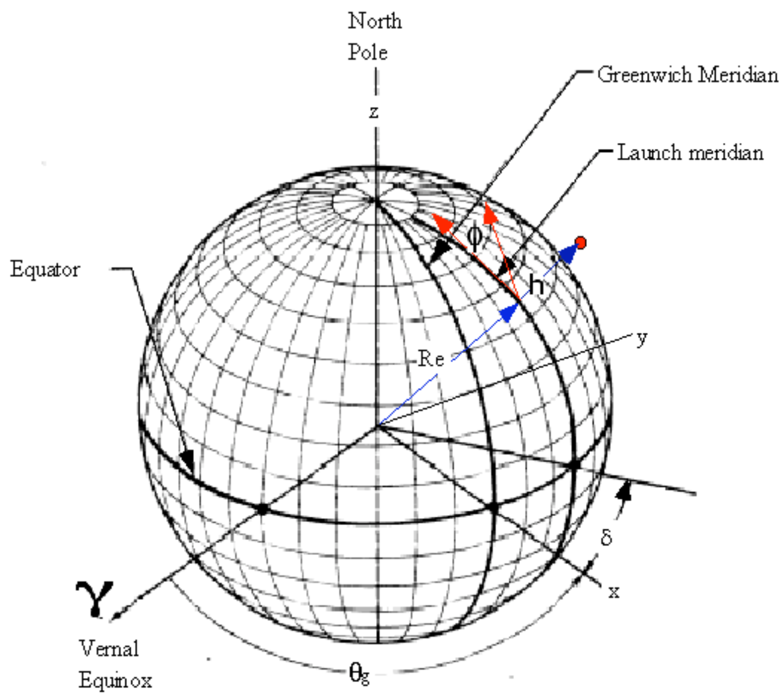
Appendix to Section 3: A brief overview of Geodetics



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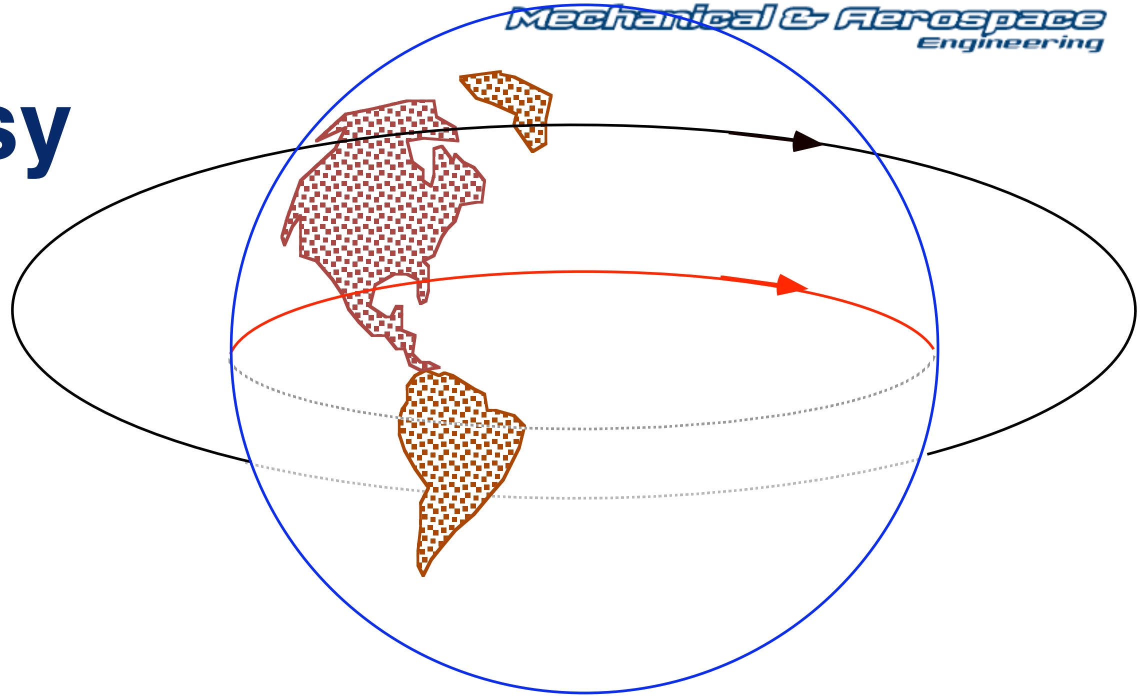
Geodesy

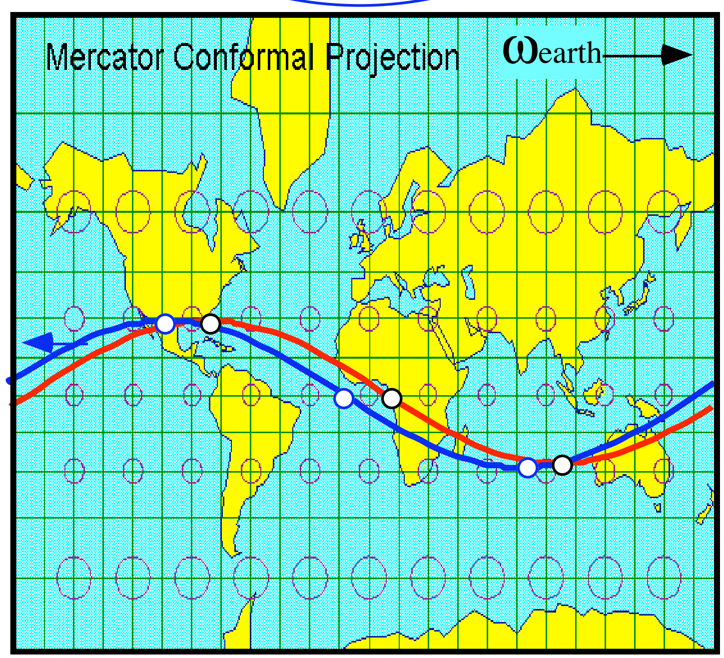


- Navigation Geeks do Calculations in Geocentric (spherical) Coordinates

- Map Makers Give Surface Data in Terms of Geodetic (elliptical) Coordinates

- Need to have some idea how to relate one to another

-- science of geodesy



How Does the Earth Radius Vary with Latitude?

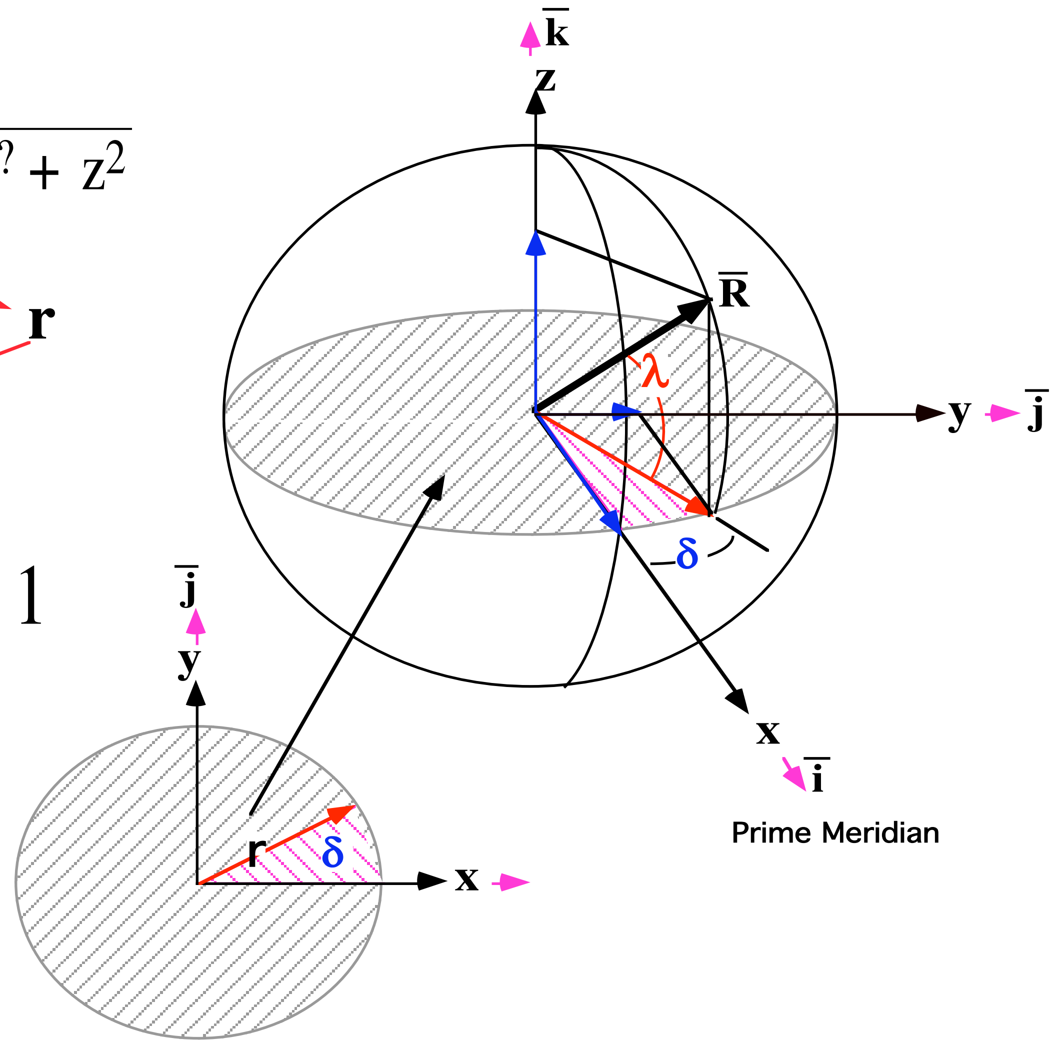
$$R_{Earth} = \sqrt{x^{?} + y_2 + z^2} = \sqrt{r^{?} + z^2}$$

r

Ellipse:

$$\left[\frac{r}{R_{eq}}\right]^2 + \left[\frac{z}{R_{eq}\sqrt{1 - e^2_{Earth}}}\right]^2 = 1$$

"a" "b"



How Does the Earth Radius vary with Latitude?

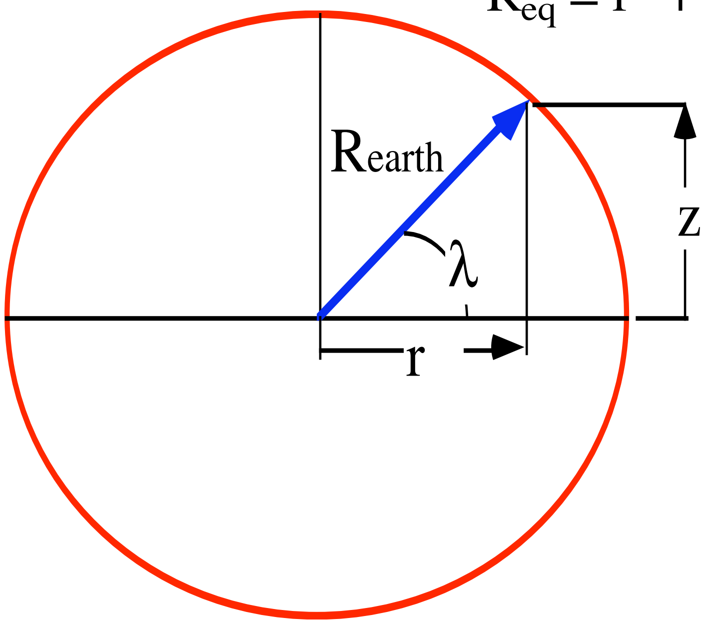
$$\left[\frac{r}{R_{eq}}\right]^2 + \left[\frac{z}{R_{eq}\sqrt{1 - e^2_{Earth}}}\right]^2 = 1$$

$$\Downarrow$$

$$\left[1 - e^2_{Earth}\right] r^2 + z^2 = R^2_{eq}\left[1 - e^2_{Earth}\right]$$

$$\Downarrow$$

$$R^2_{eq} = r^2 + \frac{z^2}{\left[1 - e^2_{Earth}\right]} = r^2\left[1 + \frac{\left[\frac{z}{r}\right]^2}{\left[1 - e^2_{Earth}\right]}\right]$$



$$r^2 = R^2_{earth}\cos^2[\lambda]$$

$$\left[\frac{z}{r}\right]^2 = \tan^2[\lambda]$$

How Does the Earth Radius vary with Latitude?

$$\frac{R_{eq}^2}{R_{earth}^2} = \cos^2[\lambda]\left[1 + \frac{\tan^2[\lambda]}{[1 - e_{Earth}^2]}\right] =$$

$$\left[\frac{[1 - e_{Earth}^2]\cos^2[\lambda] + \sin^2[\lambda]}{[1 - e_{Earth}^2]}\right] =$$

Inverting ....

$$\frac{\cos^2[\lambda] + \sin^2[\lambda] - e_{Earth}^2\cos^2[\lambda]}{[1 - e_{Earth}^2]} = \frac{1 - e_{Earth}^2\cos^2[\lambda]}{[1 - e_{Earth}^2]}$$

$$\frac{R_{earth(\lambda)}}{R_{eq}} = \sqrt{\frac{1 - e_{Earth}^2}{1 - e_{Earth}^2\cos^2[\lambda]}}$$

Earth Radius vs Geocentric Latitude

$$\frac{R_{earth(\lambda)}}{R_{eq}} = \sqrt{\frac{1 - e^2_{Earth}}{1 - e^2_{Earth}\cos^2[\lambda]}}$$

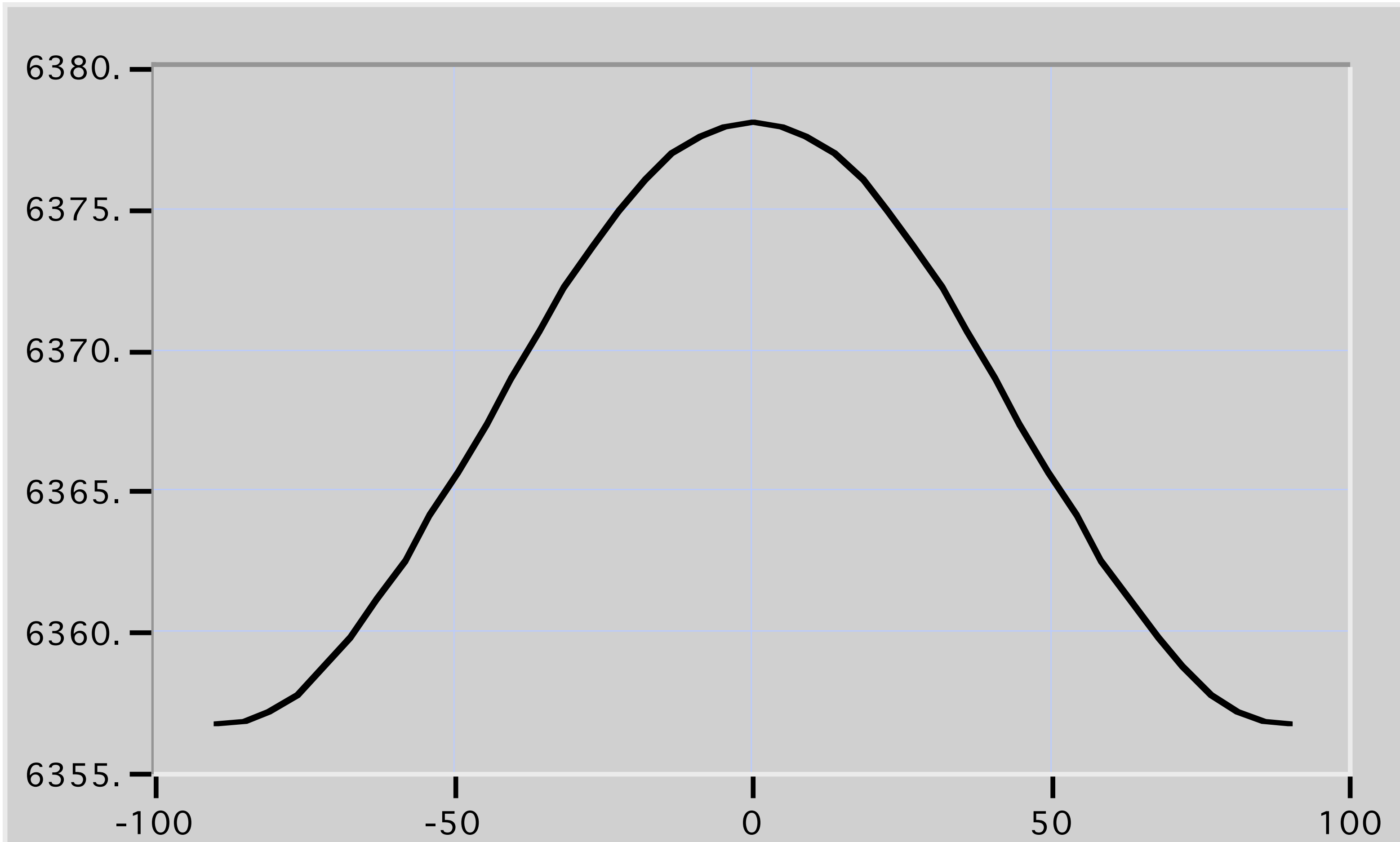
Polar Radius: 6356.75170 km

Equatorial Radius: 6378.13649 km

$$e_{Earth} = \sqrt{1 - \left[\frac{b}{a}\right]^2} = \sqrt{\frac{a^2 - b^2}{a^2}} =$$

$$\frac{\sqrt{[6378.13649]^2 - 6378.13649^2}}{[6378.13649]} = 0.08181939$$

Earth Radius vs Geocentric Latitude (concluded)



Radius, Km

Geocentric Latitude, deg.

Earth Radius … alternate formula

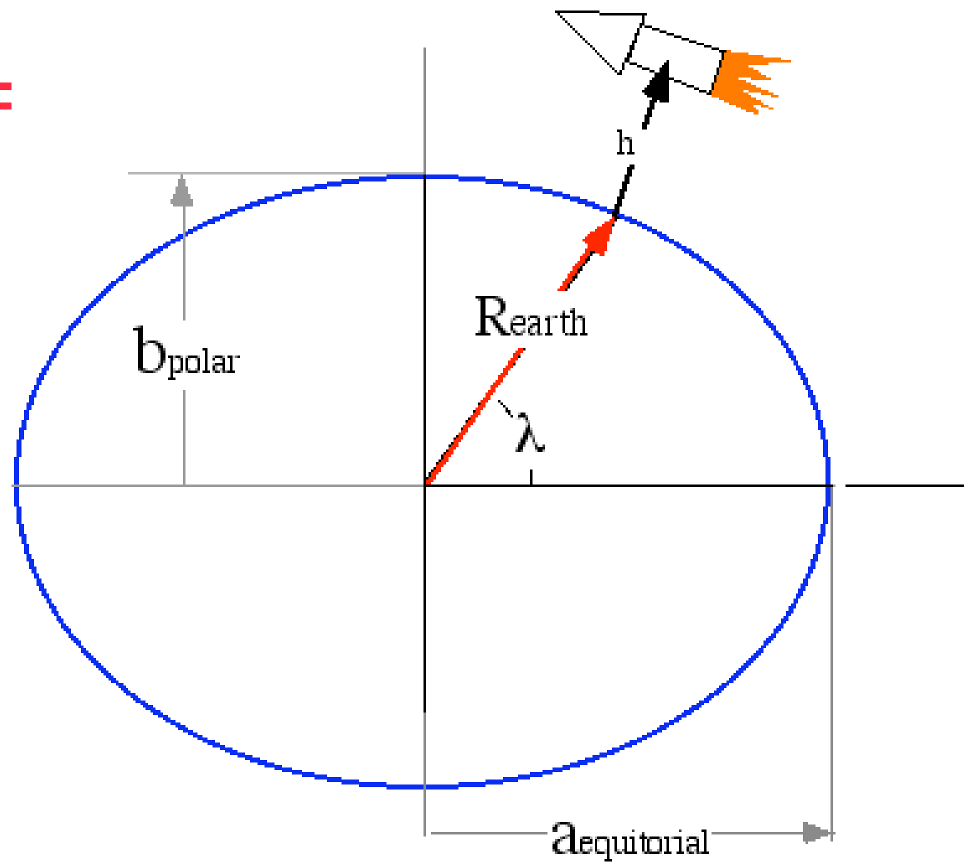
- Earth radius as Function of Latitude

$$R_{earth} = \frac{a_{equitorial}}{\sqrt{\left[1 + \frac{e^2_{earth}}{1 - e^2_{earth}} \sin^2 \lambda\right]}}$$

$$a_{equitorial} = 6378.13649 \text{ km}$$

$$b_{polar} = 6356.7515 \text{ km}$$

$$e_{earth} = \sqrt{1 - \left[\frac{b_{polar}}{a_{equitorial}}\right]^2}$$



What is the mean radius of the earth?

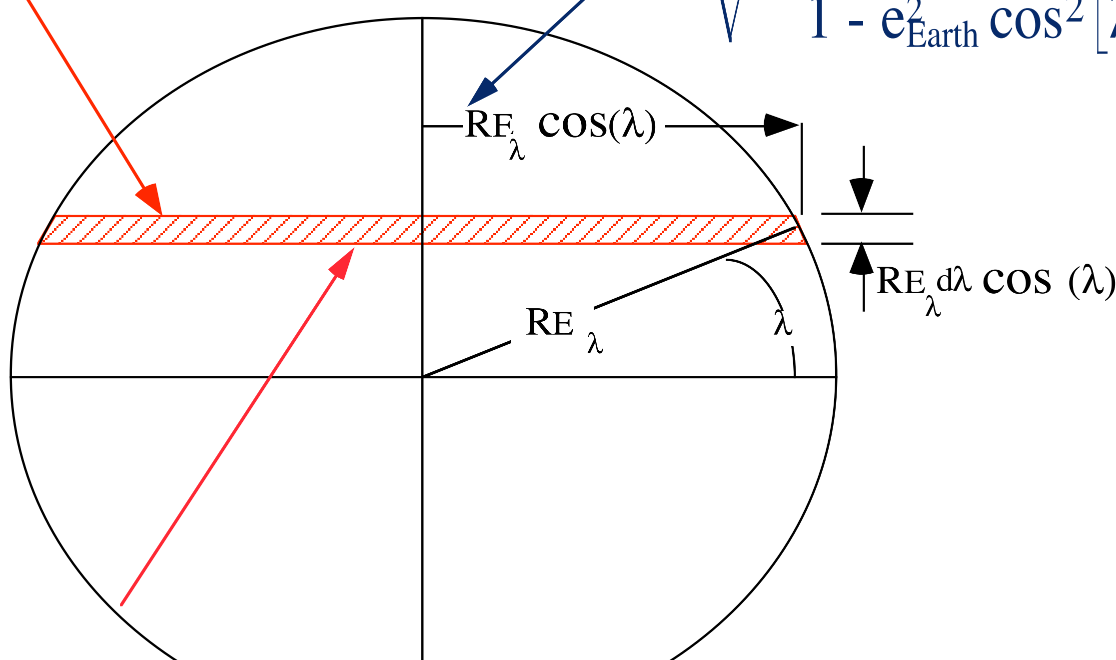
$$dA = \pi \left[R_{E_\lambda} \cos^2(\lambda) \right]$$

$$R_{E_\lambda} = R_{eq} \sqrt{\frac{1 - e_{Earth}^2}{1 - e_{Earth}^2 \cos^2[\lambda]}}$$

- **IAU Convention: Based on Earth's Volume**

Sphere Volume:

$$\frac{4\pi}{3} R_{E_{mean}}^3 = V_E$$



$$dV = \pi \left[R_E \cos(\lambda)^2 \right] \times R_{E_\lambda} d\lambda \cos(\lambda)$$

What is the Earth's Mean Radius?

(continued)

- **Earth's (ellipsoid) Volume**

$$V_E = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi \left[R_{E_\lambda} \cos(\lambda) \right]^3 d\lambda =$$

$$R_{eq}^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi \left[\frac{1 - e_{Earth}^2}{1 - e_{Earth}^2 \cos^2[\lambda]} \right]^{3/2} \cos^3(\lambda) d\lambda =$$

$$\boxed{\frac{4\pi}{3} \sqrt{1 - e_{earth}^2} \, R_{eq}^3}$$

What is the Earth's Mean Radius?
(continued)

- Based on Volume

$$\text{Ellipsoid Volume:} \frac{4\pi}{3}\sqrt{1-e^2}\; R_{eq}^3$$

$$\text{Sphere Volume:} \frac{4\pi}{3}\; R_{sphere}^3$$

$$\Downarrow$$

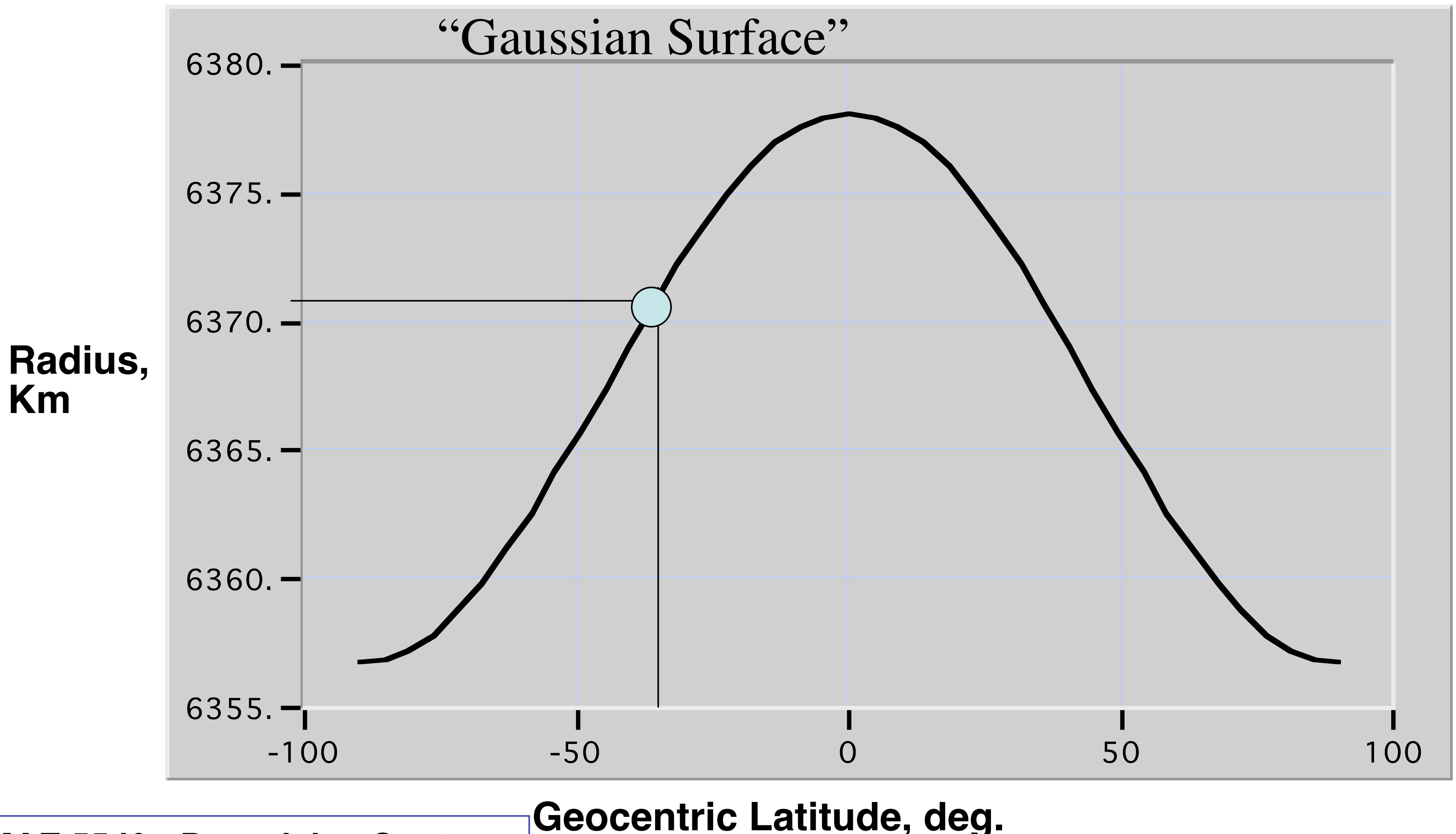
$$R_{sphere} \approx R_{mean} = \left[1 - 0.0818193 9^2\right]^{1/6} 6378.13649 = \boxed{6371.0002 \text{ km}}$$

- ***Mean Radius*** **we have been using is for a Sphere with same volume as the Earth**

$$M_E = \rho_E V_E$$

"gravitational radius"

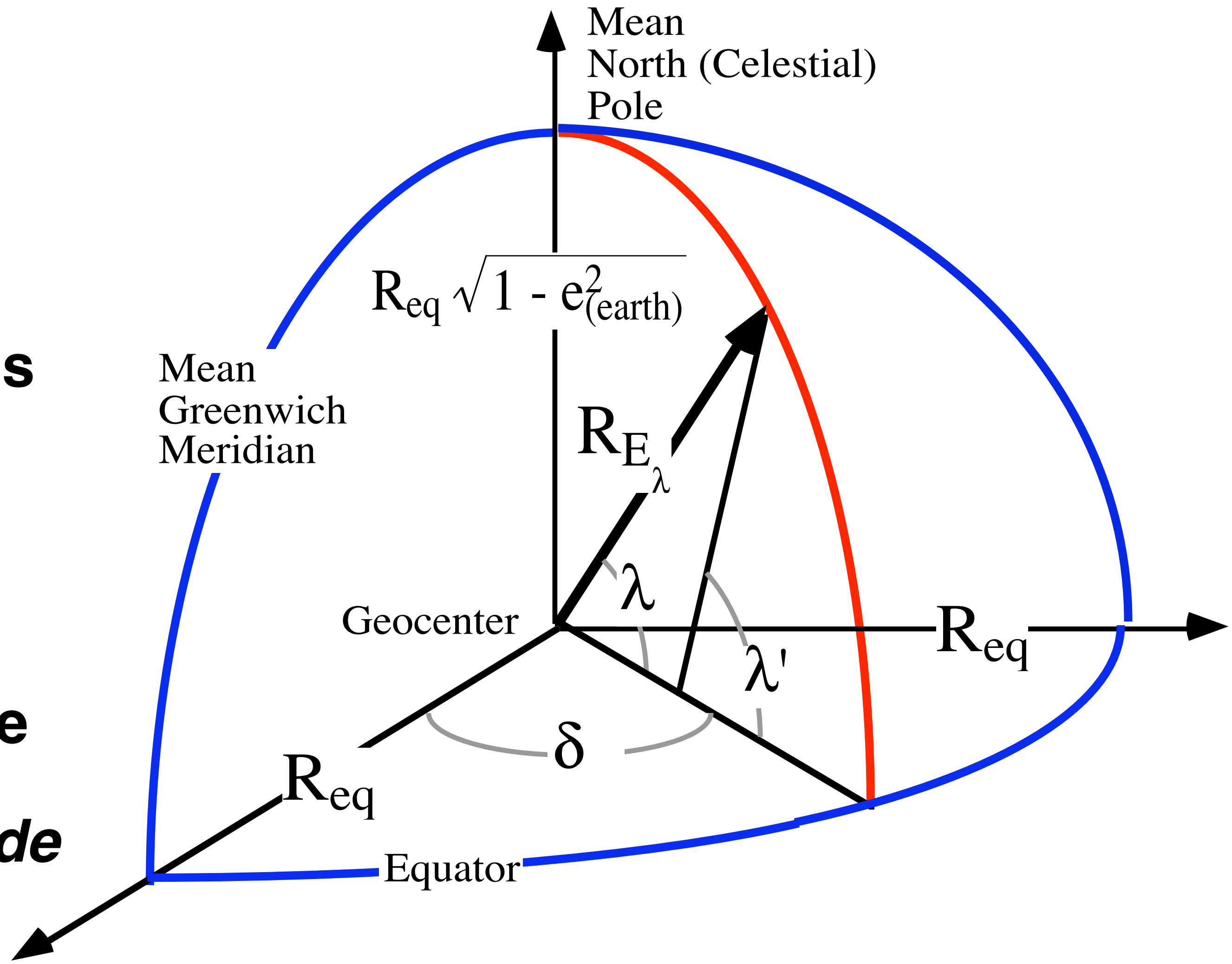
Earth Radius vs Geocentric Latitude (concluded)



Geocentric vs Geodetic Coordinates

- **Map makers define a new latitude which is the angle that normal to the Earth's surface makes with the respect to the equatorial plane**

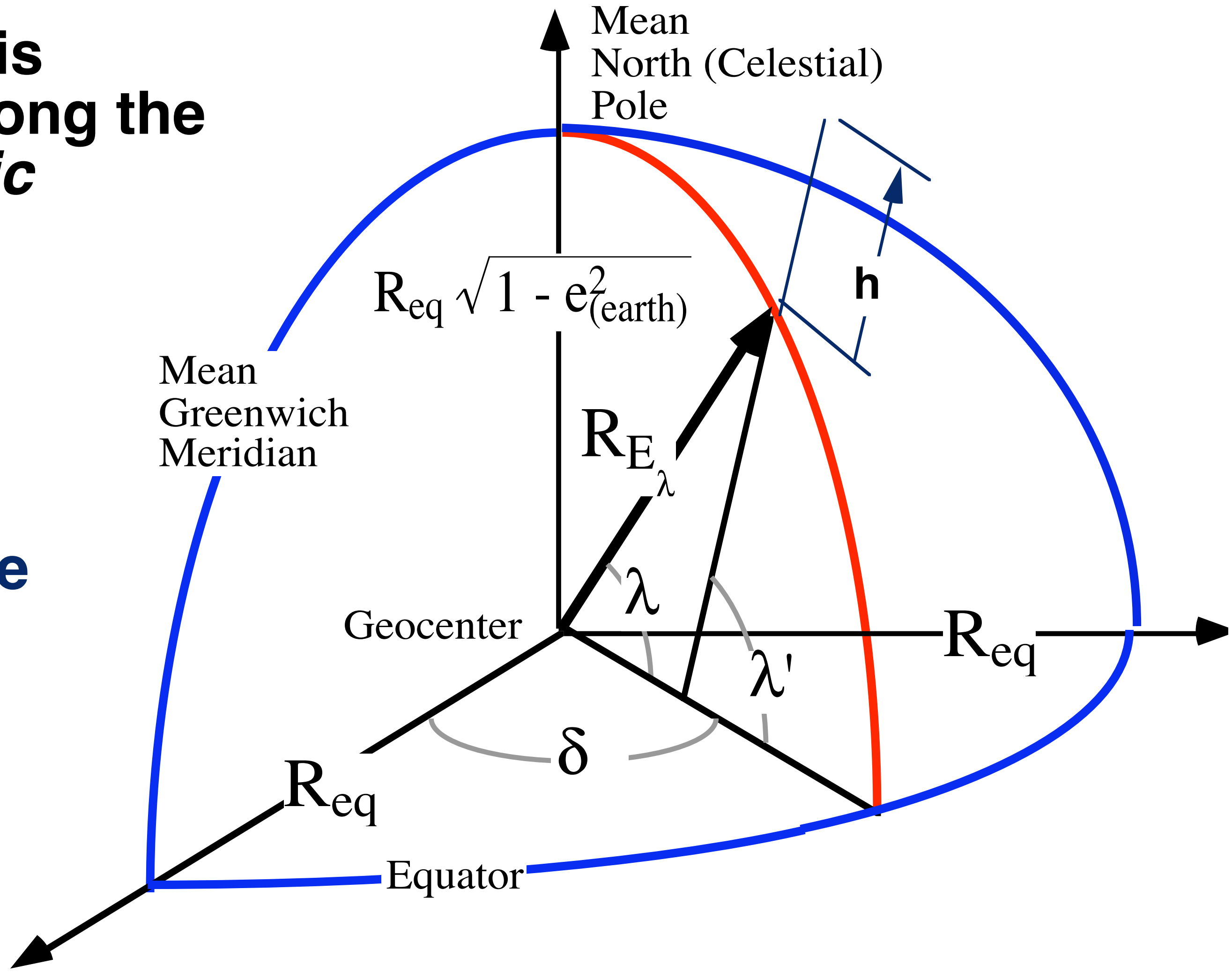
- ***Geodetic latitude***



Geocentric vs Geodetic Coordinates *(contined)*

- **Since the Earth is Elliptical only along the z-axis ... *geodetic* and *geocentric* *longitude* are identical**

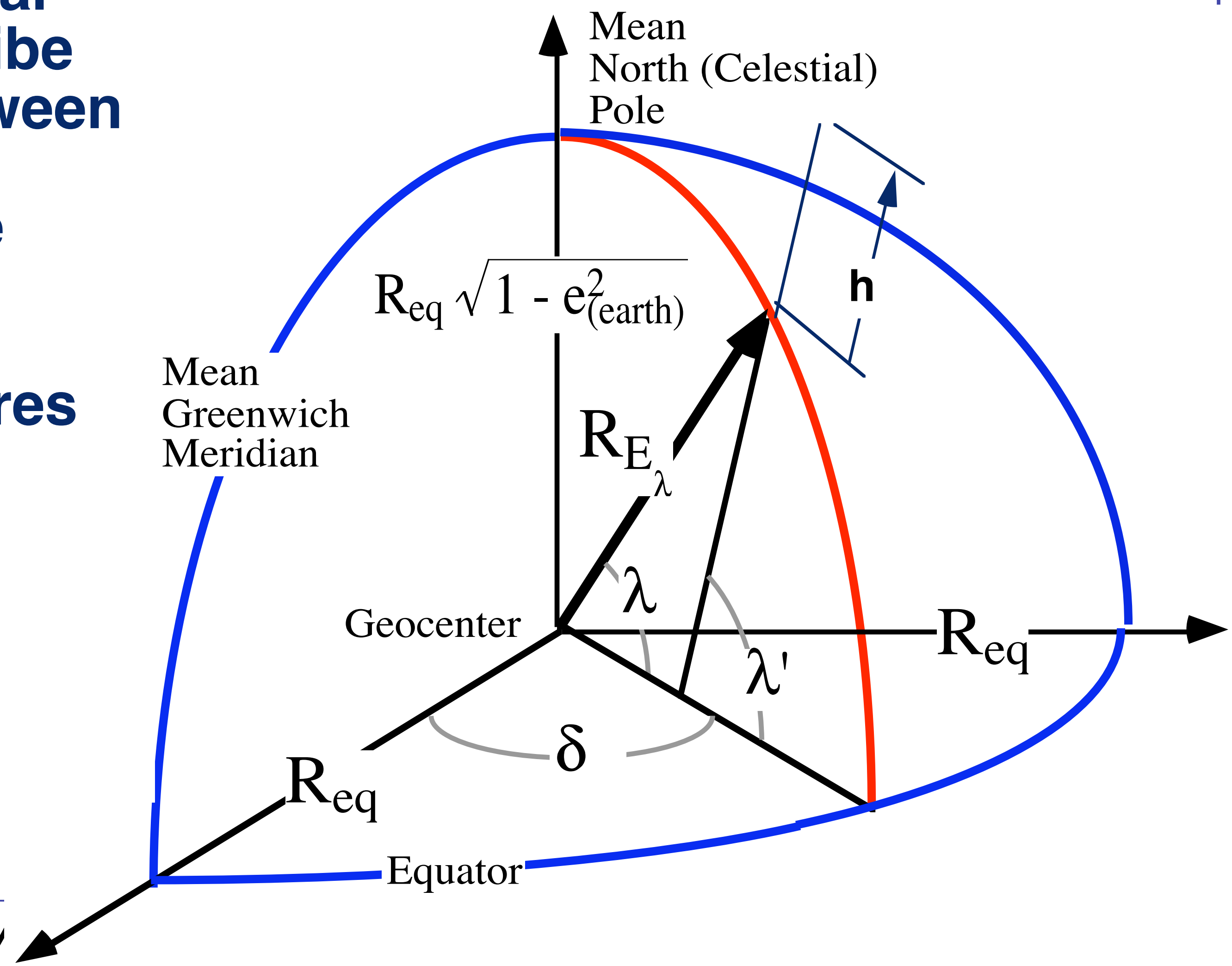
- **Altitude is an extension of the *line of latitude* geodetic)**



Geocentric vs Geodetic Coordinates *(contined)*

- Complex nonlinear equations describe relationship between geocentric and geodetic latitude

- Derivation requires Extensive Knowledge of Spherical Trigonometry



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Geocentric vs Geodetic Coordinates *(contined)*

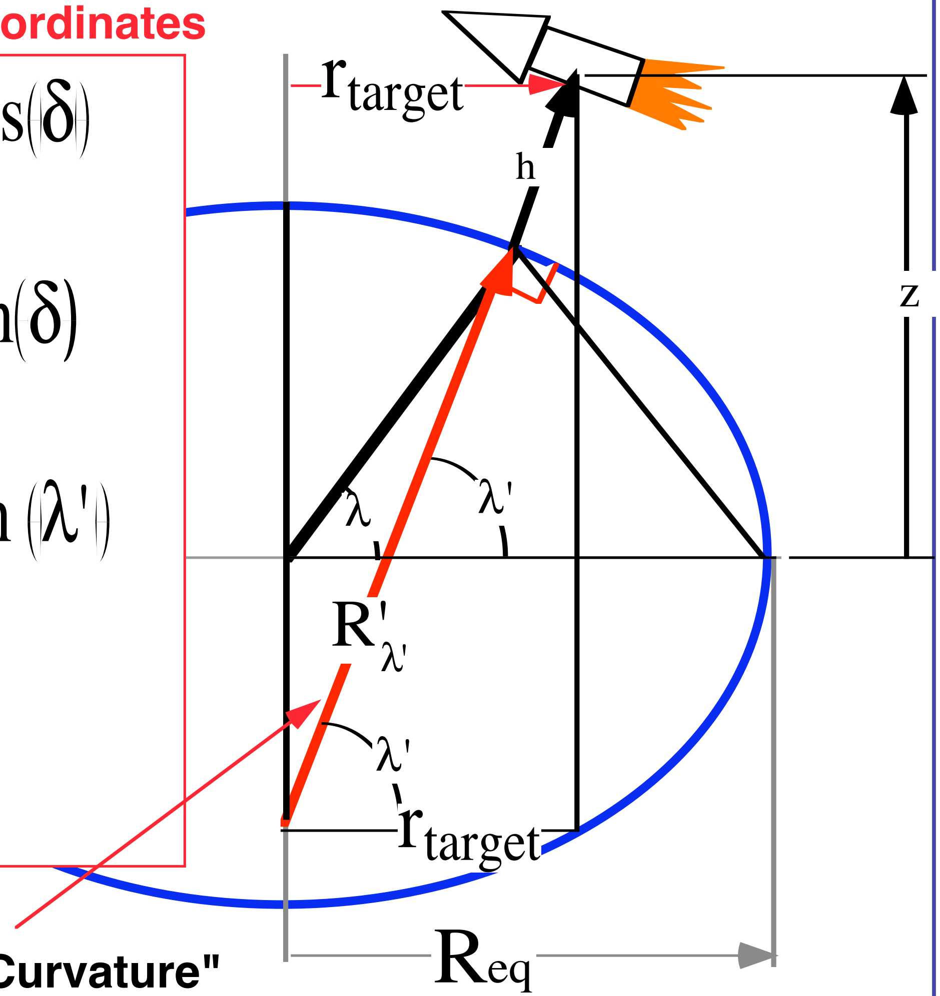
Geocentric Cartesian Coordinates

$$x_{target} = \left[R'_{\lambda'} + h\right] \cos(\lambda') \cos(\delta) \quad *$$

$$y_{target} = \left[R'_{\lambda'} + h\right] \cos(\lambda') \sin(\delta)$$

$$z_{target} = \left[R'_{\lambda'}\left[1 - e^2_{earth}\right] + h\right] \sin(\lambda')$$

$$R'_{\lambda'} = \frac{R_{eq}}{\sqrt{1 - e^2_{earth} \sin^2(\lambda')}}$$



"Radius of Curvature"

* We would be here all week if I try to derive this

Whitmore, Stephen A., and Haering, Edward A., Jr., FORTRAN Program for the Analysis of Ground Based Range Tracking Data--Usage and Derivations, NASA TM 104201, December, 1992

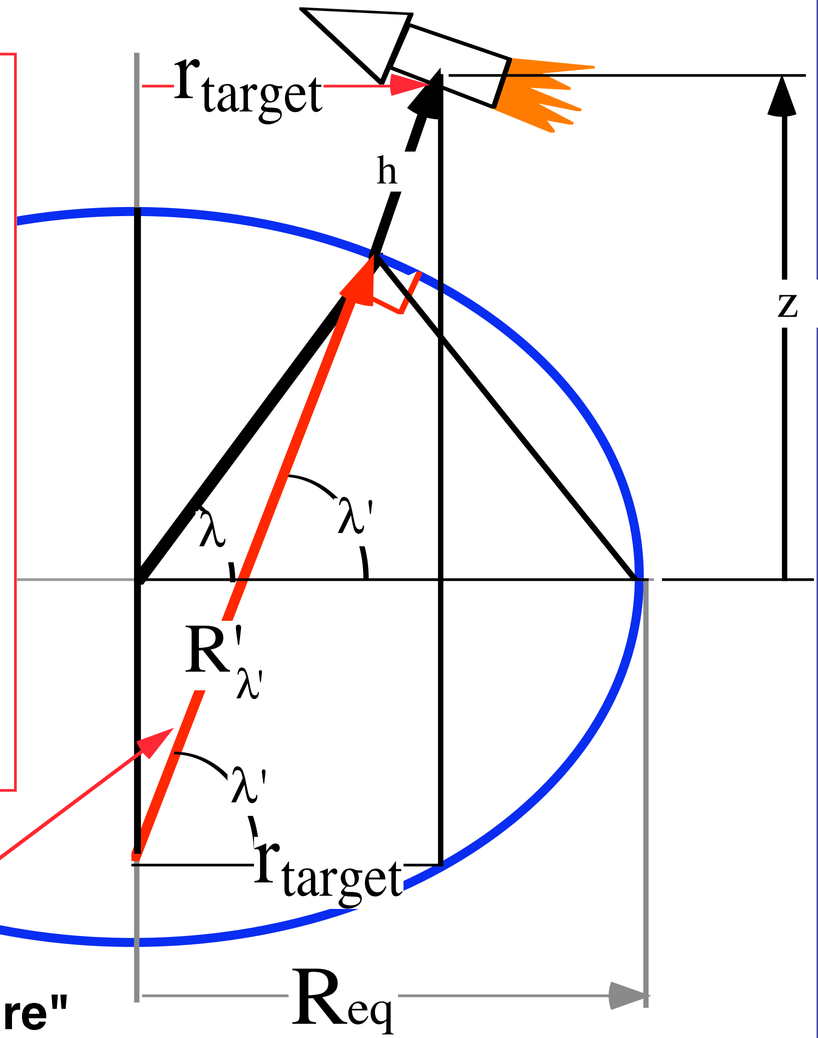
Geocentric vs Geodetic Coordinates *(contined)*

Geocentric Polar Coordinates

$$R_{target} = \sqrt{x_{target}^2 + y_{target}^2 + z_{target}^2}$$

$$\delta_{target} = \tan^{-1}\left[\frac{y_{target}}{x_{target}}\right]$$

$$\lambda_{target} = \tan^{-1}\left[\frac{z_{target}}{\sqrt{x_{target}^2 + y_{target}^2}}\right]$$



"Radius of Curvature"

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Geocentric vs Geodetic Coordinates *(concluded)*

Inverse Relationships, non-linear no direct solution

$$h = \frac{\sqrt{x_{target}^2 + y_{target}^2}}{\cos(\lambda')} - R'_{\lambda'}$$

$$\delta_{target} = \tan^{-1}\left[\frac{y_{target}}{x_{target}}\right]$$

$$\lambda'_{target} = \tan^{-1}\left[\frac{z_{target}}{\sqrt{x_{target}^2 + y_{target}^2}} \times \left[\frac{1}{1 - e_{earth}^2 \frac{R'_{\lambda'}}{R'_{\lambda'} + h_{target}}}\right]\right]$$

Whitmore, Stephen A., and Haering, Edward A., Jr., FORTRAN Program for the Analysis of Ground Based Range Tracking Data--Usage and Derivations, NASA TM 104201, December, 1992

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Pulling it all together

- **Given geodetic coordinates -- compute geocentric**

i) Compute geocedtric cartesian coordinates

Range, runway threshholds, radar antennae, beacon ....

$$x_{target} = \left[R'_{\lambda'} + h\right] \cos\left(\lambda'\right) \cos\left(\delta\right)$$

$$y_{target} = \left[R'_{\lambda'} + h\right] \cos\left(\lambda'\right) \sin\left(\delta\right)$$

$$z_{target} = \left[R'_{\lambda'}\left[1 - e^2_{earth}\right] + h\right] \sin\left(\lambda'\right)$$

$$R'_{\lambda'} = \frac{R_{eq}}{\sqrt{1 - e^2_{earth} \sin^2\left(\lambda'\right)}}$$

Pulling it all together (continued)

- **Given geodetic coordinates -- compute geocentric**

ii) Compute Geocentric polar coordinates next

$$R_{target} = \sqrt{x_{target}^2 + y_{target}^2 + z_{target}^2}$$

$$\delta_{target} = \tan^{-1}\left[\frac{y_{target}}{x_{target}}\right]$$

$$\lambda_{target} = \tan^{-1}\left[\frac{z_{target}}{\sqrt{x_{target}^2 + y_{target}^2}}\right]$$

Pulling it all together (concluded)

- **Given geocentric (usually x,y,z) coordinates -- compute geodetic**

GPS, INS, TLE's ....

$$R'_{\lambda'} = \frac{R_{eq}}{\sqrt{1 - e^2_{earth} \sin^2(\lambda')}}$$

No explicit solution: requires
1) series expansion solution,
2) numerical iteration,
3) or a special solution called "Ferrari's method"**

$$h = \frac{\sqrt{x_{target}^2 + y_{target}^2}}{\cos(\lambda')} - R'_{\lambda'}$$

$$\delta_{target} = \tan^{-1}\left[\frac{y_{target}}{x_{target}}\right]$$

$$\lambda'_{target} = \tan^{-1}\left[\frac{z_{target}}{\sqrt{x_{target}^2 + y_{target}^2}} \times \left[\frac{1}{1 - e^2_{earth}\dfrac{R'_{\lambda'}}{R'_{\lambda'} + h_{target}}}\right]\right]$$

****NASA Technical Paper 3430, Whitmore and Haering, FORTRAN Program forAnalyzing Ground-Based Tracking Data: Usage and Derivations, Version 6.2, 1995**

Numerical Example

- Edwards Air Force Base, Radar Site #34

$$\lambda' = 34.96081°$$

$$\delta = -117.91150°$$

$$h = 2563.200 \text{ ft}$$

- Find corresponding geocentric cartesian and polar coordinates

Numerical Example (cont'd)

- **Compute Local Radius of Curvature**

$$R'_{\lambda'} = \frac{R_{eq}}{\sqrt{1 - e^2_{earth} \sin^2(\lambda')}} =$$

$$\frac{6378.13649 \text{ km}}{\sqrt{1 - \left[0.08181939 \ \sin\left(34.96081 \times \frac{\pi}{180}\right)\right]^2}} =$$

$$6392.187109 \text{ km}$$

Numerical Example (cont'd)

- **Compute X and Y (geocentric)**

$$r_{target} = \left[R'_{\lambda'} + h\right]\cos\left(\lambda'\right) =$$

$$\left[6392.1871 + \left(2536.2 \times 3.048 \times 10^{-4}\right)\right]\cos\left(34.96081 \times \frac{\pi}{180}\right) =$$

5239.3131 km

$$x_{target} = r_{target}\cos(\delta) =$$

$$5239.3131 \text{ km} \times \cos\left(-117.91150 \times \frac{\pi}{180}\right) =$$

-2452.5602 km

$$y_{target} = r_{target}\sin(\delta) =$$

$$5216.0074 \text{ km} \times \sin\left(-117.91150 \times \frac{\pi}{180}\right) =$$

-4629.83218 km

Numerical Example (cont'd)

- **Compute z (geocentric)**

$$z_{target} = \left[R'_{\lambda'}\left[1 - e^2_{earth}\right] + h\right]\sin\left(\lambda'\right) =$$

$$\left[6392.1871\,\text{km}\left[1 - 0.08181939^2\right] + \left(2536.2 \times 3.048 \times 10^{-4}\right)\right]\sin\left(34.96081 \times \frac{\pi}{180}\right) =$$

$$3638.7480\,\text{km}$$

Numerical Example (cont'd)

- **Compute Geocentric Polar Coordinates**

$$R_{target} = \sqrt{2452.5602^2 + 4629.83218^2 + 3638.7480^2} = 6378.94104 km$$

$$\delta_{target} = \tan^{-1}\left[\frac{y_{target}}{x_{target}}\right] = \frac{180}{\pi} \times \tan^{-1}\left[\frac{-4629.83218}{-2452.5602}\right] = -117.9115°$$

$$\lambda_{target} = \tan^{-1}\left[\frac{z_{target}}{\sqrt{x_{target}^2 + y_{target}^2}}\right] = \frac{180}{\pi} \times \tan^{-1}\left[\frac{3638.7480}{5239.3131}\right] = 34.7803°$$

Numerical Example (cont'd)

- Compute Local Earth Radius and Geocentric Distance Above Geoid

$$R_{E_\lambda} = R_{eq}\sqrt{\frac{1 - e^2_{Earth}}{1 - e^2_{Earth}\cos^2[\lambda]}} =$$

$$6378.13649 \text{ km}\sqrt{\frac{1 - 0.08181939^2}{1 - 0.08181939^2\cos^2\left[34.7083 \times \frac{\pi}{180}\right]}} =$$

$$6364.23 \text{ km}$$

$$D_{geoid} = R_{target} - R_{E_\lambda} =$$

$$[6378.94104 - 6364.23] \text{ km} = 14.7 \text{ km} = 48228.25 \text{ ft}$$

Numerical Example (concluded)

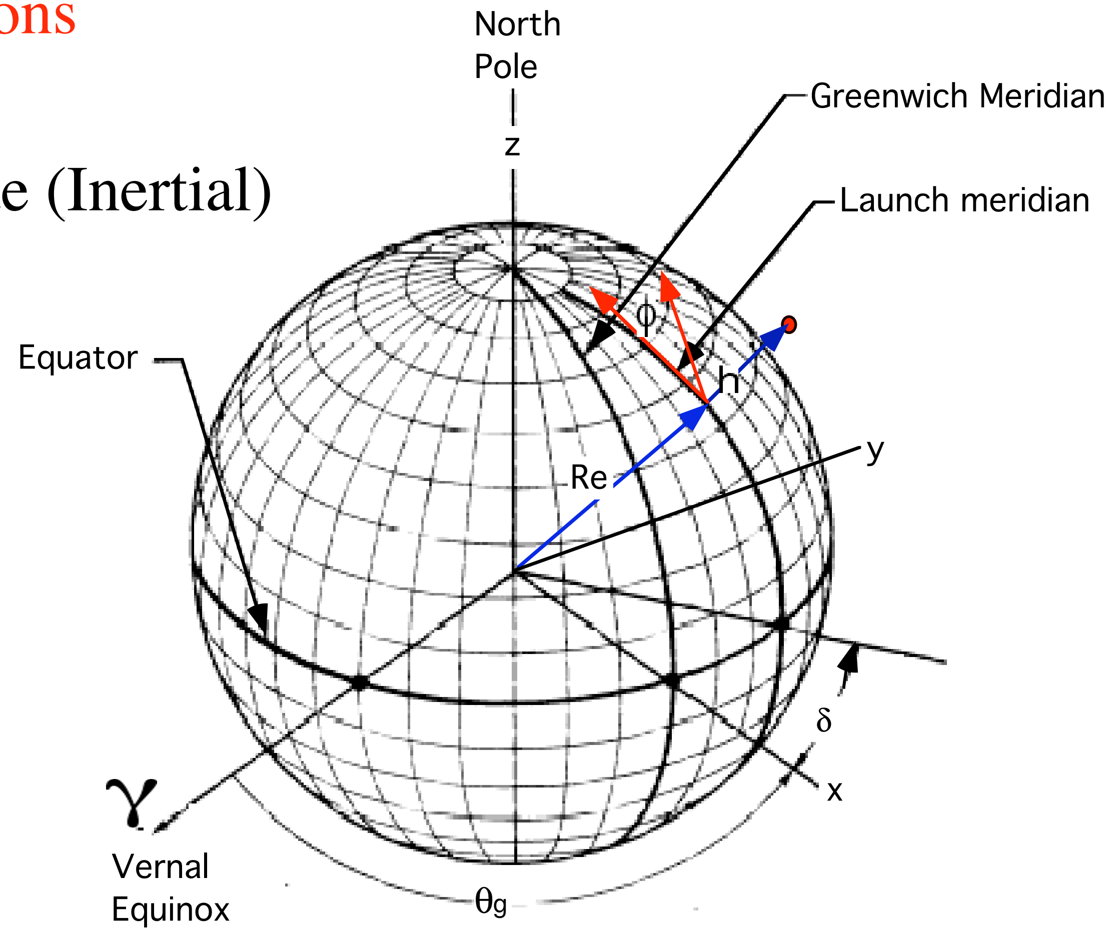
- Comparison

Geodetic	Geocentric
$\lambda' = 34.96081°$	$\lambda = 34.7803°$
$\delta = -117.91150°$	$\delta = -117.91150°$
$h = 2563.200$ ft	$D_{geoid} = 48228.25$ ft

- Earth Oblateness is NOT trivial, and in the REAL World -- it *must be* accounted for

Appendix II: Rigorous Derivation of Realizable Launch Inclination

- Launch Initial Conditions
- Position: λ, Latitude
  Ω, Longitude (Inertial)
  h, Altitude

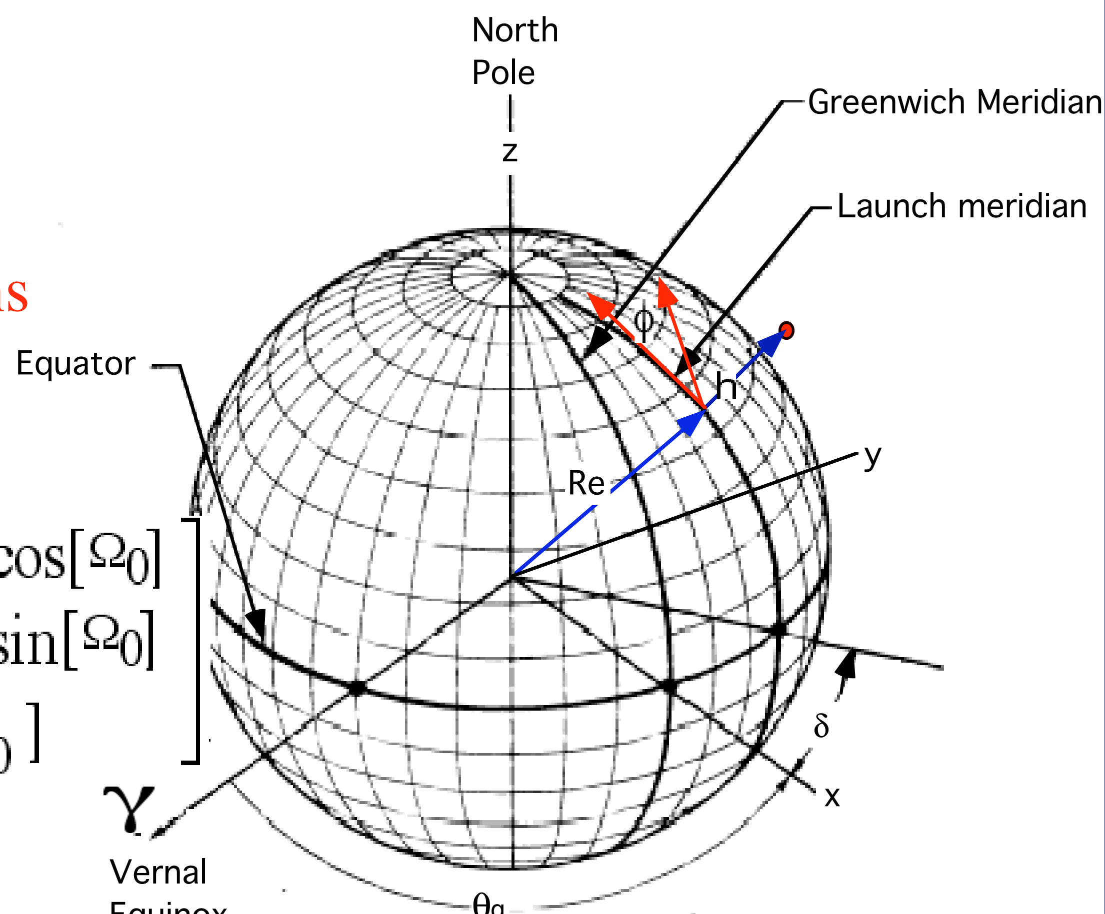


• Launch Initial Conditions

• Launch Initial Conditions (Inertial Coordinates)

$$\begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix}_0 = \begin{bmatrix} [R_{earth}+h_0]\cos[\lambda_0]\cos[\Omega_0] \\ [R_{earth}+h_0]\cos[\lambda_0]\sin[\Omega_0] \\ [R_{earth}+h_0]\sin[\lambda_0] \end{bmatrix}$$

$$\Omega_0 = \delta_0 + \theta_g$$



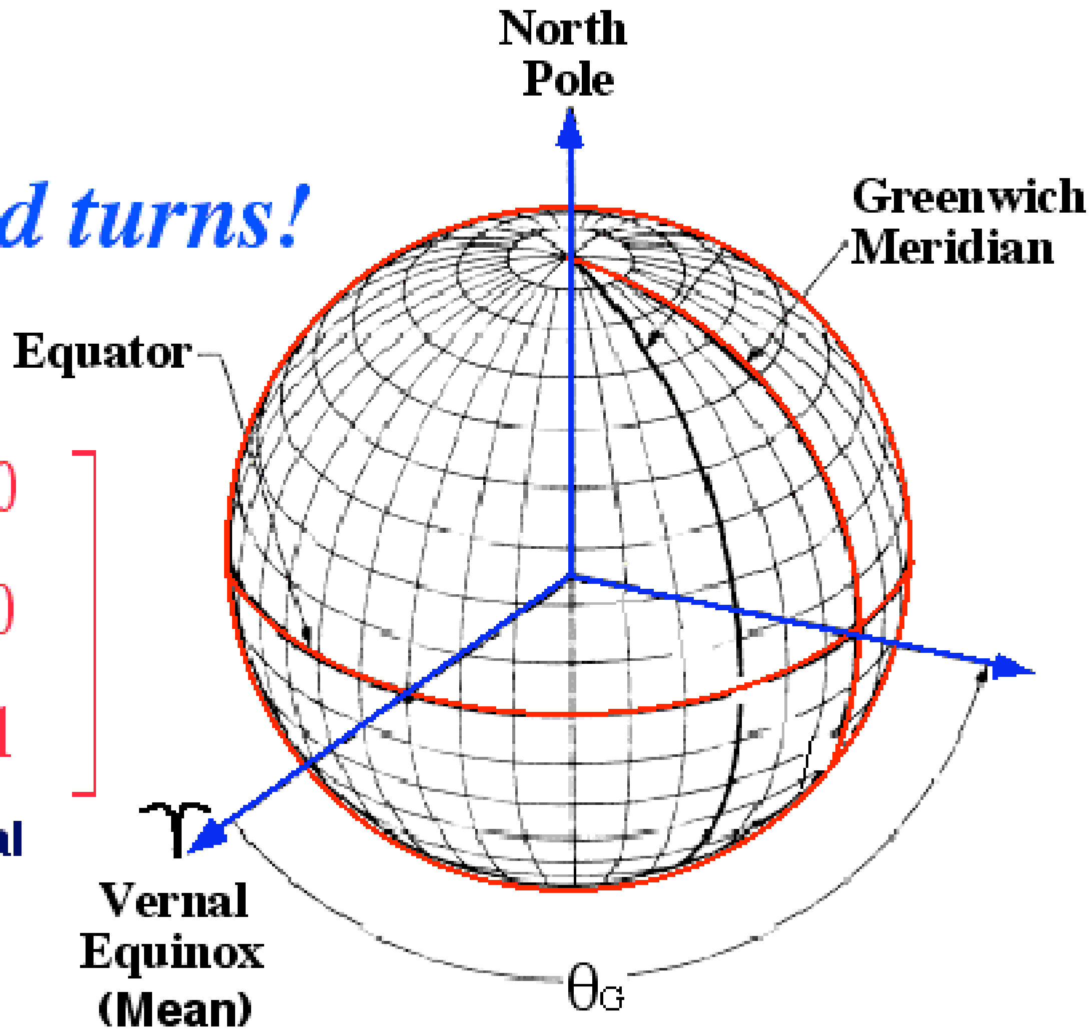
Sidereal hour angle

As the world turns!

$$M_{Rotation} = \begin{bmatrix} \cos[\theta_G] & -\sin[\theta_G] & 0 \\ \sin[\theta_G] & \cos[\theta_G] & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Greenwich Sidereal (Hour) Angle

Ignore for now!



Computing the Hour Angle

- θ_G Historically Expressed in Hours

... Sometimes referred to as Greenwich Mean Sidereal Time ... but we are going to treat it as an angle

$$\theta_G = \omega_{earth} \times \left[T_{GMST} - T_{JD2000} \right]$$

- Sidereal time is a measure of the Earth's rotation with respect to distant celestial objects.