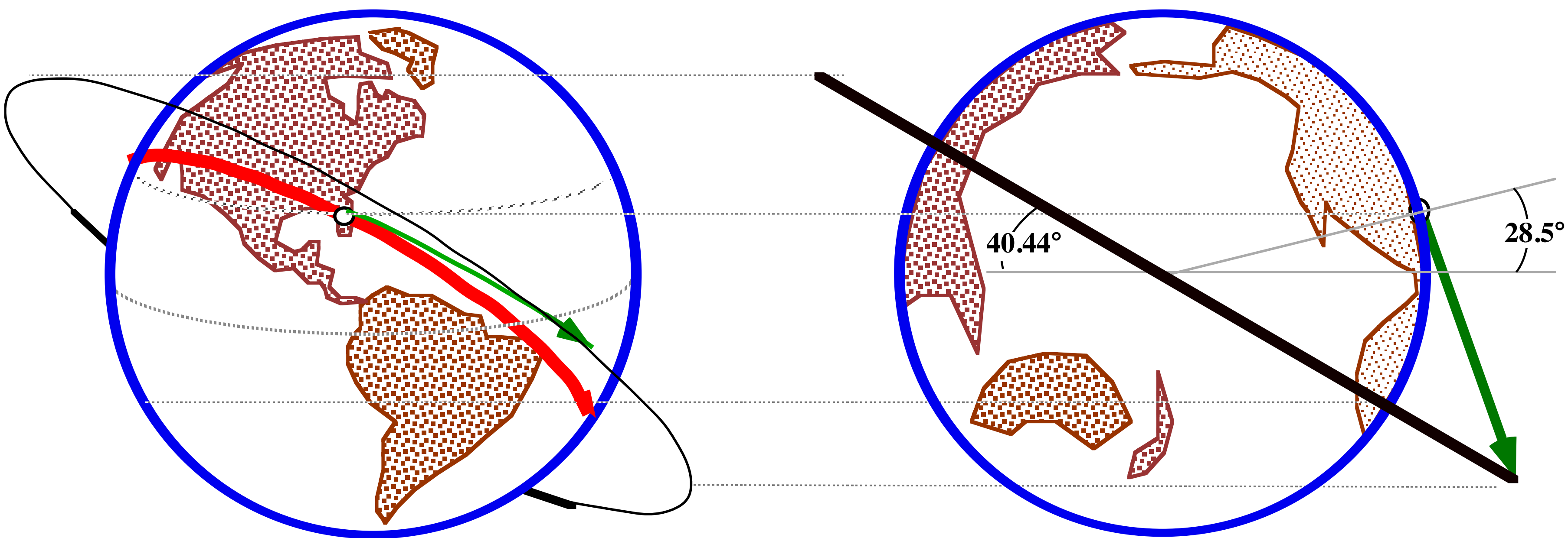
- Example

Kennedy Space Center (KSC)

30° *South* (from east)Launch Direction

40.44° Orbit Inclination

Huh?



Achieved Orbit Inclinations

$$\begin{bmatrix} 0^\circ \text{ Launch Angle} & & 28.6^\circ \text{ Inclination} \\ -90^\circ \text{ Launch Angle} & = & 90^\circ \text{ Inclination} \\ 30^\circ \text{ Launch Angle} & & 40.44^\circ \text{ Inclination} \\ -30^\circ \text{ Launch Angle} & & 40.44^\circ \text{ Inclination} \end{bmatrix}$$

What?! Doesn't add up ...

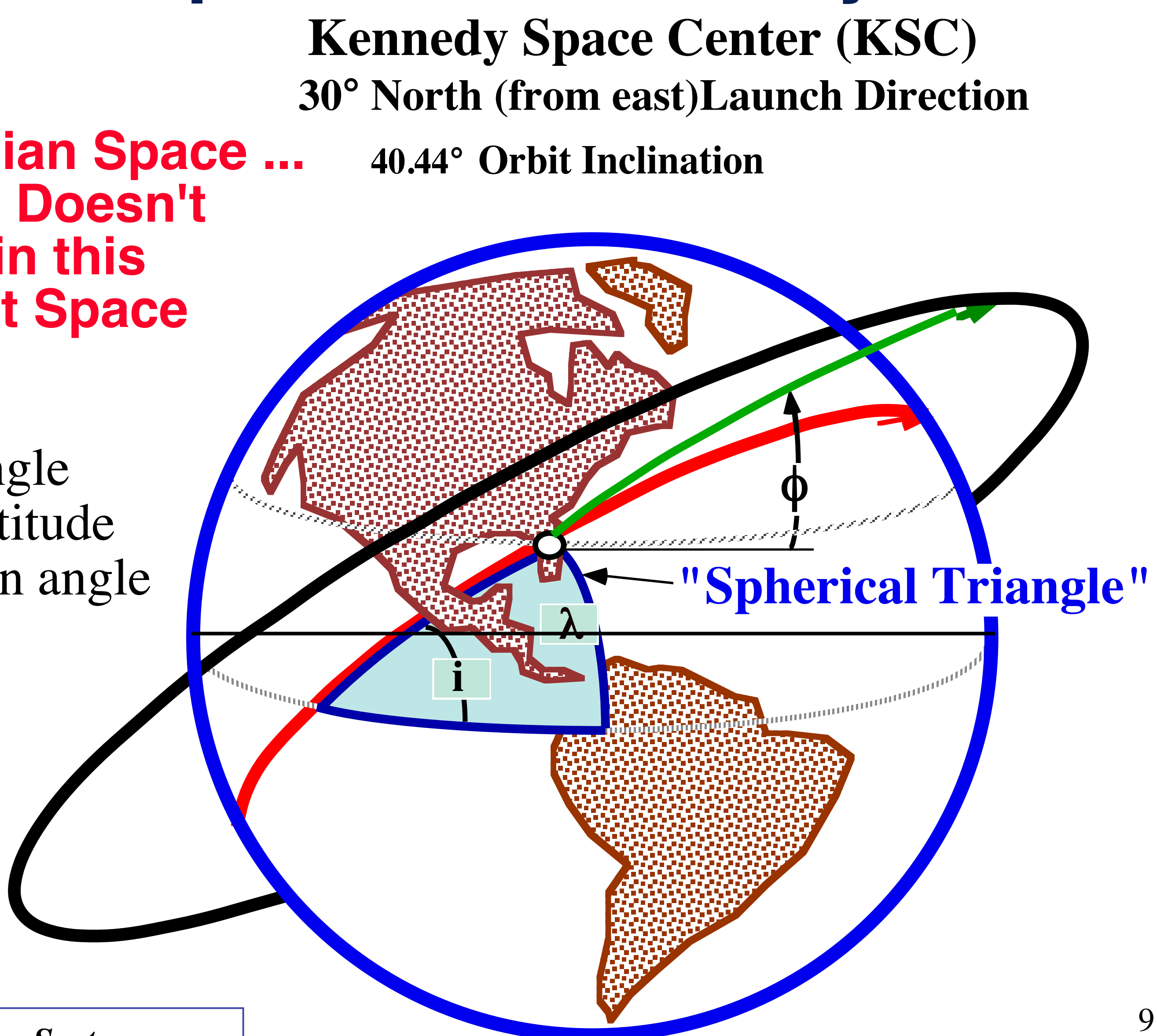
UtahState UNIVERSITY

Mechanical & Aerospace Engineering

"Spherical Geometry"

• **Non-Euclidian Space ... Addition Doesn't Exist! in this Abstract Space**

φ –– launch angle
λ –– launch latitude
i –– inclination angle



UtahState UNIVERSITY

Mechanical & Aerospace Engineering

"Spherical Geometry"

"fixed earth approximation"

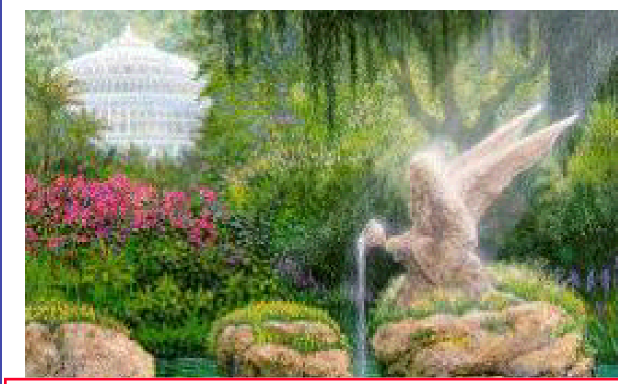
- **Non-Euclidian Space ... Addition Doesn't Exist! in this Abstract Space**

$$\cos(i) = \cos(\phi)\cos(\lambda)$$

ϕ -- launch angle
λ -- launch latitude
i -- inclination angle

Launch Angle sometimes expressed as "azimuth" ... angle from local true north

$$Az = \begin{bmatrix} 90° - \phi \Rightarrow \text{degrees} \\ \frac{\pi}{2} - \phi \Rightarrow \text{radians} \end{bmatrix}$$



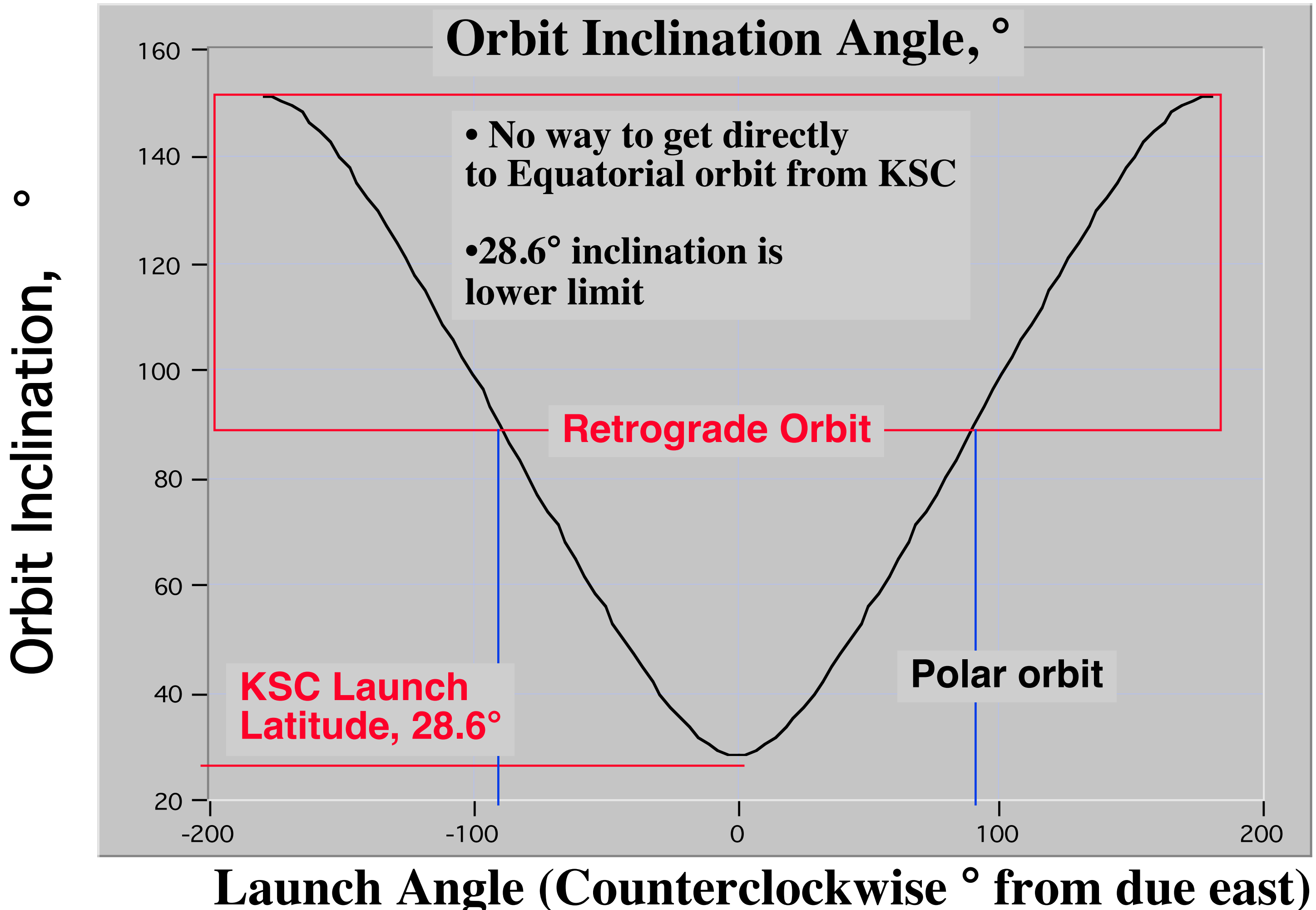
- **then a miracle occurs:**

$$\cos(i) = \cos(\lambda) \cdot \sin(Az)$$

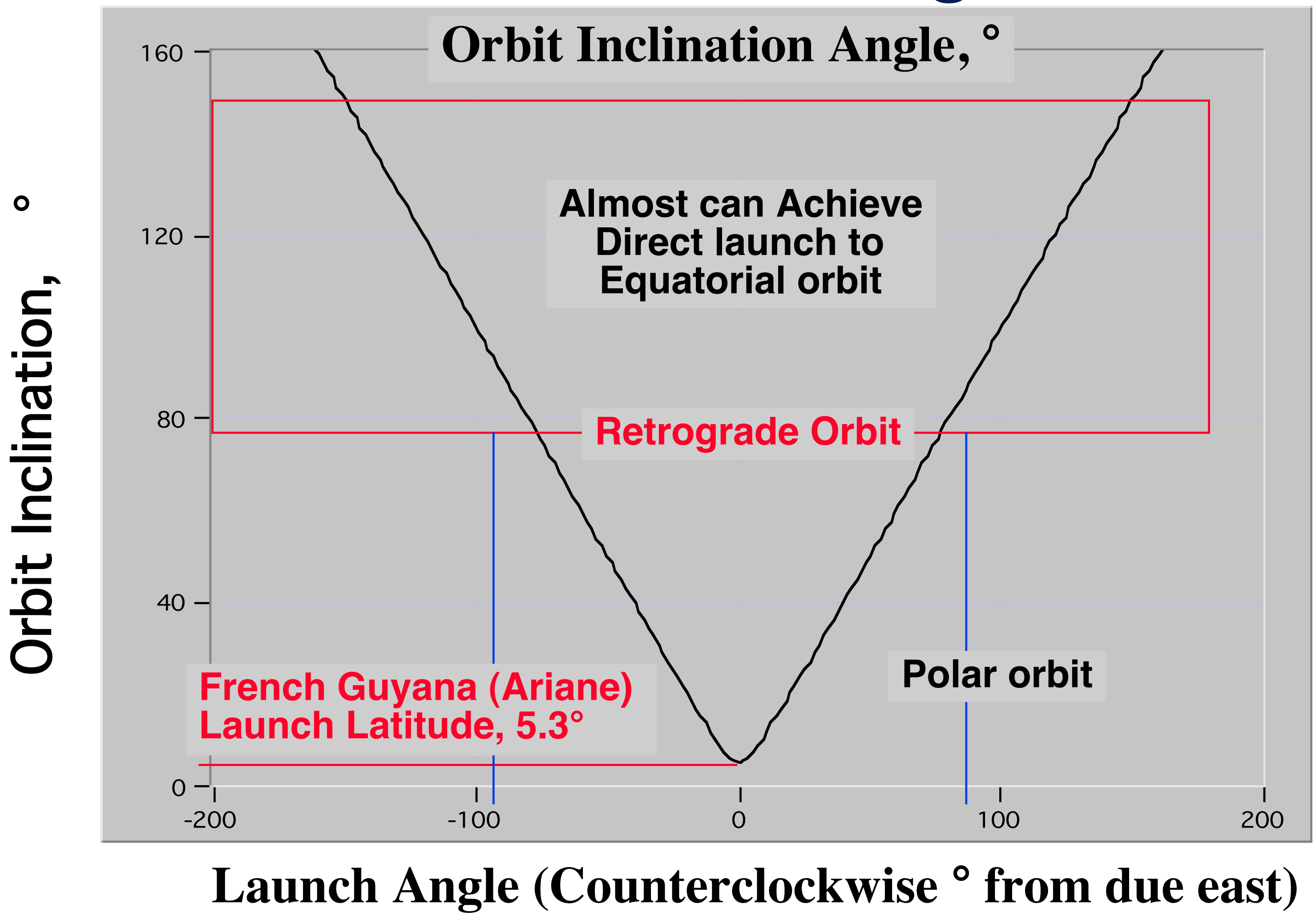
Direct Launch Inclination Angle

$$\left[\begin{array}{l} i = \dfrac{180^\circ}{\pi}\cos^{-1}\left[\cos\left[30^\circ\dfrac{\pi}{180^\circ}\right]\times\cos\left[28.6^\circ\dfrac{\pi}{180^\circ}\right]\right] = 40.44^\circ \\ i = \dfrac{180^\circ}{\pi}\cos^{-1}\left[\cos\left[-30^\circ\dfrac{\pi}{180^\circ}\right]\times\cos\left[28.6^\circ\dfrac{\pi}{180^\circ}\right]\right] = 40.44 \end{array}\right]$$

Achievable Direct-Launch Inclination Angles



Achievable Direct-Launch Inclination Angles



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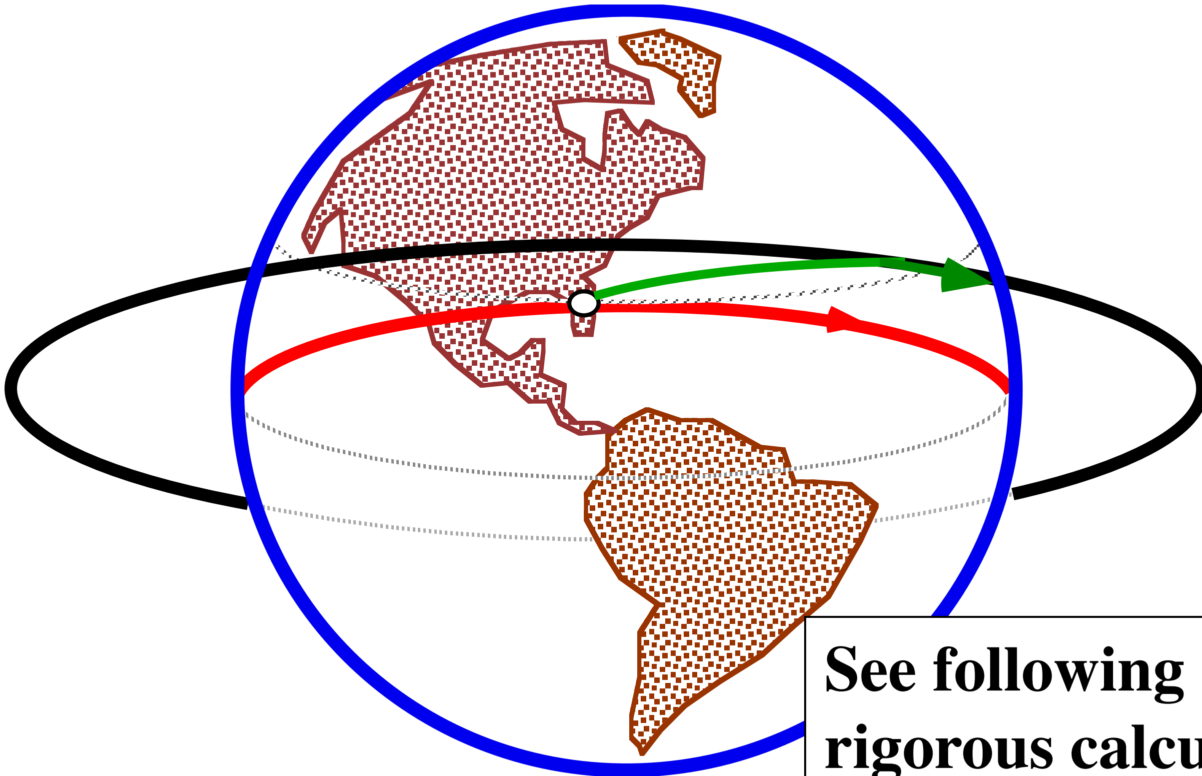
Bottom Line



See following slides for rigorous calculation

• *Physically Impossible* to Launch Directly into an orbit with a *Lower* inclination Angle than the Launch latitude

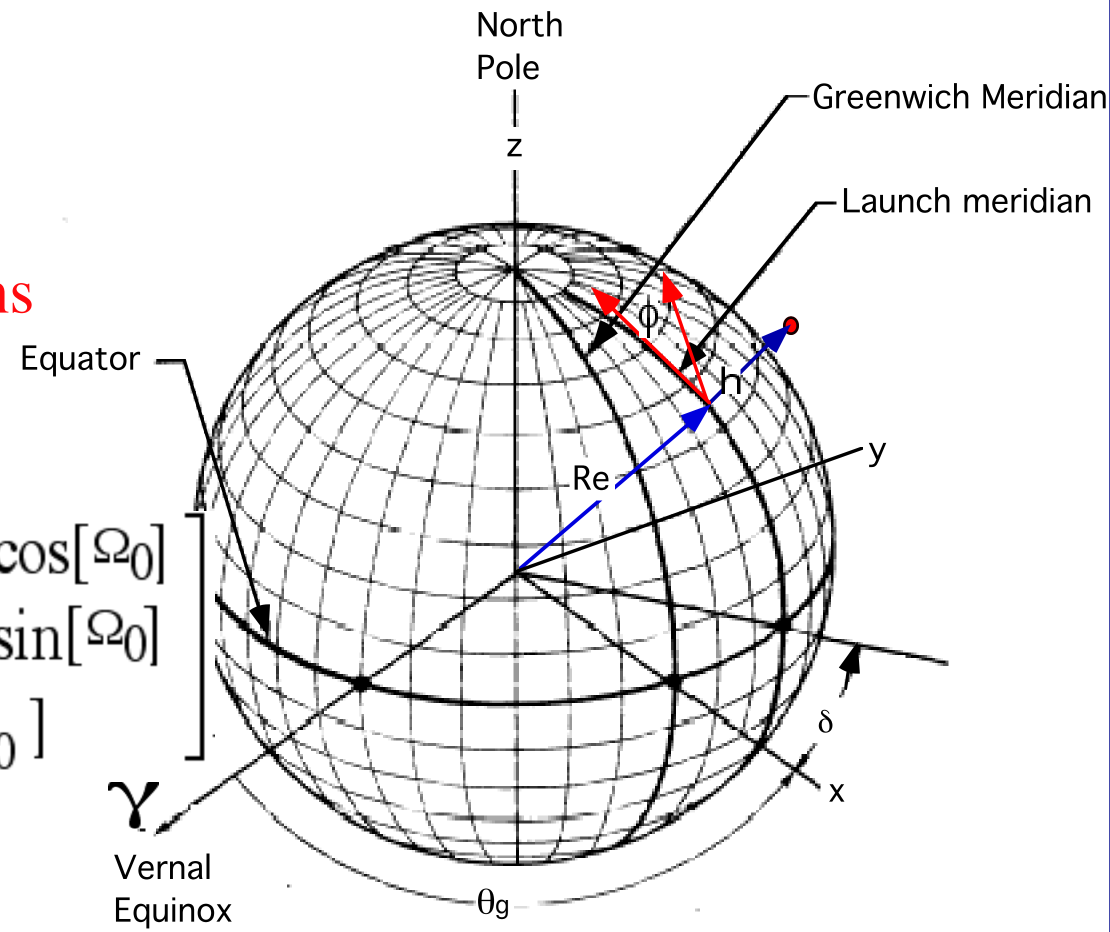
•Physically Possible to launch directly into any orbit with an inclination angle *greater than* or equal to launch latitude

• Launch Initial Conditions

• Launch Initial Conditions (Inertial Coordinates)

$$\begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix}_0 = \begin{bmatrix} [R_{earth} + h_0]\cos[\lambda_0]\cos[\Omega_0] \\ [R_{earth} + h_0]\cos[\lambda_0]\sin[\Omega_0] \\ [R_{earth} + h_0]\sin[\lambda_0] \end{bmatrix}$$

$$\Omega_0 = \delta_0 + \theta_g$$

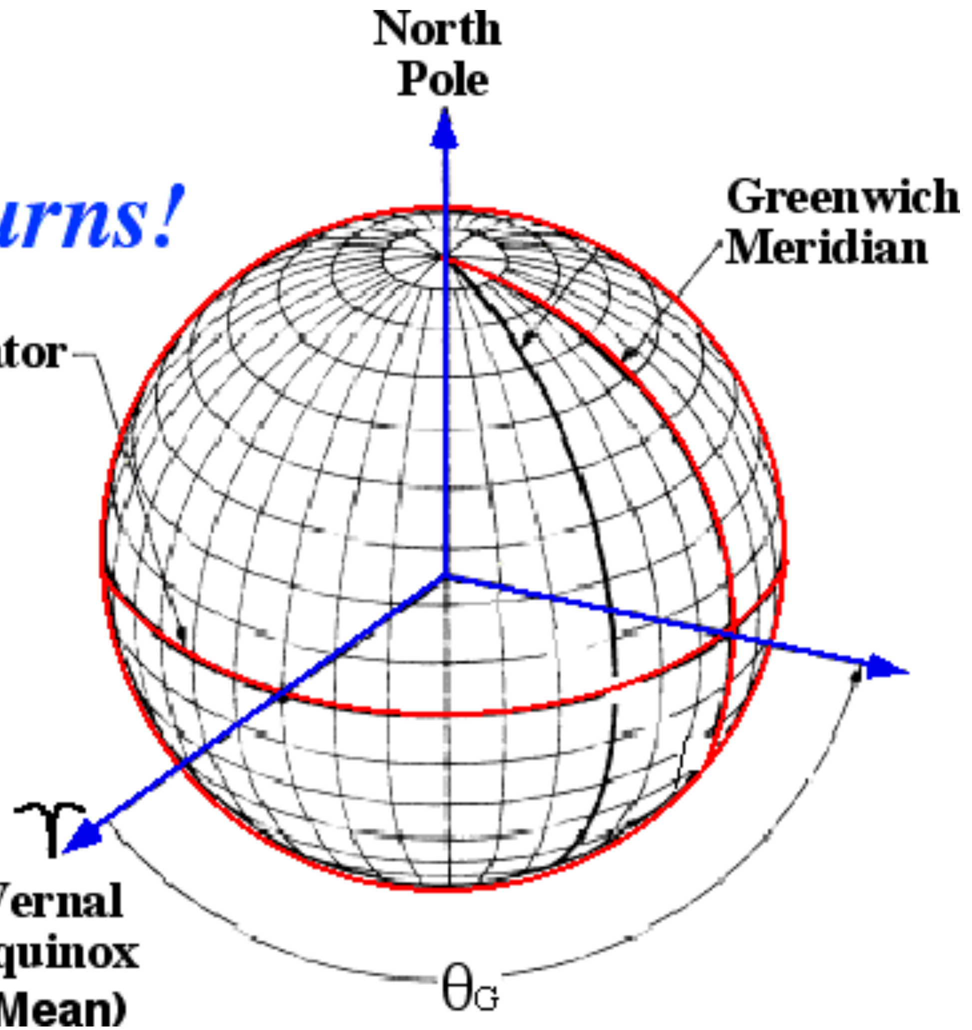


Sidereal hour angle

As the world turns!

$$M_{Rotation} = \begin{bmatrix} \cos[\theta_G] & -\sin[\theta_G] & 0 \\ \sin[\theta_G] & \cos[\theta_G] & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Greenwich Sidereal (Hour) Angle



Ignore for now!

Computing the Hour Angle

- θ_G Historically Expressed in Hours

... Sometimes referred to as Greenwich Mean Sidereal Time ... but we are going to treat it as an angle

$$\theta_G = \omega_{earth} \times \left[T_{GMST} - T_{JD2000} \right]$$

- Sidereal time is a measure of the Earth's rotation with respect to distant celestial objects.

• Earth Radius

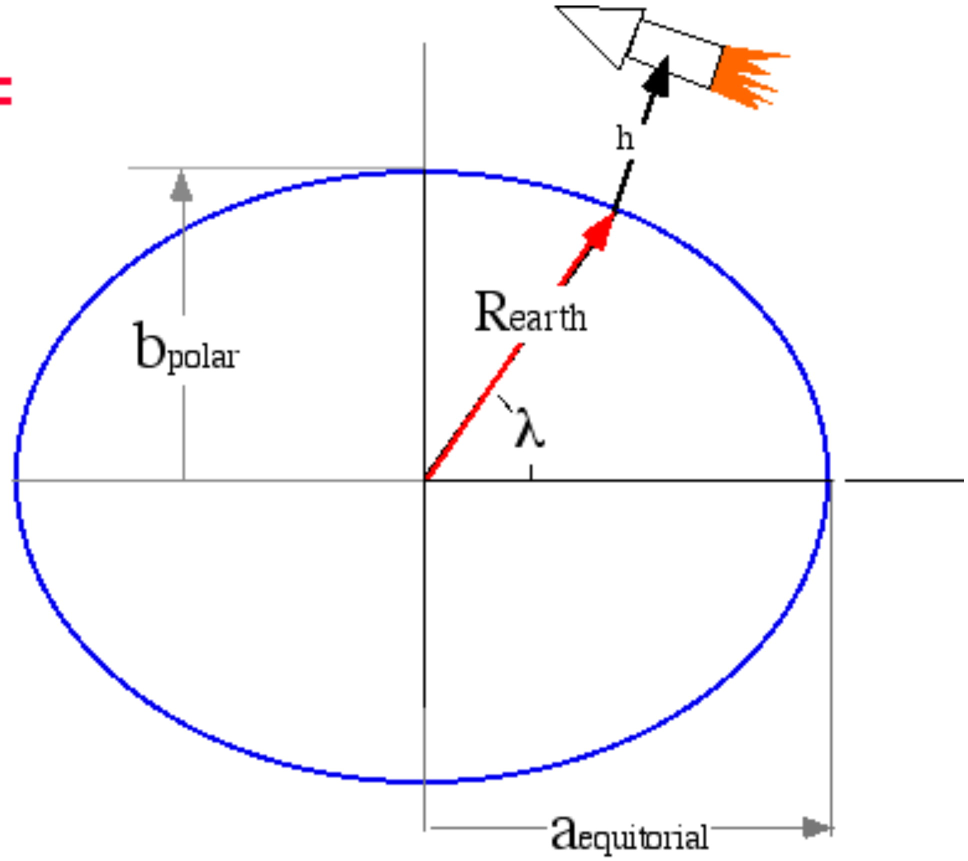
• Earth radius as Function of Latitude

$$R_{earth} = \frac{a_{equitorial}}{\sqrt{\left[1 + \frac{e_{earth}^2}{1 - e_{earth}^2} \sin^2 \lambda\right]}}$$

$$a_{equitorial} = 6378.13649 \text{ km}$$

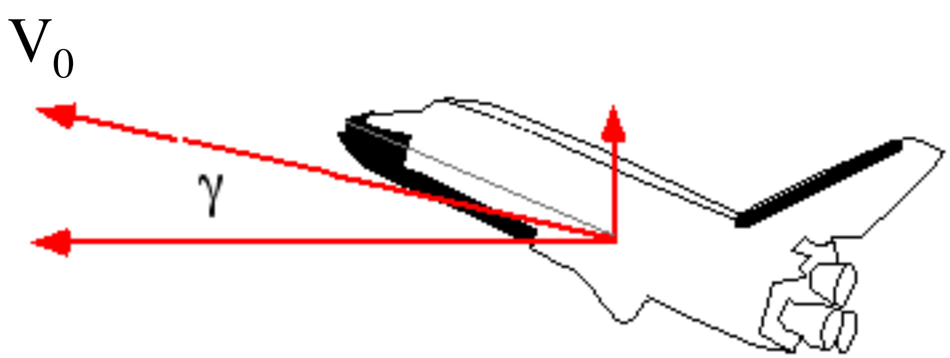
$$b_{polar} = 6356.7515 \text{ km}$$

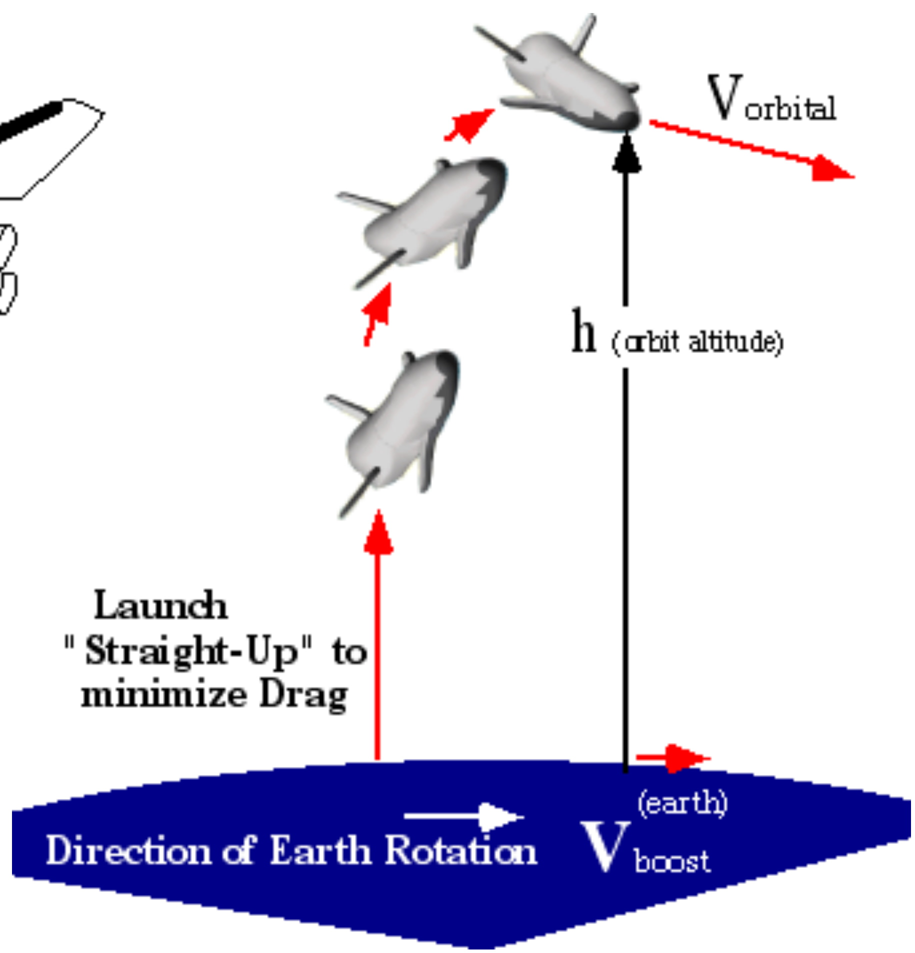
$$e_{earth} = \sqrt{1 - \left[\frac{b_{polar}}{a_{equitorial}}\right]^2}$$



• Initial Velocity Vector

after the rocket “turns the corner”





• Launch Initial Conditions

• Velocity: V_o (earth relative vel.)
  γ (flight path angle)
  ϕ (launch azimuth)

• Initial Velocity Vector

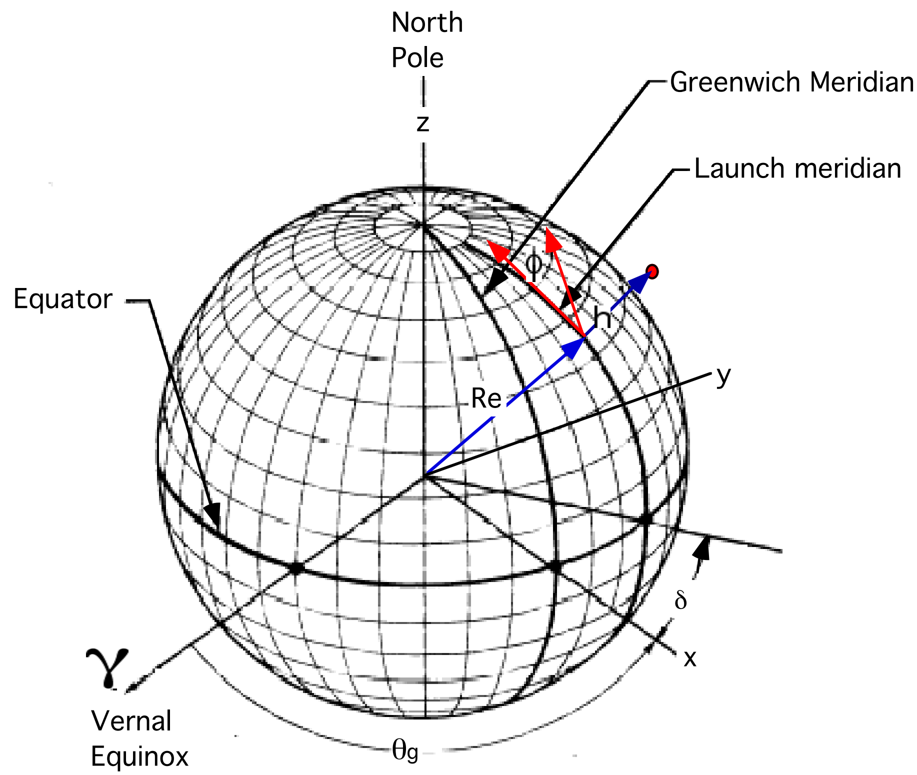
after the rocket "turns the corner"

• Launch Initial Conditions

$$\begin{bmatrix} V_{north} \\ V_{east} \\ V_{down} \end{bmatrix}_0 = \begin{bmatrix} V_0 \cos(\phi) \cos(\gamma) \\ V_0 \sin(\phi) \cos(\gamma) \\ -V_0 \sin(\gamma) \end{bmatrix}$$

+ rotational velocity of earth

V_o (earth relative vel.)
γ (flight path angle)
ϕ (launch azimuth)



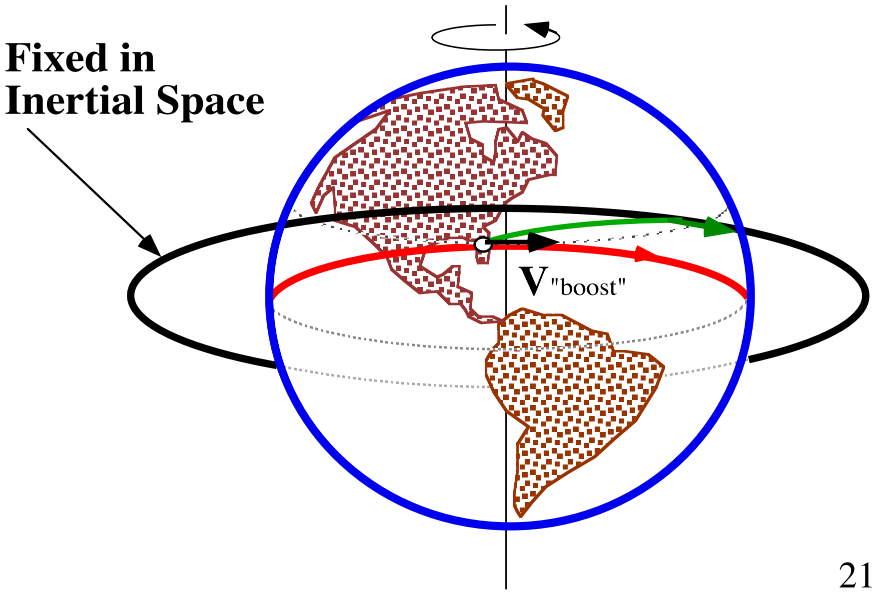
• Earth "Boost"

$$\lambda_0 \equiv \textit{geocentric latitude}$$

$$V_{boost}^{(earth)} = \left[R_{earth} + h_0\right]_{launch} \left[\Omega_{earth} \cos[\lambda_0]\right]$$

• Launch Initial Conditions

$V_{boost}^{(earth)}$ acts due east



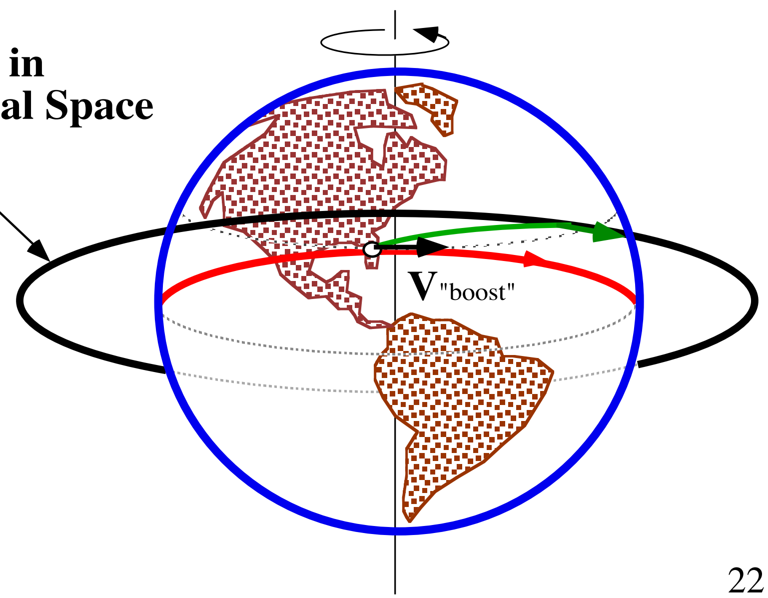
• Angular Velocity of Earth

Angular Velocity of the Earth

- 1 Solar Day = 23 hrs 56 min 4.1 seconds = 86164.1 seconds
- $\Omega_{earth} = \frac{360°}{86164.1 \text{ seconds}} \times \frac{\pi}{180°} = .00007292115 \frac{\text{rad}}{\text{sec}}$

• Launch Initial Conditions

$V_{boost}^{(earth)}$ acts due east



• Initial Velocity Vector

after the rocket "turns the corner"

• Launch Initial Conditions

$$\begin{bmatrix} V_{north} \\ V_{east} \\ V_{down} \end{bmatrix}_0 = \begin{bmatrix} V_0 \cos(\phi) \cos(\gamma) \\ V_0 \sin(\phi) \cos(\gamma) + [R_{earth} + h_0]_{launch} [\Omega_{earth} \cos[\lambda_0]] \\ -V_0 \sin(\gamma) \end{bmatrix}$$

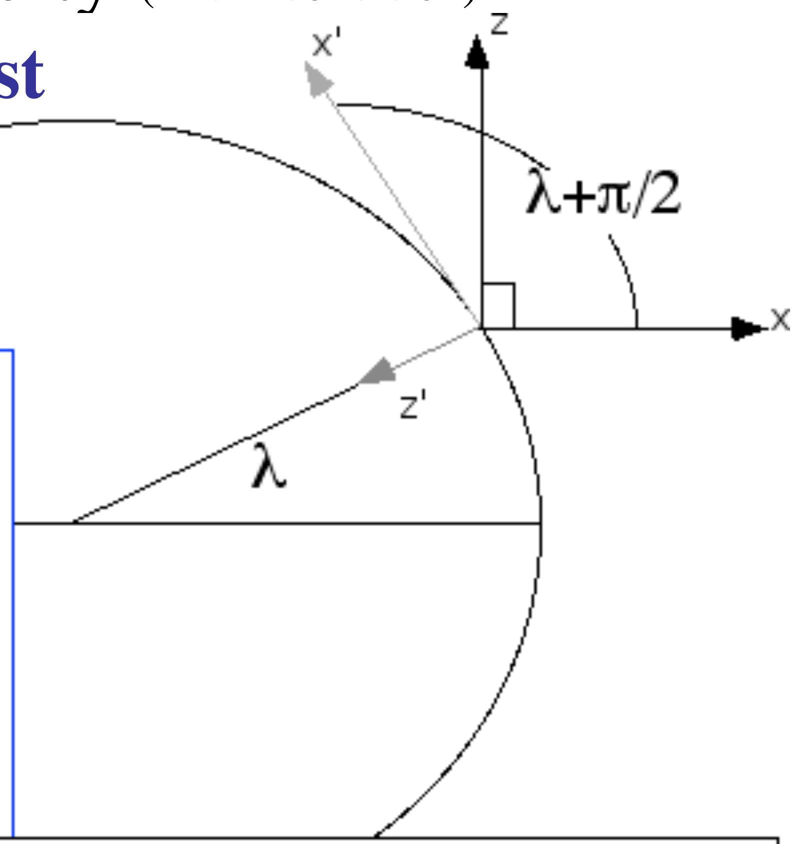
$$\begin{bmatrix} \phi \\ \lambda \end{bmatrix} \equiv \begin{bmatrix} \textit{launch azimuth (from due north)} \\ \textit{geocentric latitude} \end{bmatrix}$$

Inertial Velocity (Initial condition)

Rotate Through Latitude First

2-rotation

- **Clock-wise rotation**



$$M_\lambda = \begin{bmatrix} \cos\left[\frac{\pi}{2}+\lambda\right] & 0 & -\sin\left[\frac{\pi}{2}+\lambda\right] \\ 0 & 1 & 0 \\ \sin\left[\frac{\pi}{2}+\lambda\right] & 0 & \cos\left[\frac{\pi}{2}+\lambda\right] \end{bmatrix} =$$

$$\begin{bmatrix} -\sin[\lambda] & 0 & -\cos[\lambda] \\ 0 & 1 & 0 \\ \cos[\lambda] & 0 & -\sin[\lambda] \end{bmatrix}$$

$$\cos\left[\frac{\pi}{2}+\lambda\right] = \cos\left[\frac{\pi}{2}\right]\cos[\lambda] - \sin\left[\frac{\pi}{2}\right]\sin[\lambda] = -\sin[\lambda]$$

$$\sin\left[\frac{\pi}{2}+\lambda\right] = \cos\left[\frac{\pi}{2}\right]\sin[\lambda] + \sin\left[\frac{\pi}{2}\right]\cos[\lambda] = \cos[\lambda]$$

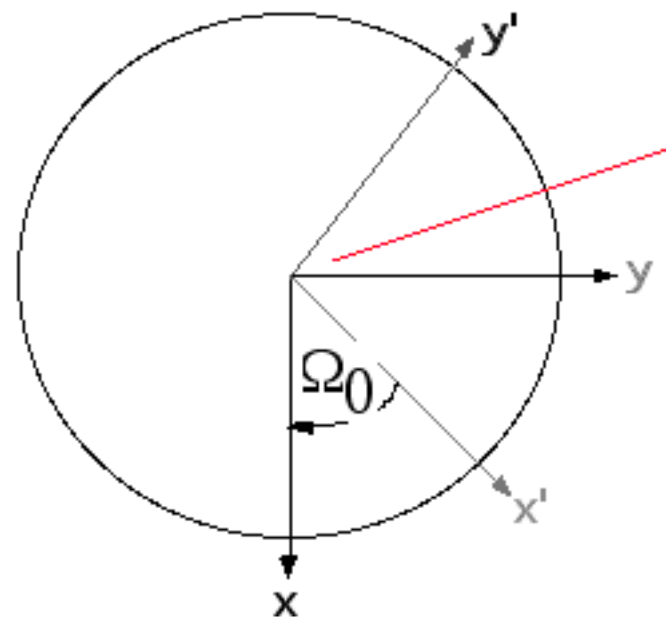
No 1 (X-axis) rotation
Needed Why? z` already points
Towards center of earth

Inertial Velocity (Initial condition)

Rotate Through Longitude Next

$$M_\delta = \begin{bmatrix} \cos[\Omega_0] & -\sin[\Omega_0] & 0 \\ \sin[\Omega_0] & \cos[\Omega_0] & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Clock-wise rotation**



Inertial Velocity (Initial condition)

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_{Inertial} =$$

$$\begin{bmatrix} -\cos[\Omega_0]\sin[\lambda_0] & -\sin[\Omega_0] & -\cos[\Omega_0]\cos[\lambda_0] \\ -\sin[\Omega_0]\sin[\lambda_0] & \cos[\Omega_0] & -\sin[\Omega_0]\cos[\lambda_0] \\ \cos[\lambda_0] & 0 & -\sin[\lambda_0] \end{bmatrix}$$

$$\begin{bmatrix} V_0 \cos(\phi)\cos(\gamma) \\ V_0 \sin(\phi)\cos(\gamma) + [R_{earth} + h_0]\,\Omega_{earth}\cos(\lambda_0) \\ -V_0 \sin(\gamma) \end{bmatrix}$$

Initial Conditions In Inertial Coordinates

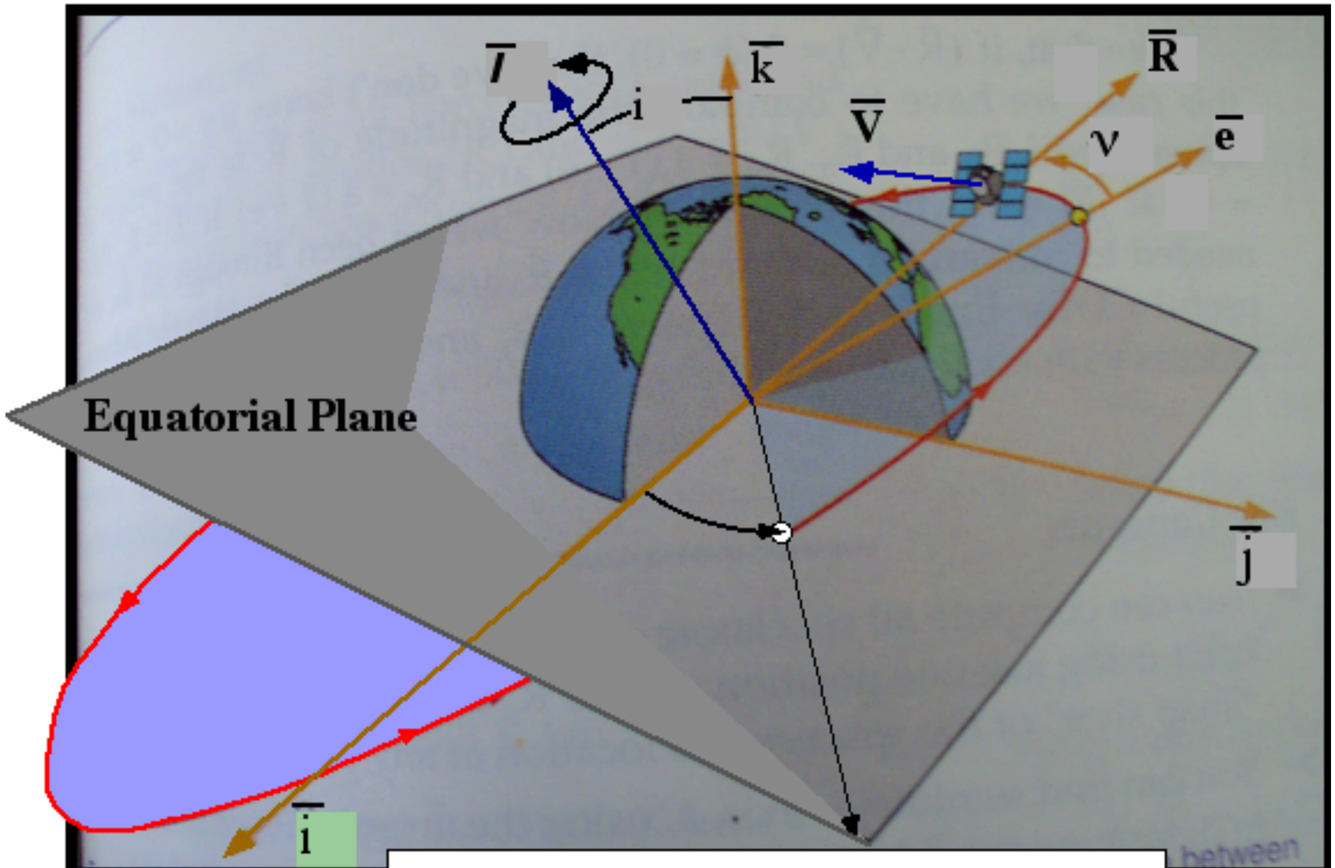
$$\begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}_{Inertial} = \begin{bmatrix} [R_{earth} + h_0] \cos[\lambda_0] \cos[\delta_0] \\ [R_{earth} + h_0] \cos[\lambda_0] \sin[\delta_0] \\ [R_{earth} + h_0] \sin[\lambda_0] \end{bmatrix}$$

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_{Inertial} =$$

$$\begin{bmatrix} -\cos[\Omega_0] \sin[\lambda_0] & -\sin[\Omega_0] & -\cos[\Omega_0] \cos[\lambda_0] \\ -\sin[\Omega_0] \sin[\lambda_0] & \cos[\Omega_0] & -\sin[\Omega_0] \cos[\lambda_0] \\ \cos[\lambda_0] & 0 & -\sin[\lambda_0] \end{bmatrix} \begin{bmatrix} V_0 \cos(\phi) \cos(\gamma) \\ V_0 \sin(\phi) \cos(\gamma) + [R_{earth} + h_0]\, \Omega_{earth} \cos(\lambda_0) \\ -V_0 \sin(\gamma) \end{bmatrix}$$

Out-of-plane orbital elements (cont'd)

Inclination Angle



$$\bar{l} = \bar{R} \times \bar{V}$$

$$\cos(i) = \frac{\left(\bar{k} \cdot \bar{l}\right)}{\left\|\bar{l}\right\|} = \frac{\bar{l}_z}{\left\|\bar{l}\right\|}$$

This VI calculates Launch Initial Conditions from user inputs for launch site and rocket data.

Launch Site Info

Latitude (deg): 28.50

Longitude (deg): -82.00

Launch Altitude, h (kn: 25.00

Launch Azimuth, ϕ (d: 0.00

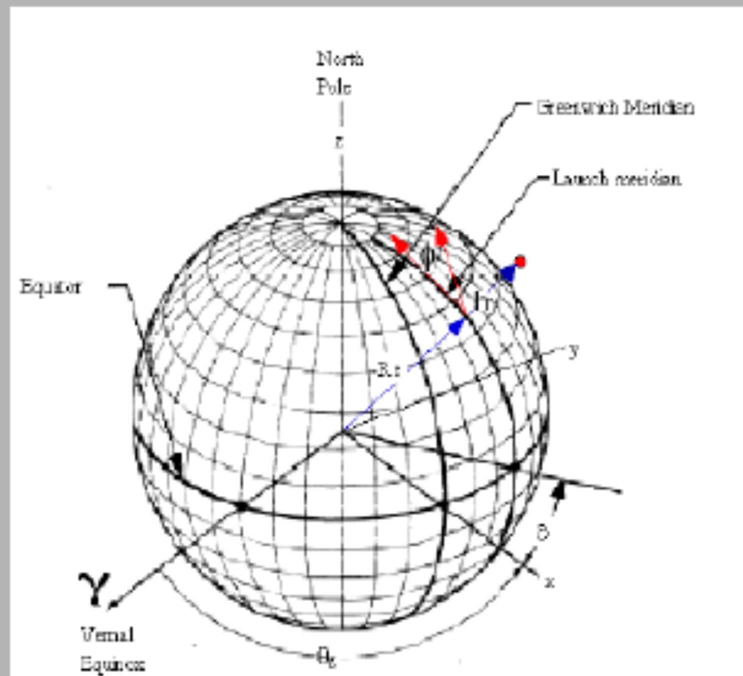
Rocket Info

Rocket Mass (kg): 0.00

Propellant Mass (kg): 0.00

Flight Path Angle, γ (d: 0.00

Earth Relative Launch Velocity, Vo (km/s): 8.00



Launch Initial Conditions

Vr (km/s): 8.010419

Vv (km/s): 0.000140

Inclination, i (deg): 97.43

Rlaunch, Rl (km): 6373.248569

True Anomaly, ν (d: 0.04

Initial Mass (kg): 0.00

Inertial Coordinates

X_I (km): 779.497392

Y_I (km): -5546.412139

Z_I (km): 3041.051396

Launch Velocities

Vnorth (km/s: 8.000000

Veast (km/s): 0.408426

Vdown (km/s: -0.000140

Inertial Velocities

Vxi: -0.126793

Vyi: 3.836841

Vzi: 7.030604

Vr: 0.00

Vnu: 8.01

This VI calculates Launch Initial Conditions from user inputs for launch site and rocket data.

Launch Site Info

Latitude (deg)
28.50

Longitude (deg)
-82.00

Launch Altitude, h (kn
25.00

Launch Azimuth, ϕ (d
60.00

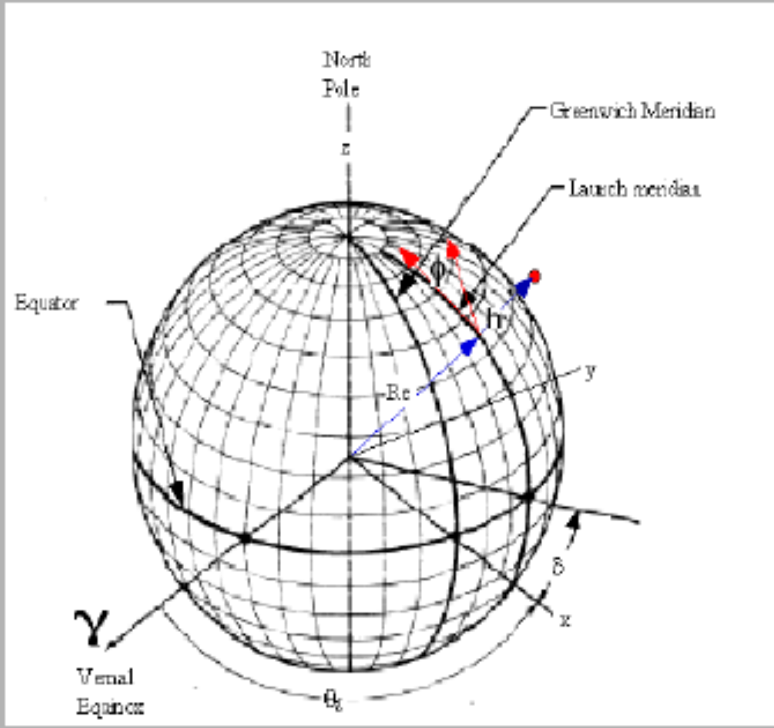
Rocket Info

Rocket Mass (kg)
0.00

Propellant Mass (kg)
0.00

Flight Path Angle, γ (d
0.00

Earth Relative Launch Velocity, Vo (km/s)
7.80



Launch Initial Conditions

Vr (km/s)
8.156264

Vv (km/s)
0.000136

Inclination, i (deg)
39.48

Rlaunch, Rl (km)
6373.248569

True Anomaly, ν (d
0.02

Initial Mass (kg)
0.00

Inertial Coordinates

X_I (km)
779.497392

Y_I (km)
-5546.412139

Z_I (km)
3041.051386

Launch Velocities

Vnorth (km/s
3.900000

Veast (km/s)
7.163424

Vdown (km/s
-0.000136

Inertial Velocities

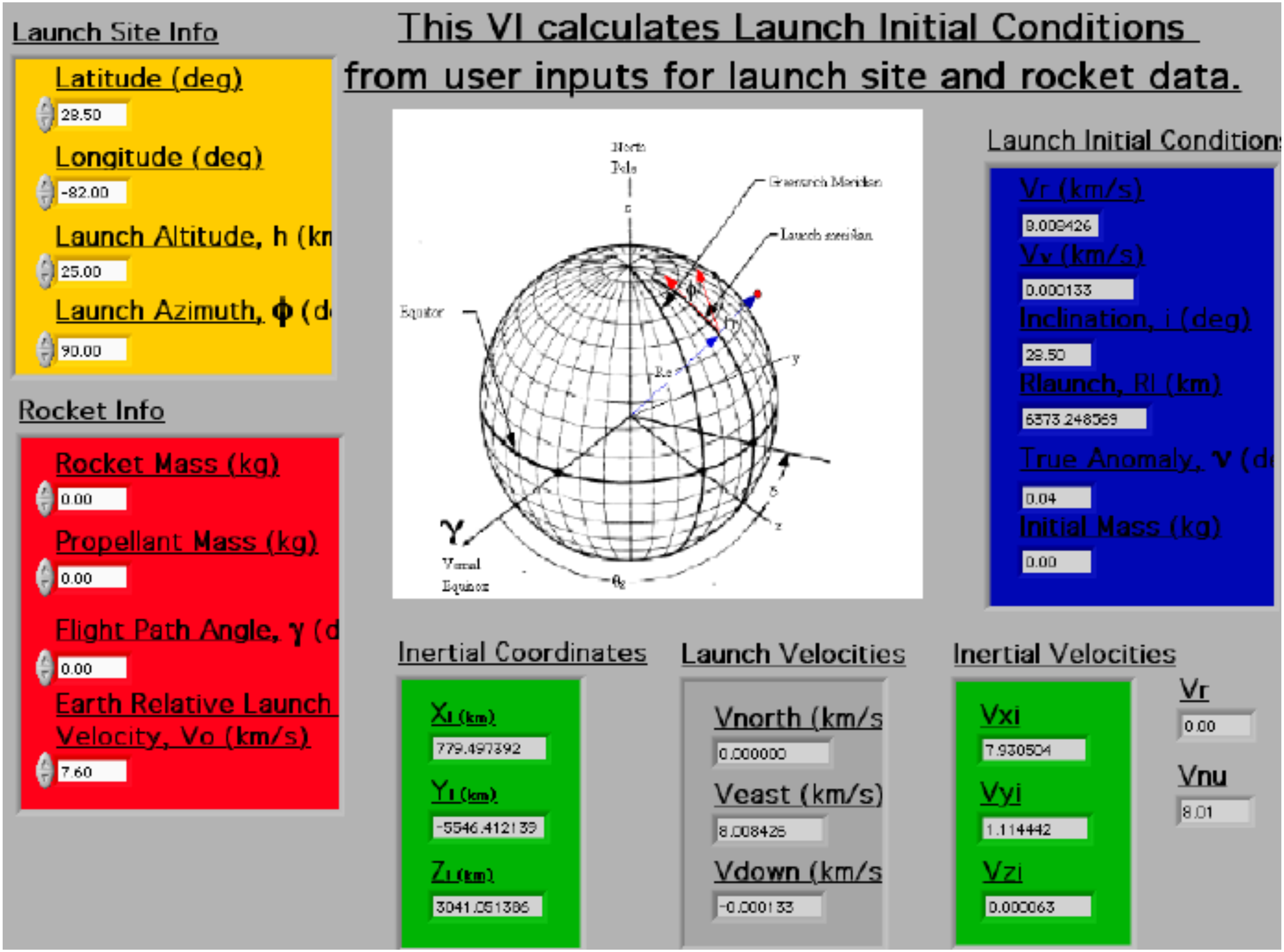
Vxi
6.834736

Vyi
2.839646

Vzi
3.427452

Vr
0.00

Vnu
8.16



This VI calculates Launch Initial Conditions
from user inputs for launch site and rocket data.
Launch Site Info
Latitude (deg)
28.50
Longitude (deg)
-82.00
Launch Altitude, h (kn
25.00
Launch Azimuth, φ (d
90.00
Rocket Info
Rocket Mass (kg)
0.00
Propellant Mass (kg)
0.00
Flight Path Angle, γ (d
0.00
Earth Relative Launch
Velocity, Vo (km/s)
7.60
North Pole
Greenwich Meridian
Launch meridian
Equator
Vernal Equinox
Launch Initial Condition
Vr (km/s)
8.008426
Vv (km/s)
0.000133
Inclination, i (deg)
28.50
Rlaunch, Rl (km)
6373.248569
True Anomaly, ν (d
0.04
Initial Mass (kg)
0.00
Inertial Coordinates
XI (km)
779.497392
YI (km)
-5546.412139
ZI (km)
3041.051386
Launch Velocities
Vnorth (km/s
0.000000
Veast (km/s)
8.008426
Vdown (km/s
-0.000133
Inertial Velocities
Vxi
7.930504
Vyi
1.114442
Vzi
0.000063
Vr
0.00
Vnu
8.01

This VI calculates Launch Initial Conditions from user inputs for launch site and rocket data.

Launch Site Info

Latitude (deg): 28.50

Longitude (deg): -82.00

Launch Altitude, h (km: 25.00

Launch Azimuth, ϕ (d: 120.00

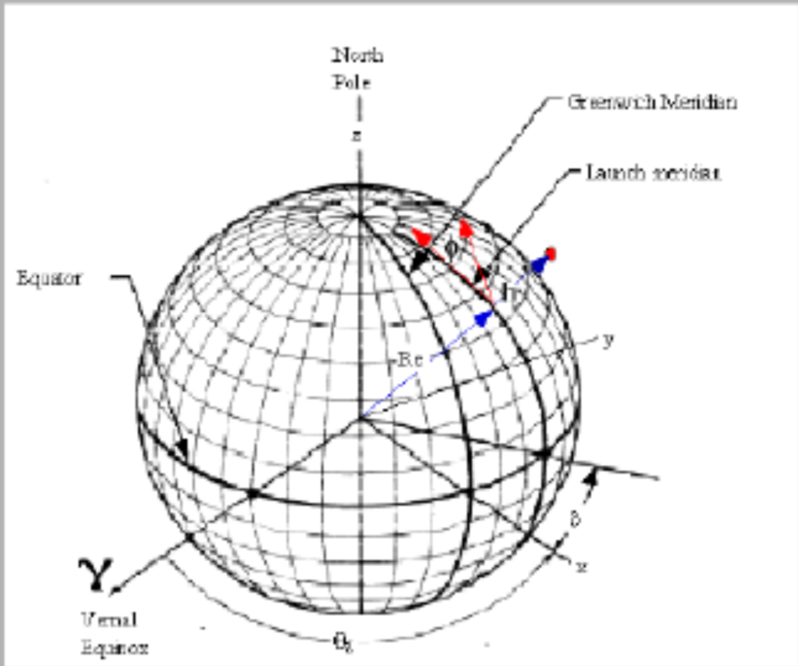
Rocket Info

Rocket Mass (kg): 0.00

Propellant Mass (kg): 0.00

Flight Path Angle, γ (d: 0.00

Earth Relative Launch Velocity, Vo (km/s): 7.80



Launch Initial Conditions

Vr (km/s): 8.156264

Vv (km/s): 0.000136

Inclination, i (deg): 39.48

Rlaunch, Rl (km): 6373.248569

True Anomaly, ν (d: 0.02

Initial Mass (kg): 0.00

Inertial Coordinates

XI (km): 779.497392

YI (km): -5546.412139

ZI (km): 3041.051386

Launch Velocities

Vnorth (km/s: -3.900000

Veast (km/s): 7.163424

Vdown (km/s: -0.000136

Inertial Velocities

Vxi: 7.352716

Vyi: -0.845971

Vzi: -3.427322

Vr: 0.00

Vnu: 8.16

This VI calculates Launch Initial Conditions from user inputs for launch site and rocket data.

Launch Site Info

Latitude (deg)
28.50

Longitude (deg)
-82.00

Launch Altitude, h (kn
25.00

Launch Azimuth, ϕ (d
180.00

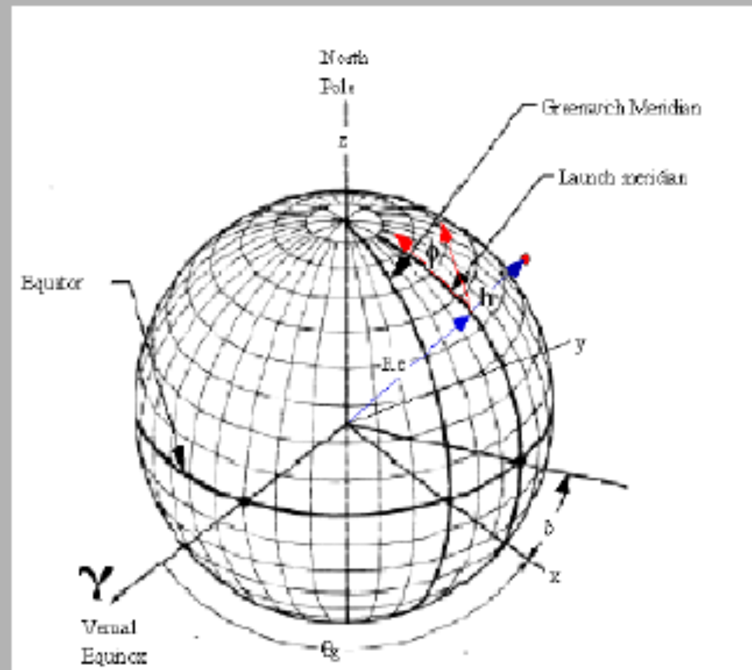
Rocket Info

Rocket Mass (kg)
0.00

Propellant Mass (kg)
0.00

Flight Path Angle, γ (d
0.00

Earth Relative Launch Velocity, Vo (km/s)
8.00



Launch Initial Conditions

Vr (km/s)
8.010419

Vv (km/s)
0.000140

Inclination, i (deg)
87.43

Rlaunch, Rl (km)
6373.248569

True Anomaly, ν (d
0.04

Initial Mass (kg)
0.00

Inertial Coordinates

Xi (km)
779.497392

Yi (km)
-5546.412139

Zi (km)
3041.051396

Launch Velocities

Vnorth (km/s
-8.000000

Veast (km/s)
0.409426

Vdown (km/s
-0.000140

Inertial Velocities

Vxi
0.935729

Vyi
-3.723400

Vzi
-7.030470

Vr
0.00

Vnu
8.01