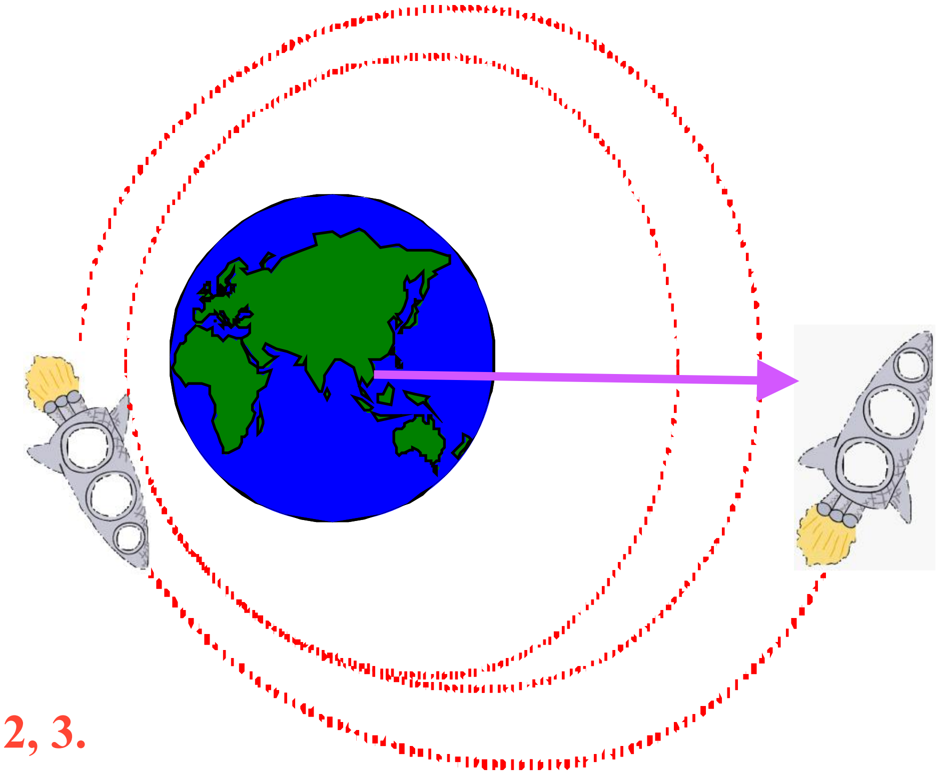
Launch Dynamics II:
2-Dimensional Equations of Motion



Taylor, Chapters 2, 3.

- ~~Sutton and Biblarz, chapter 4.~~

Real World Launch Analysis

Pegasus User's Guide

Trajectory Design Optimization

Orbital designs a ***unique mission trajectory for each Pegasus flight to maximize payload*** performance while complying with the satellite and launch vehicle constraints. Using the ***3-Degree of Freedom Program*** for Optimization of Simulation Trajectories(POST), a desired orbit is specified and a set of optimization parameters and constraints are designated. Appropriate data for mass properties, aerodynamics, and motor ballistics are input. POST then selects values for the optimization parameters that target the desired orbit with specified constraints on key parameters such as angle of attack, dynamic loading, payload thermal, and ground track. ***After POST has been used to determine the optimum launch trajectory, a Pegasus-specific six degree of Freedom simulation program*** is used to verify Trajectory acceptability with realistic attitude dynamics, including separation analysis on all stages.

Pegasus®
ORBITAL SCIENCES CORPORATION

- 6-DOF simulations Costs ***A LOT!*** To run And are typically Not used for Trajectory design!

- We are going to develop A simple ***2^+-D*** code That works well For mission profile development

Orbital Energy

- In Ideal Keplerian World ε (specific orbital energy) is constant

$$\varepsilon_{orbit} = -\frac{\mu}{2a_{orbit}}$$

- If a non-conservative force is performing work on the satellite operarting within the orbit "a", after a period of time t, the new orbit energy level is

$$\left[\varepsilon_{orbit}\right]_t = -\frac{\mu}{2a_{orbit_t}} = -\frac{\mu}{2a_{orbit_0}} + \frac{\text{Energy added}}{m_{satellite}}$$

Kepler's Laws

Kepler

• **Kepler's First Law:** *In a two body universe, orbit of a planet is a conic section with the sun (Earth) centered at one of the focii*

• **Kepler's Second Law:** *In a two body universe, radius vector from the sun (Earth) to the planet sweeps out equal areas in equal times*

• **Kepler's Third Law:** *In a two body universe, square of the period of any object revolving about the sun (Earth) is in the same ratio as the cube of its mean distance*

No Longer Apply

Orbital Dynamics

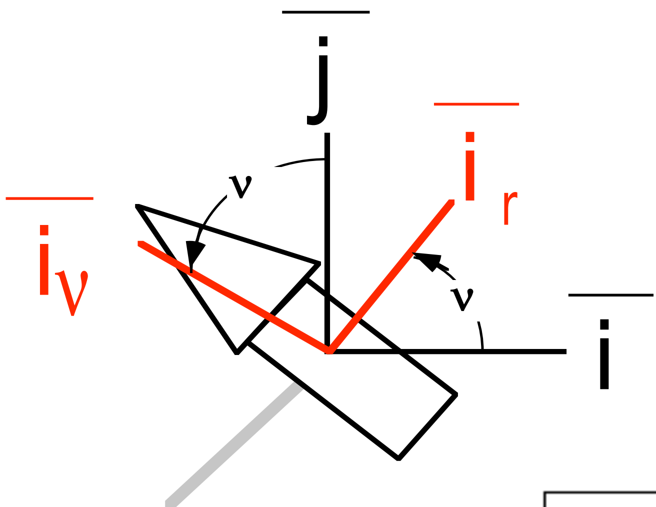
- Must resort to Newton's laws to describe these orbits

$$\overline{V} = \frac{\partial \overline{R}}{\partial t} + \overline{\omega} \times \overline{R}$$

$$\frac{\sum F_{external}}{M} = \frac{\partial \overline{V}}{\partial t} + \overline{\omega} \times \overline{V}$$

$$\dot{M}_{vehicle} = - \frac{F_{thrust}}{g_0 I_{sp}}$$

Coordinate Transformations:



{i, j} fixed in space

Transform ⇒ polar ↑ inertial

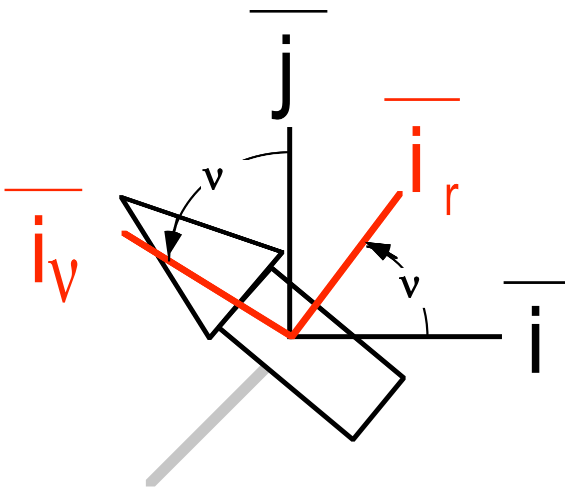
$$\bar{i} = \bar{i}_r \cos[\nu] - \bar{i}_\nu \sin[\nu]$$
$$\bar{j} = \bar{i}_r \sin[\nu] + \bar{i}_\nu \cos[\nu]$$

Transform ⇒ inertial ↑ polar

$$\bar{i}_r = \bar{i} \cos[\nu] + \bar{j} \sin[\nu]$$
$$\bar{i}_\nu = -\bar{i} \sin[\nu] + \bar{j} \cos[\nu]$$

Coordinate Transformations: (cont;d)

- A Matrix "trick" for coordinate transform in 2-D



Transform ⇒ polar ↑ inertial

$$\bar{i} = \bar{i}_r \cos[\nu] - \bar{i}_\nu \sin[\nu]$$

$$\bar{j} = \bar{i}_r \sin[\nu] + \bar{i}_\nu \cos[\nu]$$

Transform ⇒ inertial ↑ polar

$$\bar{i}_r = \bar{i} \cos[\nu] + \bar{j} \sin[\nu]$$

$$\bar{i}_\nu = -\bar{i} \sin[\nu] + \bar{j} \cos[\nu]$$

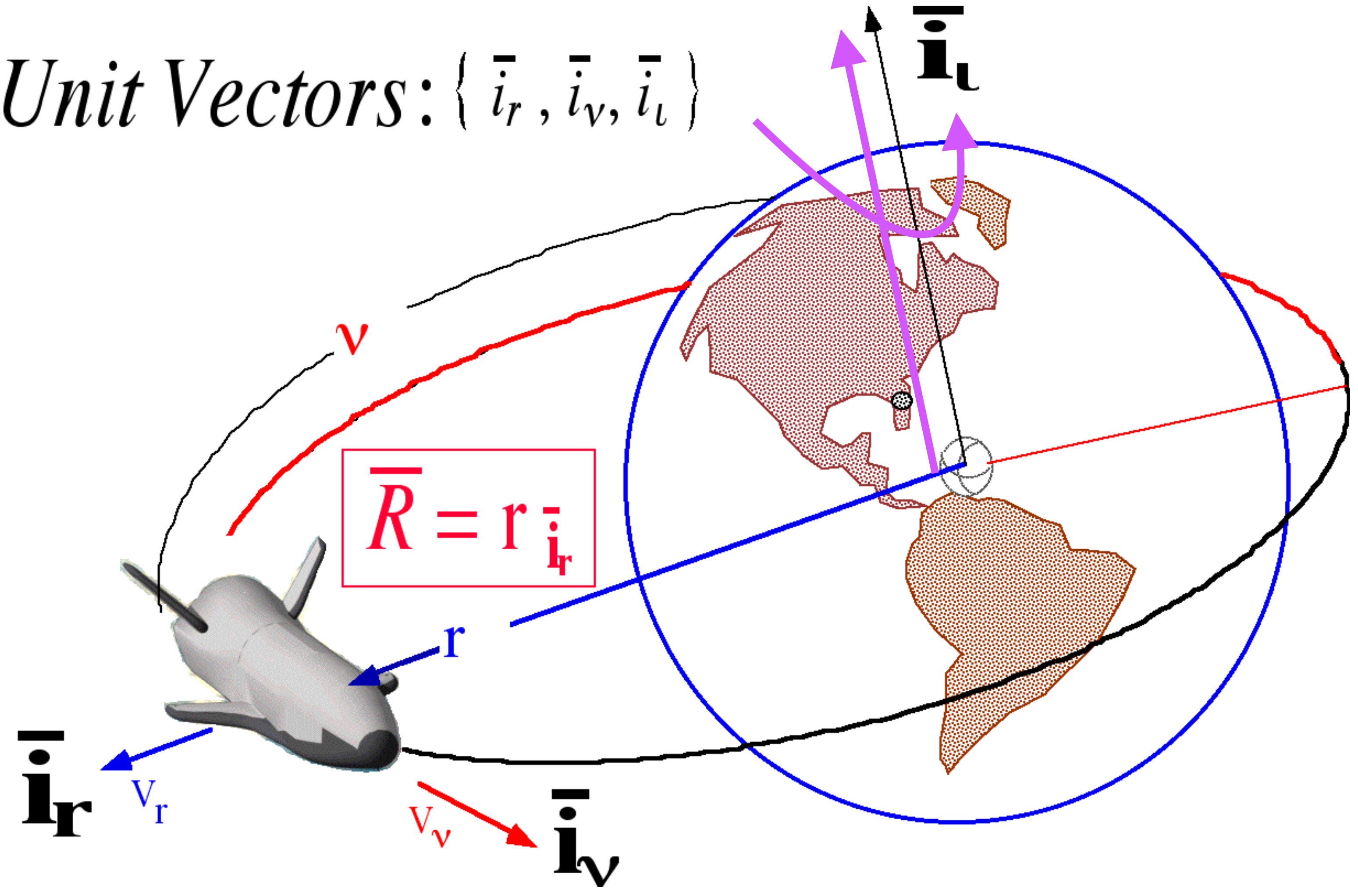
$$\begin{bmatrix} \bar{i}_r \\ \bar{i}_\nu \end{bmatrix} = \begin{bmatrix} \cos[\nu] & \sin[\nu] \\ -\sin[\nu] & \cos[\nu] \end{bmatrix} \begin{bmatrix} \bar{i} \\ \bar{j} \end{bmatrix}$$

$$\Downarrow$$

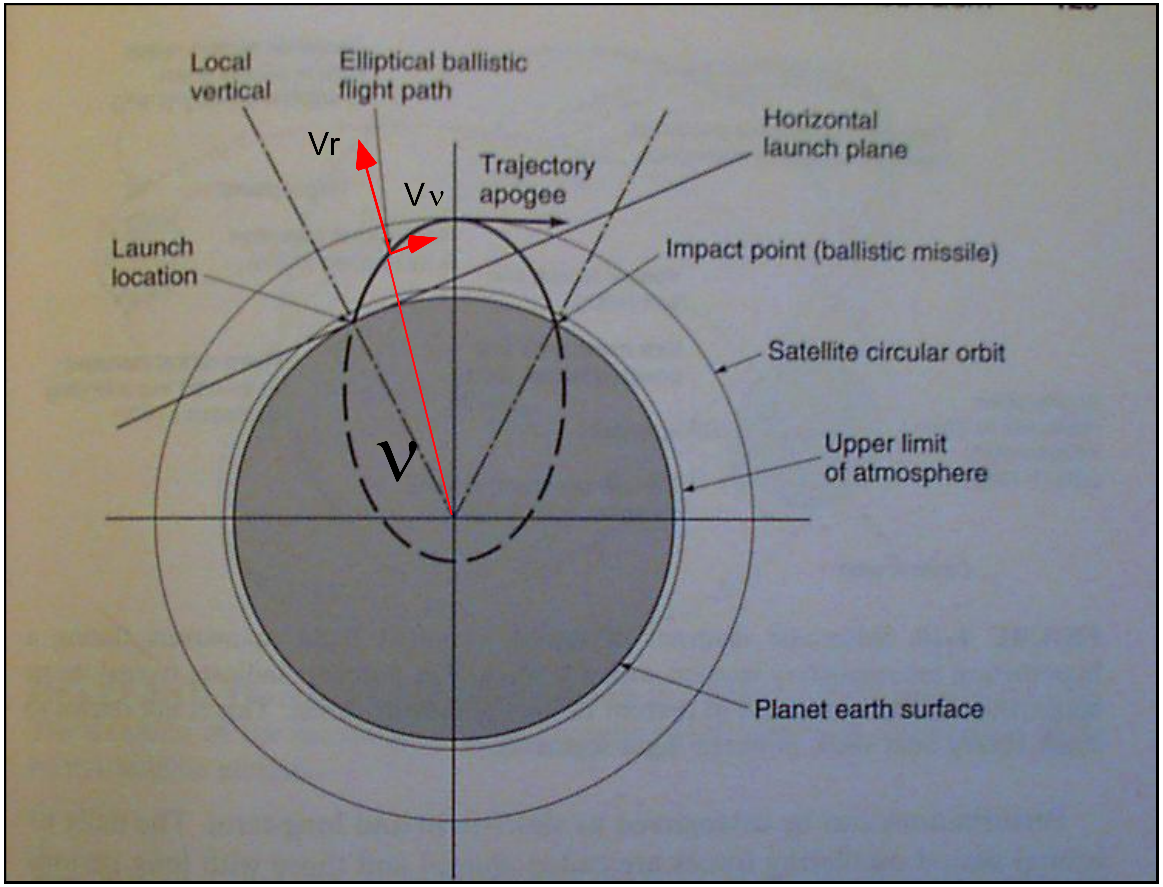
$$\begin{bmatrix} \bar{i} \\ \bar{j} \end{bmatrix} = \begin{bmatrix} \cos[\nu] & \sin[\nu] \\ -\sin[\nu] & \cos[\nu] \end{bmatrix}^T \begin{bmatrix} \bar{i}_r \\ \bar{i}_\nu \end{bmatrix} = \begin{bmatrix} \cos[\nu] & -\sin[\nu] \\ \sin[\nu] & \cos[\nu] \end{bmatrix} \begin{bmatrix} \bar{i}_r \\ \bar{i}_\nu \end{bmatrix}$$

Perifocal Coordinate System

Unit Vectors: $\{ \bar{i}_r, \bar{i}_\nu, \bar{i}_\iota \}$



Perifocal Coordinate System
Sub-orbital Image



Velocity Vector

$$\overline{V} = \frac{\partial \overline{r}}{\partial t} + \overline{\omega} \times \overline{r} = \dot{r}\,\overline{\mathbf{i}}_r + \begin{bmatrix} \overline{\mathbf{i}}_r & \overline{\mathbf{i}}_\nu & \overline{\mathbf{i}}_t \\ 0 & 0 & \dot{\nu} \\ r & 0 & 0 \end{bmatrix} =$$

$$\dot{r}\,\overline{\mathbf{i}}_r + [\dot{\nu}\,r\,]\,\overline{\mathbf{i}}_\nu = \begin{bmatrix} \dot{r} \\ \dot{\nu}\,r \\ 0 \end{bmatrix} \equiv \begin{bmatrix} V_r \\ V_\nu \\ 0 \end{bmatrix}$$

Acceleration Vector

$$\bar{A} = \frac{\partial \bar{V}}{\partial t} + \bar{\omega} \times \bar{V} = \begin{bmatrix} \dot{V}_r \\ \dot{V}_\nu \\ 0 \end{bmatrix} + \left\| \begin{bmatrix} \bar{i}_r & \bar{i}_\nu & \bar{i}_L \\ 0 & 0 & \dot{\nu} \\ V_r & V_\nu & 0 \end{bmatrix} \right\| = \begin{bmatrix} \dot{V}_r \\ \dot{V}_\nu \\ 0 \end{bmatrix} + \begin{bmatrix} -\dot{\nu} V_\nu \\ \dot{\nu} V_r \\ 0 \end{bmatrix}$$

- But from previous slide

$$\begin{bmatrix} \dot{r} \\ \dot{\nu}\, r \\ 0 \end{bmatrix} \equiv \begin{bmatrix} V_r \\ V_\nu \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} V_r \\ \frac{V_\nu}{r} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ \dot{\nu} \end{bmatrix}$$

Acceleration Vector (cont'd)

$$\bar{A} = \begin{bmatrix} \dot{V}_r \\ \dot{V}_\nu \\ 0 \end{bmatrix} + \begin{bmatrix} -\dfrac{V_\nu^{\,2}}{r} \\ \dfrac{V_\nu V_r}{r} \\ 0 \end{bmatrix}$$

- **Instantaneously**

$$\bar{F} = m\frac{d}{dt}\bar{V} \Rightarrow \bar{A} = \frac{\bar{F}}{m}$$

$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_\nu \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{V_\nu{}^2}{r} \\ \frac{V_\nu V_r}{r} \\ 0 \end{bmatrix} = \frac{\bar{F}}{m}$$

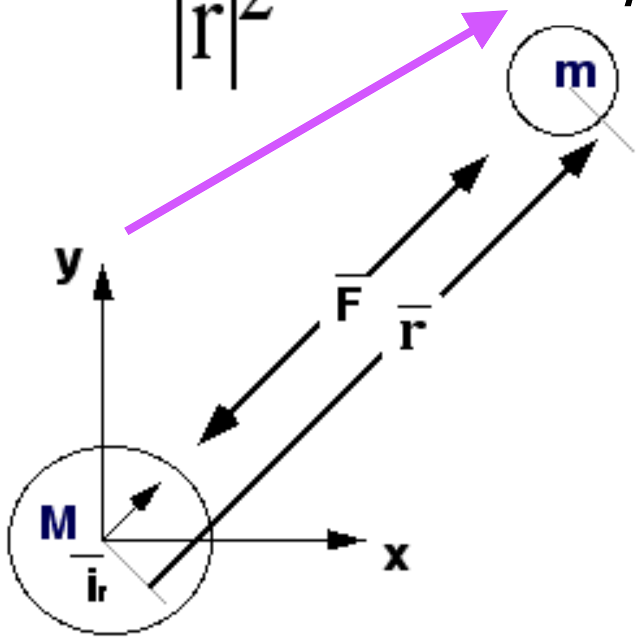
Newton's Second Law

Regrouping

$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_\nu \end{bmatrix} = \frac{\bar{F}}{m} - \begin{bmatrix} -\dfrac{V_\nu^2}{r} \\ \dfrac{V_\nu V_r}{r} \end{bmatrix}$$

Gravitational (conservative) Forces

$$\overline{F}_{grav} = - \frac{G\,M\,m}{|r|^2}\,\overline{i}_r = -\frac{\mu}{r^2}\,\overline{i}_r$$



"Inverse-square" law "potential" field

Isaac Newton, (1642-1727)

• Assume spherical earth .. Always acts in i_r direction

Vehicle Mass

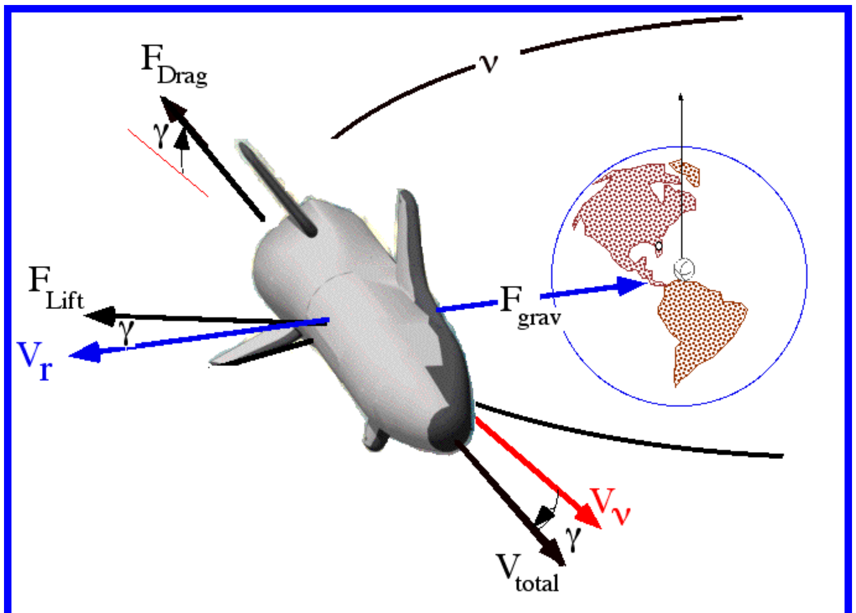
$$\dot{m}_{vehicle} = -\frac{F_{thrust}}{g_0 I_{sp}}$$

$$M_t = M_o - \int_0^t \frac{F_{thrust}}{g_o I_{sp}}$$

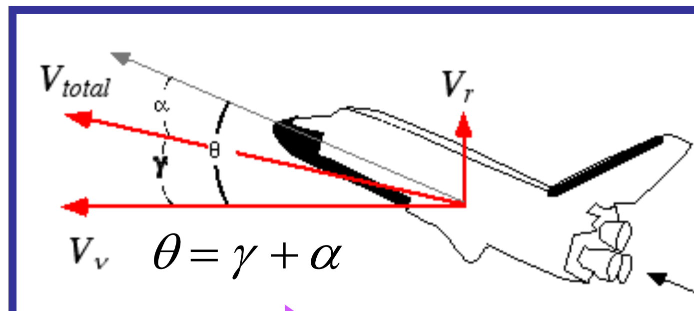
Initial mass of vehicle

Non-Conservative Forces

$$\begin{bmatrix} \dfrac{F_r}{m} \\ \dfrac{F_v}{m} \end{bmatrix} = \begin{bmatrix} \dfrac{\left(F_{lift}\cos(\gamma) - F_{drag}\sin(\gamma) + F_{thrust}\sin(\theta)\right)}{m} \\ -\dfrac{\left(F_{lift}\sin(\gamma) + F_{drag}\cos(\gamma) - F_{thrust}\cos(\theta)\right)}{m} \end{bmatrix}$$



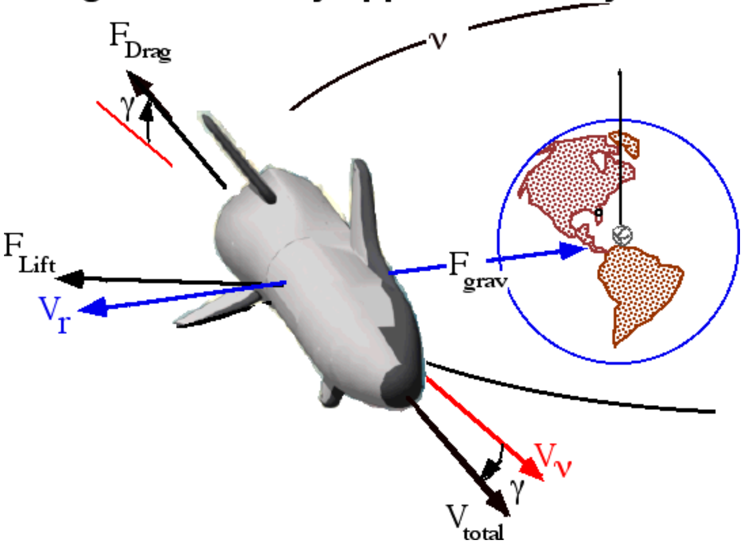
Vector Force Diagram



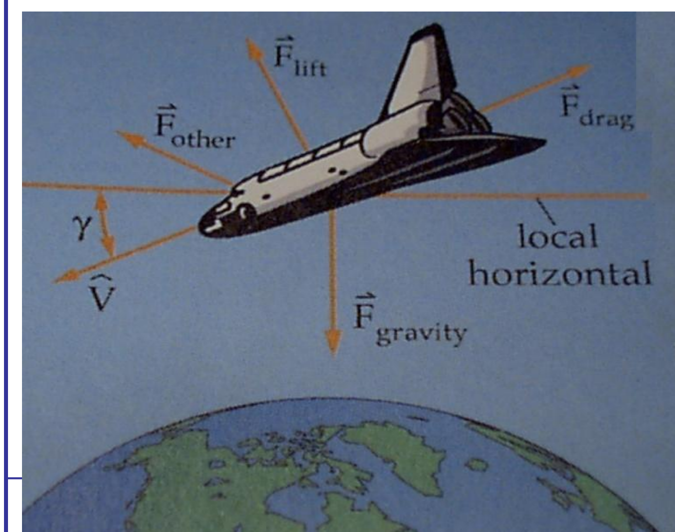
$$\gamma = \tan^{-1}\left[\frac{V_r}{V_v}\right]$$

Aerodynamic Forces

- Lift Force perpendicular to velocity vector
- Drag force directly opposes velocity vector



A_{ref} … reference
Area … planform
Or diameter based



"Dynamic Pressure"

$$F_{lift} = C_L A_{ref} \bar{q}_\infty = C_L A_{ref} \frac{1}{2} \rho_\infty V_\infty^{\;2}$$

$$F_{drag} = C_D A_{ref} \bar{q}_\infty = C_D A_{ref} \frac{1}{2} \rho_\infty V_\infty^{\;2}$$

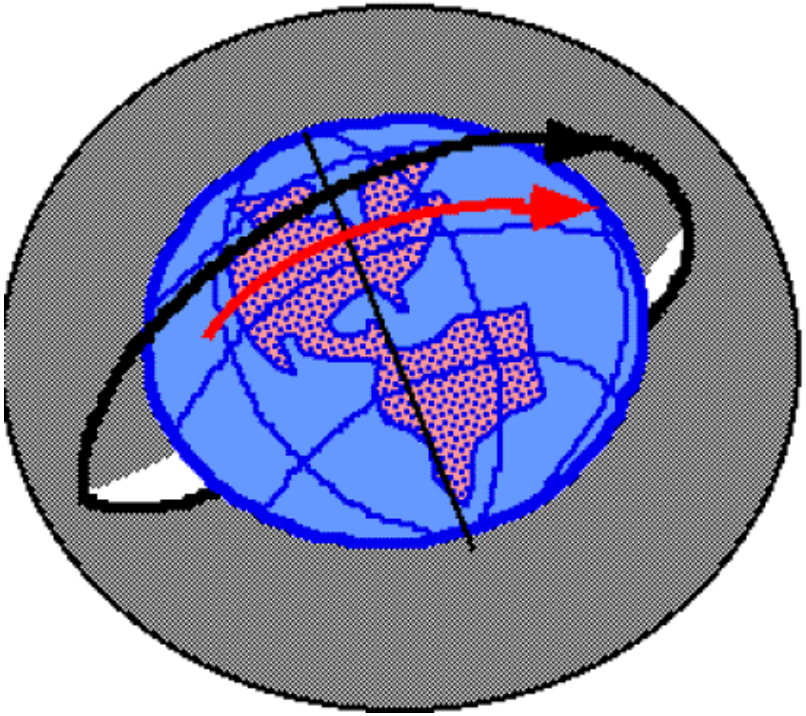
Aerodynamic Forces (cont'd)

Air "sticks" to Earth boundary

Airspeed

$$\bar{q}_\infty = \frac{1}{2}\rho_\infty V_\infty^2 \rightarrow V_\infty = \left\| \bar{V}_{inertial} - \bar{V}_{atmosphere} \right\| = \left\| \bar{V}_{inertial} - \bar{\omega}_{earth} \times \bar{R} \right\|$$

Inertial Velocity



Lower atmosphere:

$$\bar{V}_{atmosphere} = \bar{\omega}_{earth} \times \bar{r}_{satellite}$$

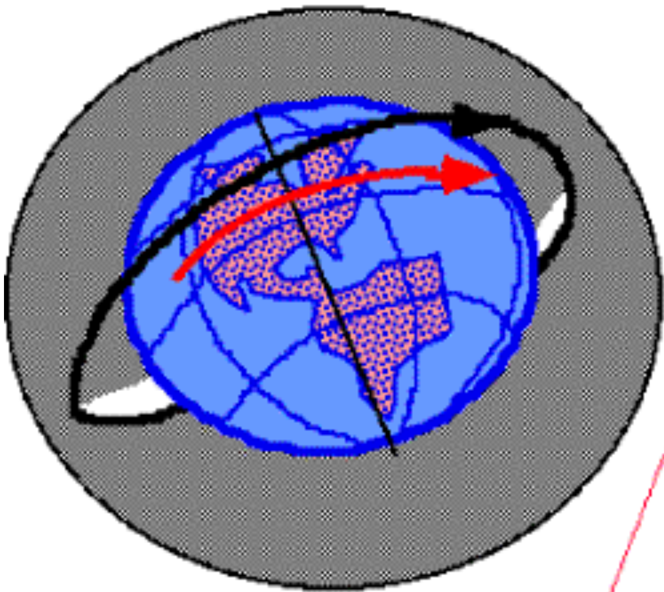
Aerodynamic Forces (cont'd)

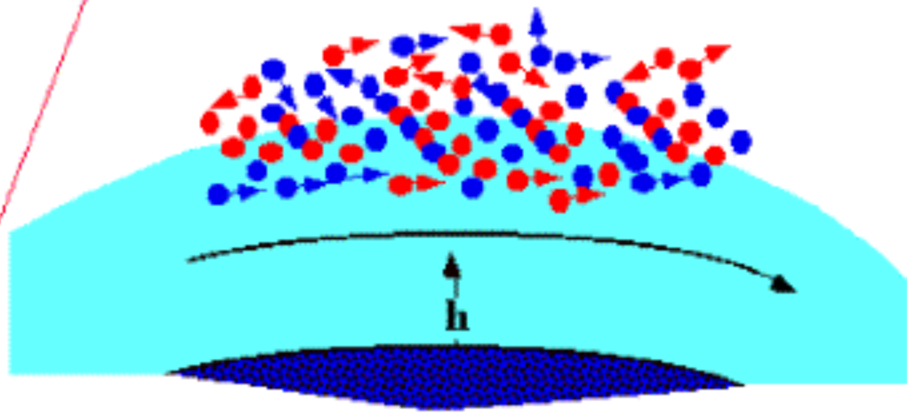
See appendix 1 at end of slides

"Cross Product"

Upper atmosphere:

$$\bar{V}_{atmosphere} << \bar{\omega}_{earth} \times \bar{r}_{satellite}$$





Rarified Upper Atmosphere "Slips" and is not "Attached" to the Earth

- Good Approximation:

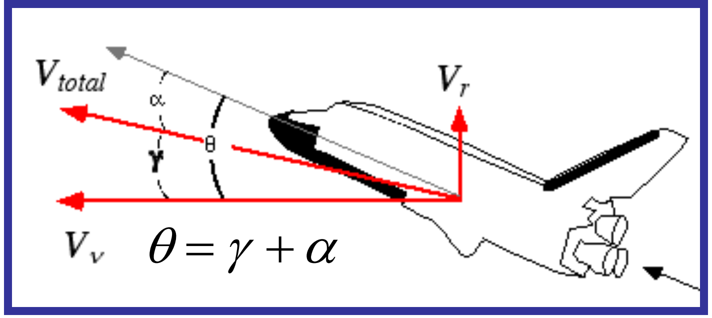
$$V_{\infty} = \left\| \bar{V}_{inertial} - \bar{V}_{earth} - \bar{V}_{wind} \right\|$$

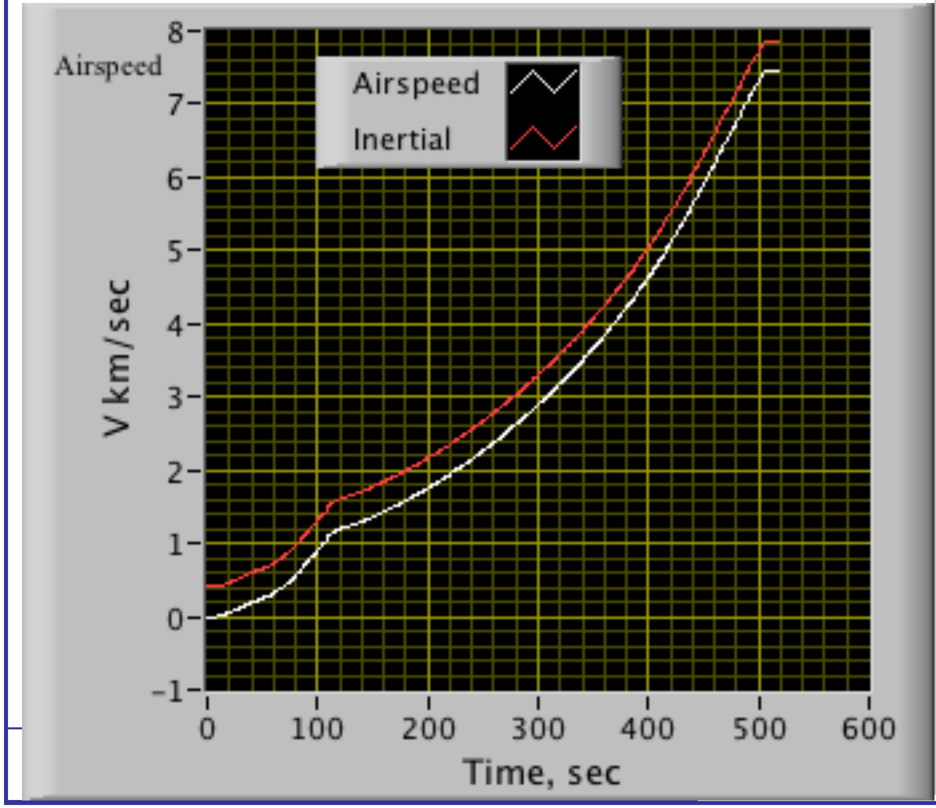
Aerodynamic Forces (revisited)

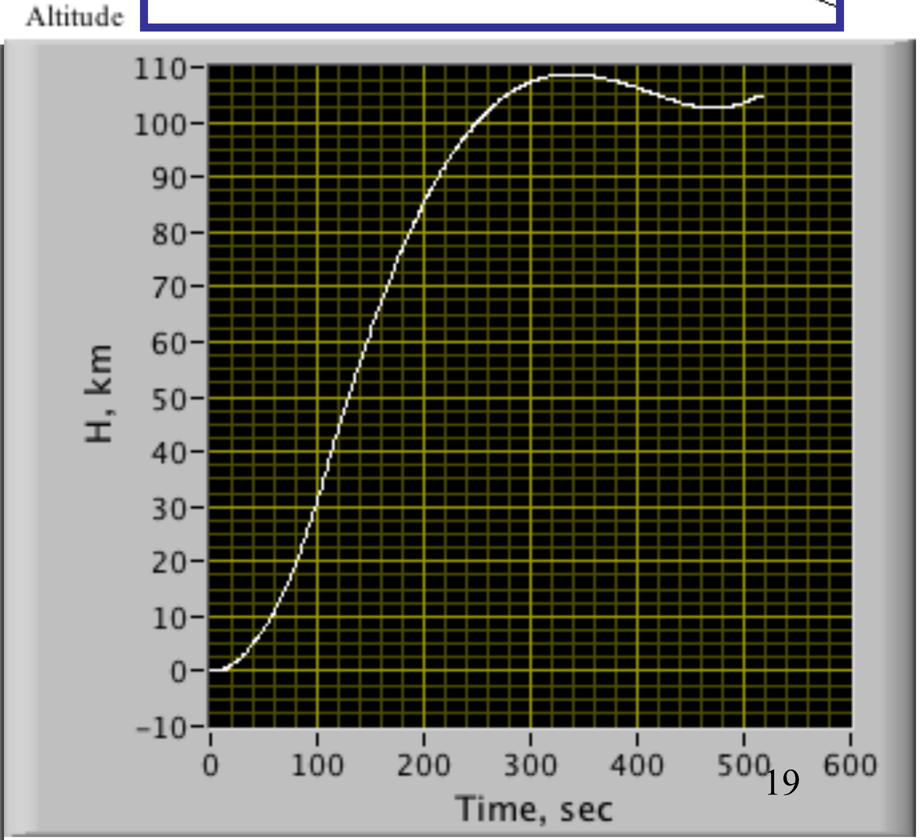
- Look at STS 114-aero example

--C_L, C_D typically function of Mach, α

--Typically implemented as table lookup

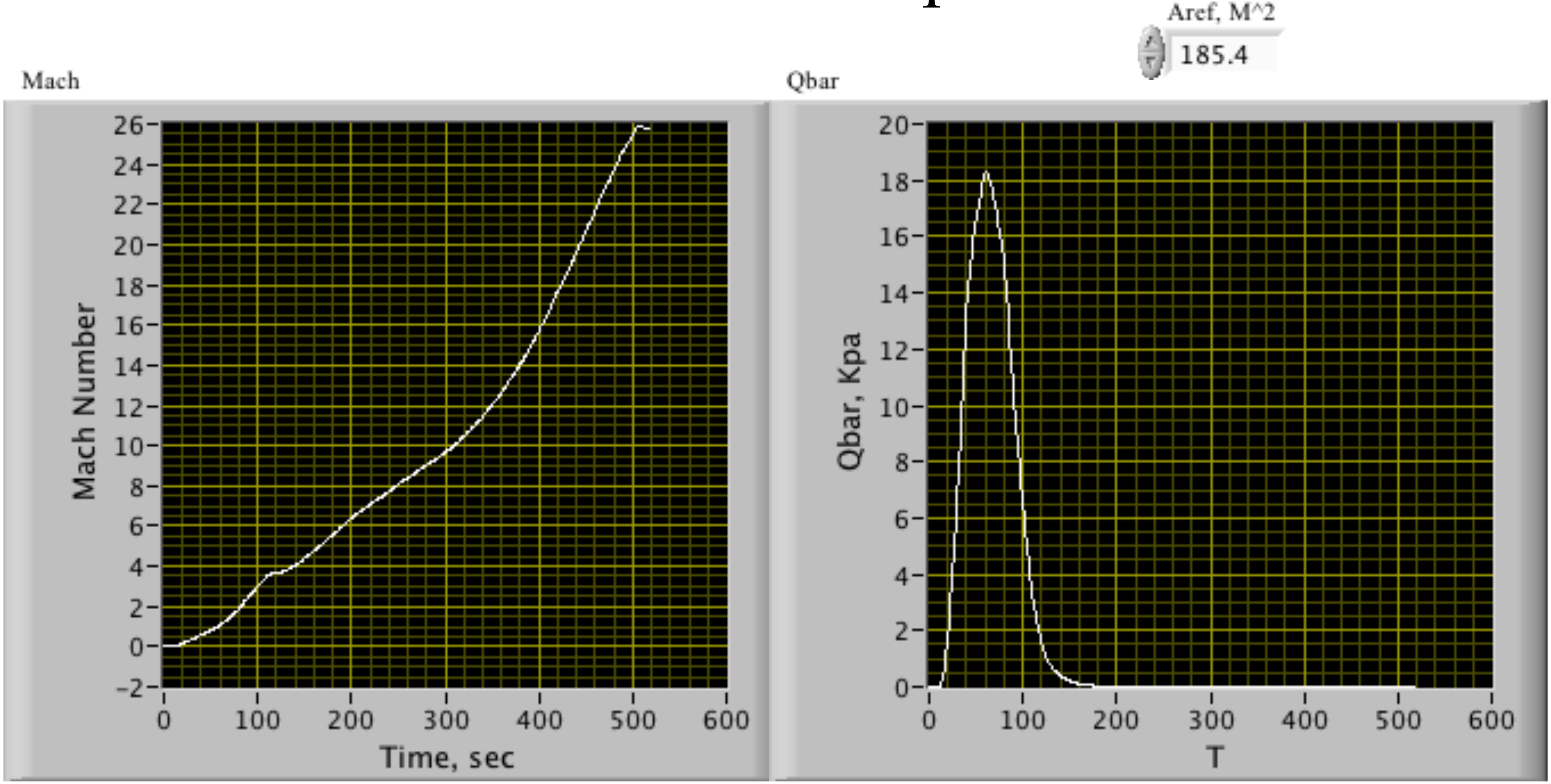






Aerodynamic Forces (cont'd)

- Look at STS 114-aero example

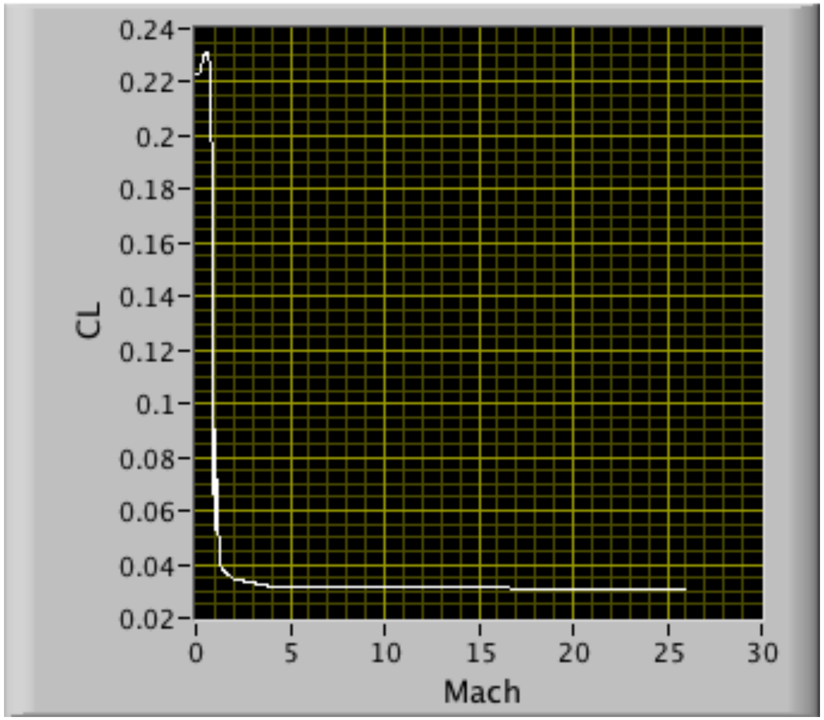


Aerodynamic Forces (cont'd)

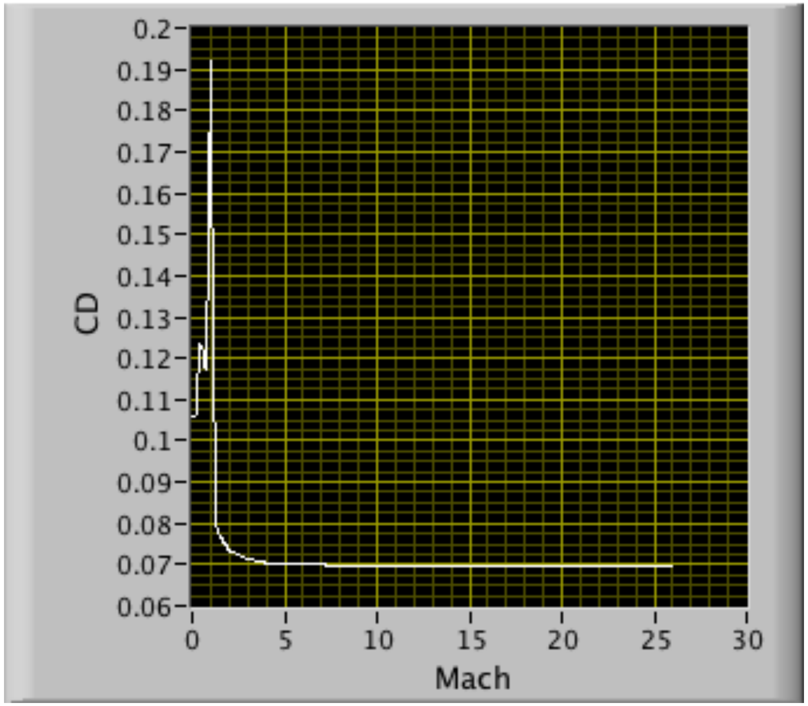
- Look at STS 114-aero example

Aref, M^2
185.4

Lift Coefficient 2

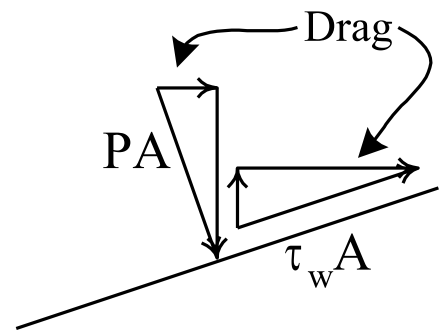


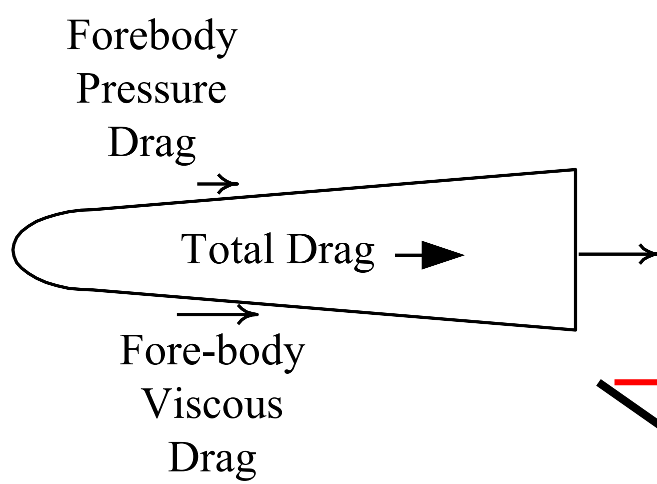
Drag Coefficient 2

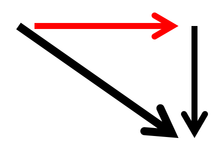


Drag

- Several Types of Drag Act on Flight Vehicles
  - Simplest case
    - Pressure drag (form drag)
      - Fore-body
      - Base
      - Wave drag
      - Induced or Compressive ***drag due to lift***
    - Viscous drag
      - Fore-body
    - Total drag

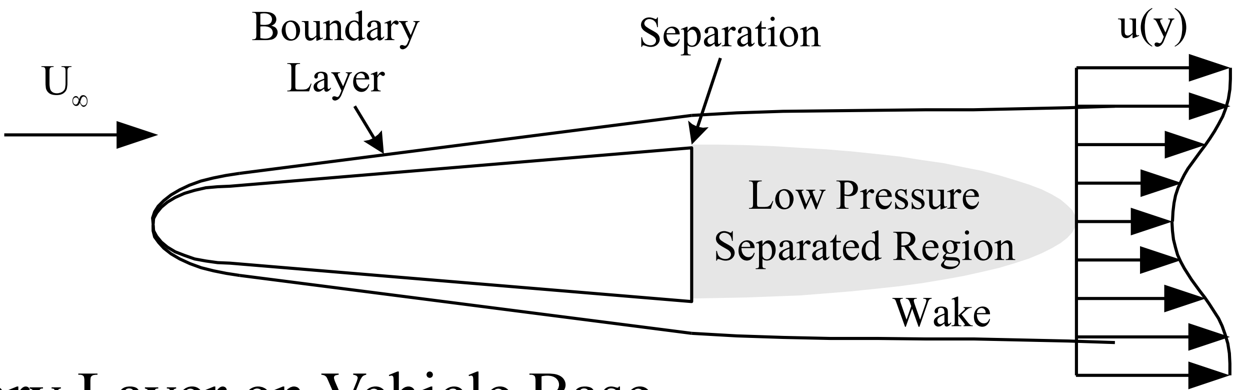




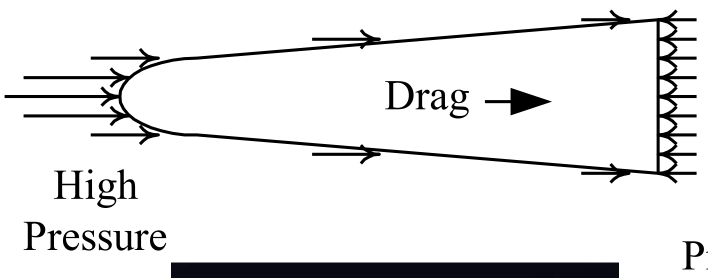


Drag due to lift

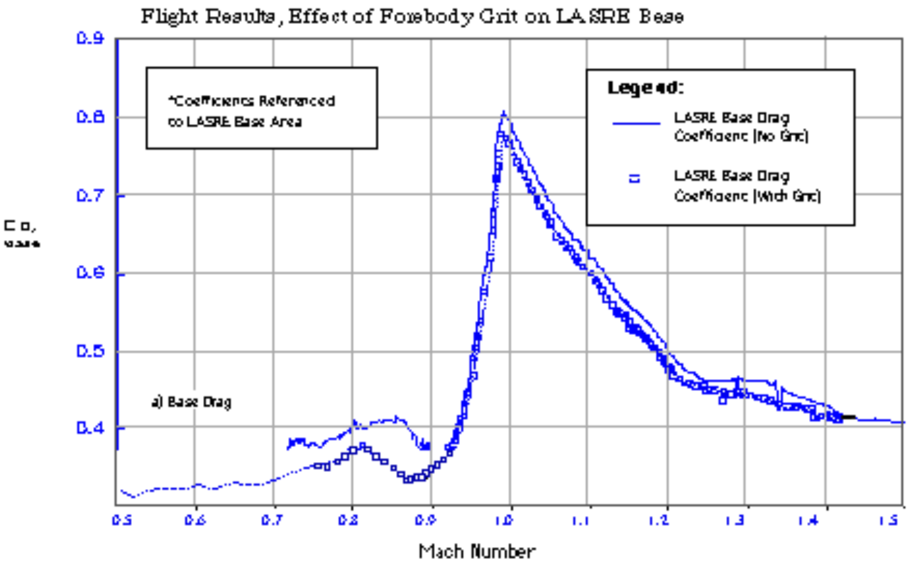
Base Drag: What is it?



- Boundary Layer on Vehicle Base Area Separates
- Low Pressure Separated Region Forms
- Low Pressure Causes a Large net Pressure Difference
- ***Especially significant on Launch vehicle after rocket burnout***



Base Drag (cont'd)



$$C_D(M, K_1, K_2, n) = \frac{(1 + K_1 M^n) C_{D_0}}{\sqrt{|M^2 - 1|} + (1 - \sqrt{|M^2 - 1|}) \frac{C_{D_0}}{K_2}}$$

Linear Aerospike Rocket Engine

Collected Equations

$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{v} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \dfrac{V_v^{\,2}}{r} + \dfrac{F_{lift}\cos(\gamma) - F_{drag}\sin(\gamma) + F_{thrust}\sin(\theta)}{m} - \dfrac{\mu}{r^2} \\ -\left[\dfrac{V_r V_v}{r} + \dfrac{F_{lift}\sin(\gamma) + F_{drag}\cos(\gamma) - F_{thrust}\cos(\theta)}{m}\right] \\ V_r \\ \dfrac{V_v}{r} \\ -\dfrac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix}$$

$$\gamma = \tan^{-1}\left[\frac{V_r}{V_v}\right]$$

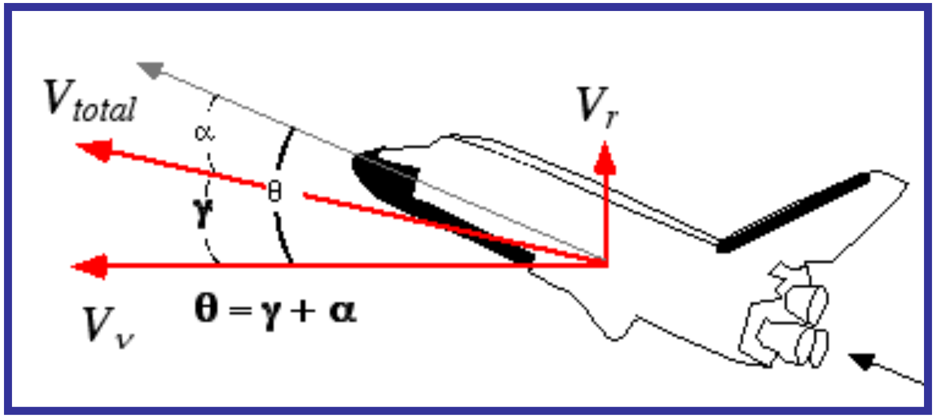
$$\dot{X} = f\left[X, F_{thrust}, \theta\right]$$

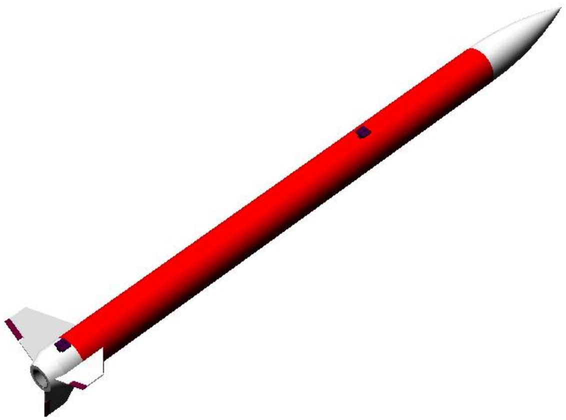
Vector Form of State Equations

$$\dot{X} = f\left[X, F_{thrust}, \theta\right]$$

$$\dot{X} = \begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{v} \\ \dot{m} \end{bmatrix} \rightarrow f\left[X, F_{thrust}, \theta\right] = \begin{bmatrix} \frac{V_v^{\,2}}{r} + \frac{F_{lift}\cos(\gamma) - F_{drag}\sin(\gamma) + F_{thrust}\sin(\theta)}{m} - \frac{\mu}{r^2} \\ -\left[\frac{V_r V_v}{r} + \frac{F_{lift}\sin(\gamma) + F_{drag}\cos(\gamma) - F_{thrust}\cos(\theta)}{m}\right] \\ V_r \\ \frac{V_v}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix} \rightarrow \boxed{\begin{matrix} \gamma = \tan^{-1}\left[\frac{V_r}{V_v}\right] \\ \theta = \gamma + \alpha \end{matrix}}$$

Ballistic versus Non -Ballistic Trajectories





- Non-ballistic trajectories sustain significantly non -zero angles of attack

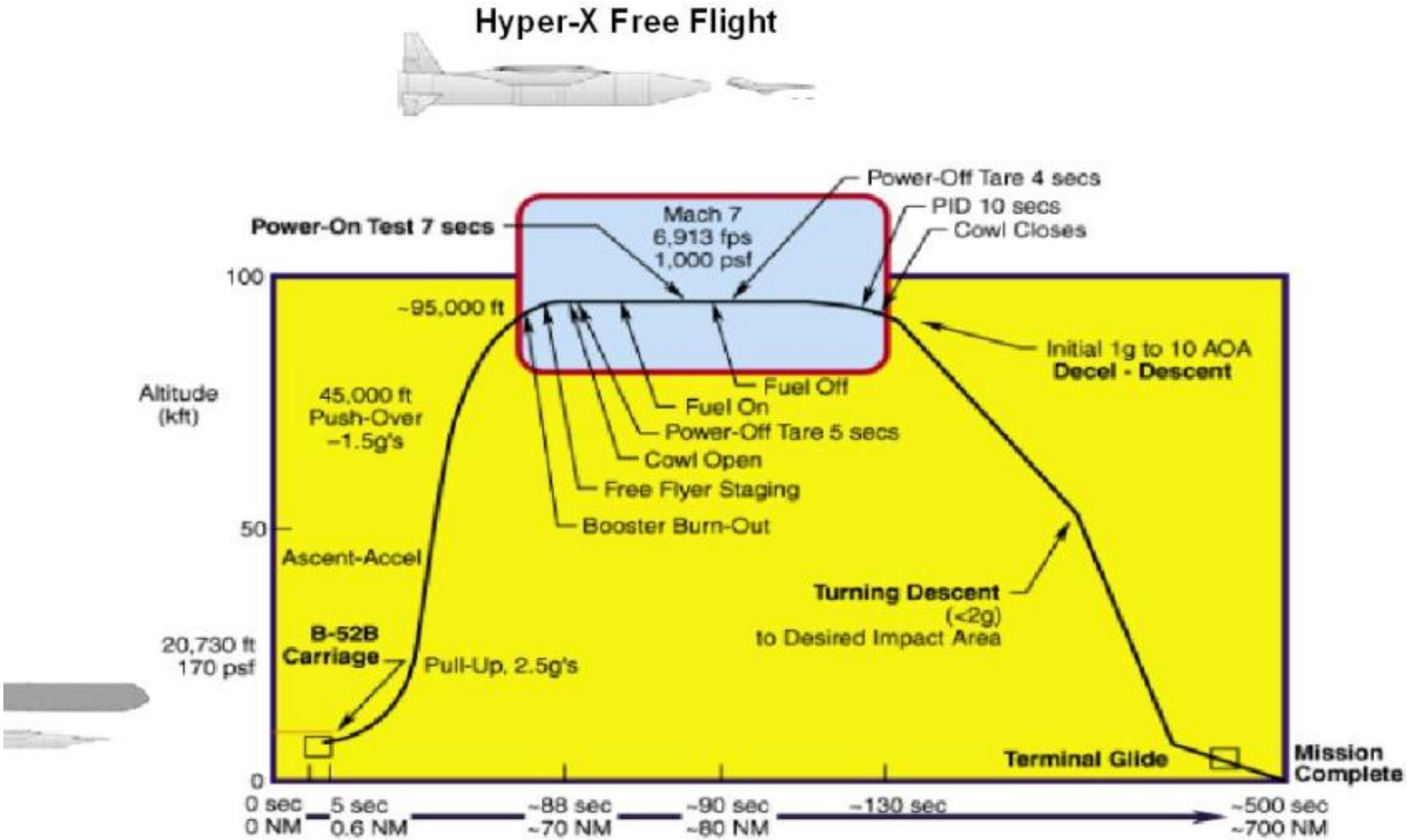
  ... lift is a factor in resulting trajectory

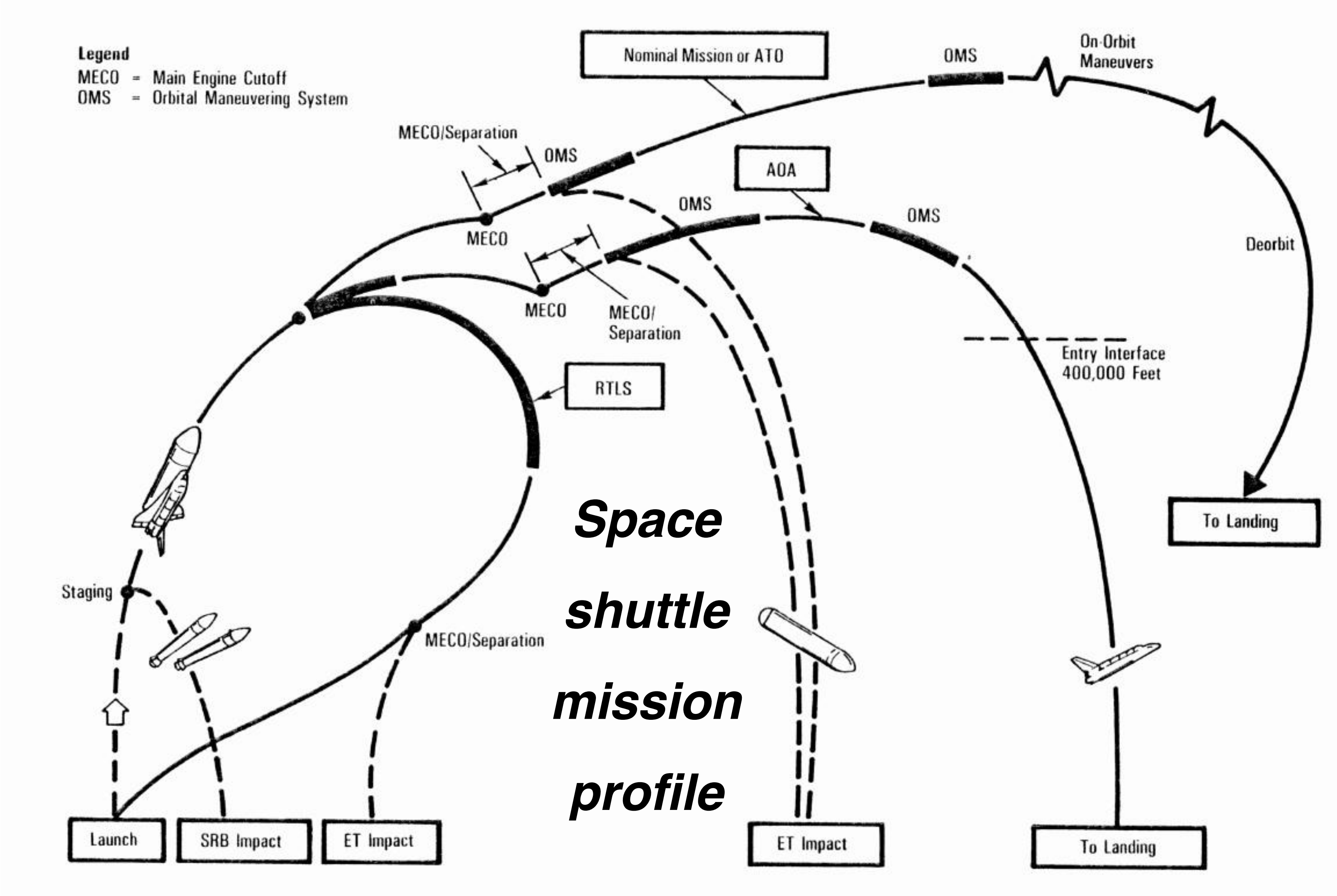
  ... so is induced drag

- *Ballistic trajectories trim rocket at ~ $0^\circ \alpha$ (θ=γ)*

... lift is a negligible factor in resulting trajectory

Example of Non-Ballistic Trajectory





Abort and Normal Mission Profile

Space shuttle mission profile

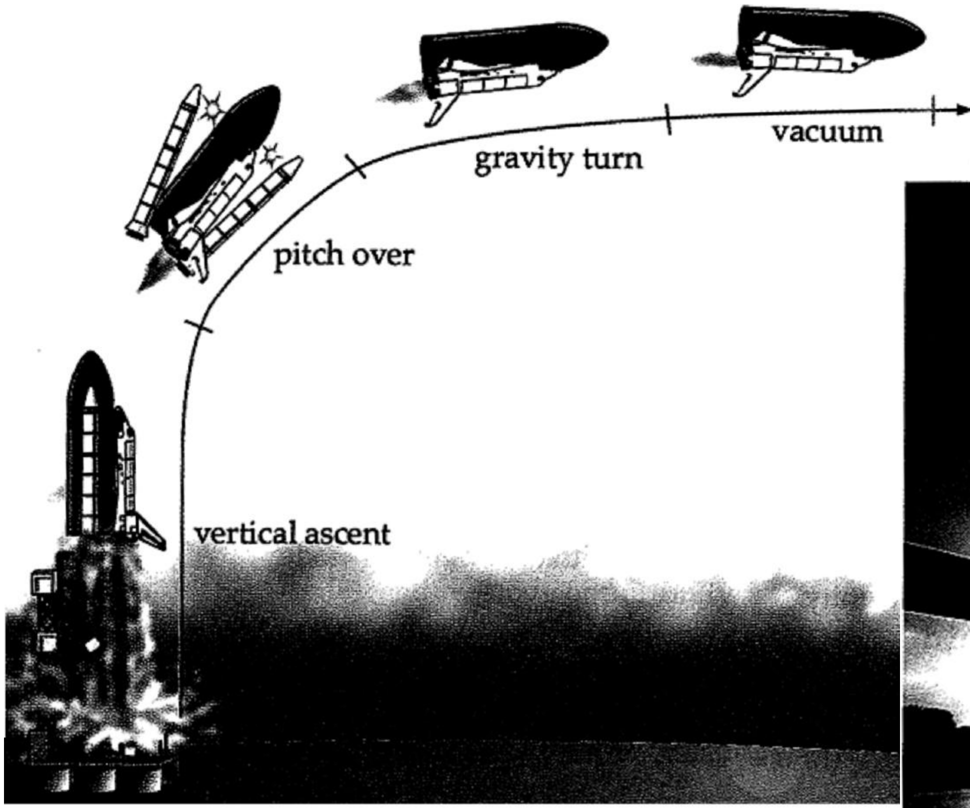
More "Gravity Turn"

Space Shuttle Launch (STS 115 – Atlantis) as seen from ISS

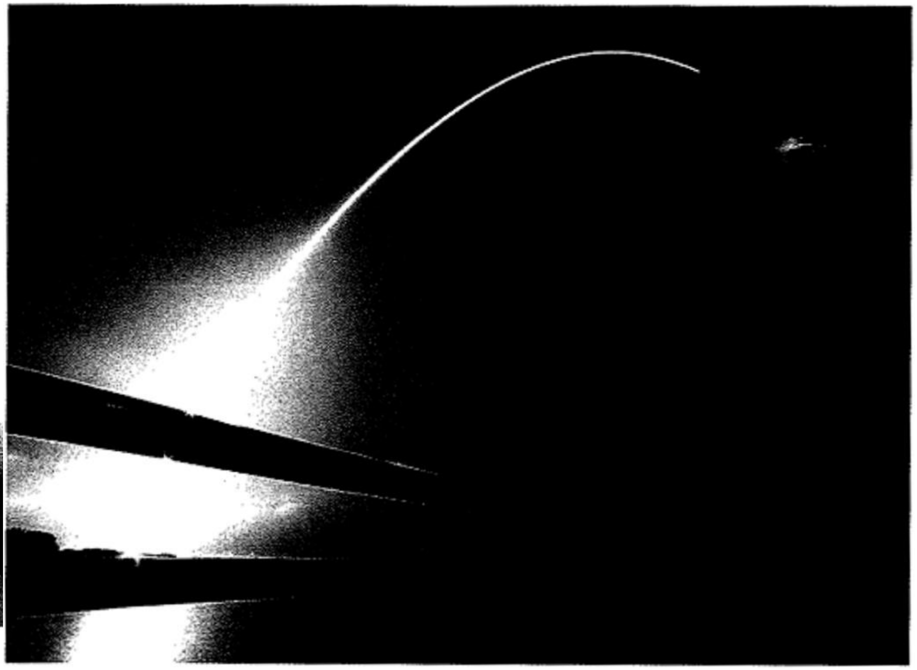
"definitely Not ballistic"

Oh That! Gravity Turn

Yup this is real!

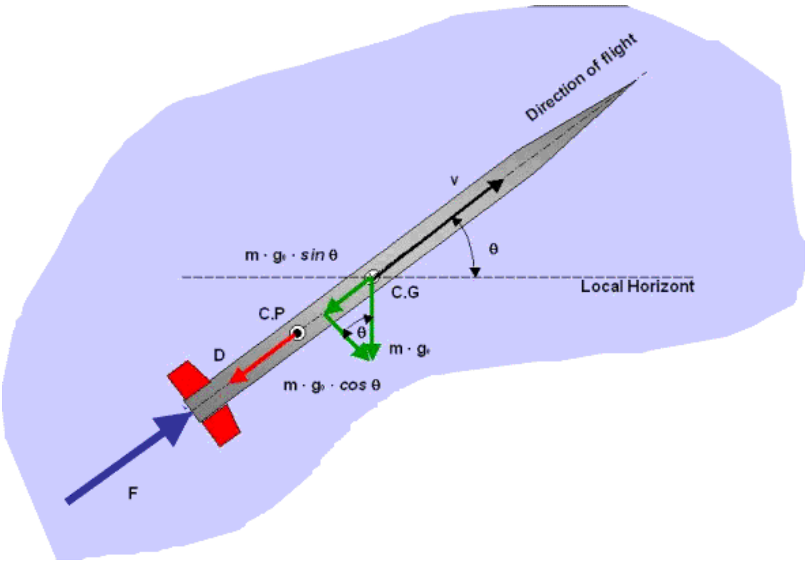


Phases of Launch Vehicle Ascent. During ascent a launch vehicle goes through four phases—vertical ascent, pitch over, gravity turn, and vacuum.

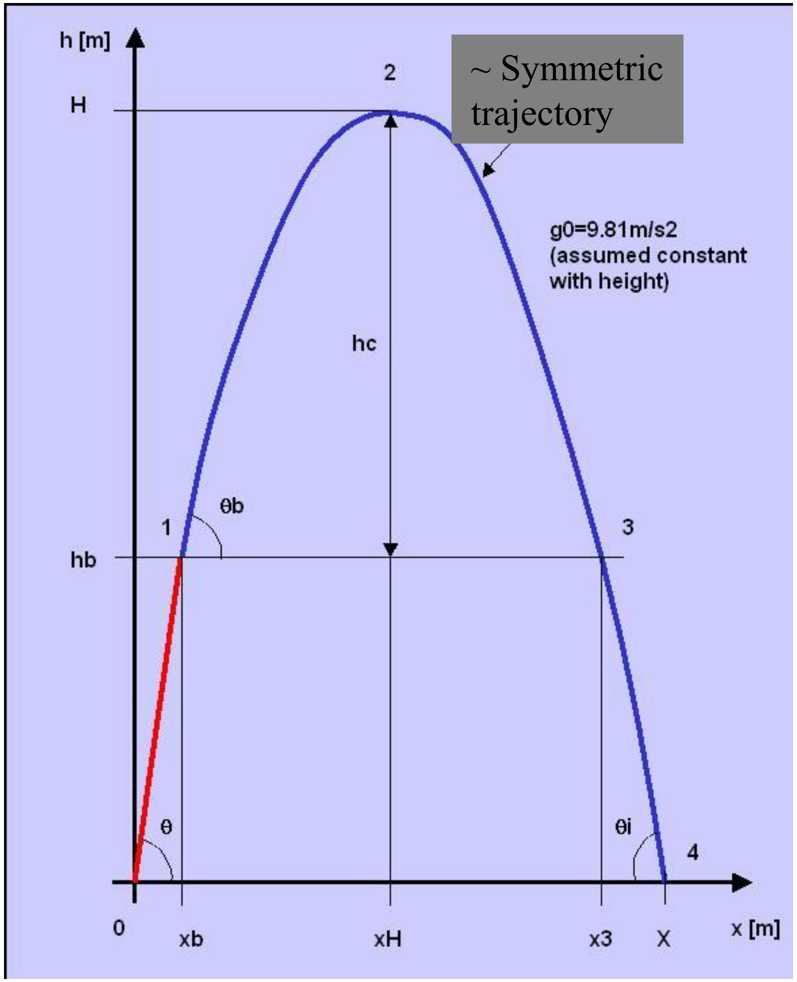


Gravity-turn maneuver of an ascending Delta II rocket with Messenger spacecraft on August 3, 2004.

Example of Ballistic Trajectory



- Ballistic Trajectories Offer minimum drag profiles (α ~ 0-->No induced drag)



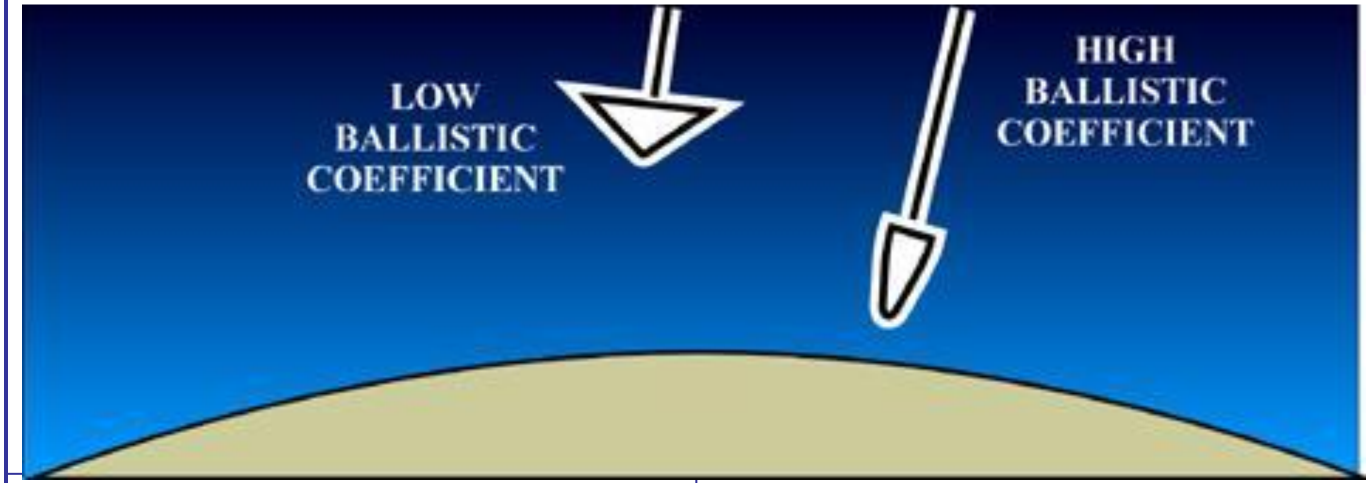
Ballistic Coefficient

- When effects of lift are negligible aerodynamic effects can be incorporated into a single parameter

…. ***Ballistic Coefficient (β)***

- β is a measure of a projectile's ability to coast. …$\beta = M/C_d A_{ref}$
… M is the projectile's mass and … $C_d A$ is the drag *form factor*.

- At any given velocity and air density, the deceleration of a rocket from drag is inversely proportional to β



- ***See Appendix 3***
For Ballistic Coefficient Calculation Examples

Collected Equations, Ballistic Trajectory

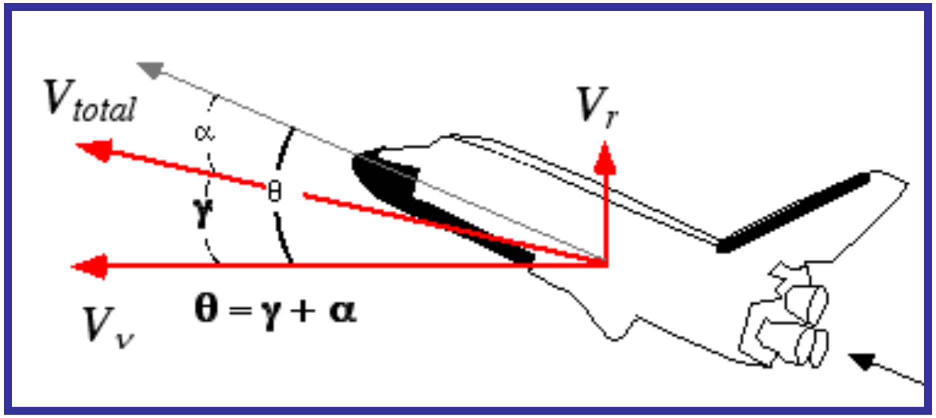
$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_\nu \\ \dot{r} \\ \dot{\nu} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \frac{V_\nu^2}{r} - \frac{\mu}{r^2} + \left[\frac{F_{thrust}}{m} - \frac{\rho V_\infty^2}{2\beta}\right]\sin(\gamma) \\ -\frac{V_r V_\nu}{r} + \left[\frac{F_{thrust}}{m} - \frac{\rho V_\infty^2}{2\beta}\right]\cos(\gamma) \\ V_r \\ \frac{V_\nu}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix}_{\alpha=0}$$

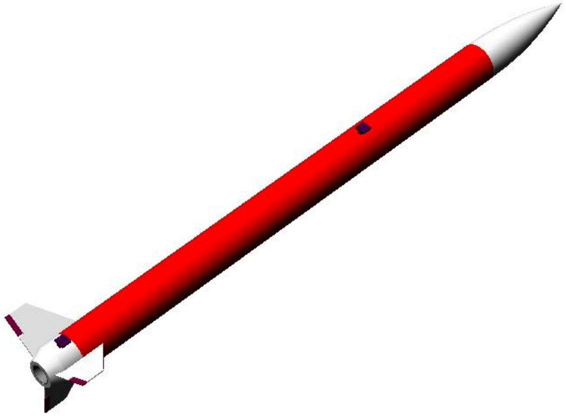
$$\gamma = \tan^{-1}\left[\frac{V_r}{V_\nu}\right]$$

$$\beta = \frac{\mathrm{m}}{\mathrm{C_D} A_{ref}}$$

$$\dot{X} = f\left[X, F_{thrust}\right]$$

Launch Ballistics Revisited

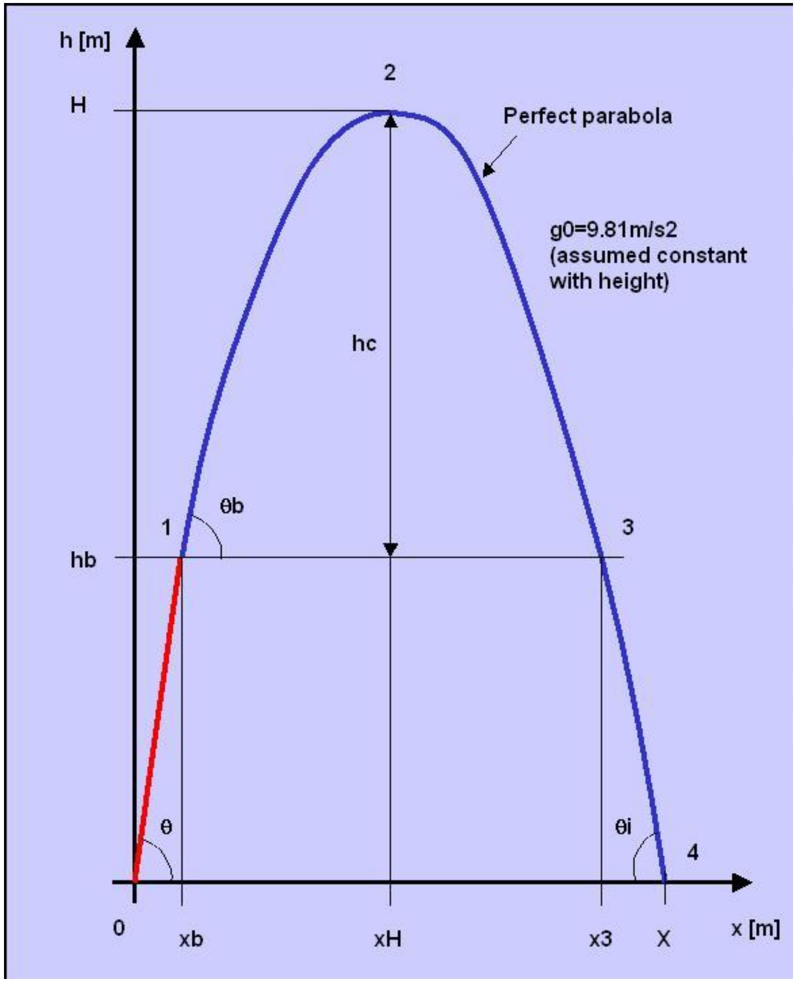




- Non-ballistic trajectories sustain significantly non -zero angles of attack
  ... lift is a factor in resulting trajectory
  ... so is induced drag

- *Ballistic trajectories trim rocket at ~ $0^\circ \alpha$ ($\theta=\gamma$)*
... lift is a negligible factor in resulting trajectory

Launch Ballistics Revisited (2)



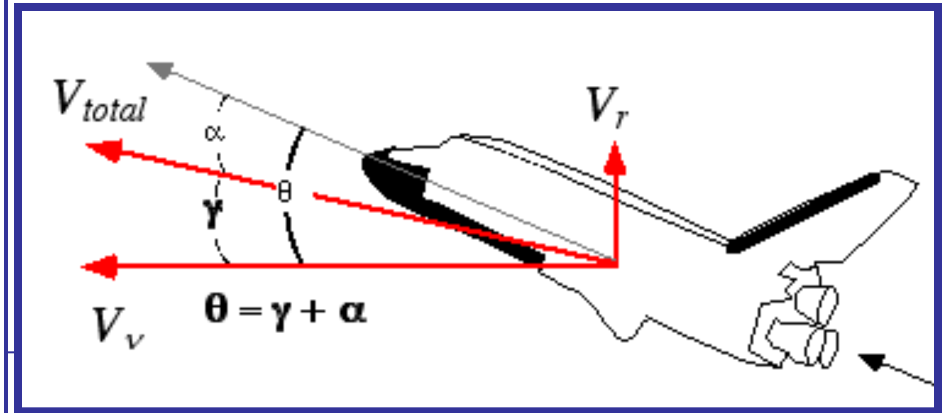
• In practice ballistic Trajectories give "lofted orbits" with ***Very high apogee Altitudes*** …

Need to ***"turn the corner"*** at some non-zero ***angle-of-attack*** to get proper Apogee/velocity phasing

Non-Ballistic Trajectories, revisited

$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{v} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \frac{V_v^2}{r} + \frac{F_{lift}\cos(\gamma) - F_{drag}\sin(\gamma) + F_{thrust}\sin(\theta)}{m} - \frac{\mu}{r^2} \\ -\left[\frac{V_r V_v}{r} + \frac{F_{lift}\sin(\gamma) + F_{drag}\cos(\gamma) - F_{thrust}\cos(\theta)}{m}\right] \\ V_r \\ \frac{V_v}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix}$$

$\theta \underline{\sim} \alpha + \gamma$



"pitch profile"
Key to accurate
Orbit insertion

Ballistic .. Bottom Line

• In practice ballistic Trajectories give "lofted orbits" with ***Very high apogee Altitudes*** …

Need to ***"turn the corner"*** at some non-zero ***angle-of-attack*** to get proper Apogee/velocity phasing

• "pitch profile" Key to accurate Orbit insertion

• Negative lift used to "turn the corner" during

• Induced Drag Penalty Accepted to achieve correct orbit parameters

• ***See Appendix 4 for numerical example***

Numerical Analysis of the 2-D Launch Equations of Motion

Integrated Equations of Motion

$$\dot{X} = f\left[X, F_{thrust}, \theta\right] \rightarrow X(t) = X(t_0) + \int_{t_0}^{t} f\left[X, F_{thrust}, \theta\right] dt$$

$\rightarrow$ approximate over fixed interval ΔT $\rightarrow$

$$X(t_0 + \Delta t) = X(t_0) + \int_{t_0}^{t_0 + \Delta t} f\left[X, F_{thrust}, \theta\right] dt$$

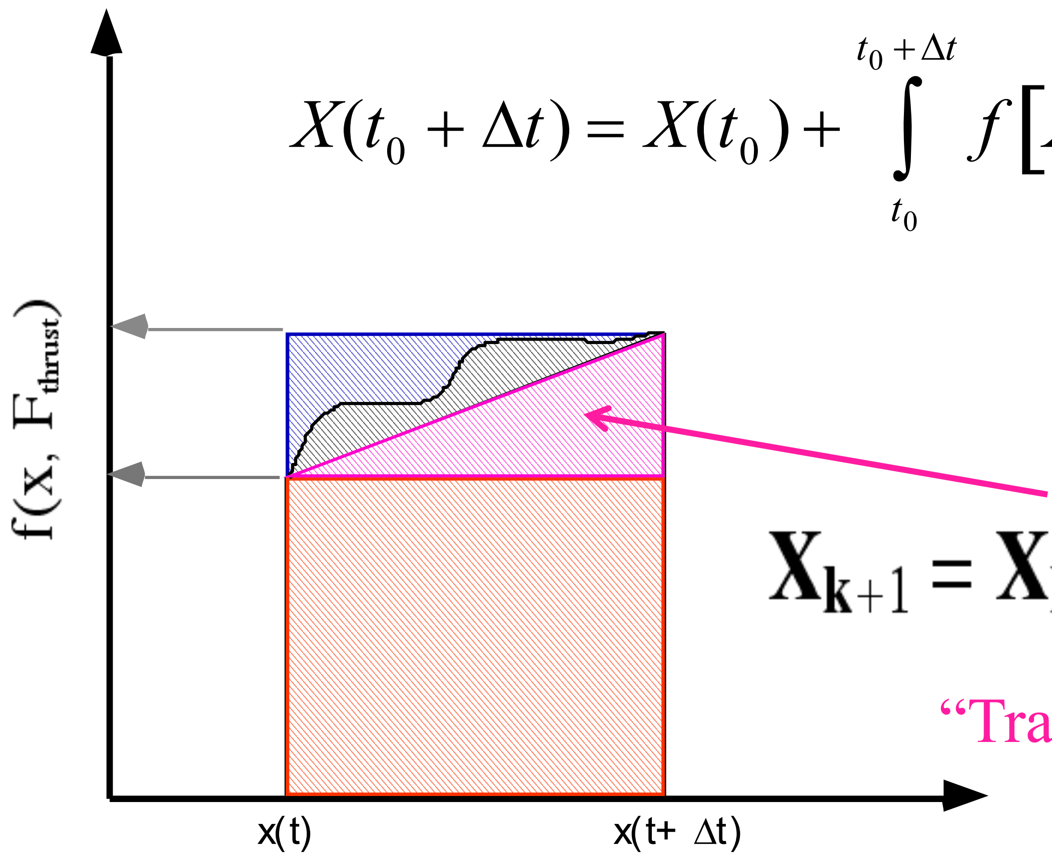
Numerical Approximation of the Integral

- **Index definitions**

$$\dot{X} = f\left(X, F_{thrust}\right) \Rightarrow \begin{bmatrix} X_k \Rightarrow X\left(t_0 + k\,\Delta t\right) \\ X_{k+1} \Rightarrow X\left(t_0 + \left(k+1\right)\Delta t\right) \\ f_k \Rightarrow f\left[X_k, F_{thrust_k}, \theta_k\right] \\ f_{k+1} \Rightarrow f\left[X_{k+1}, F_{thrust_{k+1}}, \theta_{k+1}\right] \end{bmatrix}$$

Numerical Approximation of the Integral (cont'd)

$$X(t_0 + \Delta t) = X(t_0) + \int_{t_0}^{t_0+\Delta t} f\left[X, F_{thrust}, \theta\right] dt$$



$$\mathbf{X}_{\mathbf{k}+1} = \mathbf{X}_{\mathbf{k}} + \frac{\Delta \mathbf{t}}{2}\left[\mathbf{f}_{\mathbf{k}} + \mathbf{f}_{\mathbf{k}+1}\right]$$

"Trapezoidal rule"

Or we can use Finite Differences”

- **Finite Differences**

$$\dot{X} \approx \frac{X_{k+1} - X_k}{\Delta t} \approx \frac{1}{2}\left[f_k + f_{k+1}\right]$$

- **Solving for**

$$X_{k+1} = X_k + \frac{\Delta t}{2}\left[f_k + f_{k+1}\right]$$

Numerical Approximation of the Integral (cont'd)

"Trapezoidal rule"

$$X_{k+1} = X_k + \frac{\Delta t}{2}\left[f_k + f_{k+1}\right]$$

$$f_k \Rightarrow f\left[X_k, F_{thrust_k}, \theta_k\right]$$

$$f_{k+1} \Rightarrow f\left[X_{k+1}, F_{thrust_{k+1}}, \theta_{k+1}\right]$$

- But we don't know f_{k+1}
- Soooo ... we *predict* it

$$\tilde{\hat{X}}_{k+1} \equiv X_k + \Delta t\ f_k$$

$$\tilde{\hat{f}}_{k+1} \Rightarrow f\left[\tilde{\hat{X}}_{k+1}, F_{thrust_{k+1}}, \theta_{k+1}\right]$$

Numerical Approximation of the Integral (cont'd)

- **And ... then we *correct* it**

$$\hat{X}_{k+1} = X_k + \frac{\Delta t}{2}\left[f_k + \hat{\tilde{f}}_{k+1}\right]$$

"trapezoidal rule"

Predictor/Corrector Algorithm

$$\text{Given}\left[\Delta t, \hat{X}_k, F_{thrust_k}, \theta_k\right]$$

"trapezoidal rule"

Prediction Step:

$$\Rightarrow \tilde{\hat{X}}_k = \hat{X}_k + \Delta t\, f\left[\hat{X}_k, F_{thrust_k}, \theta_k\right]$$

Correction Step:

$$\hat{\mathbf{X}}_{\mathbf{k}+1} \Rightarrow \hat{X}_{k+1} = \hat{X}_k + \frac{\Delta t}{2}\left\{f\left[\hat{X}_k, F_{thrust_k}, \theta_k\right] + f\left[\tilde{\hat{X}}_{k+1}, F_{thrust_{k+1}}, \theta_{k+1}\right]\right\}$$

Slide Indices and Repeat:

$$\hat{\mathbf{X}}_{\mathbf{k}+1} \Rightarrow \hat{\mathbf{X}}_{\mathbf{k}}$$

Higher Order Integrators

•Simple Second Order predictor/corrector works well for Small-to-moderate step sizes … but at larger step sizes can be come unstable

• Good to have a higher order integration scheme in our *bag of tools*

• 4th Order Runge-Kutta method is one most commonly used

• Lots of arcane derivations and *Mystery* with regard to This method … lets clear this up!!!

- The Runge-Kutta Method was developed by two German men Carl Runge (1856-1927), and Martin Kutta (1867- 1944) in 1901. These numerical methods are still used today.

- Carl Runge developed numerical methods for solving the differential equations that arose in his study of atomic spectra.

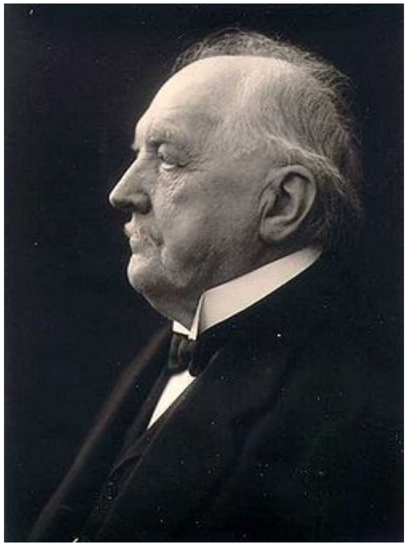
- He used so much mathematics in his research that physicists thought he was a mathematician, and he did so much physics that mathematicians thought he was a physicist.

- Today his name is associated with the Runge-Kutta methods to numerically solve differential equations.

- Kutta, another German applied mathematician, is also remembered for his contribution to the differential equations-based Kutta-Joukowski theory of airfoil lift in aerodynamics.

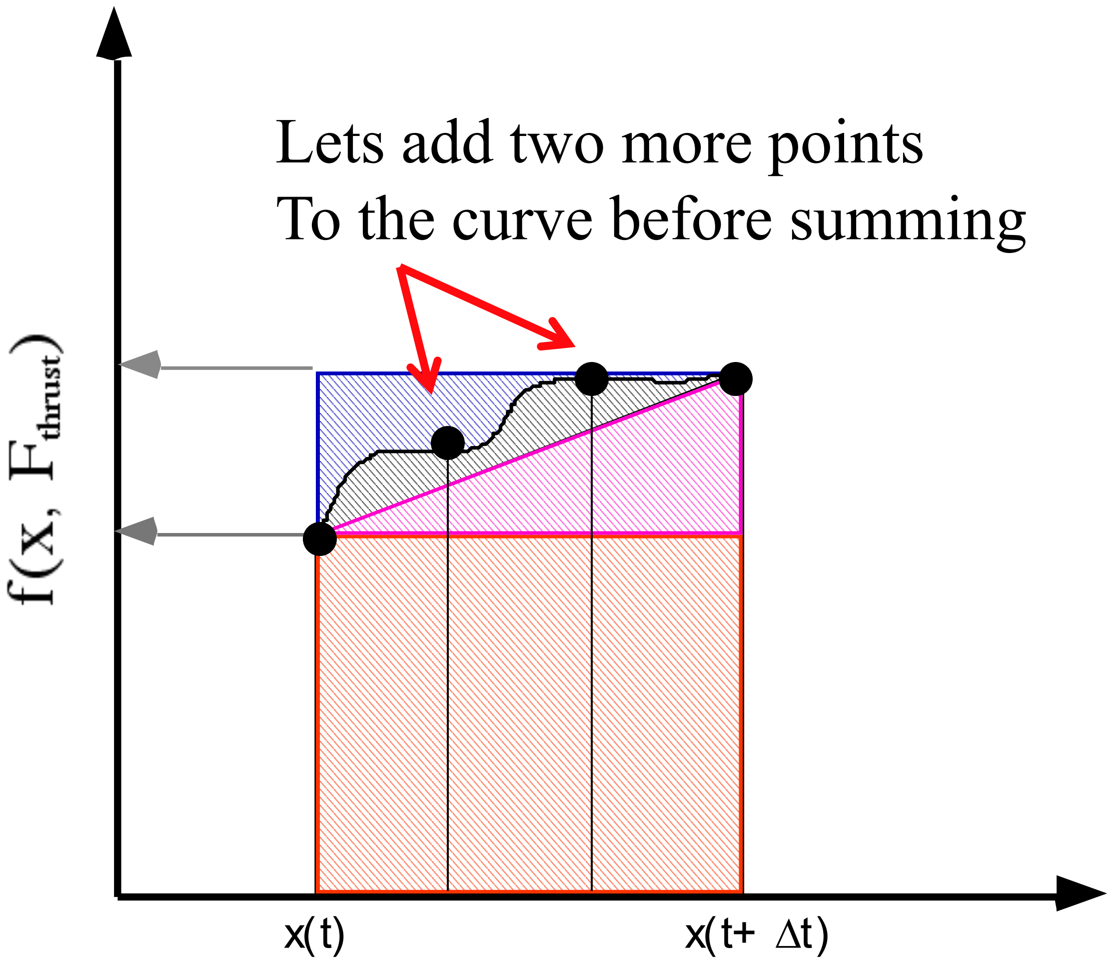
- Runge–Kutta method is an effective and widely used method for solving the initial-value problems of differential equations. Runge–Kutta method can be used to construct high order accurate numerical method by functions' self without needing the high order derivatives of functions.

Karl Runge



Martin Kutta

4th Order Runge-Kutta Method



4th Order Runge-Kutta Method

(cont'd)

- The basic Differential equation is:

$$\dot{x} = f\,[t\,,\;x]$$

- Approximate the first derivative by finite difference

$$\dot{x} \approx \hat{\dot{x}}^{(1)} = f\,[t_k+,\;x_k\,] \equiv k_1$$

4th Order Runge-Kutta Method

(cont'd)

- Now correct this derivative estimate with what we have learned

$$\dot{x} \approx \hat{\dot{x}}^{(1)} = f\left[t_k + , \; x_k\right] \equiv k_1$$

$$\hat{\dot{x}}^{(2)} = f\left[t_k + \frac{\Delta t}{2}, \; x_k + \frac{\Delta t}{2}\hat{\dot{x}}^{(1)}\right] = f\left[t_k + \frac{\Delta t}{2}, \; x_k + \frac{\Delta t}{2} k_1\right] \equiv k_2$$

- This is almost equivalent to what we have already done

4th Order Runge-Kutta Method

(cont'd)

- Repeat this process twice more to give us 4 points on the curve

$$\hat{\dot{x}}^{(3)} = f\left[t_k + \frac{\Delta t}{2},\ x_k + \frac{\Delta t}{2}\hat{\dot{x}}^{(2)}\right] = f\left[t_k + \frac{\Delta t}{2},\ x_k + \frac{\Delta t}{2} k_2\right] \equiv k_3$$

$$\hat{\dot{x}}^{(4)} = f\left[t_k + \Delta t,\ x_k + \Delta t\, \hat{\dot{x}}^{(3)}\right] = f\left[t_k + \Delta t,\ x_k + \Delta t\, k_3\right] \equiv k_4$$

4th Order Runge-Kutta Method

(cont'd)

- Finally take a weighted average of the results

$$\left[\overline{\hat{\dot{x}}}\right] = \left[\frac{\hat{\dot{x}}^{(1)} + 2\,\hat{\dot{x}}^{(2)} + 2\,\hat{\dot{x}}^{(3)} + \hat{\dot{x}}^{(4)}}{6}\right] \Rightarrow$$

$$\hat{x}_{k+1} = \hat{x}_k + \frac{\Delta t}{6}\left[k_1 + 2\,k_2 + 2k_3 + k_4\right]$$

4th Order Runge-Kutta Method

(cont'd)

- What happens if the Input (Thrust) is not Constant? …

Simply "split the difference" between $F_{thrust\ k}$ and $F_{thrust\ k+1}$
… same applies to theta (dropped for simplicity)

$$\dot{x} = f\left[t,\ x,\ F_{thrust}\right]$$

$$\dot{x} \approx \hat{\dot{x}}^{(1)} = f\left[t_k+,\ x_k,\ \underline{F_{thrust_k}}\right] \equiv k_1$$

$$\hat{\dot{x}}^{(2)} = f\left[t_k+\frac{\Delta t}{2},\ x_k + \frac{\Delta t}{2}\hat{\dot{x}}^{(1)},\ \underline{F_{thrust_k}}\right] = f\left[t_k+\frac{\Delta t}{2},\ x_k + \frac{\Delta t}{2}k_1,\ \underline{F_{thrust_k}}\right] \equiv k_2$$

4th Order Runge-Kutta Method

(cont'd)

- "split the difference" between $F_{thrust\ k}$ and $F_{thrust\ k+1}$
  … same applies to theta (dropped for simplicity)

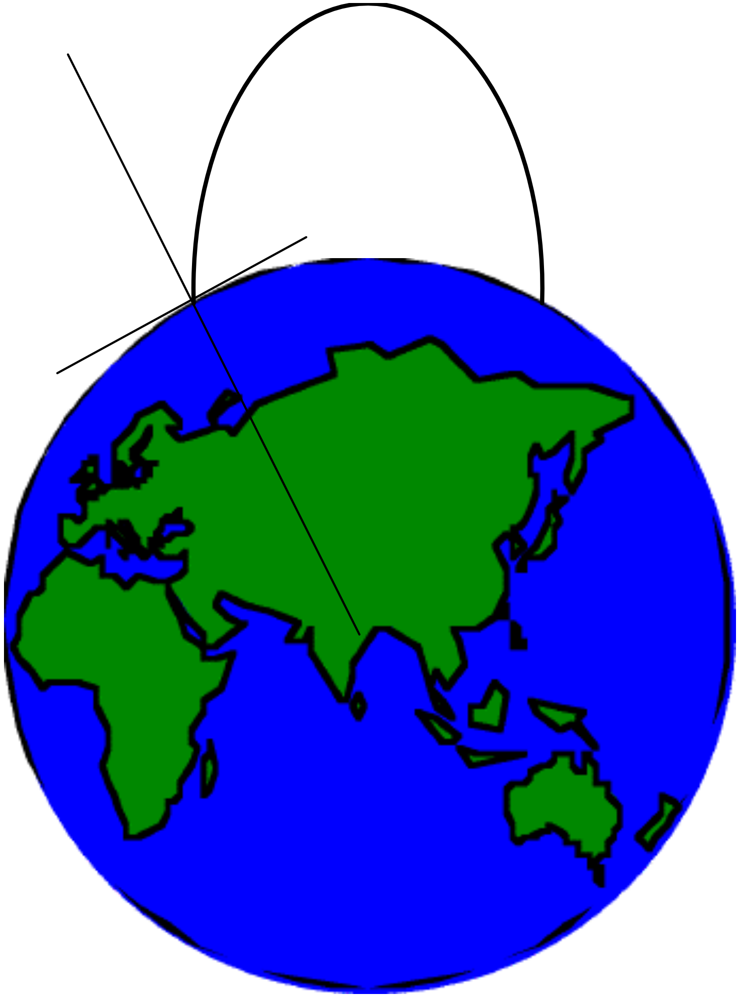
$$\hat{\dot{x}}^{(3)} = f\left[t_k+\frac{\Delta t}{2},\ x_k + \frac{\Delta t}{2}\hat{\dot{x}}^{(2)}, F_{thrust_{k+1}}\right] = f\left[t_k+\frac{\Delta t}{2}, x_k + \frac{\Delta t}{2} k_2 , F_{thrust_{k+1}}\right] \equiv k_3$$

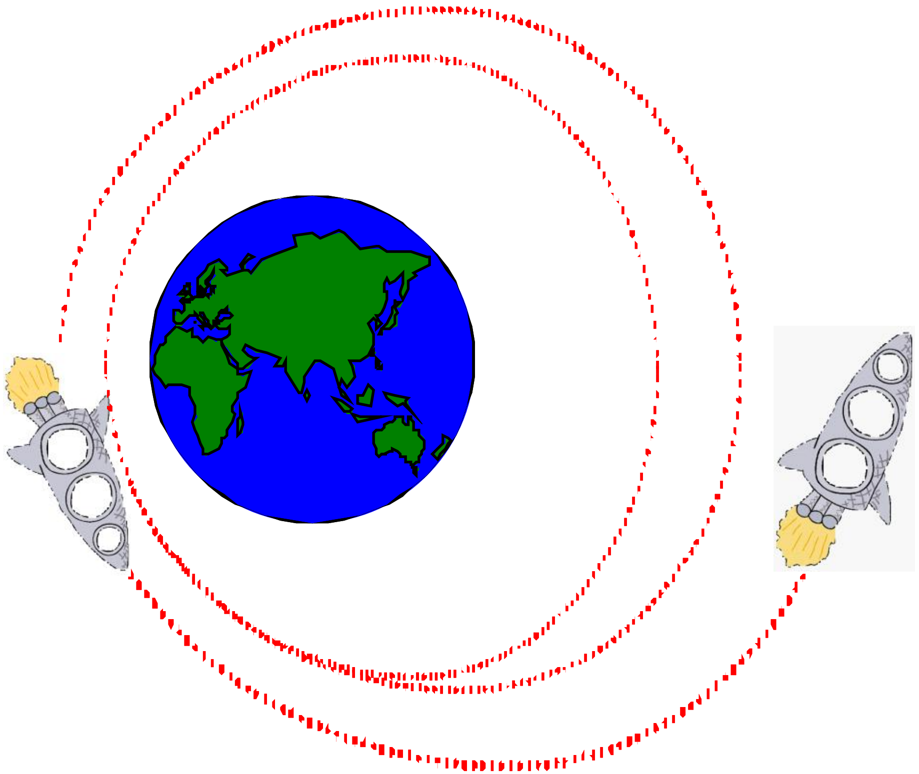
$$\hat{\dot{x}}^{(4)} = f\left[t_k+\Delta t,\ x_k + \Delta t\,\hat{\dot{x}}^{(3)}, F_{thrust_{k+1}}\right] = f\left[t_k+\Delta t, x_k + \Delta t\, k_3 , F_{thrust_{k+1}}\right] \equiv k_4$$

$$\left[\overline{\hat{\dot{x}}}\right] = \left[\frac{\hat{\dot{x}}^{(1)} + 2\,\hat{\dot{x}}^{(2)} + 2\,\hat{\dot{x}}^{(3)} + \hat{\dot{x}}^{(4)}}{6}\right] \Rightarrow$$

$$\hat{x}_{k+1} = \hat{x}_k + \frac{\Delta t}{6}\left[k_1 + 2\,k_2 + 2k_3 + k_4\right]$$

Summary Slides on E.O.M.





Collected General 2-D Equations

$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{v} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \frac{V_v^2}{r} + \frac{F_{lift}\cos(\gamma) - F_{drag}\sin(\gamma) + F_{thrust}\sin(\theta)}{m} - \frac{\mu}{r^2} \\ -\left[\frac{V_r V_v}{r} + \frac{F_{lift}\sin(\gamma) + F_{drag}\cos(\gamma) - F_{thrust}\cos(\theta)}{m} \right] \\ V_r \\ \frac{V_v}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix}$$

$$\gamma = \tan^{-1}\left[\frac{V_r}{V_v}\right]$$

$$\theta \underline{\sim} \alpha + \gamma$$

$$\dot{X} = f\left[X, F_{thrust}, \theta\right]$$

Collected Equations, Ballistic Trajectory

$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{v} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \frac{V_v^2}{r} - \frac{\mu}{r^2} + \left[\frac{F_{thrust}}{m} - \frac{\rho V_\infty^2}{2\beta}\right]\sin(\gamma) \\ -\frac{V_r V_v}{r} + \left[\frac{F_{thrust}}{m} - \frac{\rho V_\infty^2}{2\beta}\right]\cos(\gamma) \\ V_r \\ \frac{V_v}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix}_{\alpha=0}$$

$$\gamma = \tan^{-1}\left[\frac{V_r}{V_v}\right]$$

$$\beta = \frac{\mathrm{m}}{\mathrm{C_D} A_{ref}}$$

$$\dot{X} = f\left[X, F_{thrust}\right]$$

Predictor/Corrector Algorithm

$$\text{Given}\left[\Delta t, \hat{X}_k, F_{thrust_k}, \theta_k\right]$$

"trapezoidal rule"

Prediction Step:

$$\Rightarrow \tilde{\hat{X}}_{k+1} = \hat{X}_k + \Delta t\, f\left[\hat{X}_k, F_{thrust_k}, \theta_k\right]$$

Correction Step:

$$\widehat{\mathbf{X}}_{\mathbf{k}+1} \Rightarrow \hat{X}_{k+1} = \hat{X}_k + \frac{\Delta t}{2}\left\{f\left[\hat{X}_k, F_{thrust_k}, \theta_k\right] + f\left[\tilde{\hat{X}}_{k+1}, F_{thrust_{k+1}}, \theta_{k+1}\right]\right\}$$

Slide Indices and Repeat:

$$\widehat{\mathbf{X}}_{\mathbf{k}+1} \Rightarrow \widehat{\mathbf{X}}_{\mathbf{k}}$$

4th Order Runge-Kutta Method

Summary

$$\hat{\dot{x}}^{(1)} = f\left[t_k,\ x_k,\ F_{thrust_k}\right] \equiv k_1$$

$$\hat{\dot{x}}^{(2)} = f\left[t_k+\frac{\Delta t}{2},\ x_k + \frac{\Delta t}{2}\hat{\dot{x}}^{(1)},\ F_{thrust_k}\right] = f\left[t_k+\frac{\Delta t}{2},\ x_k + \frac{\Delta t}{2}k_1,\ F_{thrust_k}\right] \equiv k_2$$

$$\hat{\dot{x}}^{(3)} = f\left[t_k+\frac{\Delta t}{2},\ x_k + \frac{\Delta t}{2}\hat{\dot{x}}^{(2)},\ F_{thrust_{k+1}}\right] = f\left[t_k+\frac{\Delta t}{2},\ x_k + \frac{\Delta t}{2}k_2,\ F_{thrust_{k+1}}\right] \equiv k_3$$

$$\hat{\dot{x}}^{(4)} = f\left[t_k+\Delta t,\ x_k + \Delta t\,\hat{\dot{x}}^{(3)},\ F_{thrust_{k+1}}\right] = f\left[t_k+\Delta t,\ x_k + \Delta t\,k_3,\ F_{thrust_{k+1}}\right] \equiv k_4$$

$$\left[\overline{\hat{\dot{x}}}\right] = \left[\frac{\hat{\dot{x}}^{(1)} + 2\hat{\dot{x}}^{(2)} + 2\hat{\dot{x}}^{(3)} + \hat{\dot{x}}^{(4)}}{6}\right] \Rightarrow$$

$$\hat{x}_{k+1} = \hat{x}_k + \frac{\Delta t}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

- Slide Indices and repeat

Initial Conditions

$$\dot{X} = f\left[X, F_{thrust}, \theta\right]$$

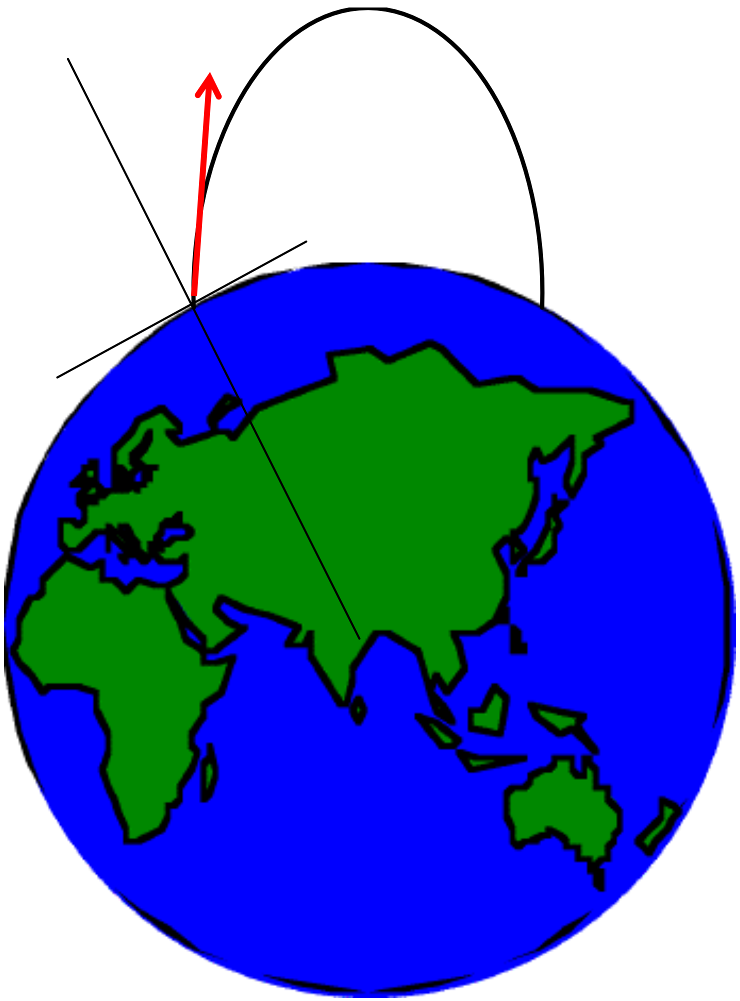
$$\dot{X} = \begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{v} \\ \dot{m} \end{bmatrix} \rightarrow f\left[X, F_{thrust}, \theta\right] = \begin{bmatrix} \frac{V_v^2}{r} + \frac{F_{lift}\cos(\gamma) - F_{drag}\sin(\gamma) + F_{thrust}\sin(\theta)}{m} - \frac{\mu}{r^2} \\ -\left[\frac{V_r V_v}{r} + \frac{F_{lift}\sin(\gamma) + F_{drag}\cos(\gamma) - F_{thrust}\cos(\theta)}{m}\right] \\ V_r \\ \frac{V_v}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix} \rightarrow$$

$$\gamma = \tan^{-1}\left[\frac{V_r}{V_v}\right]$$

$$\theta = \gamma + \alpha$$

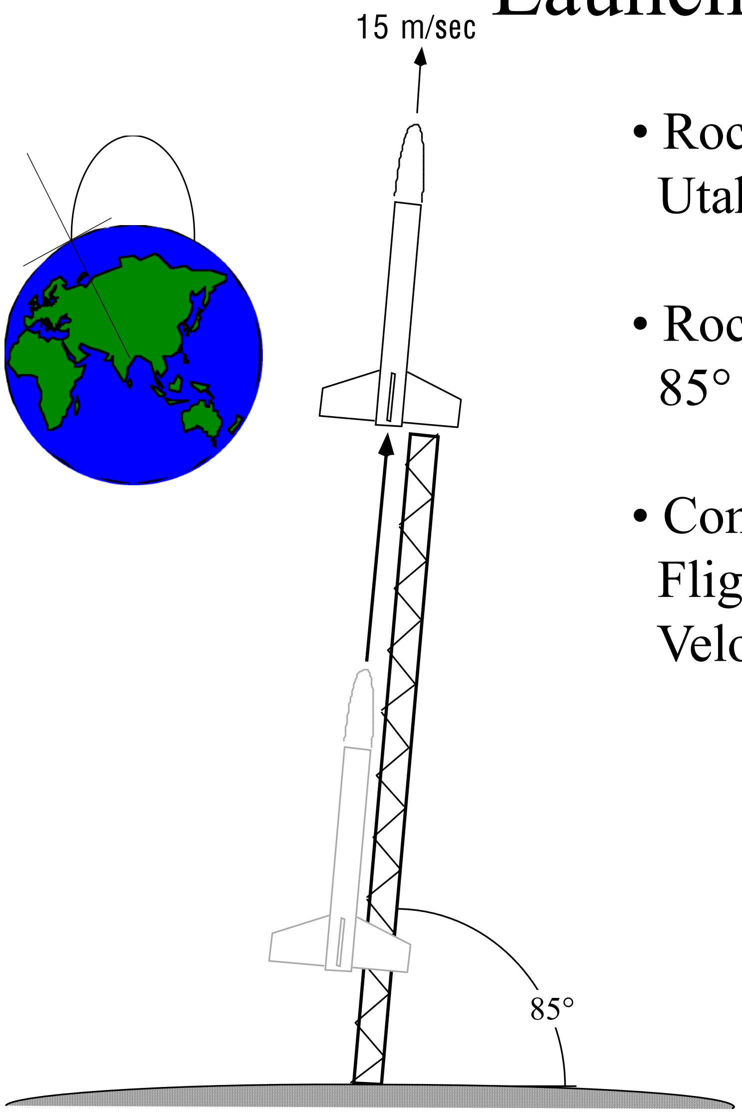
Need starting conditions for state vector X

Fixed Earth Approximation



- Ignore effects of rotation
- $V_{inertial} = V_{ground}$
- $\gamma_{inertial} = \gamma_{ground}$
- Accurate for Short Duration lower altitude flights

Launch I.C. Example:



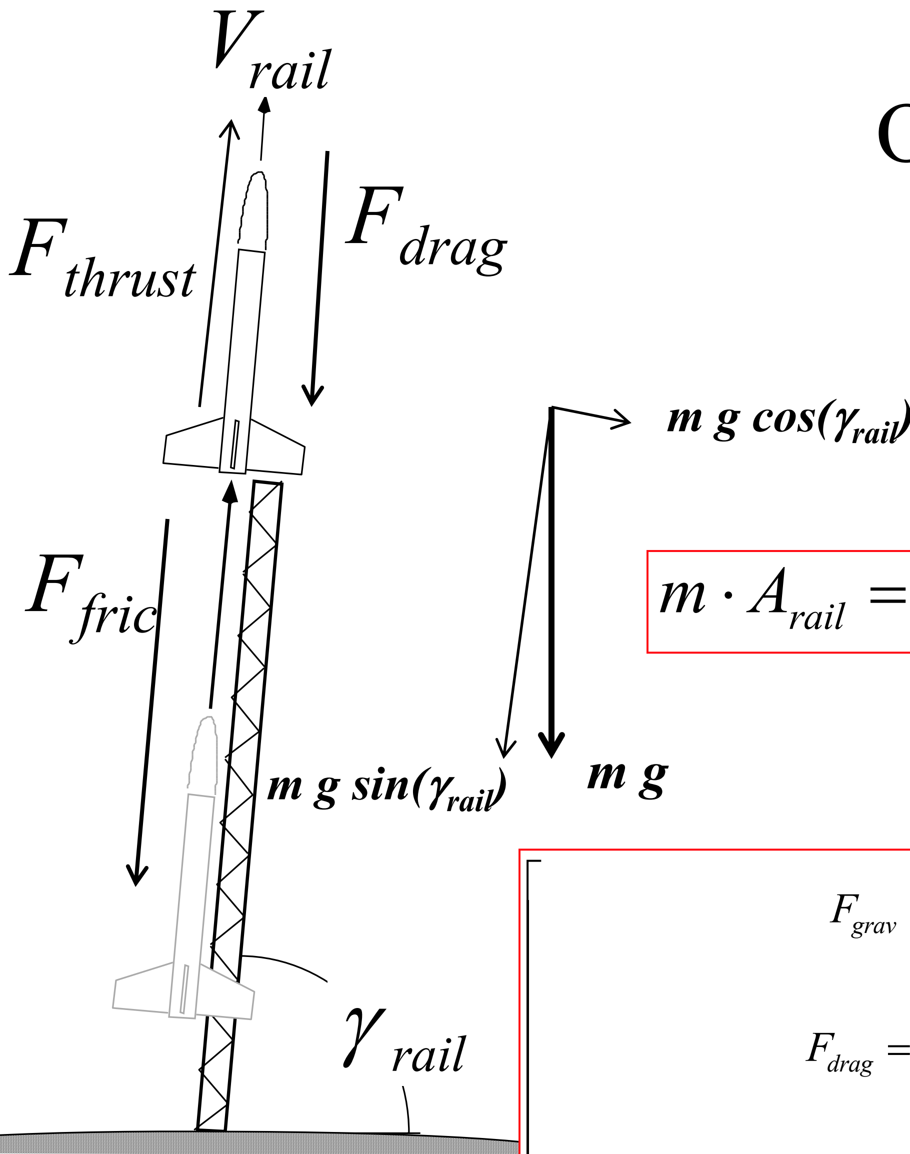
- Rocket Launch from Green River Utah -- 38° N. latitude, 3970 ft. altitude (1.21 km)

- Rocket Leaves Launch Rail at 85° angle to Local Vertical

- Compute Ground Relative, Inertial Flight path Angle, Initial Position, Velocity Vector

Solar day:86164.1 sec
Ω_{earth}: 7.292115e-0.5 rad/sec

Velocity Off of the Rail (1)



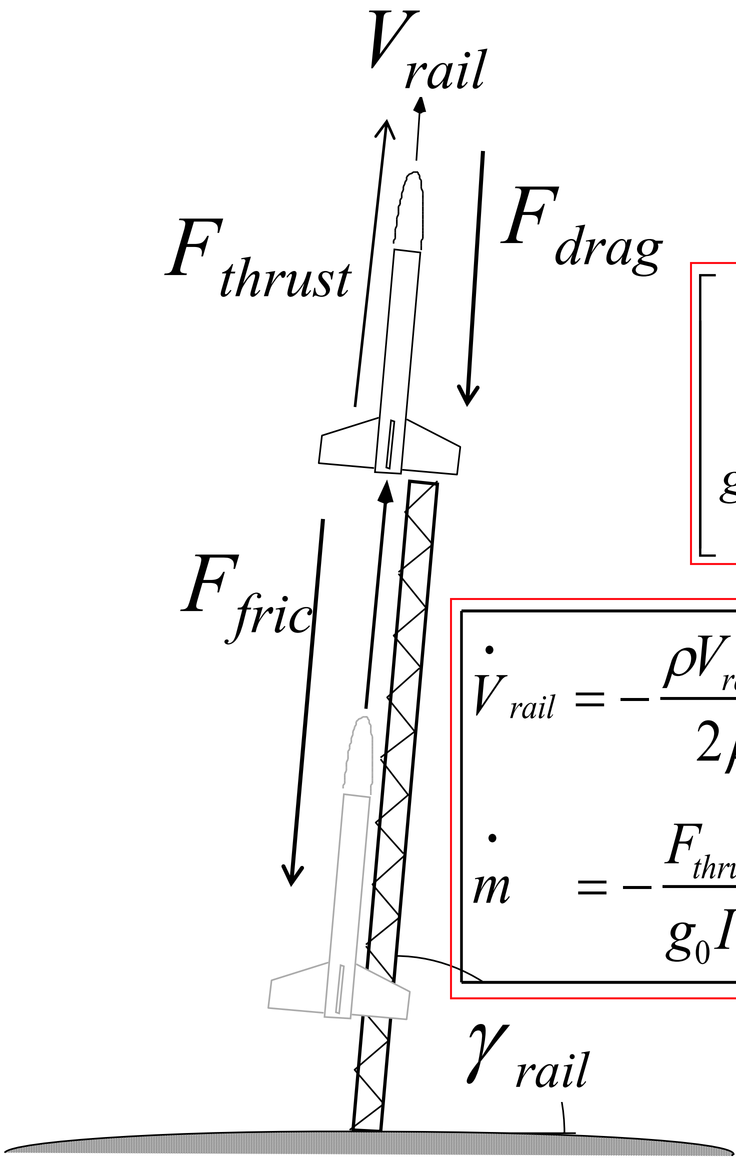
$$m \cdot A_{rail} = F_{thrust} - F_{grav} - F_{fric} - F_{drag}$$

$$F_{grav} = m \cdot g \cdot \sin(\gamma_{rail}) = m \frac{\mu}{(R_e + h)^2} \cdot \sin(\gamma_{rail})$$

$$F_{drag} = C_D A_{ref} \left(\frac{1}{2} \rho V_{rail}^{\ 2}\right) = m \frac{\rho V_{rail}^{\ 2}}{2\beta}$$

$$F_{fric} = C_f \cdot W_{norm_{rail}} = C_f \cdot m \cdot g \cos(\gamma_{rail}) = C_f \cdot m \frac{\mu}{(R_e + h)^2} \cdot \cos(\gamma_{rail})$$

Velocity Off of the Rail (2)



$$m \cdot A_{rail} = F_{thrust} - F_{grav} - F_{fric} - F_{drag}$$

$$\left[\begin{array}{l} \beta = \dfrac{m}{C_D A_{ref}} \\ g = \dfrac{\mu}{(R_e + h)^2} \end{array}\right] \rightarrow careful!\ with\ units$$

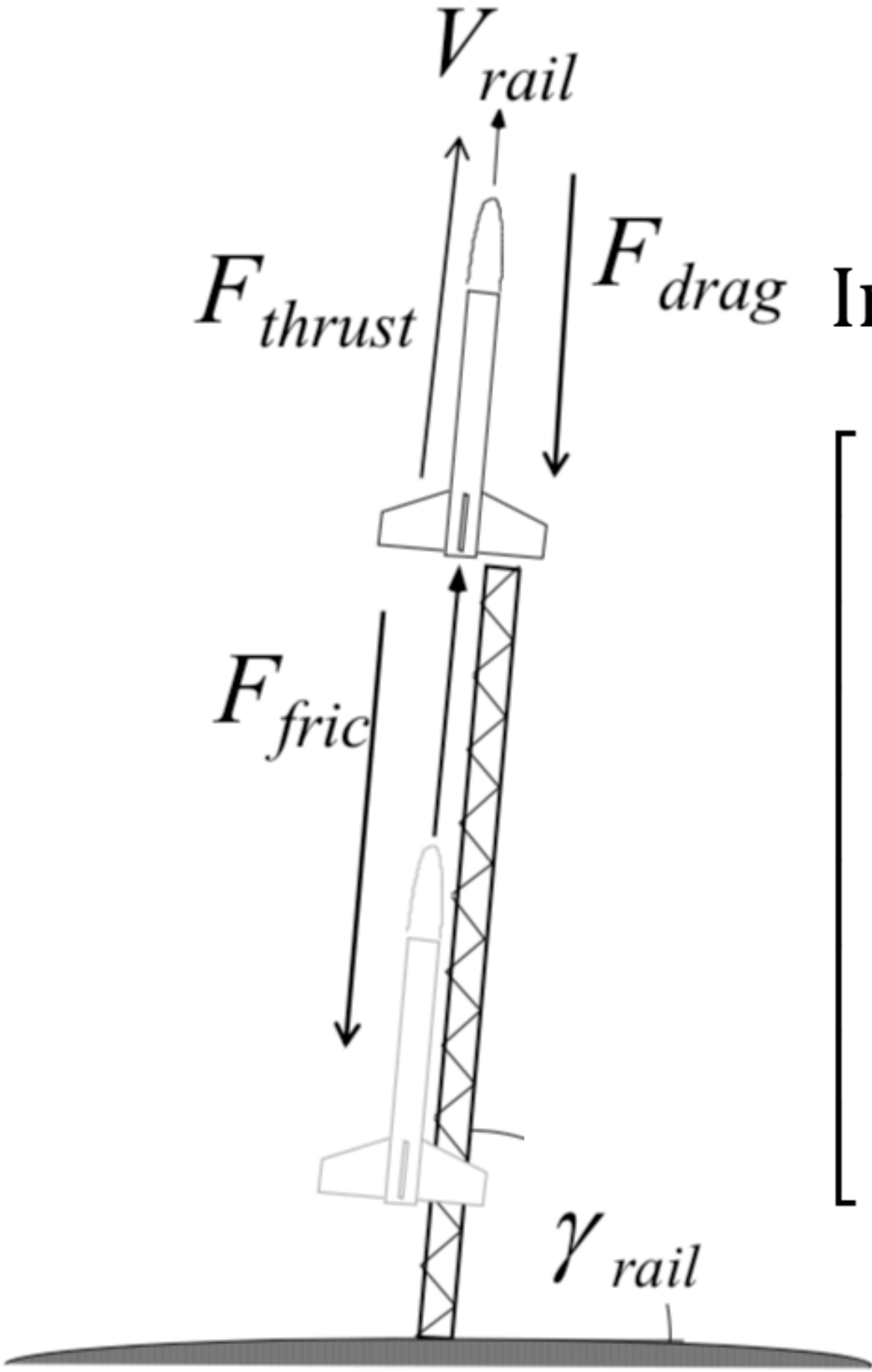
$$\dot{V}_{rail} = -\frac{\rho V_{rail}^2}{2\beta} - \frac{\mu}{(R_e + h)^2}\left[\sin(\gamma_{rail}) + C_f \cdot \cos(\gamma_{rail})\right] + \frac{F_{thrust}}{m}$$

$$\dot{m} = -\frac{F_{thrust}}{g_0 I_{sp}}$$

$\{\gamma_{rail}, V_{rail}\}$-->ground relative

$\{V_0 = 0,\ m_0 = M_{total}\}$

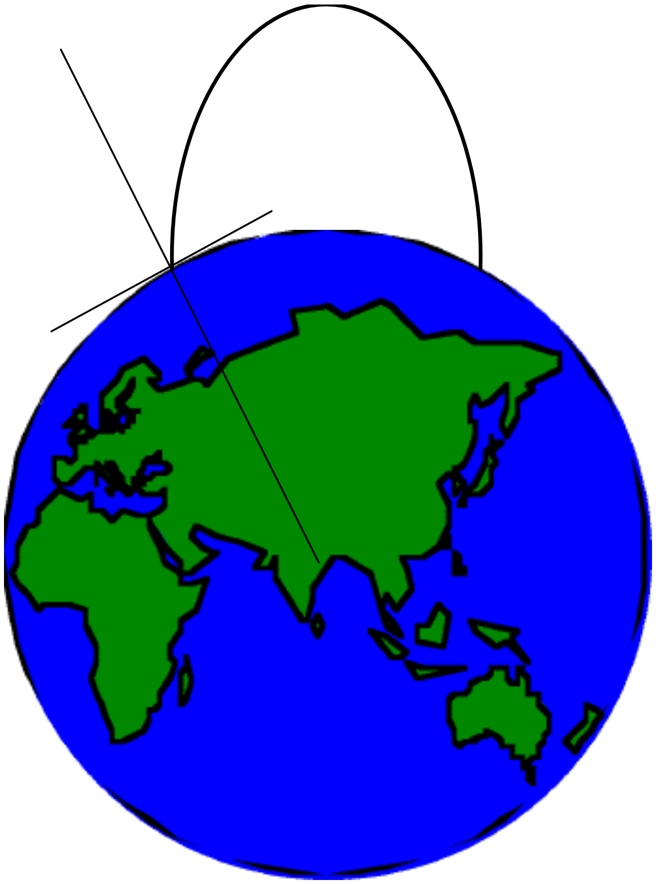
Velocity Off of the Rail *(3)*



Initial Conditions Off of Rail

$$\begin{bmatrix} V_r \\ V_v \\ r \\ v \\ m \end{bmatrix}_0 = \begin{bmatrix} V_{rail} \cdot \sin\gamma_{rail} \\ V_{rail} \cdot \cos\gamma_{rail} \\ R_\oplus + L_{rail} \cdot \sin\gamma_{rail} \\ \frac{\pi}{2} - \gamma_{rail} \\ m \end{bmatrix}_{\substack{rail \\ exit}}$$

Initial Conditions: Ground Launch, Rotating Earth



- Inertial Flight Path Angle

$$\gamma_{inertial} = \tan^{-1}\left[\frac{V_r}{V_\nu}\right]$$

- Ground Relative Flight Path Angle

$$\gamma_{ground} = \tan^{-1}\left[\frac{V_r}{V_\nu - V_{E_{eq}} \cos(Lat)}\right]$$

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Initial Conditions: Ground Launch, Rotating Earth

Initial Velocity Vector :

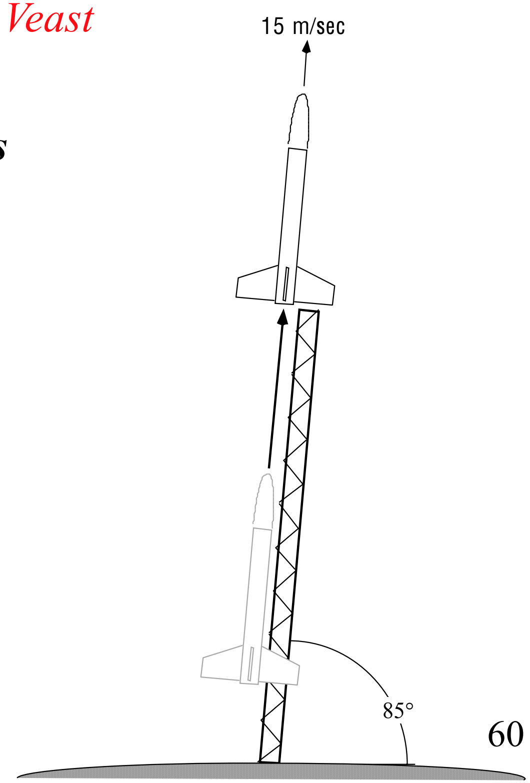
$$\begin{bmatrix} V_r = V_0 \sin(\gamma_{ground}) \rightarrow V_0 = Initial\ Groundspeed \\ V_v = \sqrt{\left[V_0 \cos(az_{launch})\cos\left(\gamma_{ground}\right)\right]^2 + \left[V_0 \sin(az_{launch})\cos\left(\gamma_{ground}\right) + V_{E_{eq}} \cos\left(Lat\right)\right]^2} \end{bmatrix}$$

Vnorth (first bracketed term) | Veast (second bracketed term)

Initial "Orbit"

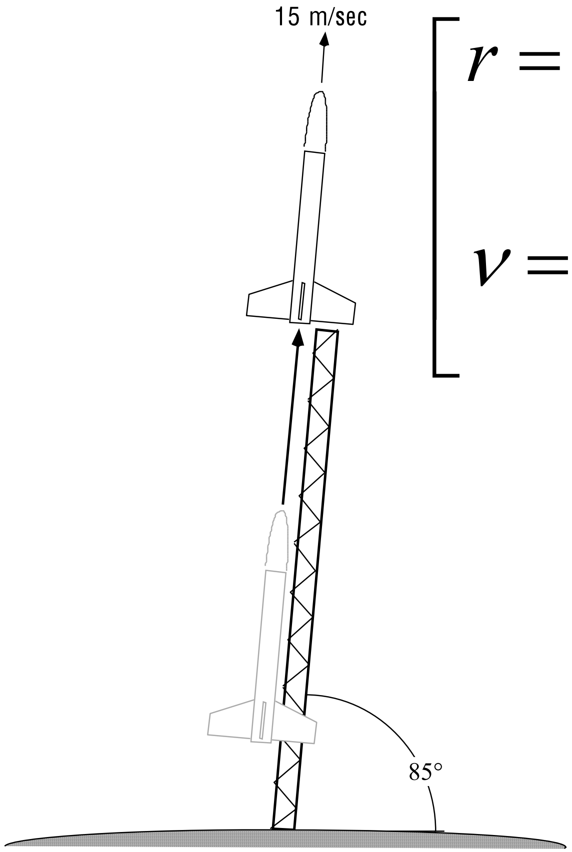
- ***See appendix 2 at end of slides***

$$\begin{bmatrix} a = \dfrac{\mu}{\left[\dfrac{2\mu}{R_{e_{(Lat)}} + h} - \left[V_r^2 + V_v^2\right]\right]} \\ e = \dfrac{R_{e_{(Lat)}} + h}{\mu}\sqrt{\left(V_v^2 - \dfrac{\mu}{R_{e_{(Lat)}} + h}\right)^2 + \left(V_r V_v\right)^2} \end{bmatrix}$$



Initial Conditions: Ground Launch, Rotating Earth (cont'd)

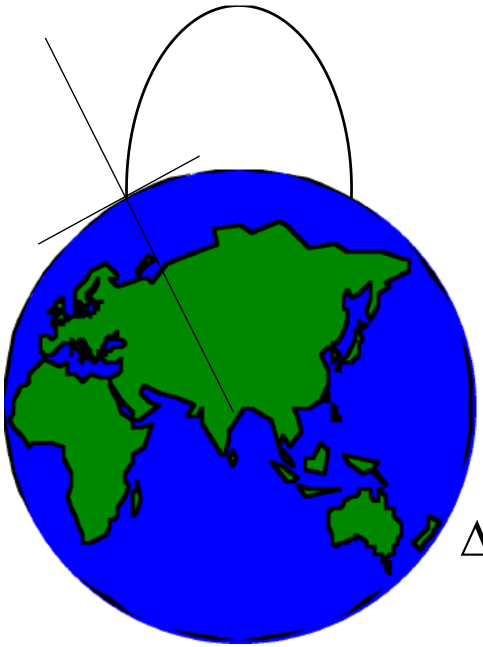
Initial Position



$$\begin{bmatrix} r = R_{e_{(Lat)}} + h \\ \nu = atan2\left\{\frac{a}{r}\left[1-e^2\right]\frac{V_r}{V_\nu}, \frac{a}{r}\left[1-e^2\right]-1\right\} \end{bmatrix}$$

- Initial Mass, m_0
  - ***See appendix 2 at end of slides***

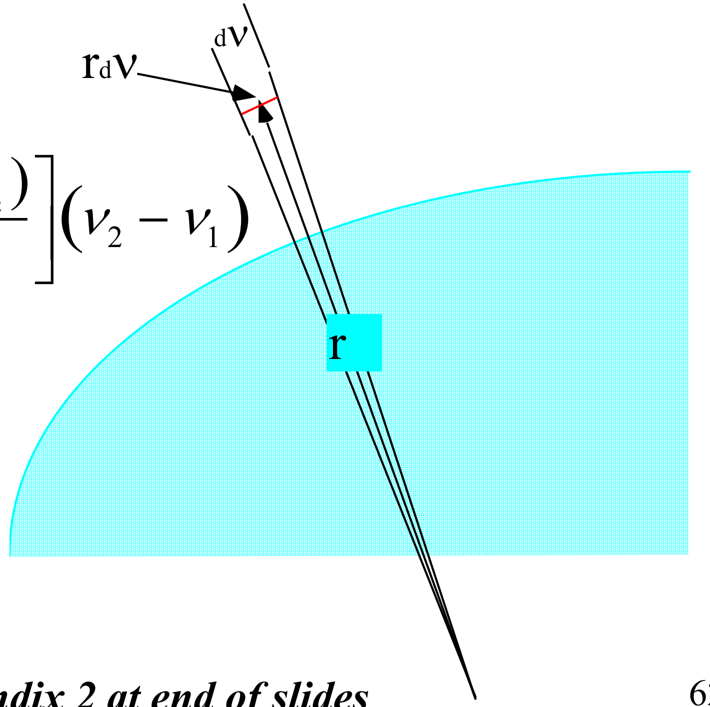
Ground Launch: Down Range Calculation



- Integrated trajectory gives

$$r, \nu$$

- Inertial Downrange



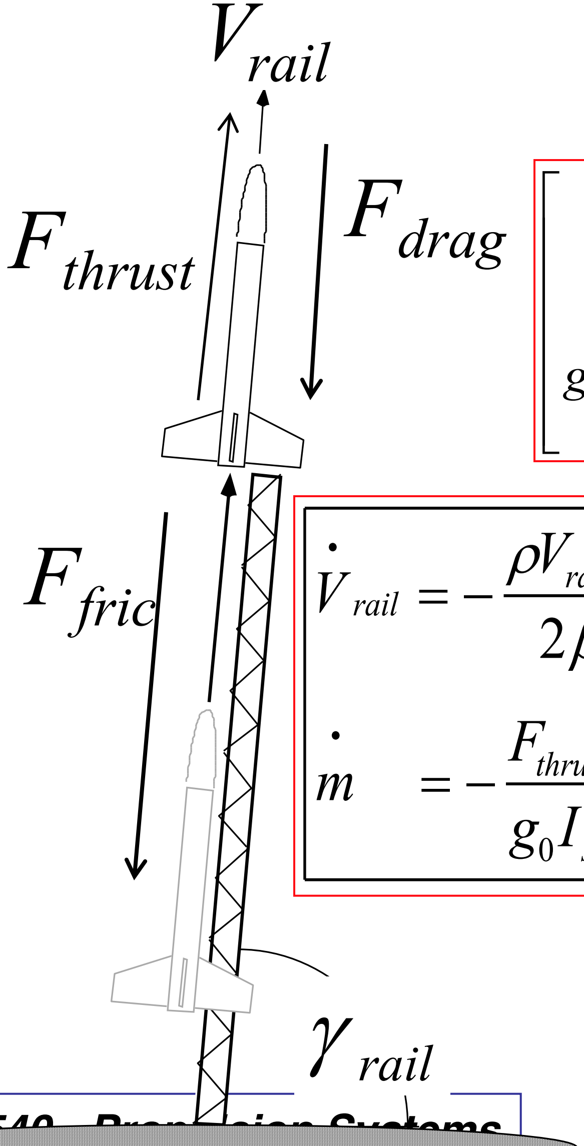
$$\Delta R = \int_{\nu_1}^{\nu_2} r d_n \approx \left[\frac{r(\nu_1) + r(\nu_2)}{2}\right](\nu_2 - \nu_1)$$

Recursive Formula

$$R_{i+1} = R_i + \left[\frac{r_{i+1} + r_i}{2}\right](\nu_{i+1} - \nu_i)$$

• See appendix 2 at end of slides

Velocity Off of the Rail



- *See appendix 2 at end of slides*

$$\left[\begin{array}{l} \beta = \dfrac{m}{C_D A_{ref}} \\ g = \dfrac{\mu}{(R_e + h)^2} \end{array}\right] \rightarrow \textit{careful! with units}$$

$$\dot{V}_{rail} = -\frac{\rho V_{rail}^2}{2\beta} - \frac{\mu}{(R_e + h)^2}\left[\sin(\gamma_{rail}) + C_f \cdot \cos(\gamma_{rail})\right] + \frac{F_{thrust}}{m}$$

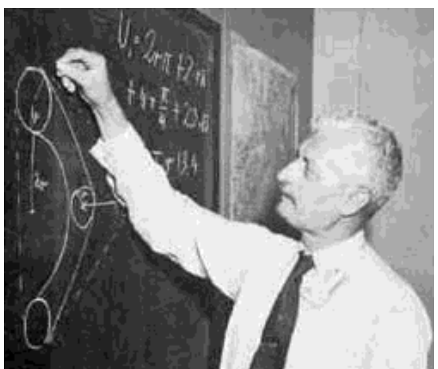
$$\dot{m} = -\frac{F_{thrust}}{g_0 I_{sp}}$$

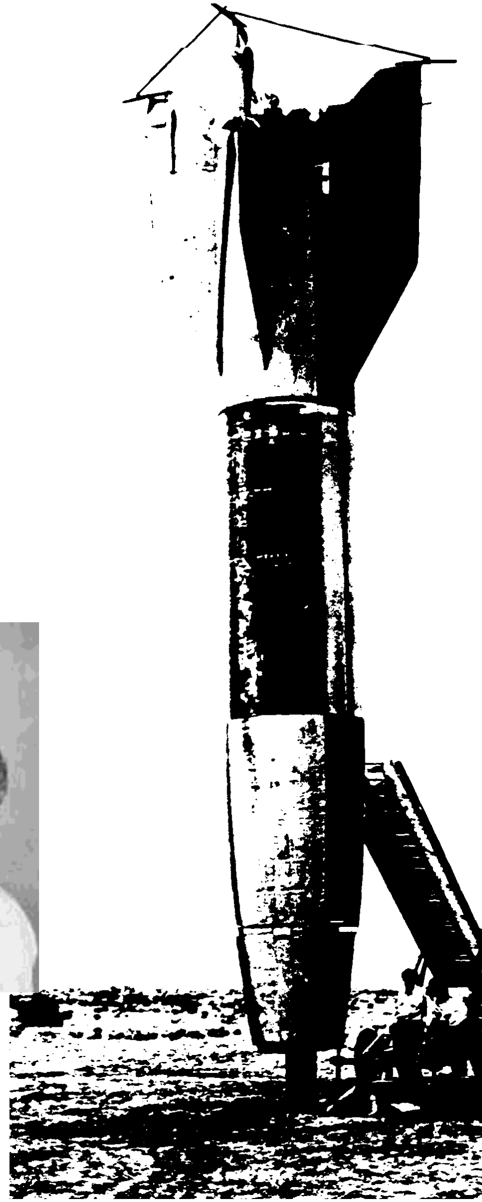
$\{\gamma_{rail}, V_{rail}\}$-->ground relative

$\{V_0=0, m_0=M_{total}\}$

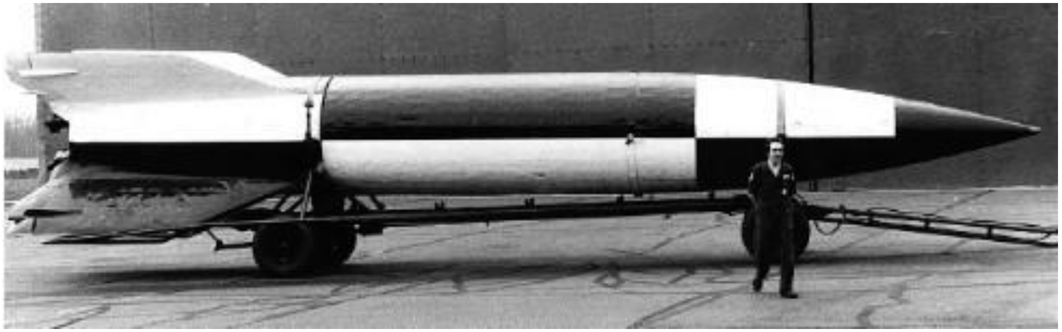
V2 Rocket Example … Ballistic Trajectory

V-2 Rocket
First Op erational System



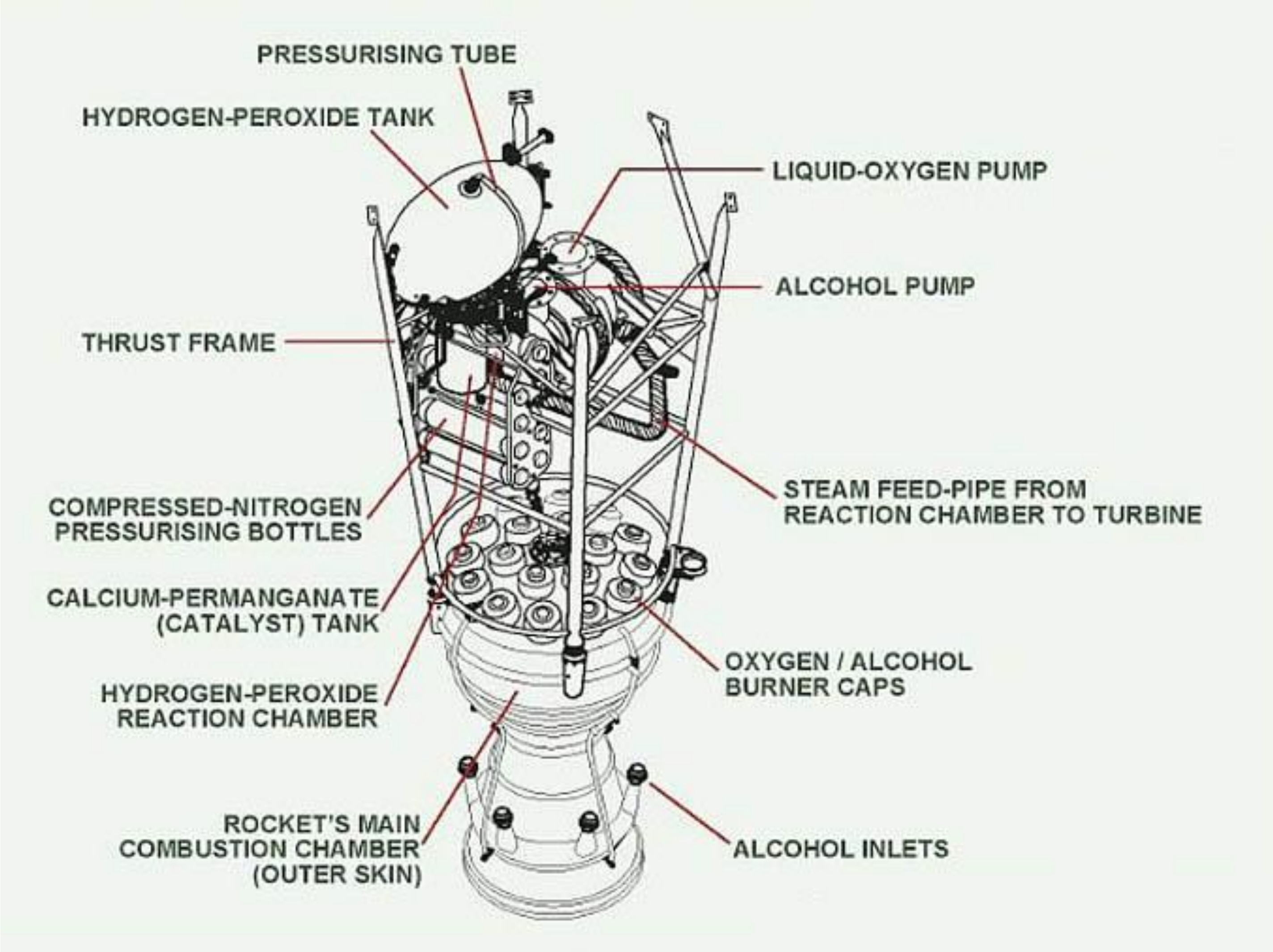


The V2



- Challenge was to deliver a one ton warhead, 180 nm range.
- Final design: Powered by A4 rocket motor, 2300 lb warhead, 190 nm (352 km) range. 47 ft long, 5.4 ft diameter, 28,229 lb takeoff weight. 59,500 lb thrust for 68 seconds.
- 6400 weapon launches
- The Americans got Von Braun and 117 other scientists, and about 100 rockets. The Soviets got the facilities and about the same number of rockets.
- 60 plus V2's and V2 mods were launched in the late 40's in US. All were sub-orbital, highest altitude was 244 miles
- LOX/Alcohol Propellants

A-4 Rocket Engine



V2 Rocket Parameters

Parameter	Value
F_{vac} (Nt)	311,800.0
F_{sl} (Nt)	264,745.3
I_{sp} (sec)	239
V_{exit} (m/sec)	2200
M_{dry}" (kg)	4008
$M_{propellant}$ (kg)	8797
GTOW (kg-f)	12805
Payload weight (warhead) (kg-f)	1043.56

V2 Rocket Parameters (2)

… Additional Specs

Exhaust velocity	V_e = 2,200 m/s
Mass ratio	m_o/m = 3.2
Gross mass	m_o = 12,805 kg
Empty mass	m = 4,008 kg
Vacuum thrust	F = 311,800 kN
Specific impulse	I_{sp} = 239 sec
Burn time	T = 68 sec
Length	L = 12 m
Diameter	D = 1.65 m
Propellants	Lox/Alcohol
Burnout Velocity	v_r = 5,500 km/hr

Additional V2 Data

(http://en.wikipedia.org/wiki/V-2)

Specifications	
Weight	12,500 kg (28,000 lb)
Length	14 m (45 ft 11 in)
Diameter	1.65 m (5 ft 5 in)
Warhead	980 kg (2,200 lb) Amatol
Wingspan	3.56 m (11 ft 8 in)
Propellant	3,810 kg (8,400 lb) of 75% ethanol and 25% water + 4,910 kg (10,800 lb) of liquid oxygen
Operational range	320 km (200 mi)
Flight altitude	88 km (55 mi) maximum altitude on long range trajectory, 206 km (128 mi) maximum altitude if launched vertically.

Specifications	
Speed	maximum: 1,600 m/s (5,200 ft/s) 5,760 km/h (3,580 mph) at impact: 800 m/s (2,600 ft/s) 2,880 km/h (1,790 mph)
Guidance system	Gyroscopes for attitude control Müller-type pendulous gyroscopic accelerometer for engine cutoff on most production rockets (10% of the Mittelwerk rockets used a guide beam for cutoff.)[2]:225
Launch platform	Mobile (Meillerwagen)

V2 1-D Thrust Model

$$F_{thrust} = \dot{m} \cdot V_{exit} + \left(P_{exit} - P_{ambient} \right) \cdot A_{exit}$$

$$\rightarrow F_{vac} - F_{sl} = \left[\dot{m} \cdot V_{exit} + \left(P_{exit} - 0 \right) \cdot A_{exit} \right] - \left[\dot{m} \cdot V_{exit} + \left(P_{exit} - P_{sl} \right) \cdot A_{exit} \right]$$

$$\rightarrow F_{vac} - F_{sl} = P_{sl} \cdot A_{exit}$$

$$A_{exit} = \frac{F_{vac} - F_{sl}}{P_{sl}} = \frac{311800.0 - 264745.3}{101325} = 0.46439 \text{ m}^2$$

=> 0.76895 m effective exit diameter

V2 1-D Thrust Model (2)

$$\left(I_{sp}\right)_{vac} = \frac{F_{vac}}{g_0 \cdot \dot{m}} \rightarrow \dot{m} = \frac{F_{vac}}{g_0 \cdot \left(I_{sp}\right)_{vac}} = \frac{311800}{9.8067 \cdot 239}$$

$$= 133.032 \text{ kg/sec}$$

$$T_{burn} = \frac{M_{propellant}}{\dot{m}} = \frac{8797}{133.032} = 66.13 \text{ sec}$$

Slight inconsistency in problem specification (68 sec given)

Need Vacuum I_{sp} = *245.77 sec* to give *68 sec* burn time for given propellant mass *(8797 kg)*

V2 1-D Thrust Model (3)

$$\rightarrow F_{vac} = \left[\dot{m} \cdot V_{exit} + P_{exit} \cdot A_{exit}\right] \rightarrow P_{exit} = \frac{F_{vac} - \dot{m} \cdot V_{exit}}{A_{exit}} ==$$

$$\frac{311800 - 133.032 \cdot 2200}{0.46439} = 41{,}192.96 \text{ Pa}$$

V2 1-D Thrust Model (4)

$$F_{thrust} = \dot{m} \cdot V_{exit} + \left(P_{exit} - P_{ambient} \right) \cdot A_{exit}$$

$$\Rightarrow \begin{bmatrix} \dot{m} \\ V_{exit} \\ P_{exit} \\ A_{exit} \end{bmatrix} = \begin{bmatrix} 133.032_{\,kg/\text{sec}} \\ 2200_{\,m/\text{sec}} \\ 41{,}192.96_{\,Pa} \\ 0.46439_{\,m^2} \end{bmatrix}$$

Rocket Equation Calculation(s)

Based on $I_{sp} = 239$ *sec*

$$\Delta V = g_0 \cdot I_{sp} \cdot \ln\left(\frac{M_{initial}}{M_{final}}\right) = 9.8067 \cdot 239 \ln\left(\frac{12805}{4008}\right)$$

$$= 2722.42 \text{ m/sec } (9800.74 \text{ km/hr})$$

Ignores drag and gravity losses

Based on $I_{sp} = 245.77$ *sec*

$$\Delta V = g_0 \cdot I_{sp} \cdot \ln\left(\frac{M_{initial}}{M_{final}}\right) = 9.8067 \cdot 245.77 \ln\left(\frac{12805}{4008}\right)$$

$$= 2799.54 \text{ m/sec } (10{,}078.36 \text{ km/hr})$$

Collected Equations, Ballistic Trajectory

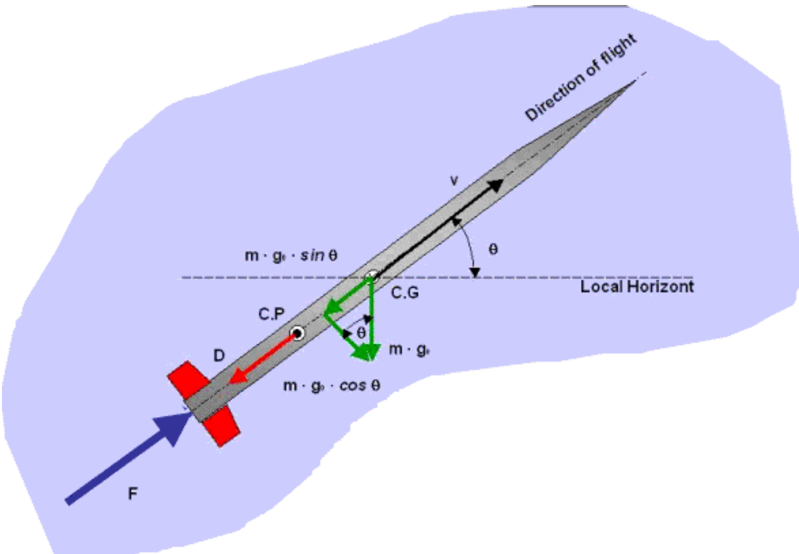
$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_\nu \\ \dot{r} \\ \dot{\nu} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \frac{V_\nu^{\,2}}{r} - \frac{\mu}{r^2} + \left[\frac{F_{thrust}}{m} - \frac{\rho V_\infty^{\,2}}{2\beta}\right]\sin(\gamma) \\ -\frac{V_r V_\nu}{r} + \left[\frac{F_{thrust}}{m} - \frac{\rho V_\infty^{\,2}}{2\beta}\right]\cos(\gamma) \\ V_r \\ \frac{V_\nu}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix}_{\alpha=0}$$

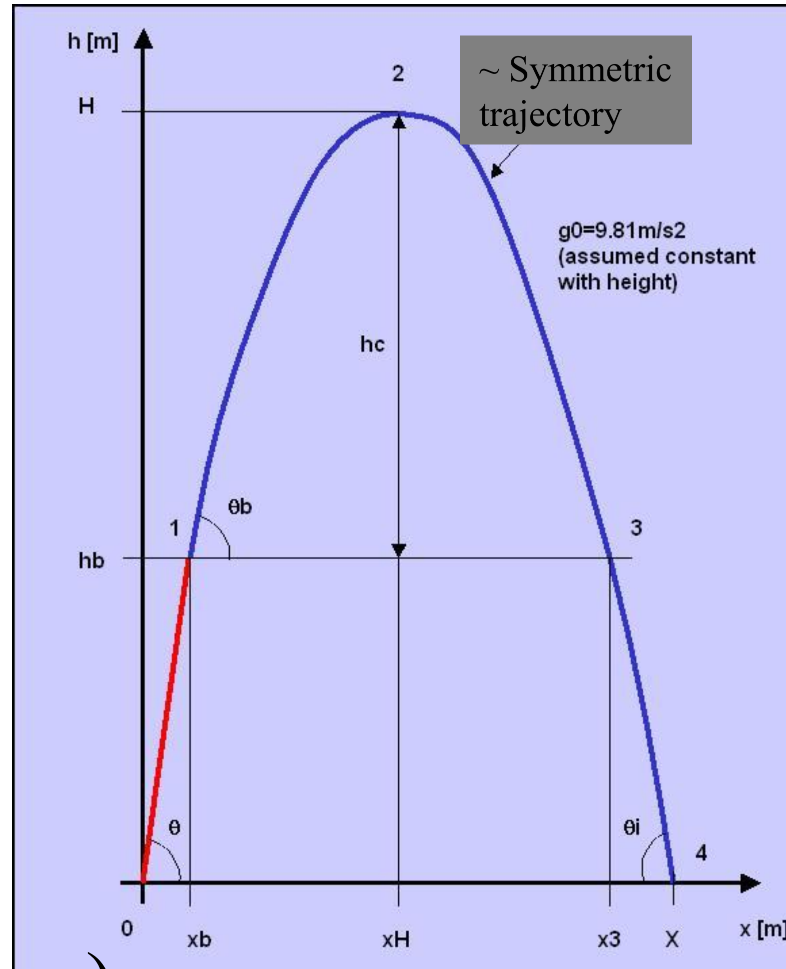
$$\gamma = \tan^{-1}\left[\frac{V_r}{V_\nu}\right]$$

$$\beta = \frac{\mathrm{m}}{\mathrm{C_D} A_{ref}}$$

$$\dot{X} = f\left[X, F_{thrust}\right]$$

Example of Ballistic Trajectory





- Ballistic Trajectories Offer minimum drag profiles (α ~ 0-->No induced drag)

V2 Computational Example

Assume constant (only marginally correct) drag coefficient

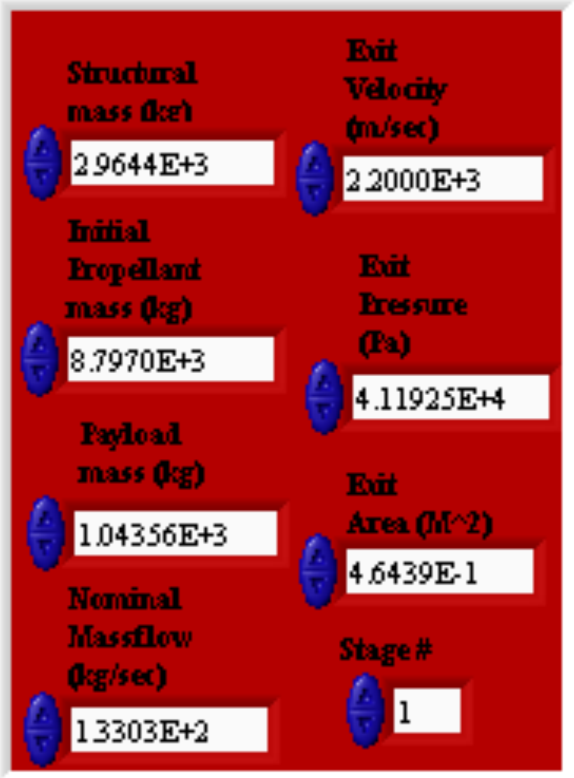
$C_D \sim 0.44,\ A_{ref} = 2.1382\ m^2$

→ $\beta(t) = M(t)/(C_D\, A_{ref})$

$$F_{thrust} = \dot{m} \cdot V_{exit} + \left(P_{exit} - P_{ambient}\right) \cdot A_{exit}$$

$$D_{rag} = C_D \cdot A_{ref} \cdot \left(\frac{1}{2}\rho \cdot V^2\right) = \frac{M(t) \cdot \left(\frac{1}{2}\rho \cdot V^2\right)}{\beta(t)}$$

Engine / Launch Mass Data



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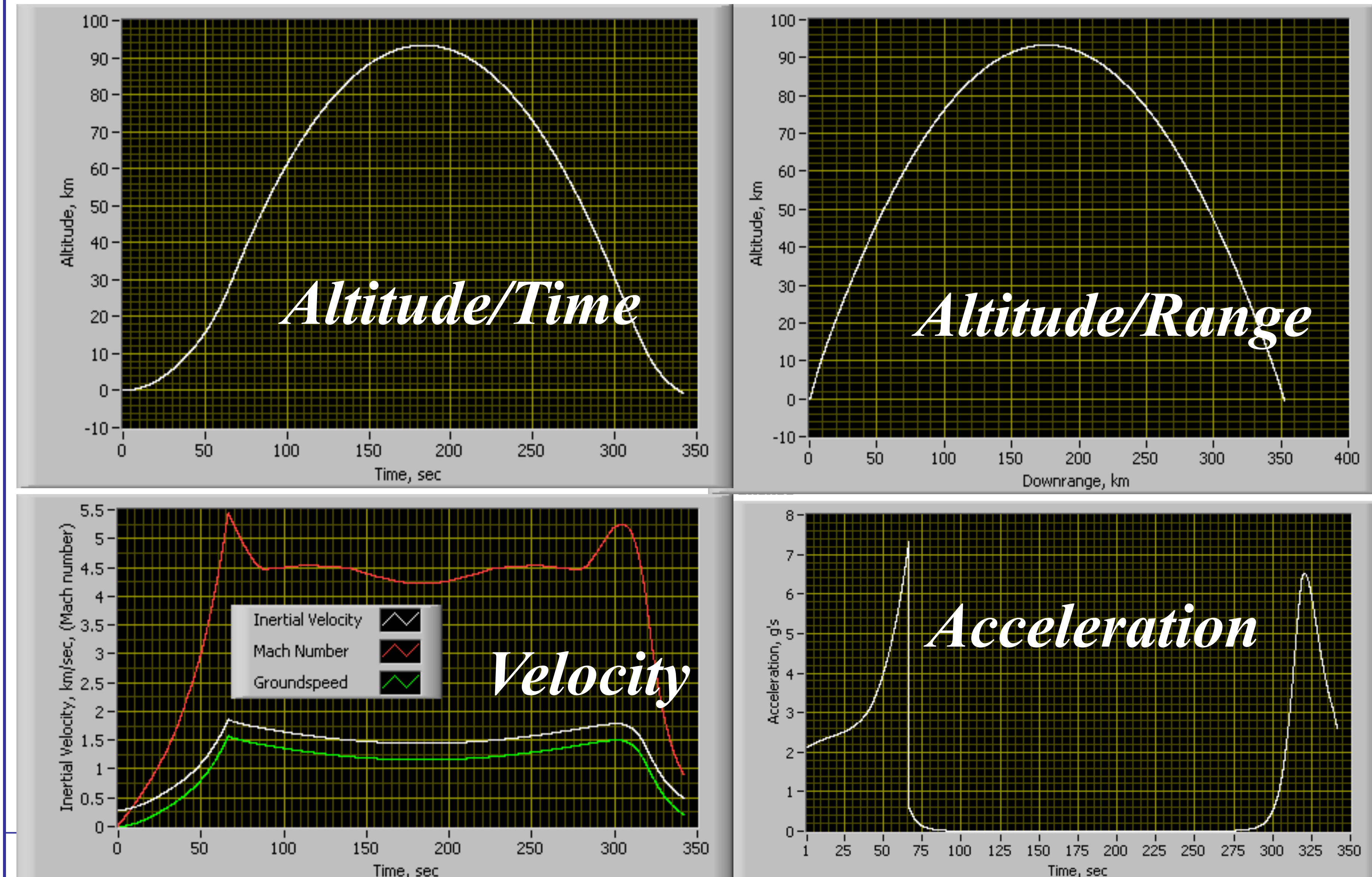
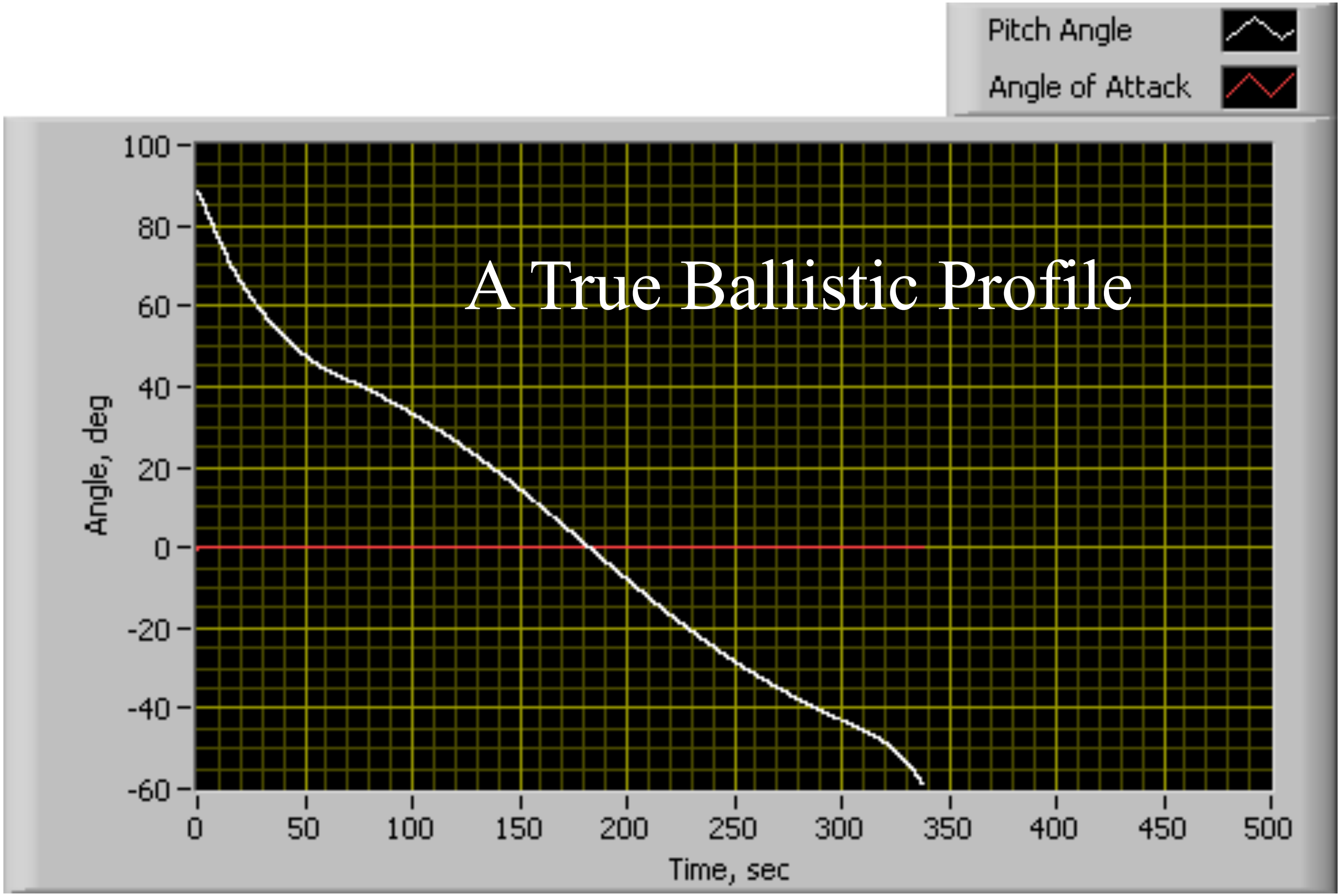
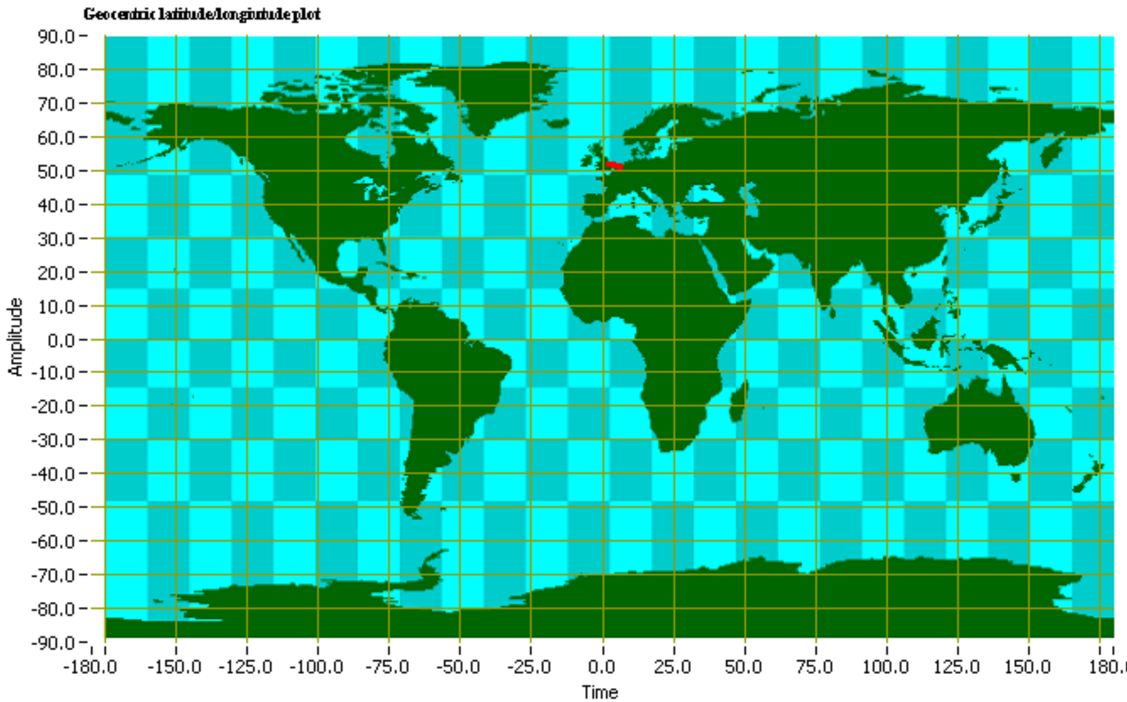
V2 Computational Example (2)

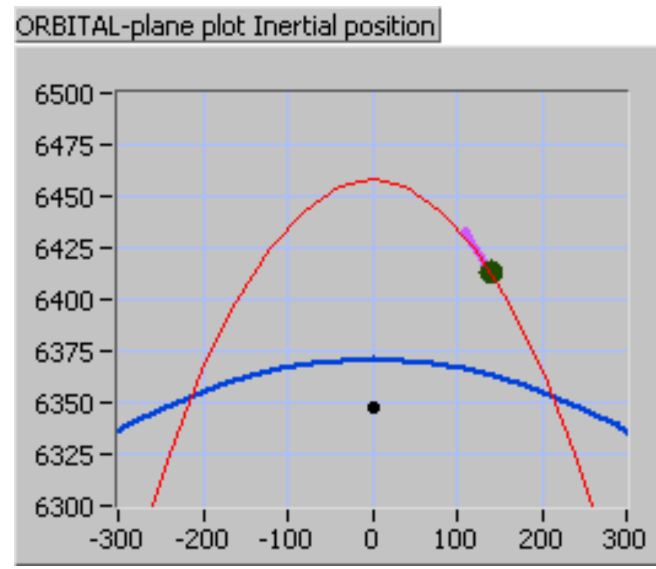


V2 Computational Example (3)



V2 Computational Example (4)





Misc. INSTANTANEOUS Orbit Parameters

Orbit perigee (km): 111.18

Perigee altitude (km) OblateEARTH: -6253.81

Orbit apogee (km): 6458.44

Apogee altitude (km) Oblate EARTH: 93.45

Mean Orbit Parameters

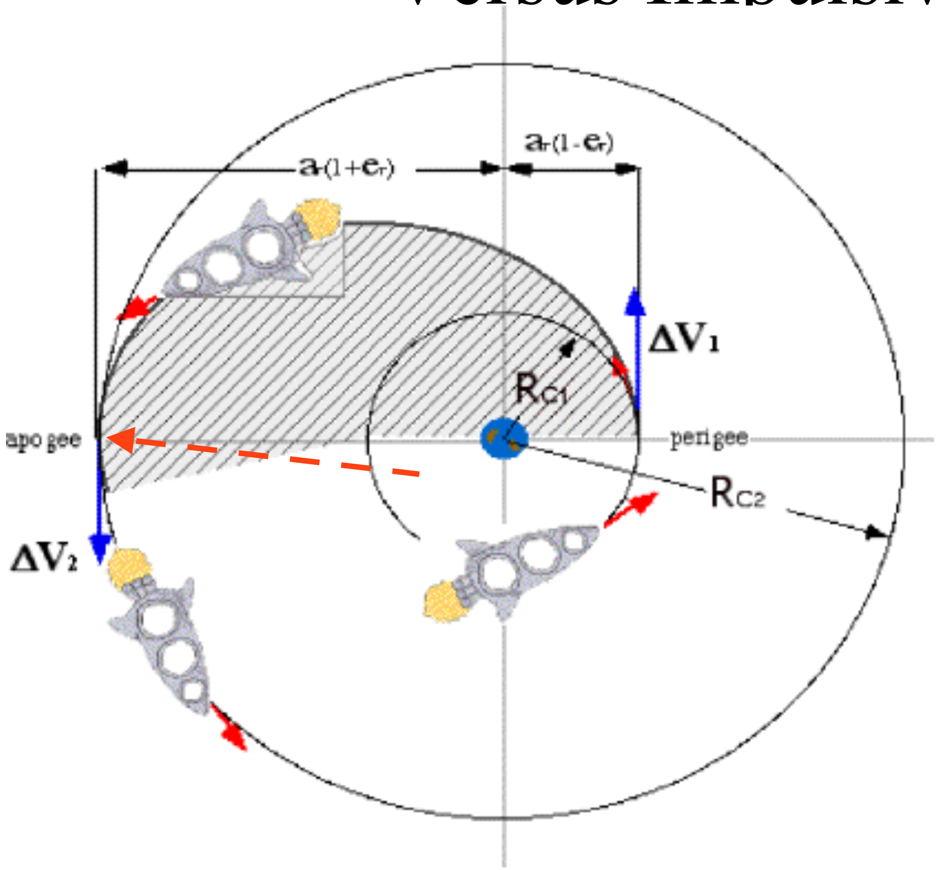
Perigee Altitude (km): -6259.82

Remaining Propellant Mass for stage: 1.00

Apogee Altitude (km): 87.44

Current time to apogee, sec: 95.62

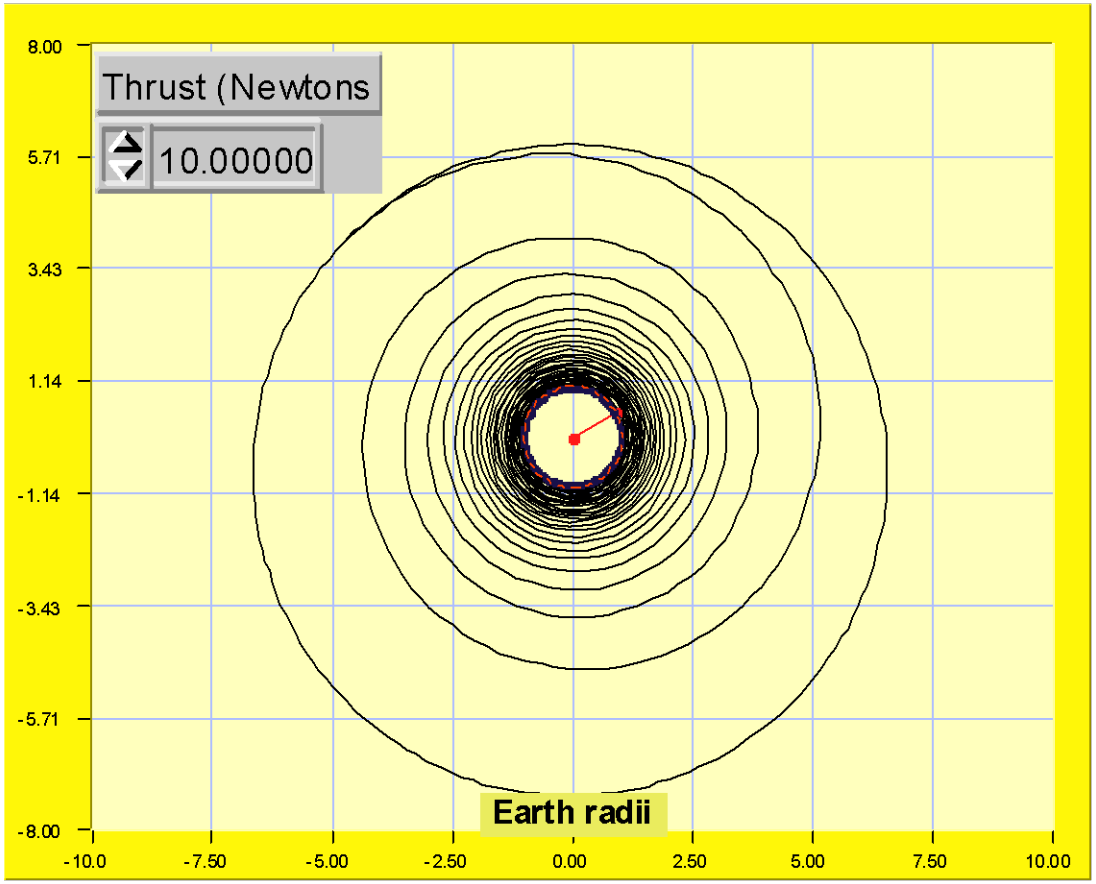
Example II: Comparison of Constant Thrust Maneuver Versus Impulsive Maneuver



- ***Hohmann transfer***
  … elliptical trajectory

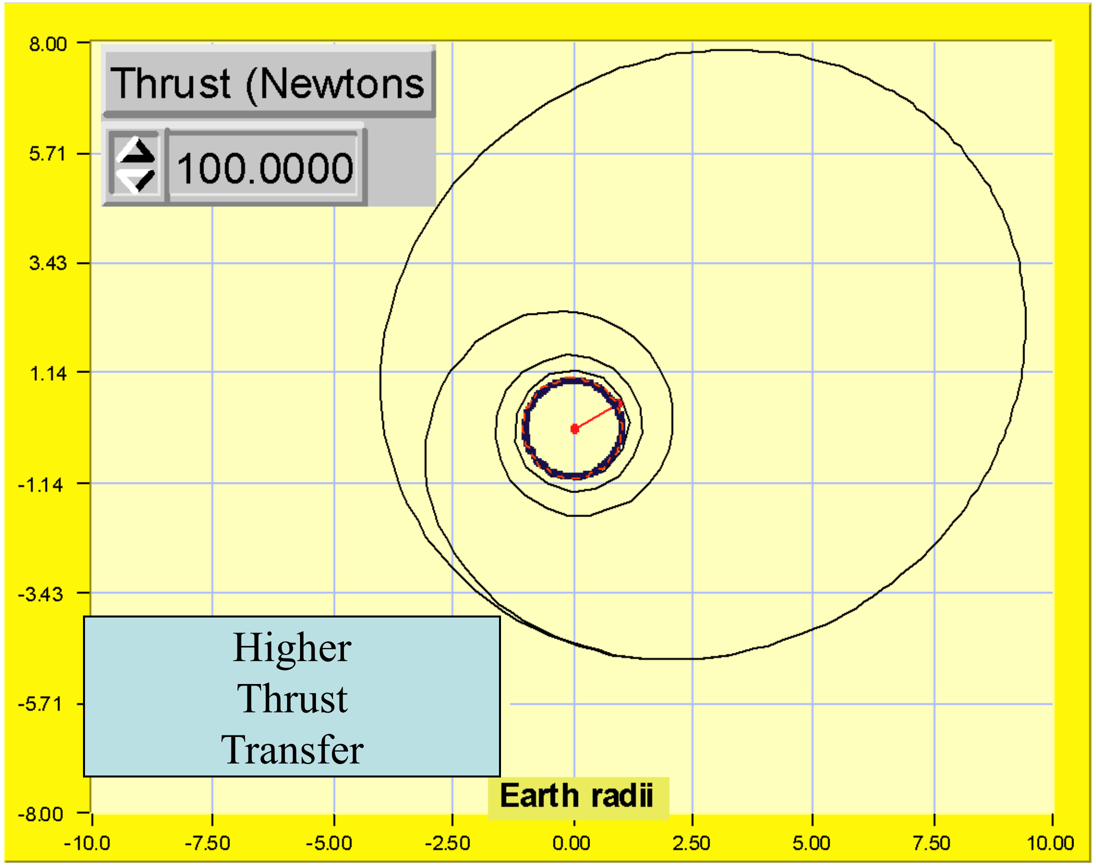
  … Kepler's laws

Comparison of Constant Thrust Maneuver Versus Impulsive Maneuver (cont'd)



- Continuous Thrust transfer

Comparison of Constant Thrust Maneuver Versus Impulsive Maneuver (cont'd)



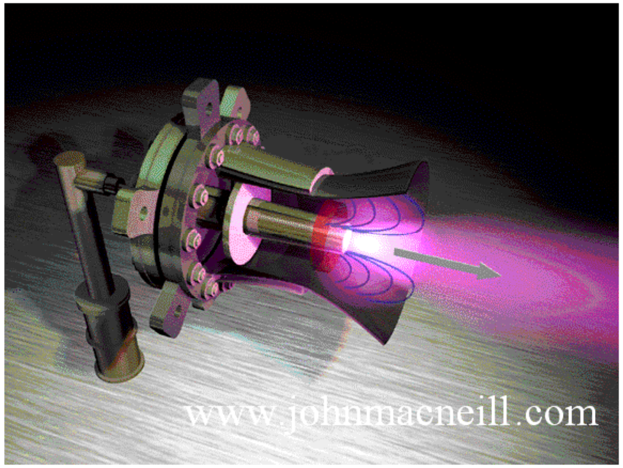
- Continuous Thrust transfer

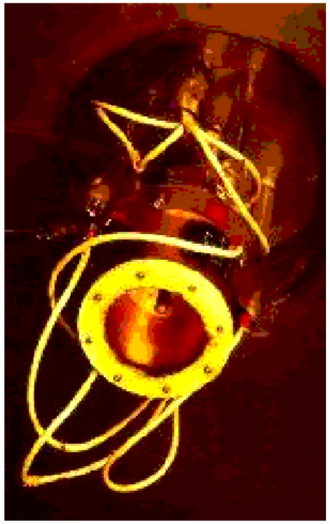
Worked EP Example

• Continuous Thrust GTO

Magnetoplasmadynamic (MPD) Thruster

Isp = ~4500 sec
η = 30%
Thrust = ~1 N (Steady)
~10 N (Pulsed)





lets look at the extreme case (cause I don't want to wait all day for my code to run)

Orbital Initial Conditions

- **Initial Orbit** **$\{t_0, a_0, e_0, \nu_0\}$**

$$\begin{bmatrix} r_0 \\ \nu_0 \end{bmatrix} = \begin{bmatrix} \left[\dfrac{a_0 \left[1 - e_0^2\right]}{1 + e_0 \cos(\nu_0)} \right] \\ \nu_0 \end{bmatrix}$$

- **Initial Velocity**

$$\begin{bmatrix} V_r \\ V_v \end{bmatrix}_0 = r_0\, \omega_0 \begin{bmatrix} \dfrac{\left[e_0 \sin(\nu_0) \right]}{\left[1 + e_0 \cos(\nu_0)\right]} \\ 1 \end{bmatrix}$$

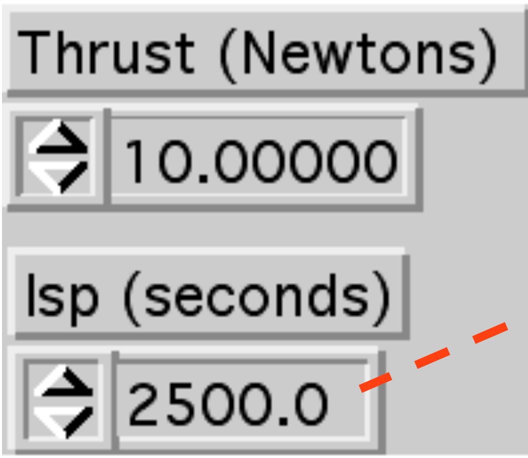
Orbital Initial Conditions

- **Initial Angular Velocity**

$$\omega_0 = \frac{\sqrt{\mu}}{\left[a_0\left[1 - e_0^2\right]\right]^{3/2}}\left[1 + e_0 \,\mathrm{Cos}\left(\nu_0\right)\right]^2$$

- $M_0 \equiv$ Initial Mass

• Continuous Thrust GTO



• MPD Thruster

• **Initial Spacecraft Mass**

1000kg

• **Initial Orbit**

6571 km, e=0.0

• **Initial Orbit Velocity**

7.7885 km/sec

Worked Example (cont'd)

Thrust (Newtons)

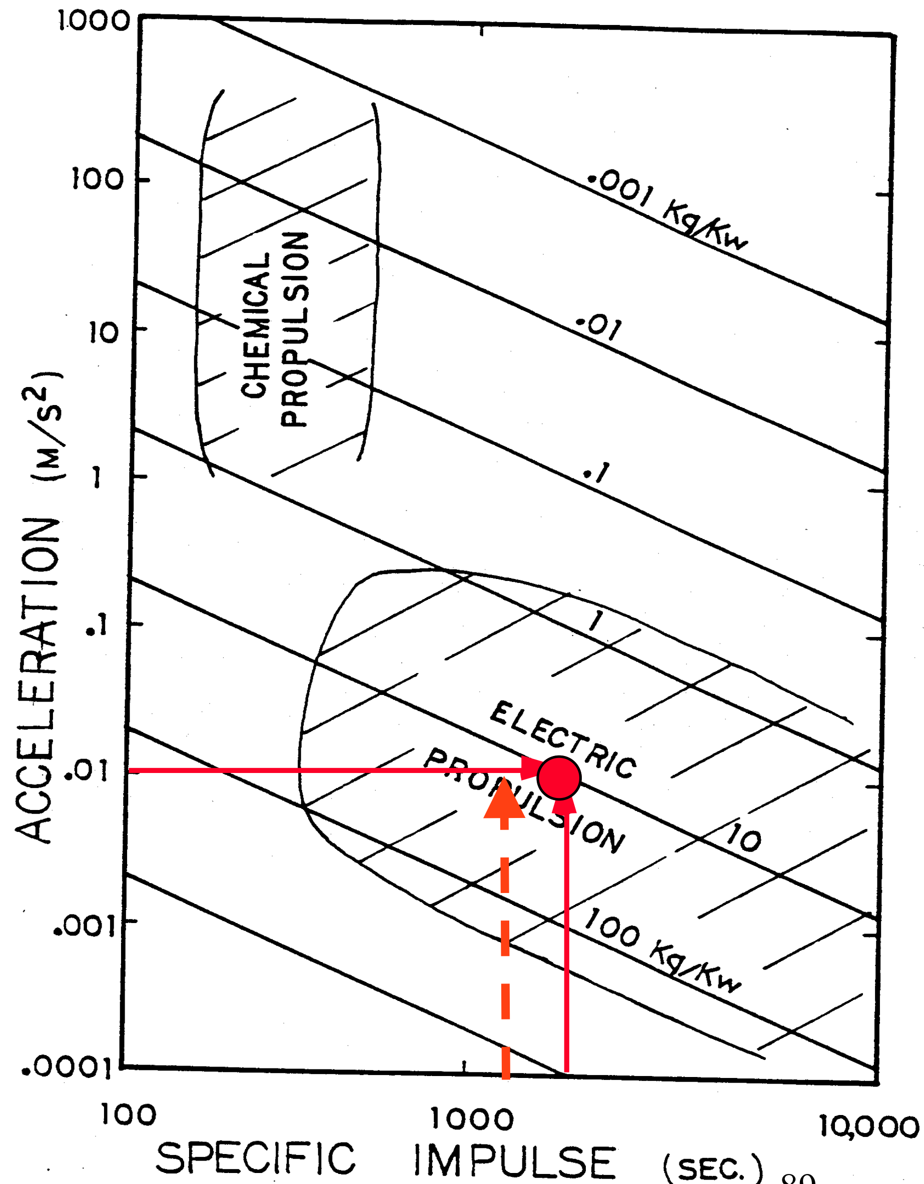
10.00000

Isp (seconds)

2500.0

- **Initial Spacecraft Mass**

  1000kg



Worked Example (cont'd)

- **Continuous Thrust GTO**

Thrust (Newtons)
10.00000

Isp (seconds)
2500.0

- MPD Thruster

Accumulated burn time (sec.)	
418000.00	116 hrs

- **Terminate Thrust when** $a(1+e)$ **Instantaneous** $= 42164.2$ **km** (Geo radius)

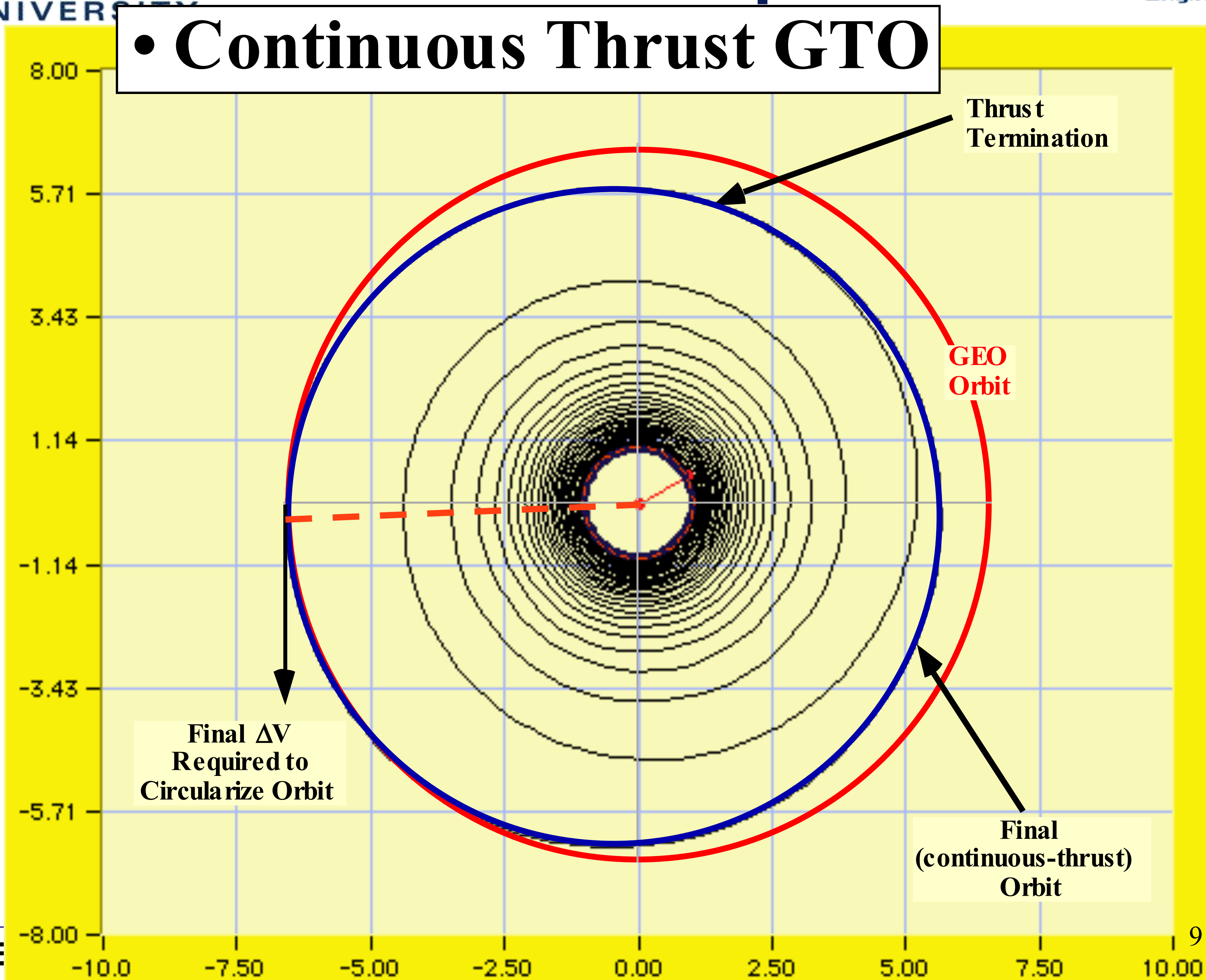
- **Final Orbit**

  $a = 38830$ km

- **Final Orbit**

  $e = 0.08584$

Worked Example (cont'd)



Worked Example (cont'd)

- **Continuous Thrust GTO**

Thrust (Newtons)

10.00000

Isp (seconds)

2500.0

- MPD Thruster

- **Propellant Required to Reach Final GTO (elliptical)**

$$M_{initial} = 1000 \text{ kg}$$
$$M_{final} = 829.5 \text{ kg}$$

$$P_{ropellant} M_{ass} = 170.5 \text{ kg}$$

Worked Example (cont’d)

- **Continuous Thrust GTO**

Thrust (Newtons)

10.00000

Isp (seconds)

2500.0

- **MPD Thruster**

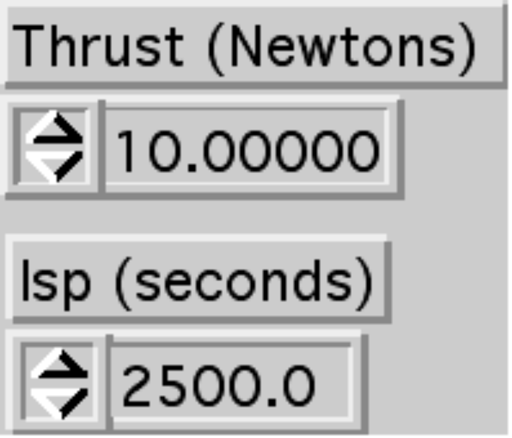
- **ΔV required to circularize final orbit**

$$V_{GEO} = \sqrt{\frac{\mu}{r}} =$$

$$\sqrt{\frac{3.986 \times 10^5 \frac{km^3}{sec^2}}{42164.2\ km}} = 3.0746 \frac{km}{sec}$$

Worked Example (cont'd)

• Continuous Thrust GTO



• MPD Thruster

• ΔV required to circularize final orbit

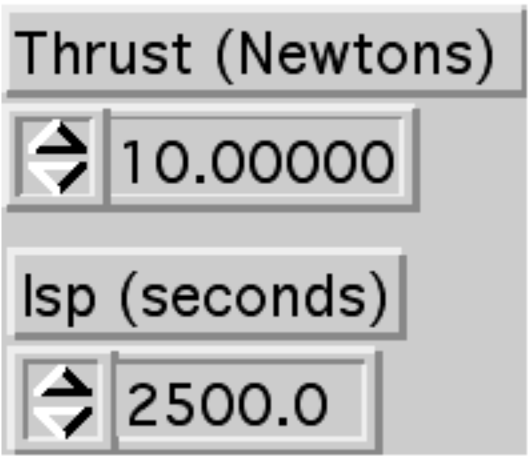
$$V_{GTO}^{(apogee)} = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} =$$

$$\sqrt{\frac{2 \times 3.986 \times 10^5 \frac{km^3}{sec^2}}{42164.2\ km} - \frac{3.986 \times 10^5 \frac{km^3}{sec^2}}{38830\ km}} = 2.9397 \frac{km}{sec}$$

$$\Delta V = 3.0746 \frac{km}{sec} - 2.9397 \frac{km}{sec} = 0.135 \frac{km}{sec}$$

Worked Example (cont'd)

- **Likely Need Conventional Propulsion for Final Burn**



- **I_{sp} 270 sec**

- MPD Thruster

$$P_{mf} = e^{\frac{\Delta V}{g_0 I_{sp}}} - 1 =$$

$$e^{\left[\frac{135.0 \frac{m}{sec}}{9.806 \frac{m}{sec^2} 270 \, sec}\right]} - 1 = .0523$$

Worked Example (cont'd)

Thrust (Newtons)

10.00000

Isp (seconds)

2500.0

- MPD Thruster

- **Conventional Propulsion Final Burn**

$$P_{mf} + 1 = \frac{M_{propellant}}{M_{final}} + \frac{M_{final}}{M_{final}} =$$

$$\frac{M_{propellant} + M_{final}}{M_{final}} \Rightarrow 1.0523 = \frac{829.5\ \text{kg}}{M_{final}}$$

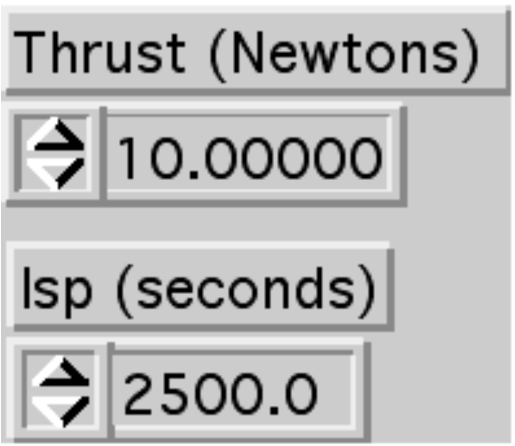
$$\Downarrow$$

$$M_{final} = \frac{829.5\ \text{kg}}{1.0523} = 788.2\ \text{kg} \Rightarrow$$

$$M_{propellant} = 829.5\ \text{kg} - 788.2\ \text{kg} = 41.2\ \text{kg}$$

Worked Example (cont'd)

- **Total Propellant Mass Fraction for GEO Transfer**



- Continuous Thrust

$$P_{mf} = \frac{M_{propellant}}{M_{final}} =$$

- MPD Thruster

$$\frac{41.2\ \text{kg} + 170.5\ \text{kg}}{788.2\ \text{kg}} = 0.26858\ \textit{wow!}$$

Compare to Hohmann transfer using Conventional Propulsion

- I_{sp} 270 sec

"delta Vee" data

DV Orbit 1 (KM/sec)

2.45536

DV Orbit 2 (KM/sec)

1.47723

DV Total (KM/sec)

3.93259

- **WHAT IS PROPELLANT FRACTION?**

$$P_{mf} = e^{\frac{\Delta V}{g_0 I_{sp}}} - 1 =$$

$$e^{\frac{3932.59 \frac{m}{sec}}{9.806 \frac{m}{sec^2} 270 \text{ sec}}} - 1 = 3.4164$$

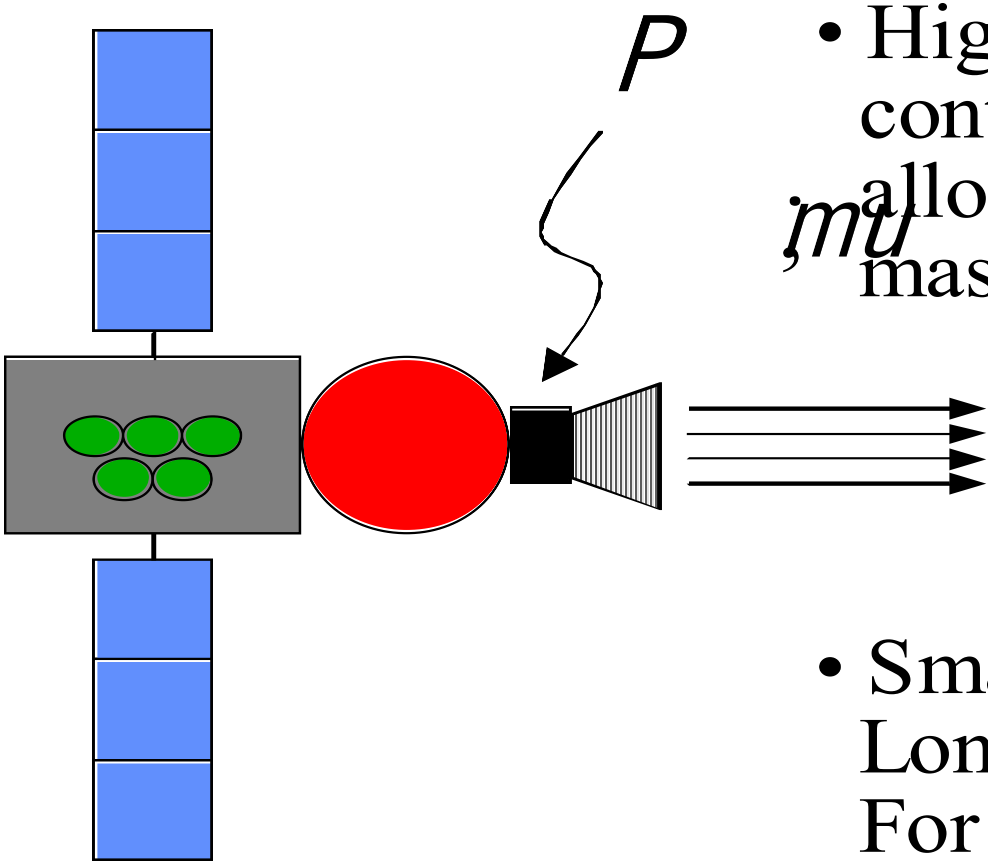
- **Final Mass 788.2 kg**

  requires ... 2692.8 of propellant!

  Versus 211.7 kg for EP

EP, in the Right Circumstances

Big Advantages

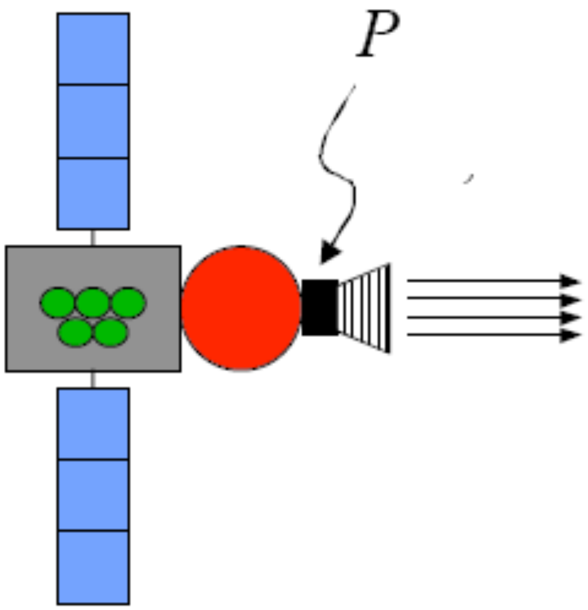


- High Isp continuous thust allows small propellant mass fractions

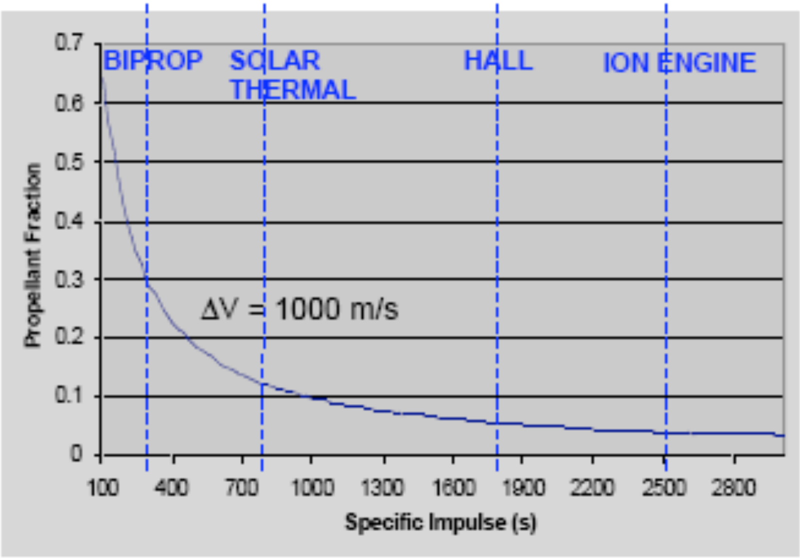
- Small Thrust Requires Long Operating Life For Engine

EP, in the Right Circumstances

Benefits of Electric Propulsion



Chemical	**400**
Solar Thermal	**800**
Nuclear Thermal	**800+**
Electric	**ANY**



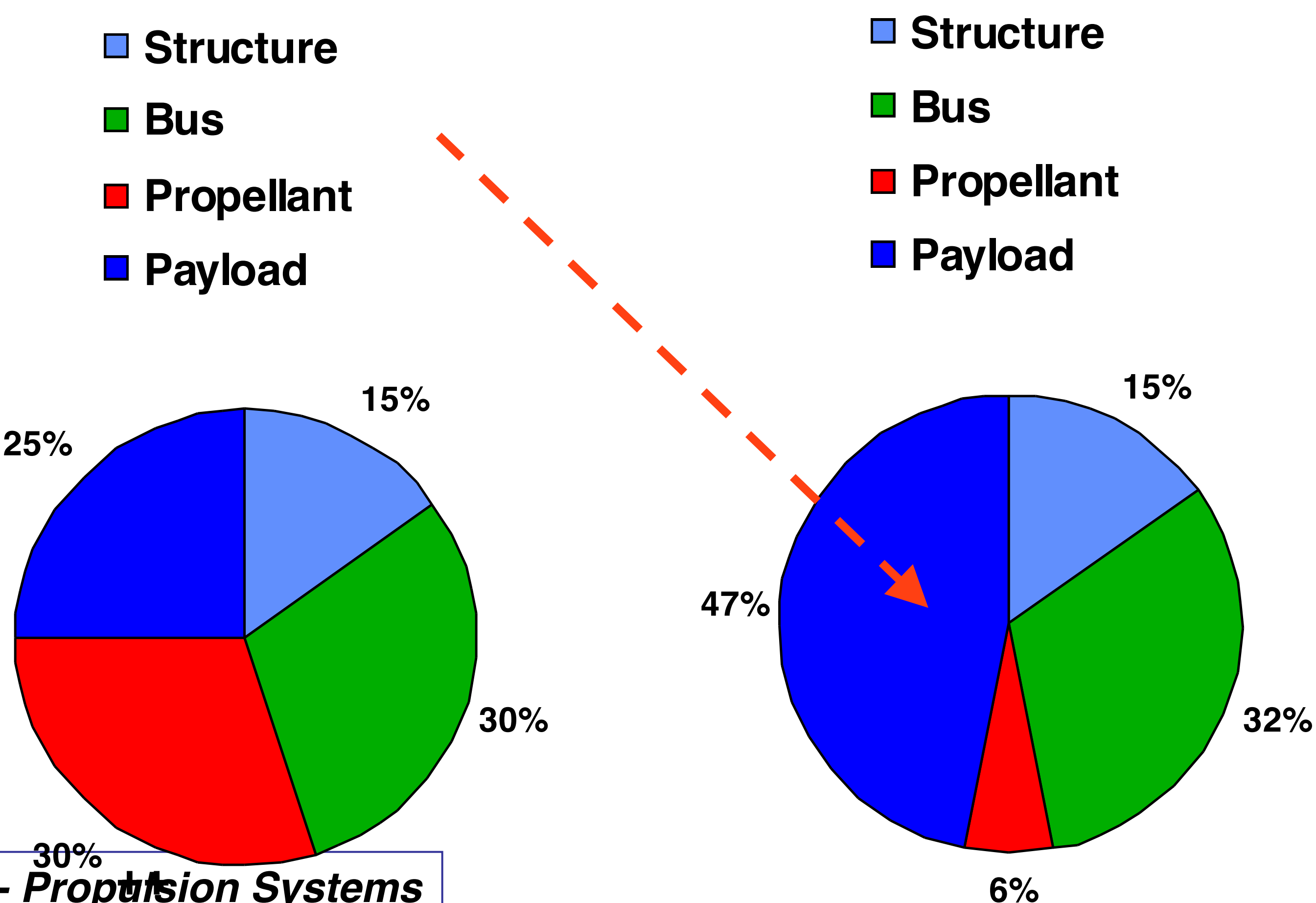
Typical Parameters of Small Thrusters

Thruster	I_{sp} (s)	η	Thrust
Solid Rocket Motor	185	90+%	100+ N
Chemical Bipropellant	315	95+%	>2 N
Arcjets	~500 - 700s >1000 s (H_2)	~30%	0.1-1 N
Pulsed Plasma Thruster	200-1500	~15%	2 μN – 4.5 mN
Colloid Thruster	450-1350	~50%	20 μN
Hall Thruster	1500-3000	~50-60%	1.8–500 mN
Ion Thrusters	1700-3900 s	~65%	1-100 mN
Field Emission Thruster	6000-9000 s	~90%	40 μN – 1.4 mN

Electric Propulsion Thrusters

Benefits of Electric Propulsion

Chemical | **Electric**



Major Project 1 …

- **Look at problem of transferring satellite to MEO (GPS) from Initial LEO Orbit**
  - Code Continuous Thrust Example

- a_{LEO} = 8530km, a_{MEO} = 13,200 km
-Thrust (electric) ~ 10 Nt, I_{sp} (electric) = 2000 sec
- Kick motor I_{sp} = 270 sec F_{kick}= 2000 Nt,

- Include a *Thrust Termination* Criterion which puts you in the proper final transfer orbit (apogee tangent to desired MEO Orbit)

- Calculate ΔV ***+ propellant mass*** required to circularize final orbit

Assume 270 Isp For Apogee Kick Motor

- For continuous thrust problem .. assume final Orbit insertion *ΔV* is delivered impulsively with Apogee Kick Motor Isp = 270 sec ….. Ignore atmospheric drag

Part a)

• Continuous Small Thrust Problem

Terminate thrust when

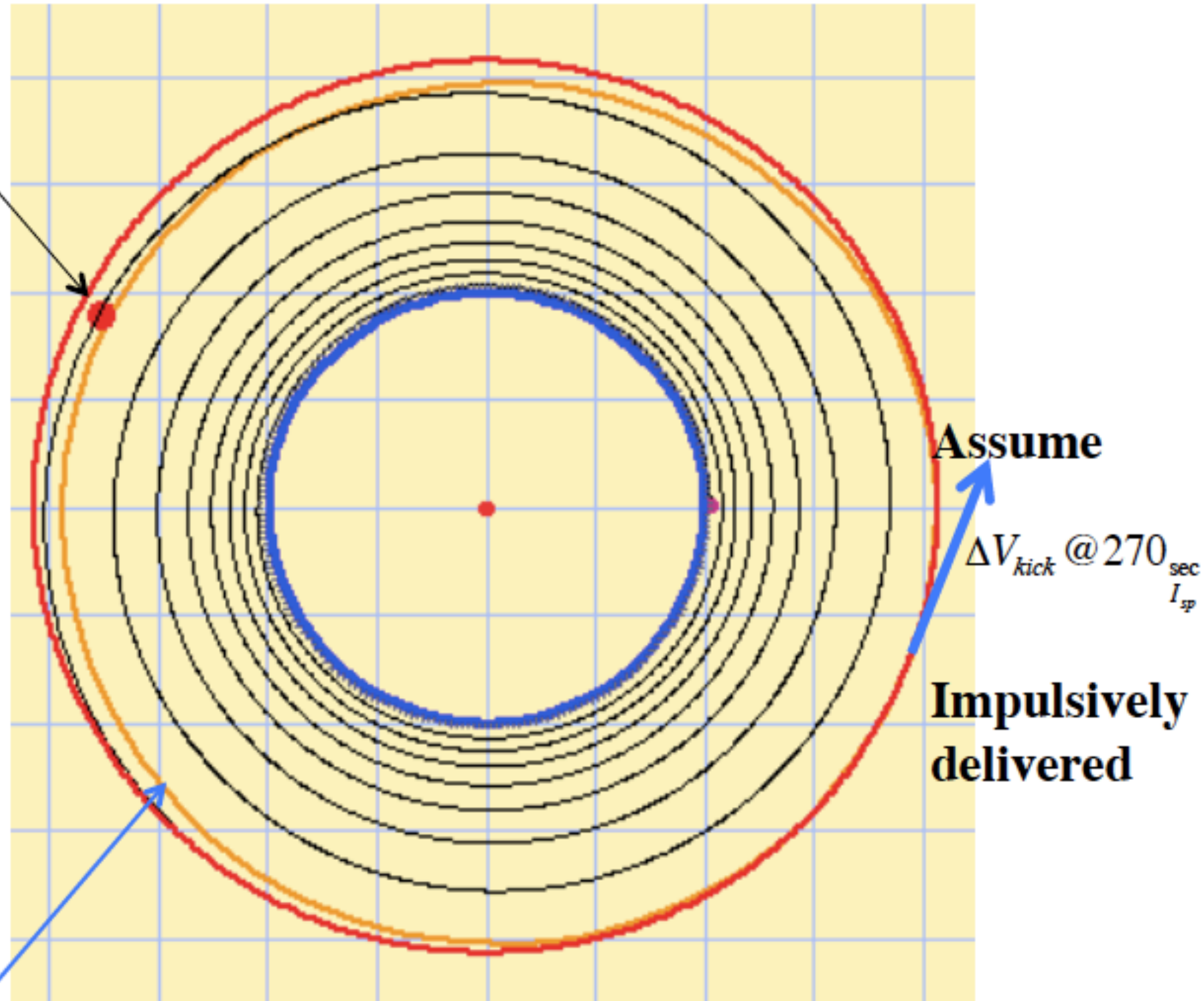
$$R_{apogee} = a \cdot (1 + e)$$
$$= 13{,}200\,km$$

Calculate:

1) Propellant mass req.
For continuous transfer

2) Propellant mass req.
For kick delta ***V (impulsive)***
(orbit circularization)

3) Final mass = 1000 kg



• ***Calculate Mass Consumed Including Kick***

Part a)

• *Continuous Small Thrust Problem*

… compare continuous thrust propellant mass calculations against Hohmann transfer calculations .. Assuming impulsively delivered Delta V for each burn

Burn 1: Isp = 2000 sec

Burn 2: Isp = 270 sec

… what can you conclude about the accuracy of the rocket equations and the impulsive Delta V assumption when applied to a long duration non-impulsive burn?

Part a) Continuous Small Thrust

… Implement *both* Trapezoidal and Runge-Kutta Integration schemes

… Assume continuous thrust transfer to transfer orbit apogee using EP device, final orbit insertion using high thrust kick motor

… compare algorithm performance as
Time interval ΔT becomes progressively larger

… Is there a point where algorithm blows up?

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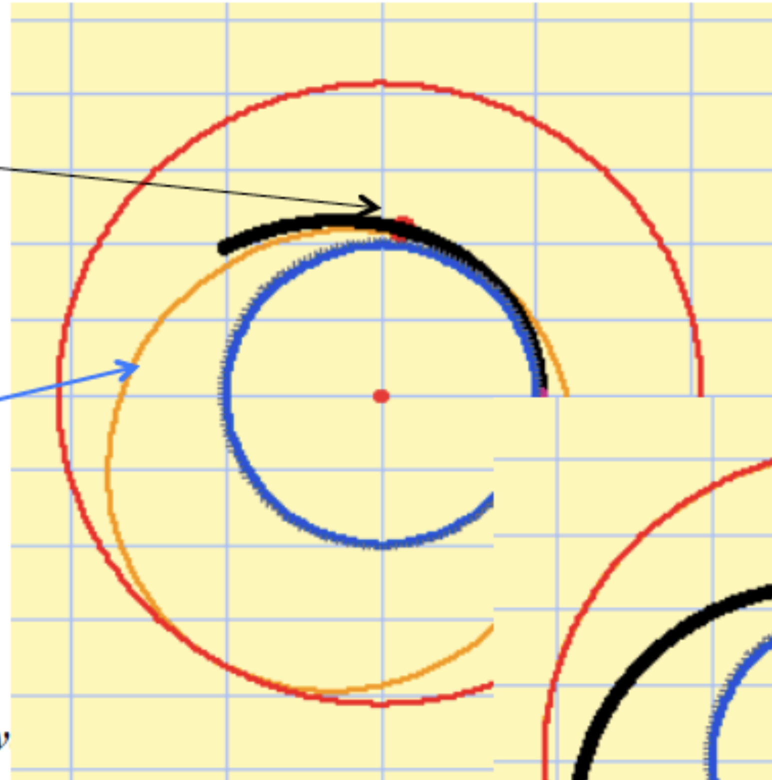
Part b) Continuous Large Thrust Analysis

Terminate thrust when

$$R_{apogee} = a \cdot (1 + e)$$

$$= 13,200\,km$$

Orbit coast

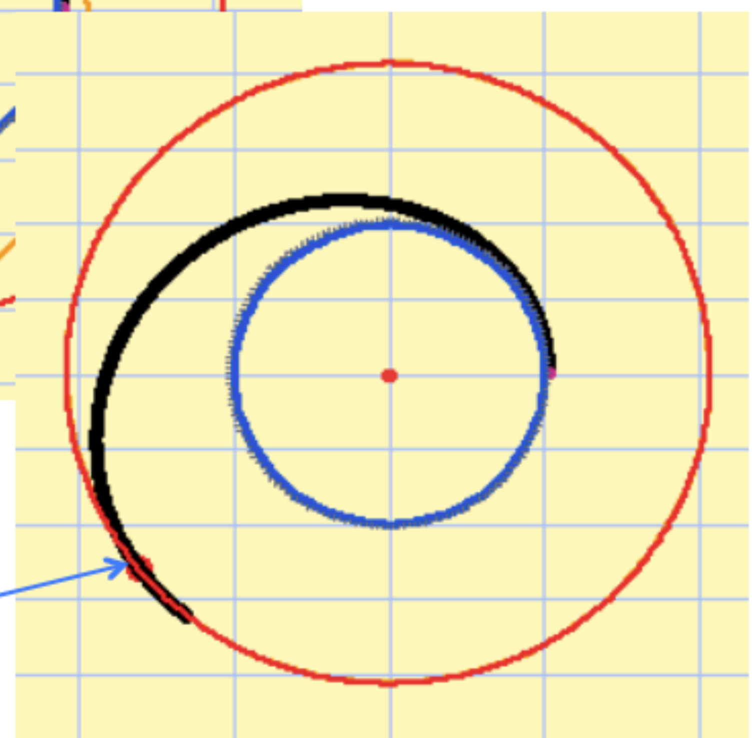


Calculate:

1) Propellant mass req.
For continuous transfer

2) Propellant mass req.
For kick delta V ***(non - impulsiv***
(orbit circularization)

3) Final mass = 1000 kg



Final Delta V delivered Non-impulsively

MEO Orbit

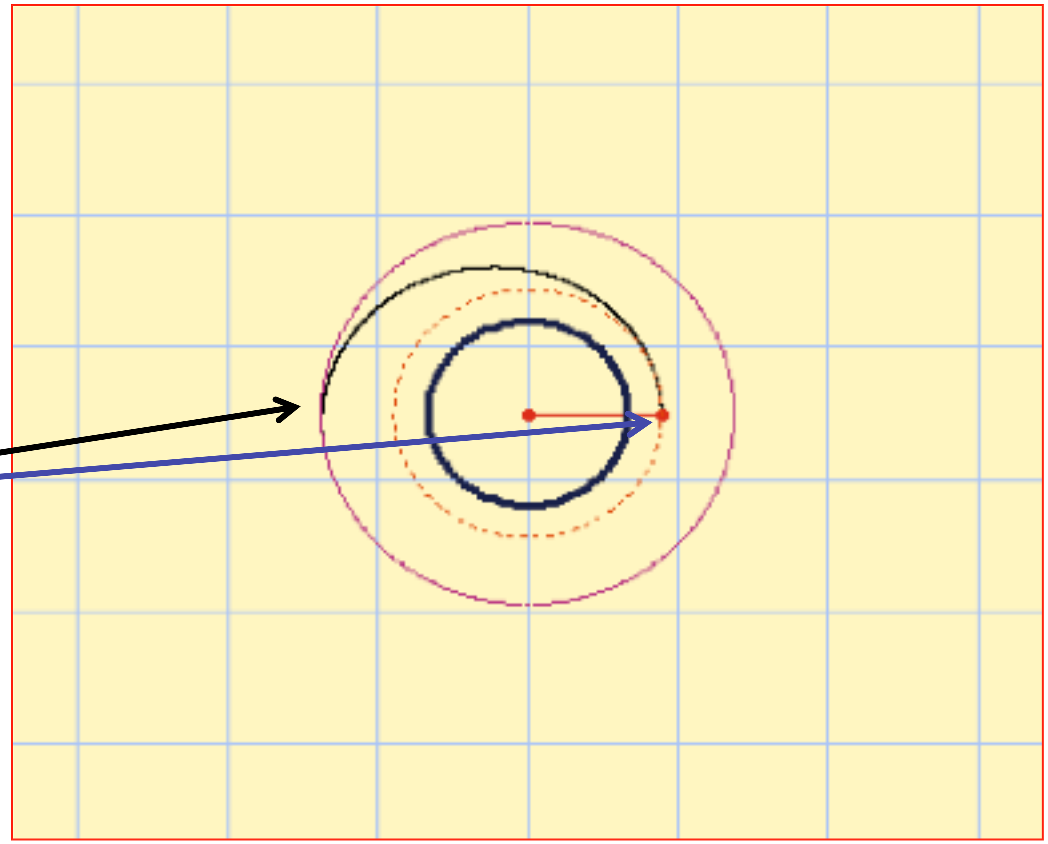
• Part b, Hohmann Transfer Calculations

Hohmann Transfer:

I_{sp}=270 sec

F_{thrust}=2000 Nt

• ***Impulsive Burn Calculations***



• Calculate Mass Consumed Including Kick

• Continuous Large Thrust Problem

… compare Hohmann Transfer for 2000 Nt Rocket (assuming impulsive thrust) Versus 2000 Nt rocket with Non Impulsive Thrust …. Also compare consumed masses to High I_{sp} Continuous Thrust transfer

… what can you conclude about the accuracy of the rocket equation and the impulsive Delta V assumption when applied to a short duration non-impulsive burn?

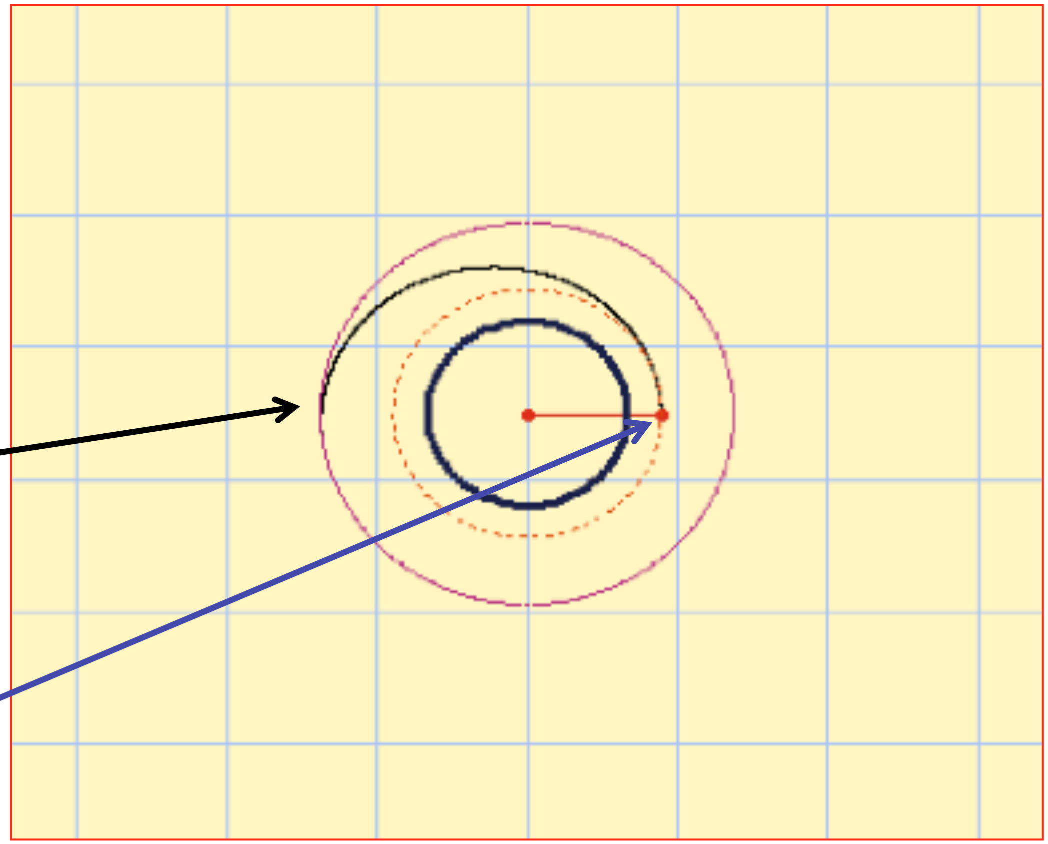
… what can you conclude about the effect of I_{sp} on required propellant mass?

• Part c … ***Bonus (1 point) ..*** Work continuous large Thrust problem with non-impulsive burns at both ends

Hohmann Transfer:
I_{sp}=270 sec
F_{thrust}=2000 Nt

• ***Non-Impulsive Burn Calculations***

• ***Continuous Thrust Burn Calculations***



• Continuous Large Thrust Problem

- **Assume BOTH burns are performed non-impulsively**
Terminate burn thrust when

$$R_{apogee} = a \cdot (1 + e)$$
$$= 13{,}200\, km$$

- **You decide when and how long to initiate the second burn to circularize the orbit**

- **Assume for large thrust …. 2000 Nt thrust (both burns) … Isp = 270 sec**

- **Calculate required propellant mass for Burn1, Burn2 (and Total)**

- **Use integrator of your choice … calculate actual delivered Delta V**
Based on consumed mass … using rocket equation

Project Hints (1)

Position within initial orbit:

$$\begin{bmatrix} r \\ \nu \end{bmatrix}_0 = \begin{bmatrix} \dfrac{a_0\left(1-e_0^{\ 2}\right)}{1+e_0\cos\left(\nu_0\right)} \\ \nu_0 \end{bmatrix} \rightarrow \begin{bmatrix} \text{circular orbit} \rightarrow e_0 = 0 \\ \text{can assume} \rightarrow \nu_0 = 0 \rightarrow a_0 = r_0 \end{bmatrix}$$

Angular velocity within initial orbit:

$$\omega_0 = \frac{\sqrt{\mu}\left[1+e_0\cos\left(\nu_0\right)\right]^2}{\left[a_0\left(1-e_0^{\ 2}\right)\right]^{3/2}} \rightarrow \begin{bmatrix} \text{circular orbit} \rightarrow e_0 = 0 \\ \text{can assume} \rightarrow \nu_0 = 0, a_0 = r_0 \end{bmatrix}$$

$$\omega_0 = \frac{\sqrt{\mu}\left[1+e_0\cos\left(\nu_0\right)\right]^2}{\left[a_0\left(1-e_0^{\ 2}\right)\right]^{3/2}} = \frac{1}{r_0}\sqrt{\frac{\mu}{r_0}}$$

Project Hints (2)

Linear Velocity within initial orbit:

$$\begin{bmatrix} V_r \\ V_\nu \end{bmatrix}_0 = r_0\omega_0 \begin{bmatrix} \dfrac{e_0 \sin\left[\nu_o\right]}{\left[1 + e_0 \cos\left(\nu_0\right)\right]} \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \text{circular orbit} \rightarrow e_0 = 0 \\ \text{can assume} \rightarrow \nu_0 = 0, a_0 = r_0 \end{bmatrix}$$

$$\begin{bmatrix} V_r \\ V_\nu \end{bmatrix}_0 = \begin{bmatrix} 0 \\ r_0\omega_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{\dfrac{\mu}{r_0}} \end{bmatrix}$$

Project Hints (3)

Instantaneous $_{\text{(no-nonconservative foreces acting)}}$ Keplerian orbit $\rightarrow$ $given : \begin{bmatrix} V_r \\ V_\nu \end{bmatrix}, \begin{bmatrix} r \\ \nu \end{bmatrix}$

$$a = \frac{\mu}{\left[\frac{2\mu}{r} - \left[V_r^{\,2} + V_\nu^{\,2} \right] \right]}$$

$$e = \frac{r}{\mu} \sqrt{\left(V_\nu^{\,2} - \frac{\mu}{r} \right)^2 + \left(V_r V_\nu \right)^2}$$

$$r_{perigee} = a(1 - e)$$

$$r_{apogee} = a(1 + e)$$

Appendix 1: Airspeed Calculation Examples

Airspeed, Qbar, Example

- Example:

A Rocket launches at an inertial flight path angle of
85° at an azimuth angle of 65°from true north

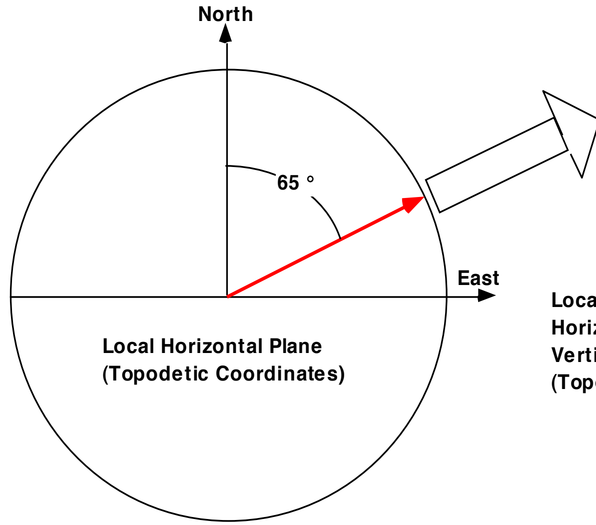
The inertial velocity is 1 km/sec, the altitude is 10 km, and

The upper level winds at 10 km are blowing from the east at 75°
From true north at 50 m/sec

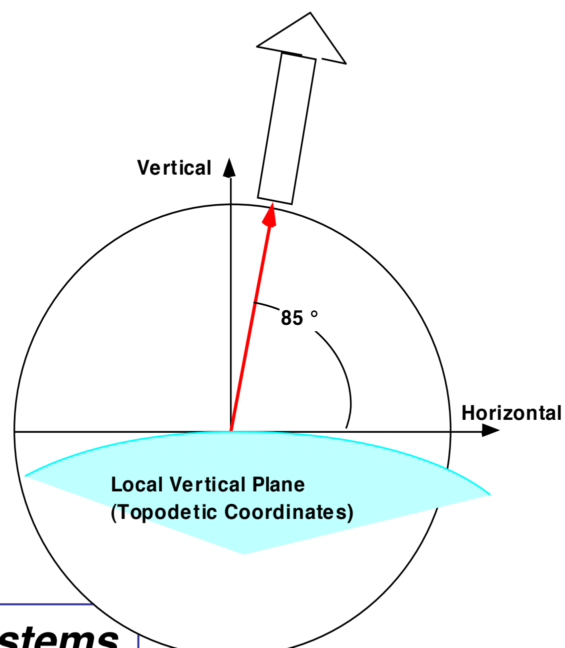
The current launch latitude is 35° north

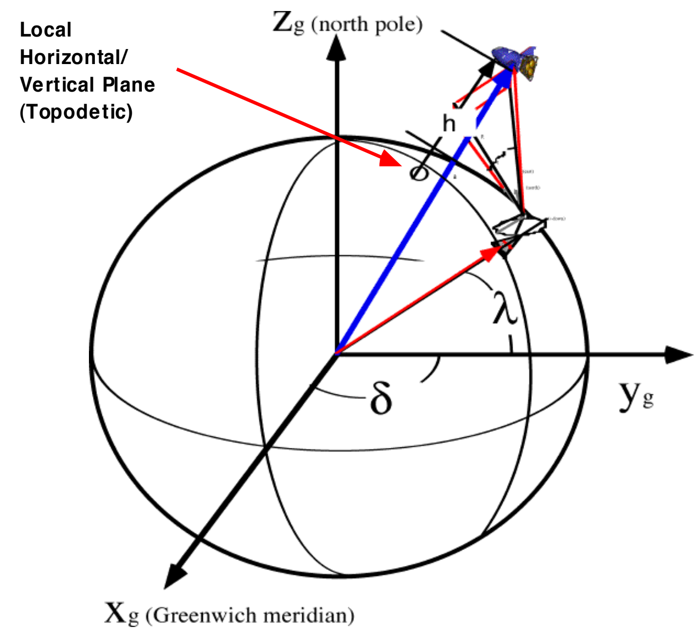
What is the airspeed? Dynamic Pressure? What is mach number?

Airspeed, Qbar, Example (cont'd)



- Look
At local earth
Tangent plane
Velocities





Airspeed, Qbar, Example (cont'd)

- Calculate North, east, and vertical Inertial Velocity Components

$$\left(\bar{V}_{inertial}\right)_{vertical} = \bar{V}_{inertial}\sin(\gamma) =$$

$$1000\sin\left(\frac{\pi}{180}85\right) = 996.2 \text{ m/sec}$$

$$\left(\bar{V}_{inertial}\right)_{north} = \bar{V}_{inertial}\cos(\gamma)\cos(Az) =$$

$$1000\cos\left(\frac{\pi}{180}85\right)\cdot\cos\left(\frac{\pi}{180}65\right) = 36.83 \text{ m/sec}$$

$$\left(\bar{V}_{inertial}\right)_{east} = \bar{V}_{inertial}\cos(\gamma)\sin(Az) =$$

$$1000\cos\left(\frac{\pi}{180}85\right)\cdot\sin\left(\frac{\pi}{180}65\right) = 78.99 \text{ m/sec}$$

Airspeed, Qbar, Example (cont'd)

- Calculate V_{earth} (Inertial Boost)

$$\text{Velocity} = V_e * \cos(lat) = 0.4638\cos\left(\frac{\pi}{180}35\right) = 379.9 \text{ m/sec}$$

Latitude	cos(lat)	velocity (km/sec)	velocity (ft/sec)
0	1	0.4638	1521
10	0.98481	0.45675	1497.89259
20	0.93969	0.43583	1429.27248
30	0.86603	0.40166	1317.22464
40	0.76604	0.35529	1165.15360
50	0.64279	0.29812	977.67995
60	0.50000	0.23190	760.50000
70	0.34202	0.15863	520.21264
80	0.17365	0.08054	264.11888
90	0.00000	0.00000	0.00000

"east" Direction

Airspeed, Qbar, Example (cont'd)

• Calculate Wind velocities

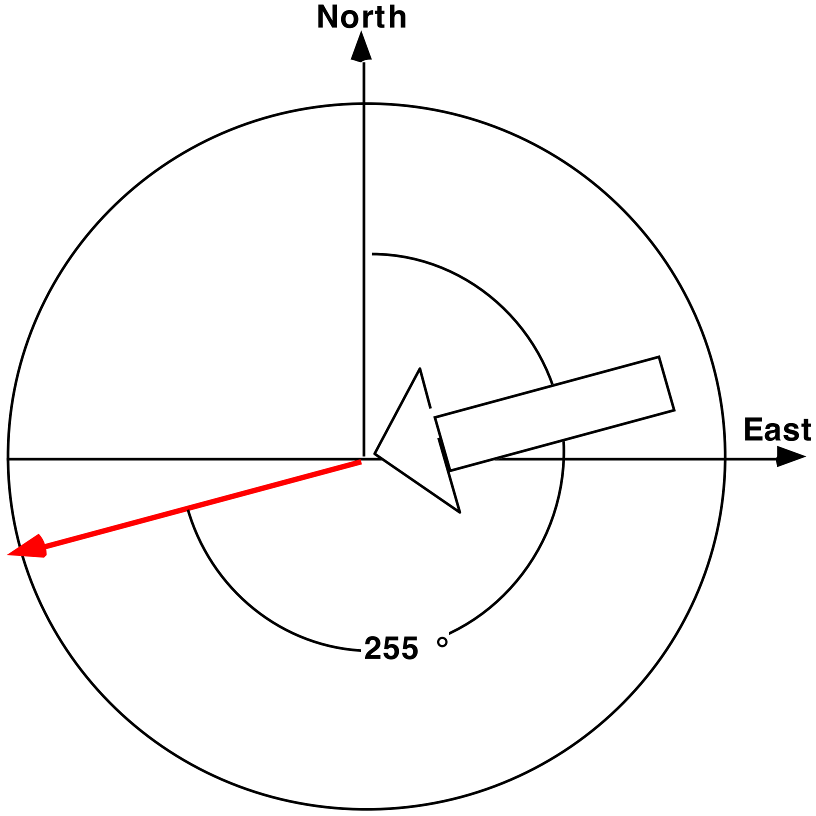
The upper level winds at 10 km are blowing *from the east* at 75°
From true north at 50 m/sec

$$\left(\bar{V}_{wind}\right)_{north} = \bar{V}_{wind}\cos(\Psi_{wind}) =$$

$$50\cos\left(\frac{\pi}{180}255\right) = -12.941 \text{ m/sec}$$

$$\left(\bar{V}_{wind}\right)_{east} = \bar{V}_{wind}\sin(\Psi_{wind}) =$$

$$50\sin\left(\frac{\pi}{180}255\right) = -48.296 \text{ m/sec}$$



Local Horizontal Plane
(Topodetic Coordinates)

Airspeed, Qbar, Example (cont'd)

- Add up components and take airspeed magnitude

$$V_\infty = \left\| \bar{V}_{inertial} - \bar{V}_{earth} - \bar{V}_{wind} \right\|$$

$$V_\infty =$$

$$\sqrt{\left(\bar{V}_{inertial} - \bar{V}_{earth} - \bar{V}_{wind}\right)^2_{north} + \left(\bar{V}_{inertial} - \bar{V}_{earth} - \bar{V}_{wind}\right)^2_{east} + \left(\bar{V}_{inertial} - \bar{V}_{earth} - \bar{V}_{wind}\right)^2_{vertical}} =$$

$$\left((36.83 - 0 - (-12.941))^2 + (78.99 - 379.9 - (-48.296))^2 + 996.2^2 \right)^{0.5}$$

=1028.93 m/sec ---> Airspeed is actually greater than the Inertial speed in this case

Airspeed, Qbar, Example (cont'd)

- Compute Dynamic Pressure

• US 1977 Standard Atmosphere

Altitude (km)	Density (kg/m^3)	Temperature, deg. K
10.0000	0.41270	223.15

$$\bar{q} = \frac{1}{2}\rho_\infty V_\infty^{\ 2} = \frac{\frac{1}{2}0.4127(1028.93^2)}{1000}$$ =218.46 kpa (4562.6 psf Ouch!)

- Compute Mach number

$$c = \sqrt{\gamma R_g T_\infty} = (1.4 \cdot 287.055 \cdot 223.15)^{0.5}$$ =299.46 m/sec

$M_\infty = V_\infty / c = 1028.93/299.46 = 3.44$

Airspeed, Qbar, Example (cont'd)

- How about a due east launch under same conditions with inertial velocity of 2 km/sec and flight path angle of 45°

$$\left(\bar{V}_{inertial}\right)_{vertical} = \bar{V}_{inertial}\sin(\gamma) =$$

$$2000\sin\left(\frac{\pi}{180}45\right) = 1414.2 \text{ m/sec}$$

$$\left(\bar{V}_{inertial}\right)_{north} = \bar{V}_{inertial}\cos(\gamma)\cos(Az) =$$

$$2000\cos\left(\frac{\pi}{180}45\right)\cdot\cos\left(\frac{\pi}{180}90\right) = 0.0 \text{ m/sec}$$

$$\left(\bar{V}_{inertial}\right)_{east} = \bar{V}_{inertial}\cos(\gamma)\sin(Az) =$$

$$2000\cos\left(\frac{\pi}{180}45\right)\cdot\sin\left(\frac{\pi}{180}90\right) = 1414.2 \text{ m/sec}$$

Airspeed, Qbar, Example (cont'd)

$$V_\infty = \left\| \bar{V}_{inertial} - \bar{V}_{earth} - \bar{V}_{wind} \right\|$$

$V_\infty =$

$$\sqrt{\left(\bar{V}_{inertial} - \bar{V}_{earth} - \bar{V}_{wind}\right)^2_{north} + \left(\bar{V}_{inertial} - \bar{V}_{earth} - \bar{V}_{wind}\right)^2_{east} + \left(\bar{V}_{inertial} - \bar{V}_{earth} - \bar{V}_{wind}\right)^2_{vertical}} =$$

$$\left((0 - 0 - (-12.941))^2 + (1414.2 - 379.9 - (-48.296))^2 + 1414.2^2 \right)^{0.5}$$

=1781.05 m/sec ---> Airspeed in this case is less than airspeed … it all depends on the direction that you launch!

Appendix 2: Launch Initial Conditions

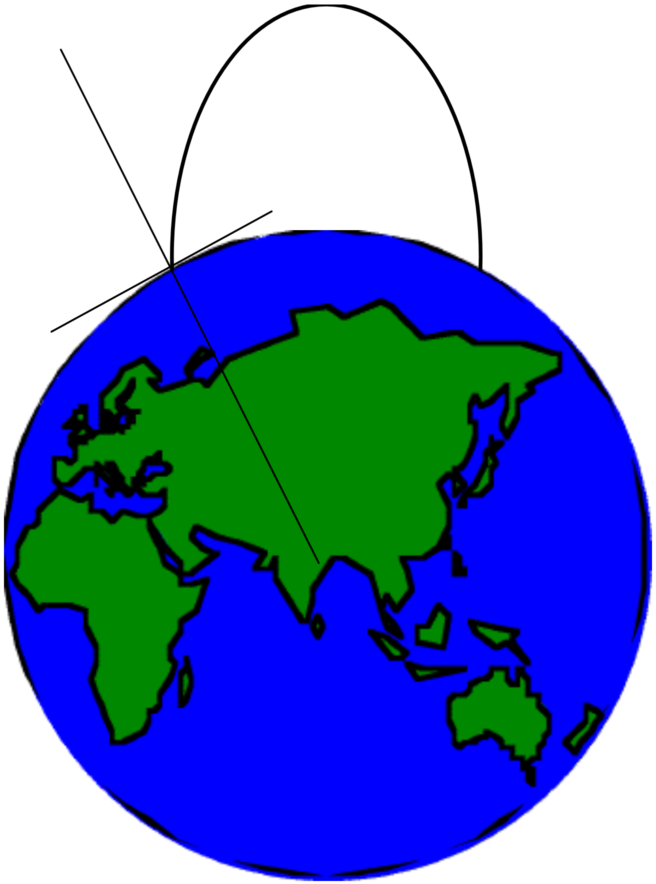
Initial Conditions

$$\dot{X} = f\left[X, F_{thrust}, \theta\right]$$

$$\dot{X} = \begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \\ \dot{r} \\ \dot{v} \\ \dot{m} \end{bmatrix} \rightarrow f\left[X, F_{thrust}, \theta\right] = \begin{bmatrix} \frac{V_v^2}{r} + \frac{F_{lift}\cos(\gamma) - F_{drag}\sin(\gamma) + F_{thrust}\sin(\theta)}{m} - \frac{\mu}{r^2} \\ -\left[\frac{V_r V_v}{r} + \frac{F_{lift}\sin(\gamma) + F_{drag}\cos(\gamma) - F_{thrust}\cos(\theta)}{m}\right] \\ \\ V_r \\ \frac{V_v}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix} \rightarrow \boxed{\begin{array}{c} \gamma = \tan^{-1}\left[\frac{V_r}{V_v}\right] \\ \theta = \gamma + \alpha \end{array}}$$

Need starting conditions for state vector X

Initial Conditions: Ground Launch, Rotating Earth



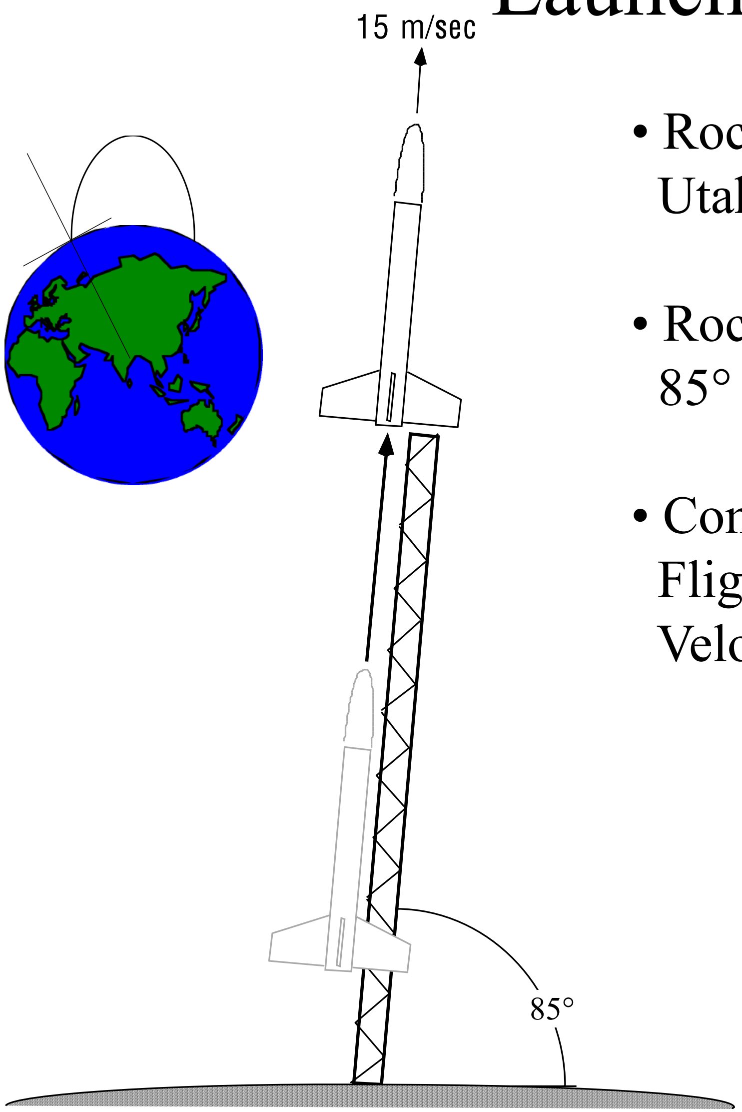
- Inertial Flight Path Angle

$$\gamma_{inertial} = \tan^{-1}\left[\frac{V_r}{V_\nu}\right]$$

- Ground Relative Flight Path Angle

$$\gamma_{ground} = \tan^{-1}\left[\frac{V_r}{V_\nu - V_{E_{eq}}\cos(Lat)}\right]$$

Launch I.C. Example:



- Rocket Launch from Green River Utah -- 38° N. latitude, 3970 ft. altitude (1.21 km)

- Rocket Leaves Launch Rail at 85° angle to Local Vertical

- Compute Ground Relative, Inertial Flight path Angle, Initial Position, Velocity Vector

Solar day:86164.1 sec
Ω_{earth}: 7.292115e-0.5 rad/sec

Launch I.C. Example: (cont'd)

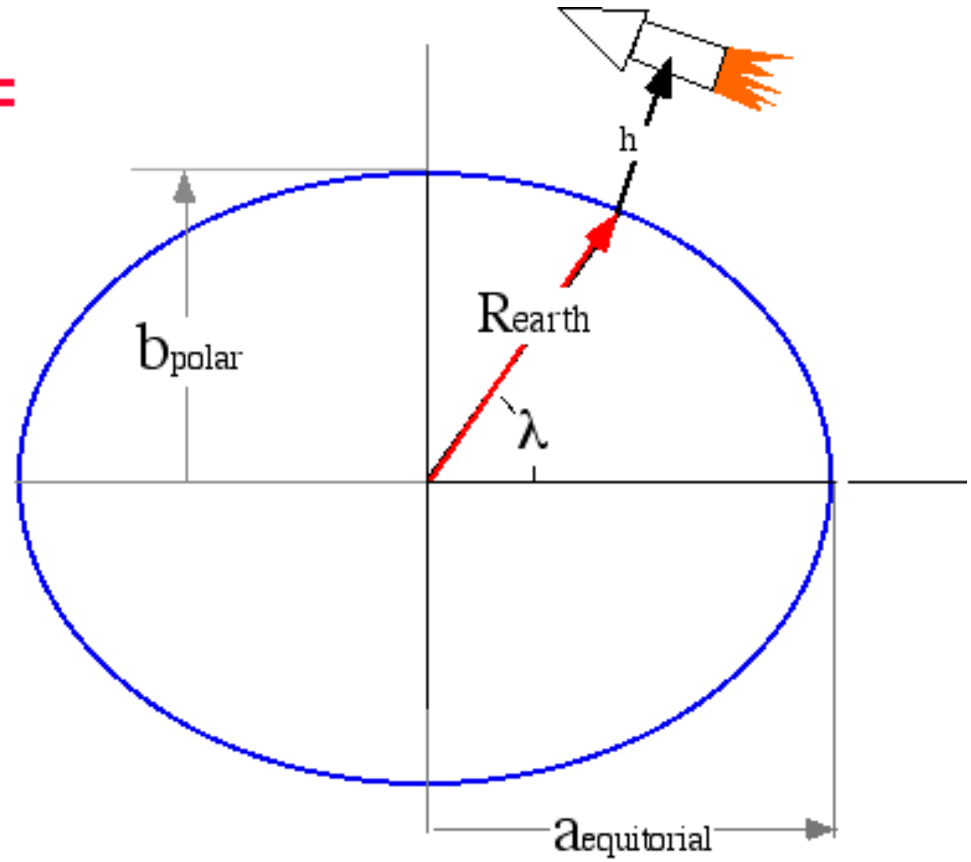
- Earth radius as Function of Latitude

$$R_{earth} = \frac{a_{equitorial}}{\sqrt{\left[1 + \frac{e_{earth}^2}{1 - e_{earth}^2}\sin^2\lambda\right]}}$$

$$a_{equitorial} = 6378.13649 \text{ km}$$

$$b_{polar} = 6356.7515 \text{ km}$$

$$e_{earth} = \sqrt{1 - \left[\frac{b_{polar}}{a_{equitorial}}\right]^2}$$



Launch I.C. Example: (cont'd)

- Earth radius at launch latitude

$$R_{earth} = \frac{a_{equitorial}}{\sqrt{\left[1 + \frac{e_{earth}^2}{1 - e_{earth}^2} \sin^2 \lambda\right]}}$$

$$e_{Earth} = \sqrt{1 - \left[\frac{b}{a}\right]^2} = \sqrt{\frac{a^2 - b^2}{a^2}} =$$

$$\frac{\sqrt{[6378.13649]^2 - 6378.13649^2}}{[6378.13649]} = 0.08181939$$

$$a_{equitorial} = 6378.13649 \text{ km}$$

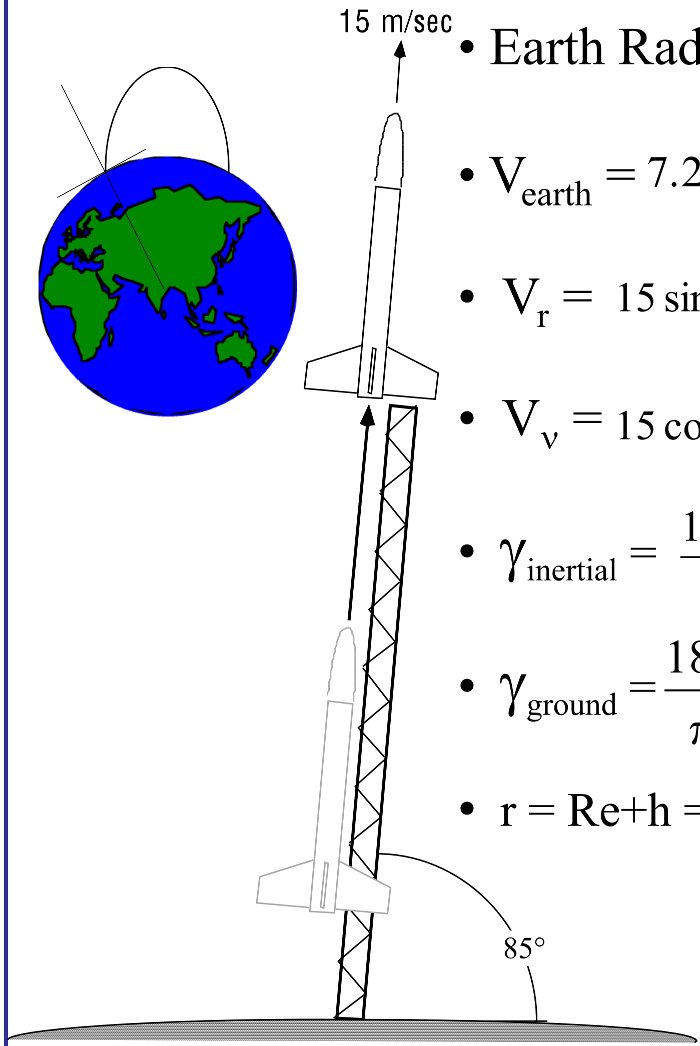
$$b_{polar} = 6356.7515 \text{ km}$$

$$e_{earth} = \sqrt{1 - \left[\frac{b_{polar}}{a_{equitorial}}\right]^2}$$

$$R_{earth} =$$

$$\frac{6378.13649}{\left(1 + \frac{0.08181939^2}{1 - 0.08181939^2} \sin^2\left(\frac{\pi}{180} 38\right)\right)^{0.5}} = 6370.01 \text{ km}$$

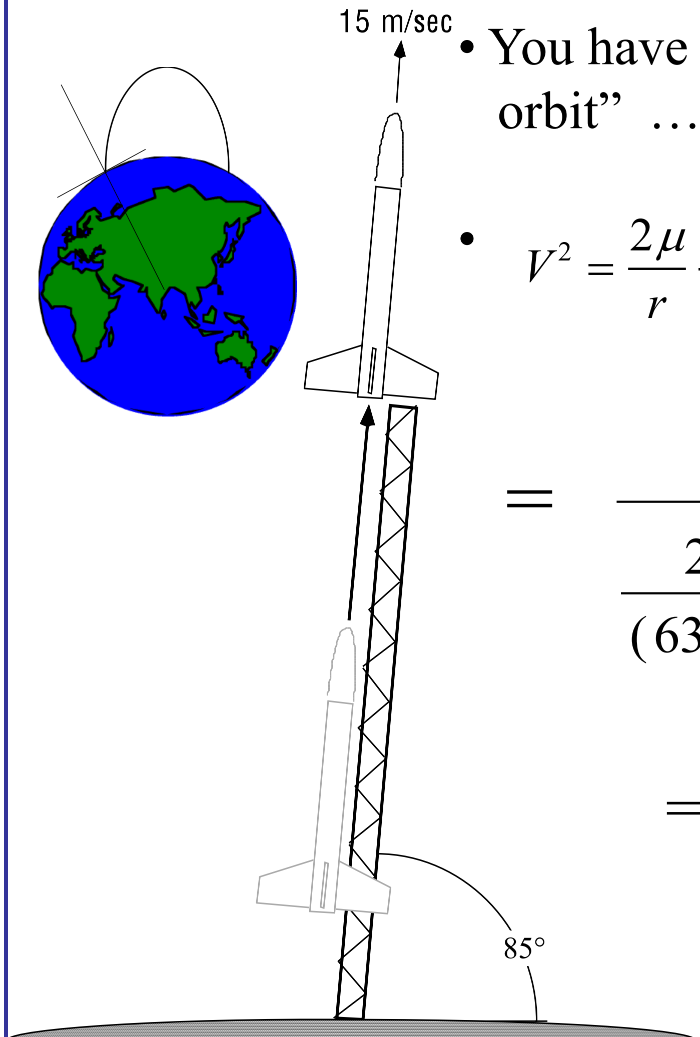
Launch I.C. Example: (cont'd)



- Earth Radius: 6370.01
- $V_{earth} = 7.292115 \cdot 10^{-5} (6370.01 + 1.21) \cos\left(38 \frac{\pi}{180}\right) = 0.36611$ km/sec
- $V_r = 15 \sin\left(\frac{\pi}{180} 85\right) = 14.942$ m/sec
- $V_\nu = 15 \cos\left(\frac{\pi}{180} 85\right) = 1.3073$ m/sec + 366.11 m/sec = 367.42 m/sec
- $\gamma_{inertial} = \frac{180}{\pi} \operatorname{atan}\left(\frac{14.942}{367.42}\right) = 2.329°$
- $\gamma_{ground} = \frac{180}{\pi} \operatorname{atan}\left(\frac{14.942}{367.42 - 366.11}\right) = 84.98°$
- r = Re+h = 6370.01+1.21= 6371.22 km

- How do we compute initial value for ν? …

Launch I.C. Example: (cont'd)



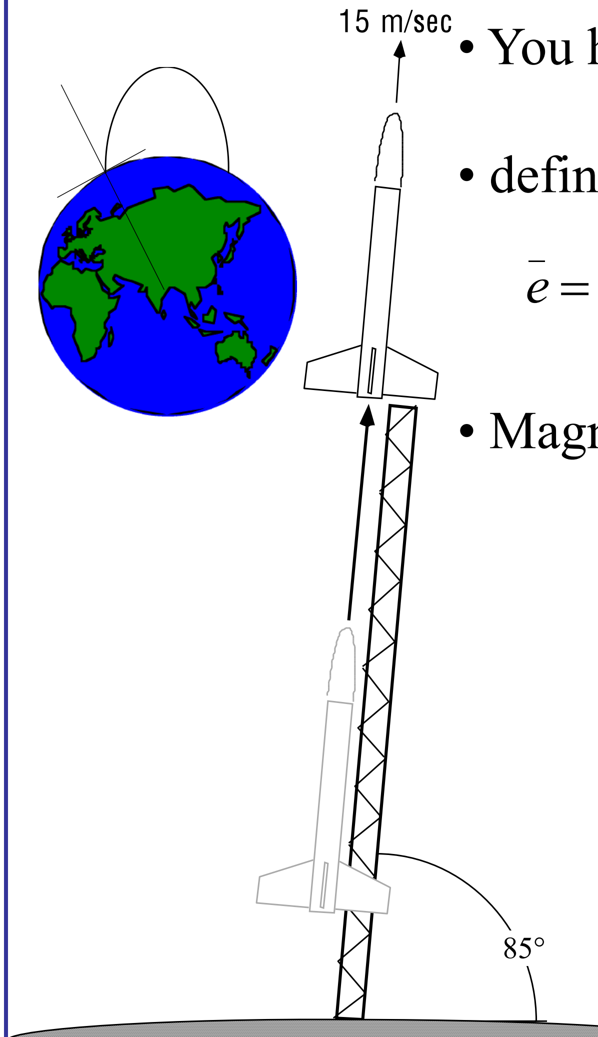
- You have to solve for the "instantaneous orbit" … *semi major axis first*

- $$V^2 = \frac{2\mu}{r} - \frac{\mu}{a} \rightarrow a = \frac{\mu}{\left[\frac{2\mu}{r} - V^2\right]} \rightarrow \boxed{a = \frac{\mu}{\left[\frac{2\mu}{r} - \left[V_r^2 + V_v^2\right]\right]}}$$

$$= \frac{3.986 \cdot 10^5}{\frac{2 \cdot 3.986 \cdot 10^5}{(6370.01 + 1.21)} - \left(\left(\frac{14.942}{1000}\right)^2 + \left(\frac{367.42}{1000}\right)^2\right)}$$

$$= 3189.056 \text{ km}$$

Launch I.C. Example: (cont'd)



- You have to solve for the "orbit" … *eccentricity next*

- define "eccentricity vector"

$$\bar{e} = \frac{1}{\mu}\left(\left[V^2 - \frac{\mu}{r}\right]\bar{R} - \left[\bar{R}\bullet\bar{V}\right]\bar{V}\right)$$

- Magnitude = Eccentricity

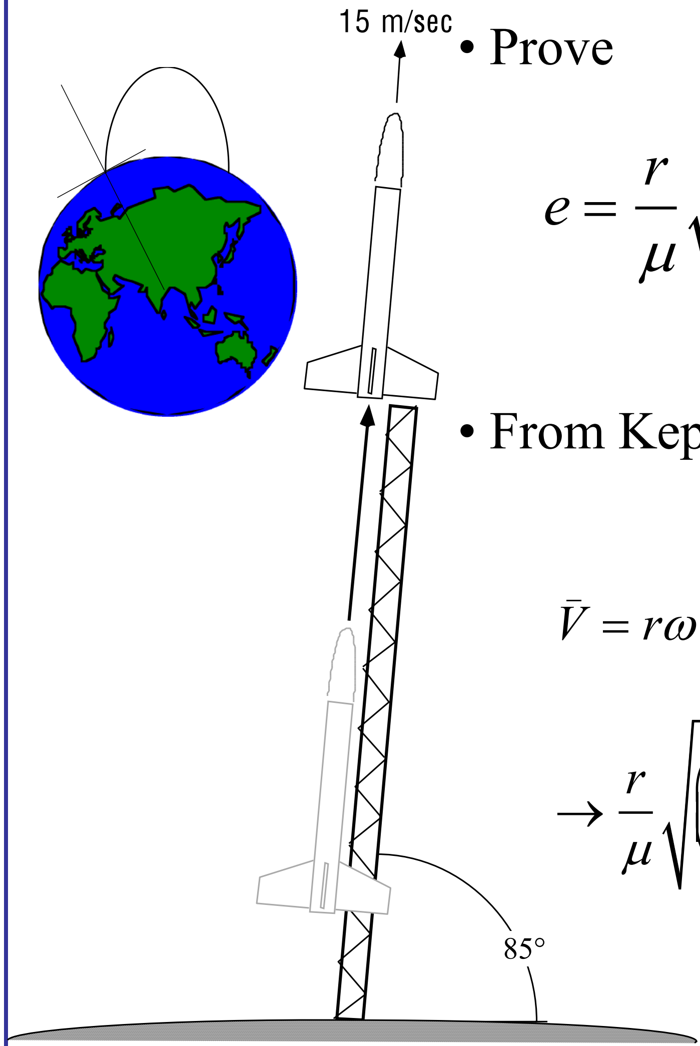
$$\left|\bar{e}\right|^2 = \left|\frac{1}{\mu}\left(\left[V^2 - \frac{\mu}{r}\right]r\,\bar{i}_r - \left[rV_r\right]\left[V_r\,\bar{i}_r + V_v\,\bar{i}_v\right]\right)\right|^2 =$$

$$\left|\bar{e}\right|^2 = \left(\frac{1}{\mu}\left[V_r^2 + V_v^2 - \frac{\mu}{r}\right]r - \frac{rV_r^2}{\mu}\right)^2 + \left(\frac{rV_rV_v}{\mu}\right)^2 =$$

$$\left(\frac{rV_r^2}{\mu} + \frac{rV_v^2}{\mu} - 1 - \frac{rV_r^2}{\mu}\right)^2 + \left(\frac{rV_rV_v}{\mu}\right)^2 = \left(\frac{rV_v^2}{\mu} - 1\right)^2 + \left(\frac{rV_rV_v}{\mu}\right)^2 =$$

$$\boxed{e = \frac{r}{\mu}\sqrt{\left(V_v^2 - \frac{\mu}{r}\right)^2 + \left(V_rV_v\right)^2}}$$

Launch I.C. Example: (cont'd)



- Prove

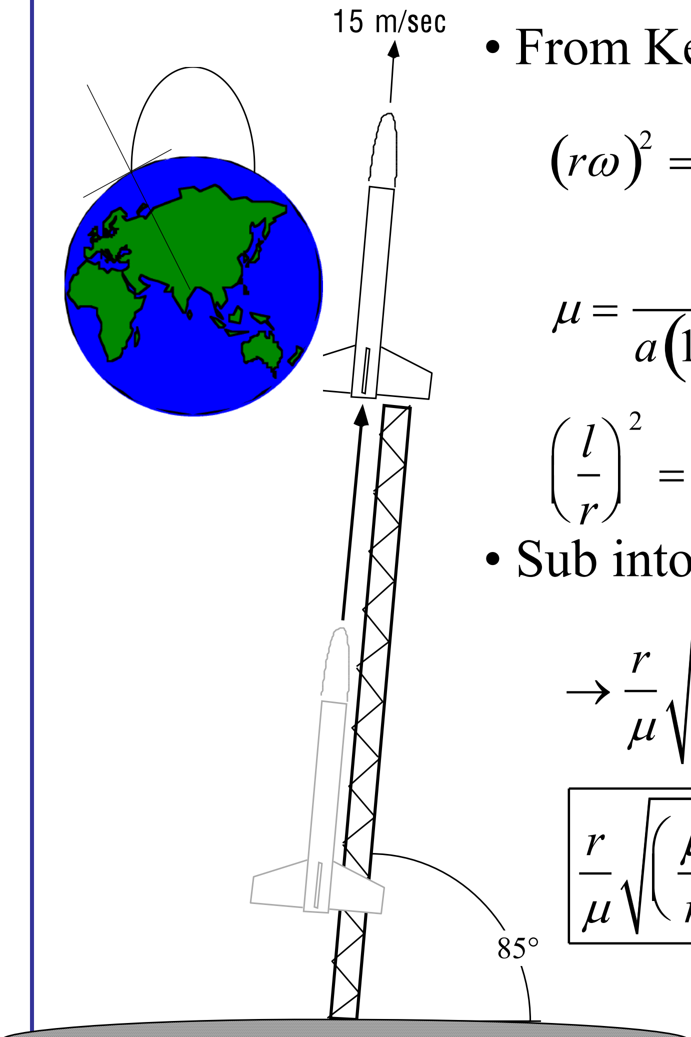
$$e = \frac{r}{\mu}\sqrt{\left(V_v^{\,2} - \frac{\mu}{r}\right)^2 + \left(V_r V_v\right)^2}$$

- From Kepler's Second law

$$\bar{V} = r\omega\left[\frac{e\sin(\nu)}{1+e\cos(\nu)}\bar{i}_r + \bar{i}_v\right] \rightarrow \text{sub into above} \rightarrow$$

$$\rightarrow \frac{r}{\mu}\sqrt{\left((r\omega)^2 - \frac{\mu}{r}\right)^2 + \left((r\omega)^2\left(\frac{e\sin(\nu)}{1+e\cos(\nu)}\right)\right)^2}$$

Launch I.C. Example: (cont'd)



- From Kepler's Third law

$$(r\omega)^2 = \left(\frac{r^2\omega}{r}\right)^2 = \left(\frac{l}{r}\right)^2$$

$$\mu = \frac{l^2}{a\left(1-e^2\right)} \rightarrow (r\omega)^2 =$$

$$\left(\frac{l}{r}\right)^2 = \frac{\mu a\left(1-e^2\right)}{r^2} = \frac{\mu}{r}\frac{a\left(1-e^2\right)}{r} = \frac{\mu}{r}\left[1+e\cos(\nu)\right]$$

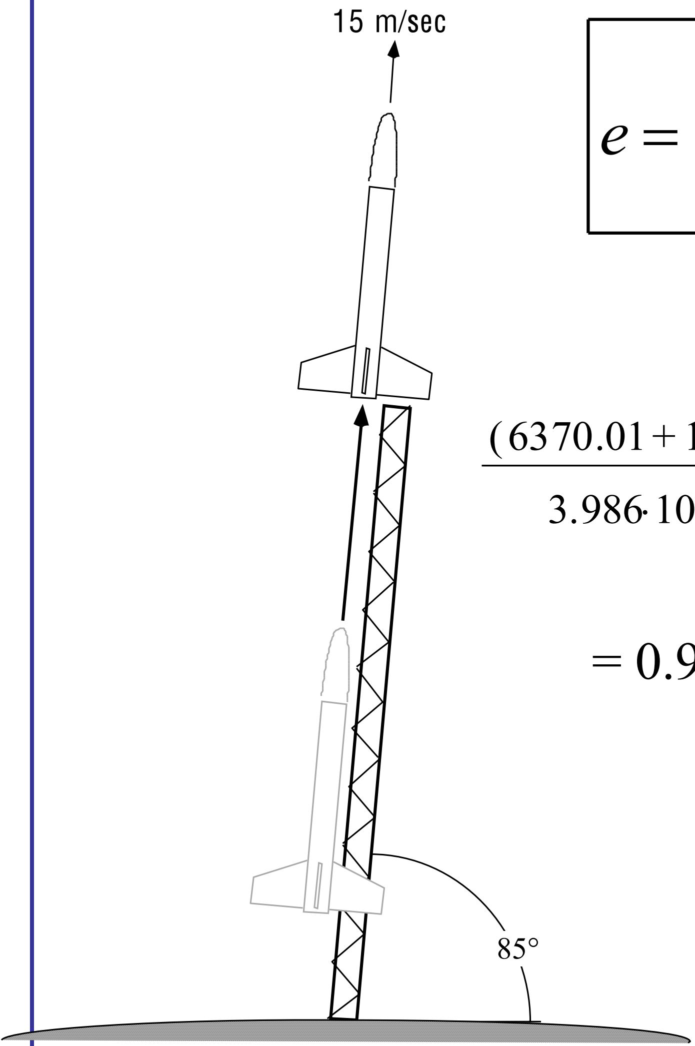
- Sub into previous

$$\rightarrow \frac{r}{\mu}\sqrt{\left(\frac{\mu}{r}+\frac{\mu}{r}e\cos(\nu)-\frac{\mu}{r}\right)^2+\left(\frac{\mu}{r}\left[1+e\cos(\nu)\right]\left(\frac{e\sin(\nu)}{1+e\cos(\nu)}\right)\right)^2} =$$

$$\boxed{\frac{r}{\mu}\sqrt{\left(\frac{\mu}{r}e\cos(\nu)\right)^2+\left(\frac{\mu}{r}e\sin(\nu)\right)^2} = \frac{r}{\mu}\frac{\mu}{r}e\sqrt{\cos(\nu)^2+\sin(\nu)^2} = e}$$

Q.E.D.

Launch I.C. Example: (cont'd)

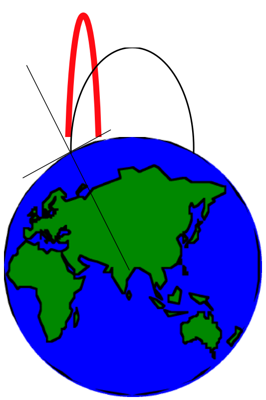


$$e = \frac{r}{\mu}\sqrt{\left(V_v^{\,2} - \frac{\mu}{r}\right)^2 + \left(V_r V_v\right)^2} =$$

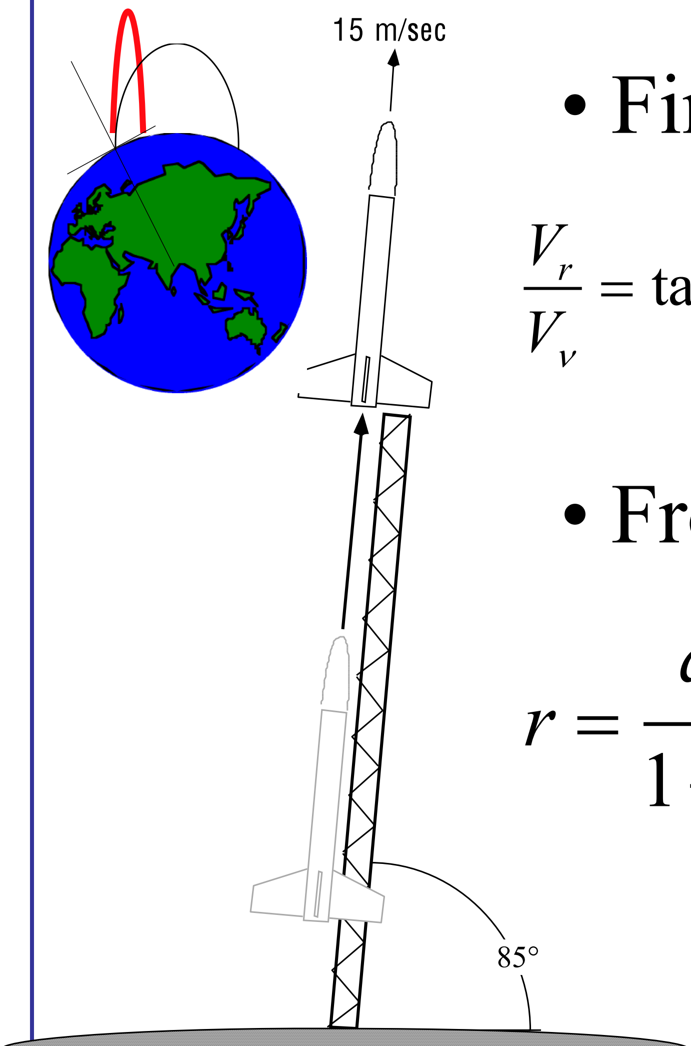
$$\frac{(6370.01+1.21)}{3.986 \cdot 10^5}\left(\left(\left(\frac{367.42}{1000}\right)^2 - \frac{3.986 \cdot 10^5}{(6370.01+1.21)}\right)^2 + \left(\frac{367.42}{1000}\frac{14.942}{1000}\right)^2\right)^{0.5}$$

$$= 0.9978422$$

"Really Skinny Orbit"



Launch I.C. Example: (cont'd)



- Finally … solve for ν

$$\frac{V_r}{V_\nu} = \tan(\gamma) = \frac{e\sin(\nu)}{1+e\cos(\nu)}$$

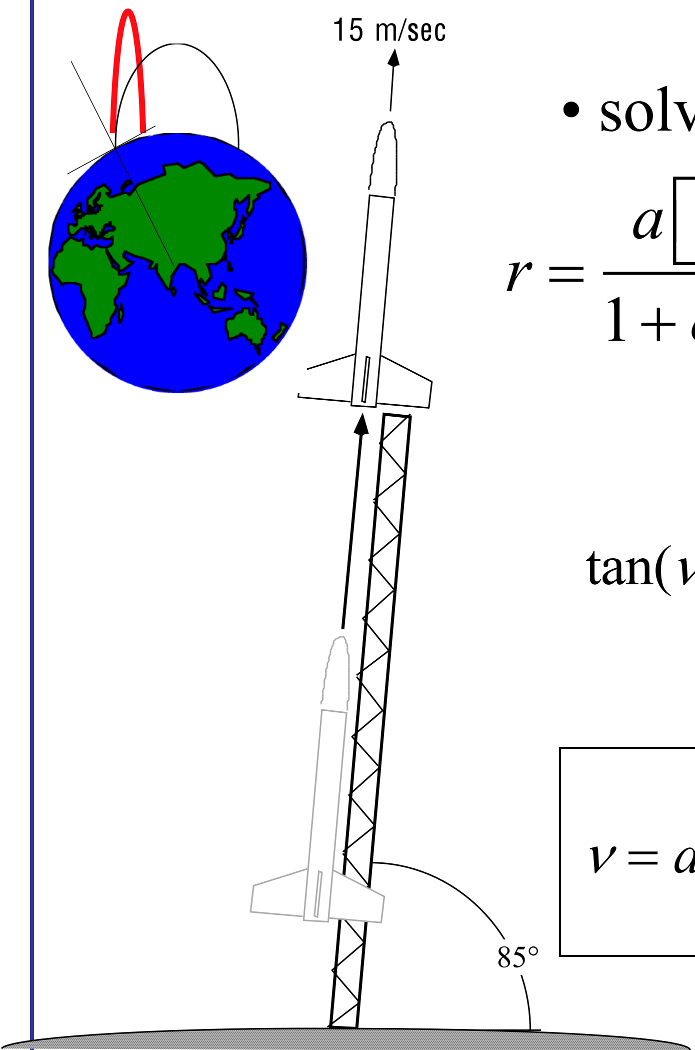
- From Kepler's First law

$$r = \frac{a\left[1-e^2\right]}{1+e\cos(\nu)}$$

- Solve for sin(ν)

$$\rightarrow \frac{a\left[1-e^2\right]}{er}\tan(\gamma) = \sin(\nu)$$

Launch I.C. Example: (cont'd)



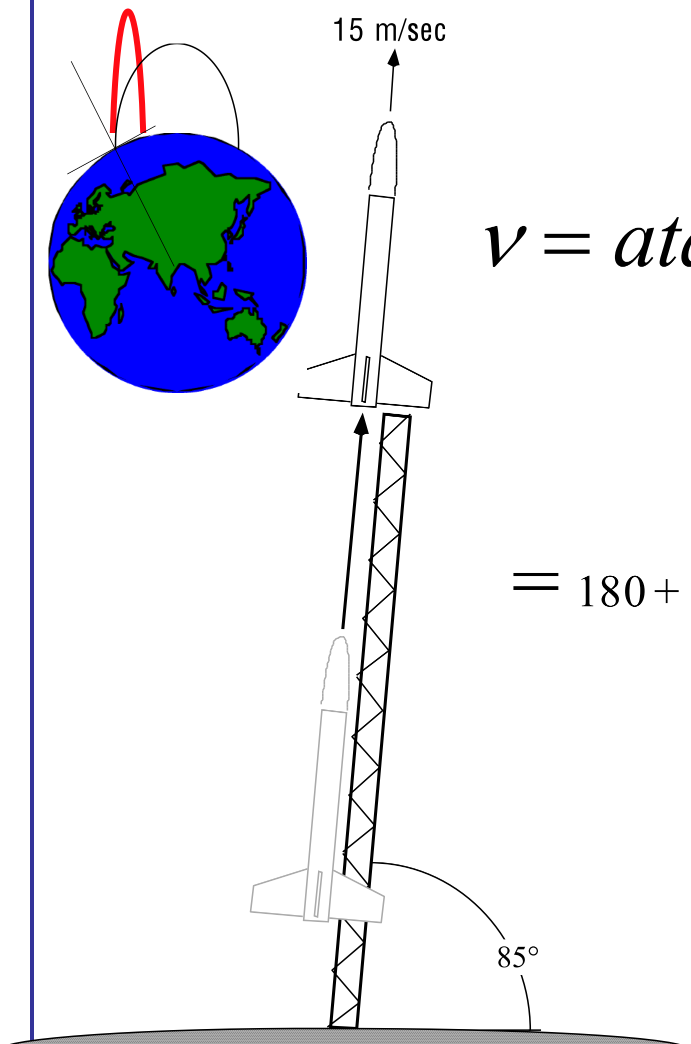
- solve for cos(ν) From Kepler's First law

$$r = \frac{a\left[1-e^2\right]}{1+e\cos(\nu)} \rightarrow \cos(\nu) = \frac{1}{e}\left[\frac{a\left[1-e^2\right]}{r} - 1\right]$$

$$\tan(\nu) = \frac{\sin(\nu)}{\cos(\nu)} = \frac{\frac{a\left[1-e^2\right]}{er}\tan(\gamma)}{\frac{1}{e}\left[\frac{a\left[1-e^2\right]}{r} - 1\right]} = \frac{\frac{a}{r}\left[1-e^2\right]\frac{V_r}{V_\nu}}{\left[\frac{a}{r}\left[1-e^2\right]-1\right]} \rightarrow$$

$$\boxed{\nu = atan2\left\{\frac{a}{r}\left[1-e^2\right]\frac{V_r}{V_\nu}, \frac{a}{r}\left[1-e^2\right]-1\right\}}$$

Launch I.C. Example: (cont'd)

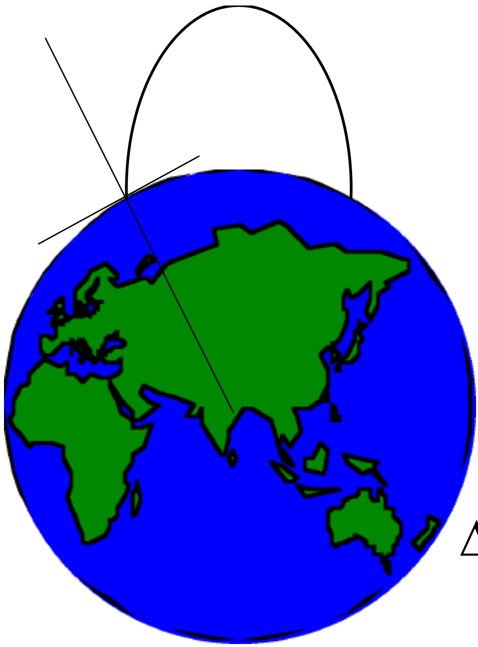


$$\nu = atan2\left\{\frac{a}{r}\left[1-e^2\right]\frac{V_r}{V_v}, \frac{a}{r}\left[1-e^2\right]-1\right\}$$

$$= 180 + \frac{180}{\pi}\,\text{atan}\left(\frac{\dfrac{\dfrac{3189.056}{(6370.01+1.21)}(1-0.997842205^2)\,14.942}{367.42}}{\dfrac{3189.056}{(6370.01+1.21)}(1-0.997842205^2)-1}\right)$$

$$= 179.995^\circ$$

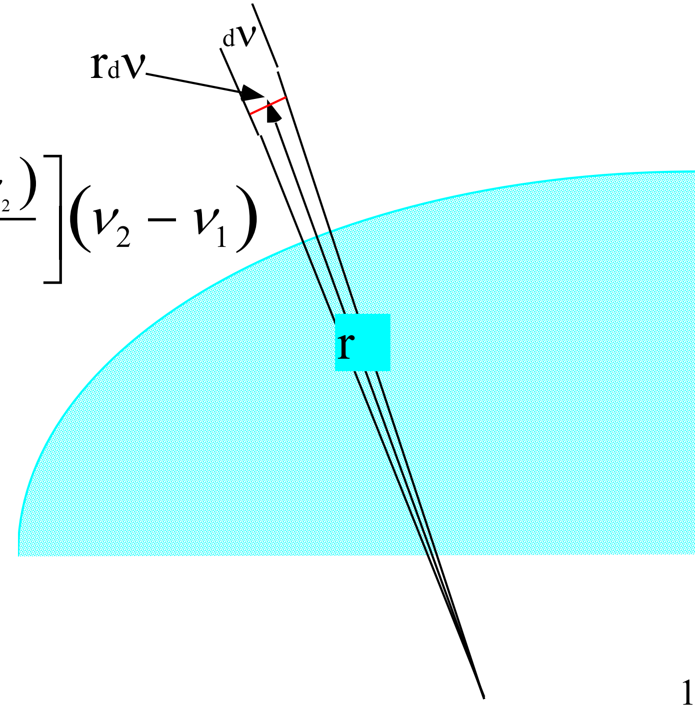
Ground Launch: Down Range Calculation



- Integrated trajectory gives

$$r, \nu$$

- Inertial Downrange



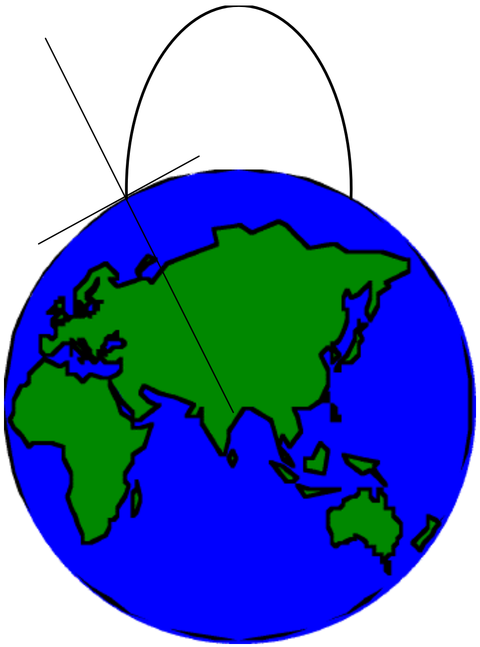
$$\Delta R = \int_{\nu_1}^{\nu_2} r d_n \approx \left[\frac{r(\nu_1) + r(\nu_2)}{2} \right] (\nu_2 - \nu_1)$$

Recursive Formula

$$R_{i+1} = R_i + \left[\frac{r_{i+1} + r_i}{2} \right] (\nu_{i+1} - \nu_i)$$

Ground Launch: Down Range Calculation

(cont'd)

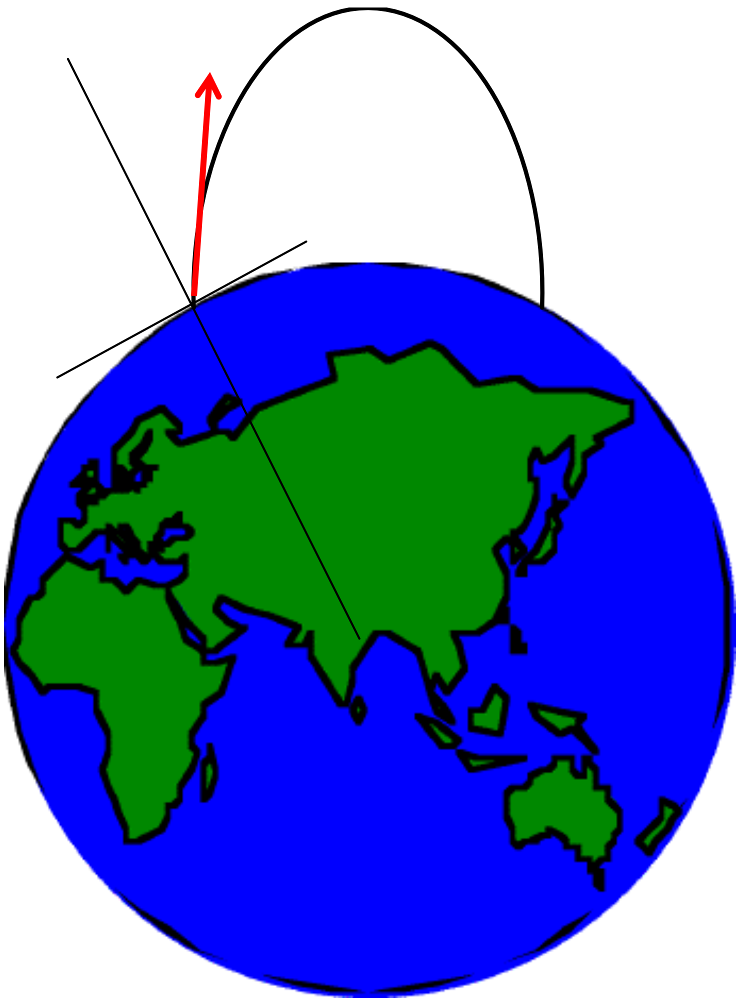


- Ground Relative Downrange
- Account for Earth Rotation

$$R_{earth} = \sqrt{\left[R\cos(Az)\right]^2 + \left[R\sin(Az) - V_{earth} \times T.O.F\right]^2}$$

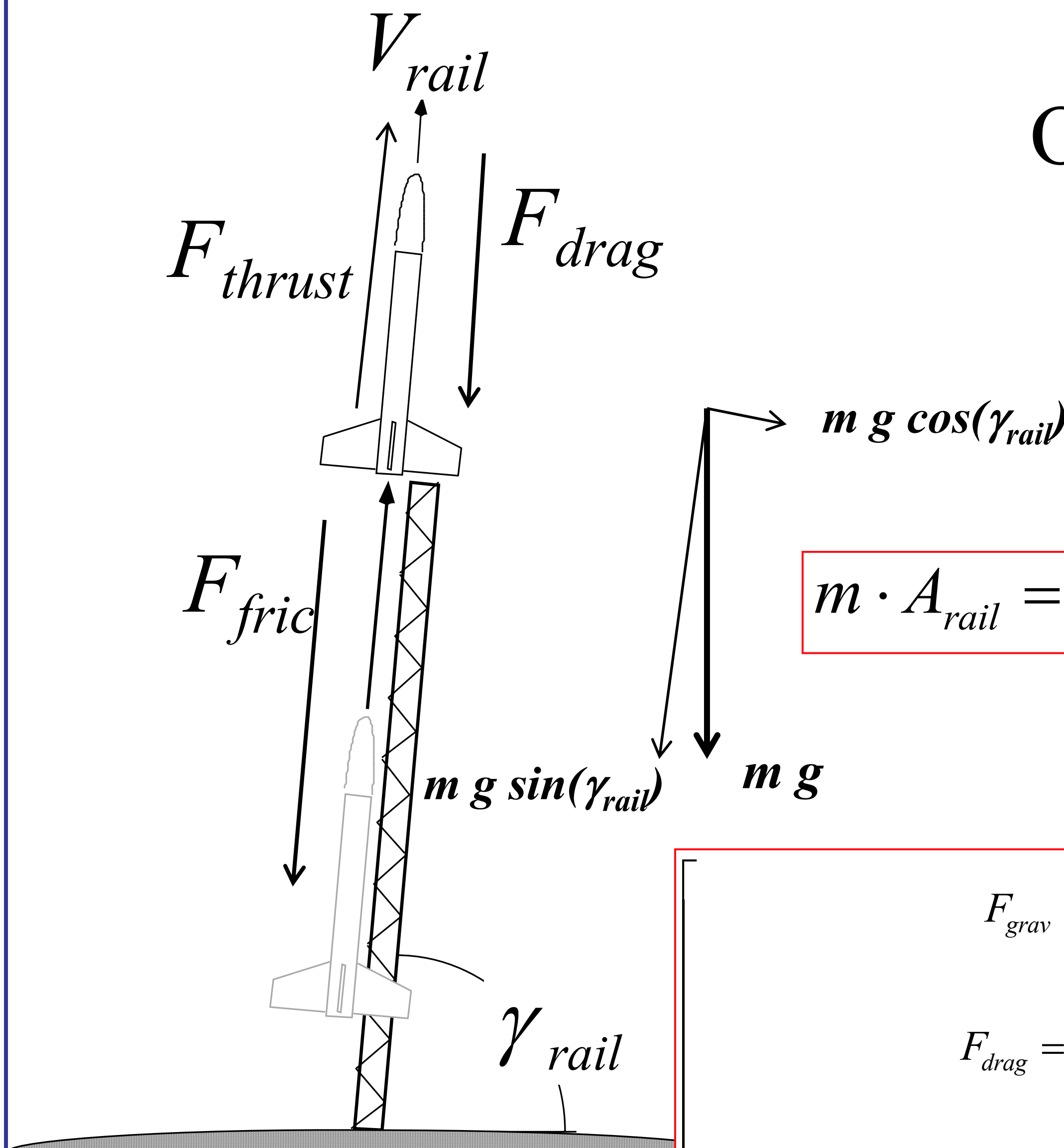
Time of flight …time from Launch to impact altitude

Fixed Earth Approximation



- Ignore effects of rotation
- $V_{inertial}=V_{ground}$
- $\gamma_{inertial}=\gamma_{ground}$
- $\square_{inertial}=\square_{ground}$
- Accurate for Short Duration lower altitude flights

Velocity Off of the Rail (1)



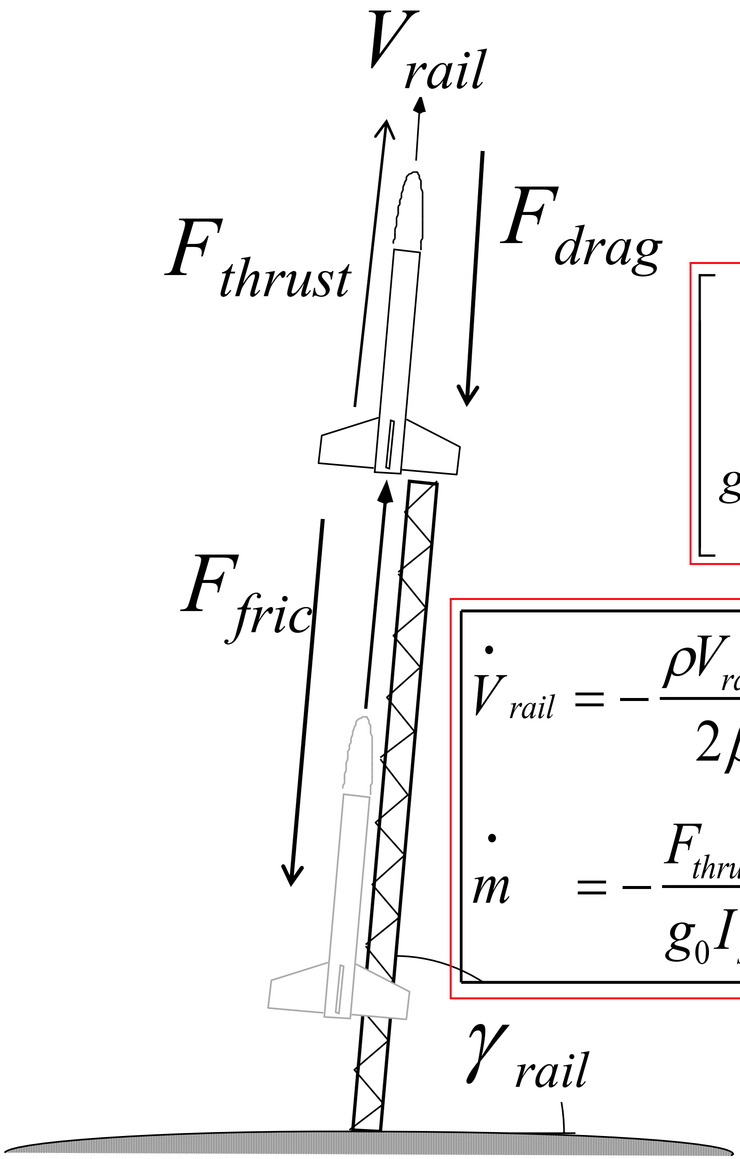
$$m \cdot A_{rail} = F_{thrust} - F_{grav} - F_{fric} - F_{drag}$$

$$F_{grav} = m \cdot g \cdot \sin(\gamma_{rail}) = m \frac{\mu}{(R_e + h)^2} \cdot \sin(\gamma_{rail})$$

$$F_{drag} = C_D A_{ref} \left(\frac{1}{2} \rho V_{rail}^{\ 2} \right) = m \frac{\rho V_{rail}^{\ 2}}{2\beta}$$

$$F_{fric} = C_f \cdot W_{norm_{rail}} = C_f \cdot m \cdot g \cos(\gamma_{rail}) = C_f \cdot m \frac{\mu}{(R_e + h)^2} \cdot \cos(\gamma_{rail})$$

Velocity Off of the Rail (2)



$$m \cdot A_{rail} = F_{thrust} - F_{grav} - F_{fric} - F_{drag}$$

$$\left[\begin{array}{l} \beta = \dfrac{m}{C_D A_{ref}} \\ g = \dfrac{\mu}{(R_e + h)^2} \end{array}\right] \rightarrow careful!\ with\ units$$

$$\dot{V}_{rail} = -\frac{\rho V_{rail}^2}{2\beta} - \frac{\mu}{(R_e + h)^2}\left[\sin(\gamma_{rail}) + C_f \cdot \cos(\gamma_{rail})\right] + \frac{F_{thrust}}{m}$$

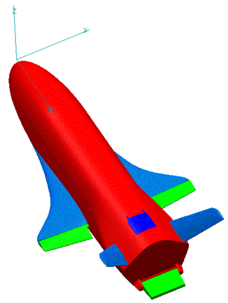
$$\dot{m} = -\frac{F_{thrust}}{g_0 I_{sp}}$$

$\{\gamma_{rail}, V_{rail}\}$-->ground relative

$\{V_0=0, m_0=M_{total}\}$

Appendix 3: Ballistic Coefficient Examples

Example
Ballistic Coefficients



- X-37 Weight = 7000 lb
- $A_{ref} = 11{,}386\ in^2$
- Drag Coefficient
  - $C_d = 0.1$ (zero alpha)

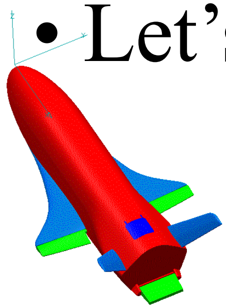
- Ballistic Coef (β) = $W/(C_d A_{ref})$
  - $\beta = 6.148\ lb/in^2$ (zero alpha)

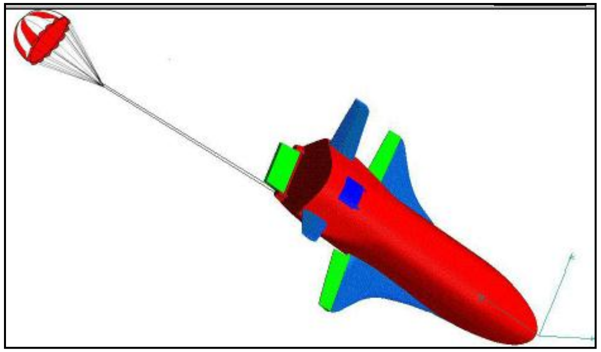
- **Individual parachute factors:**
  - **Diameter 7.4 ft (conical ribbon parachute)**
  - **Area (S) 42.99 ft^2**
  - **Free steam drag coefficient**
  - **(C_d) .55**
  - **Drag loss factor due for two**
  - **parachute cluster .95 (1.0**
  - **for single parachute)**
  - **Chute weight 60 lbs/ea**

- X-37 Orbital Transfer Vehicle

Example Ballistic Coefficients (cont'd)

- Let's deploy 1 chute





- X-37 Weight = 7000 lb
- Xc.g. = 178.75
- S_{ref} = 11,386 in^2
- Assume Drag Coefficient
  - C_d = 0.1 (zero alpha)

- Ballistic Coef (β) = $W/(C_d S_{ref})$
  - β = 6.148 lb/in^2 (zero alpha)

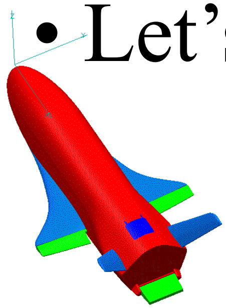
- <u>Vehicle With 1- Parachutes</u>
- Weight = 7120 lb
- A_{ref} = 11,386 in^2

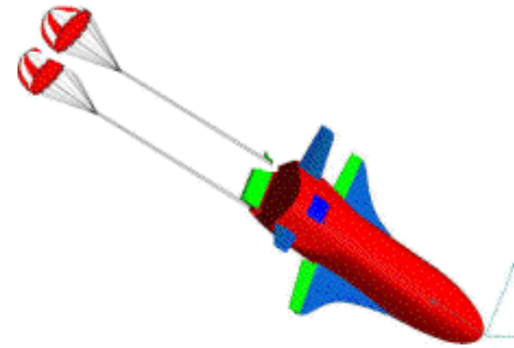
- Ballistic Coef (β) = $W/(C_d S_{ref})$ lb/in^2
  - $\beta_{1\ chute}$ = 1.554 lb/in^2

$$\beta = 7060\left(\left(\frac{144\,(42.99\cdot 0.55)}{11386} + 0.1\right) 11386\right)^{-1}$$

Example Ballistic Coefficients (cont'd)

- Let's deploy 2 chutes





- X-37 Weight = 7000 lb
- Xc.g. = 178.75
- S_{ref} = 11,386 in^2
- Assume Drag Coefficient
  - C_d = 0.1 (zero alpha)
- Ballistic Coef (β) = $W/(C_d S_{ref})$
  - β = 6.148 lb/in^2 (zero alpha)

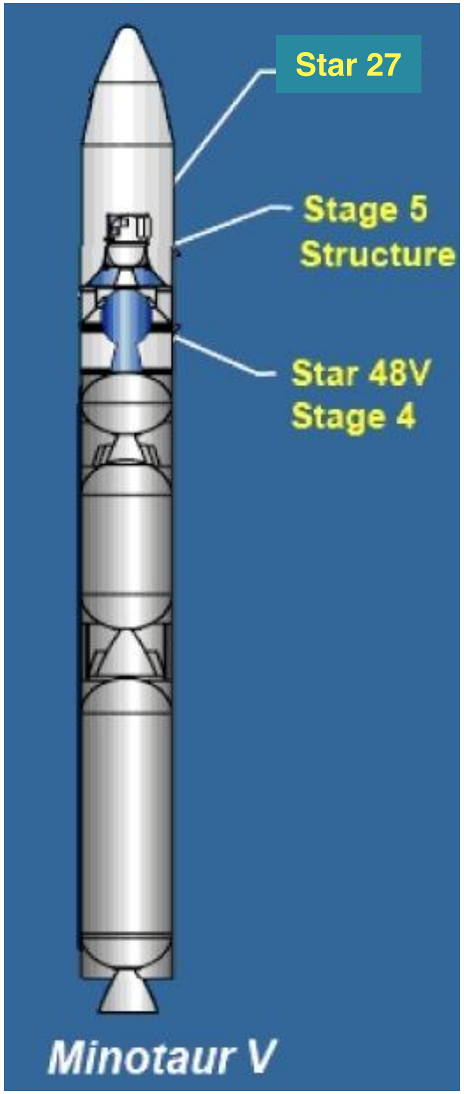
- <u>Vehicle With 2- Parachutes</u>
- Weight = 7120 lb
- A_{ref} = 11,386 in^2

- Ballistic Coef (β) = $W/(C_d S_{ref})$ lb/in^2
  - $\beta_{1\ chute}$ = 0.9359 lb/in^2

$$\beta = 7120\left(\left(\frac{2\cdot 144\,(42.99\cdot 0.55)\;(0.95)}{11386} + 0.1\right)11386\right)^{-1}$$

Appendix 4: Comparison of Ballistic and Non-Ballistic trajectories

Example I: Minotaur V Launch to Medium Earth Transfer Orbit (MTO)



1st Stage – TU-903

2nd Stage – SR-119

3rd Stage – SR-120

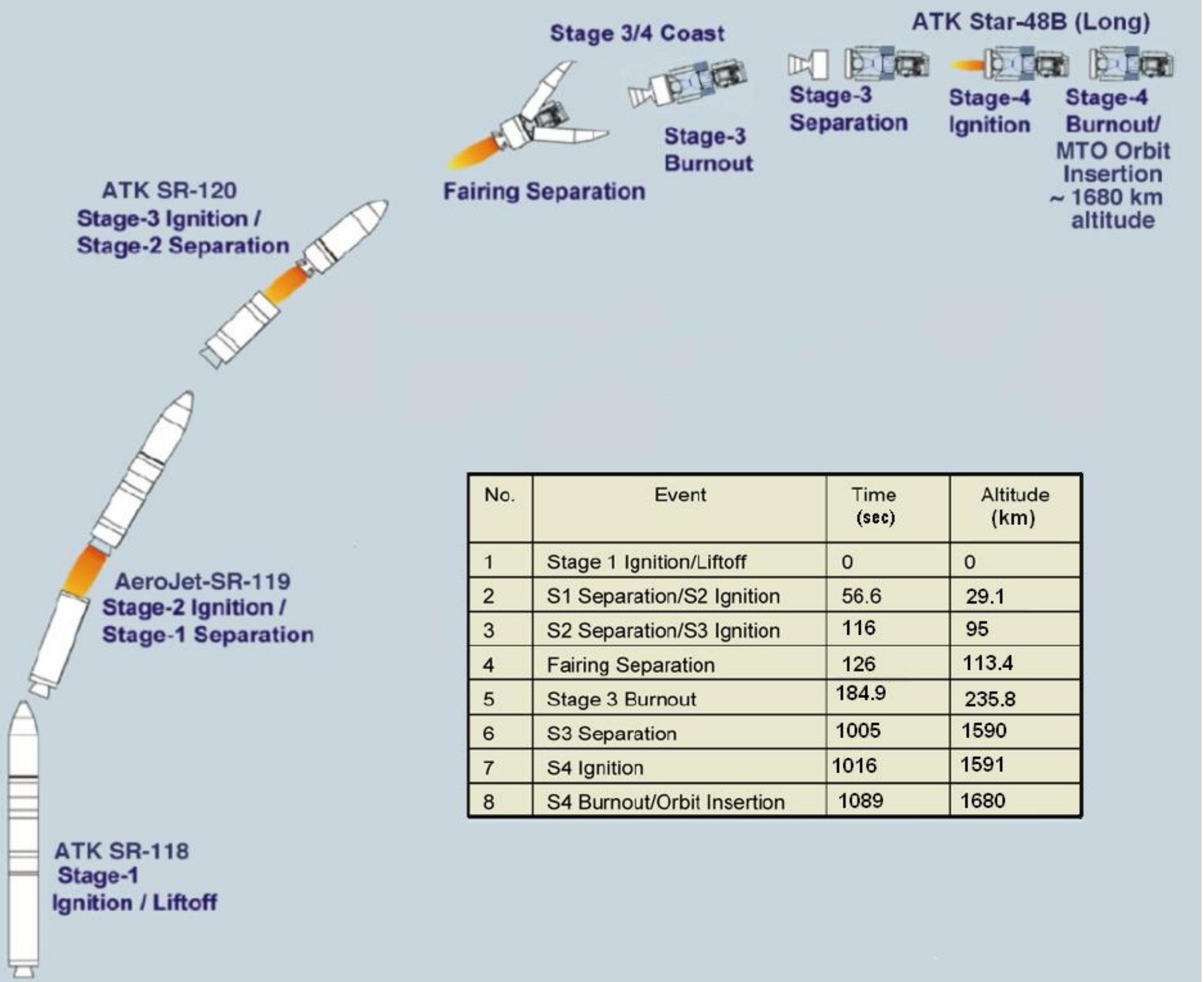
4th Stage – Star 48B long

5th Stage – Star 27 with 25-30% propellant offload *(depending on final payload mass)*

- *Required Orbit 13,000 by 19,000 km altitude*

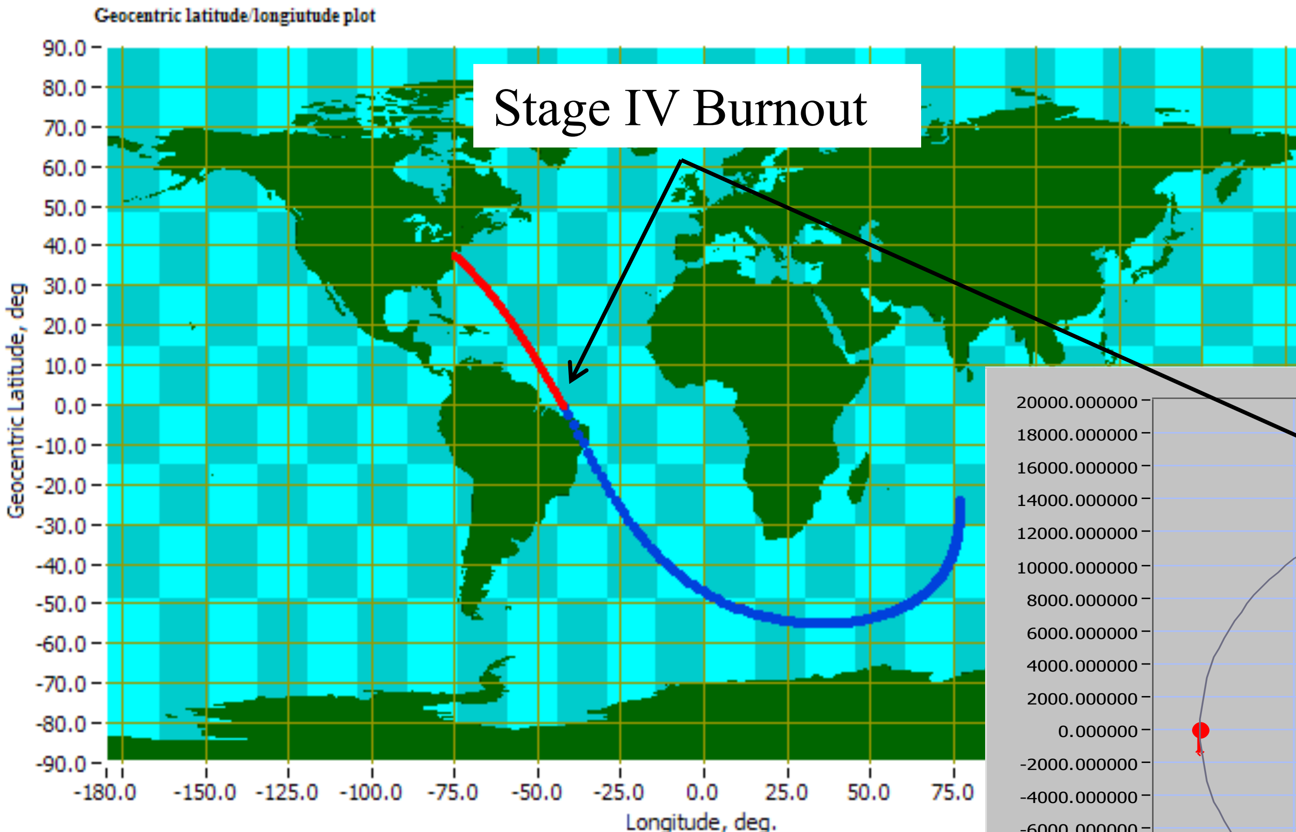
- Proposed configuration allows *400+ kg* payload delivery to 19,000 km altitude MEO orbit *without 6th stage*

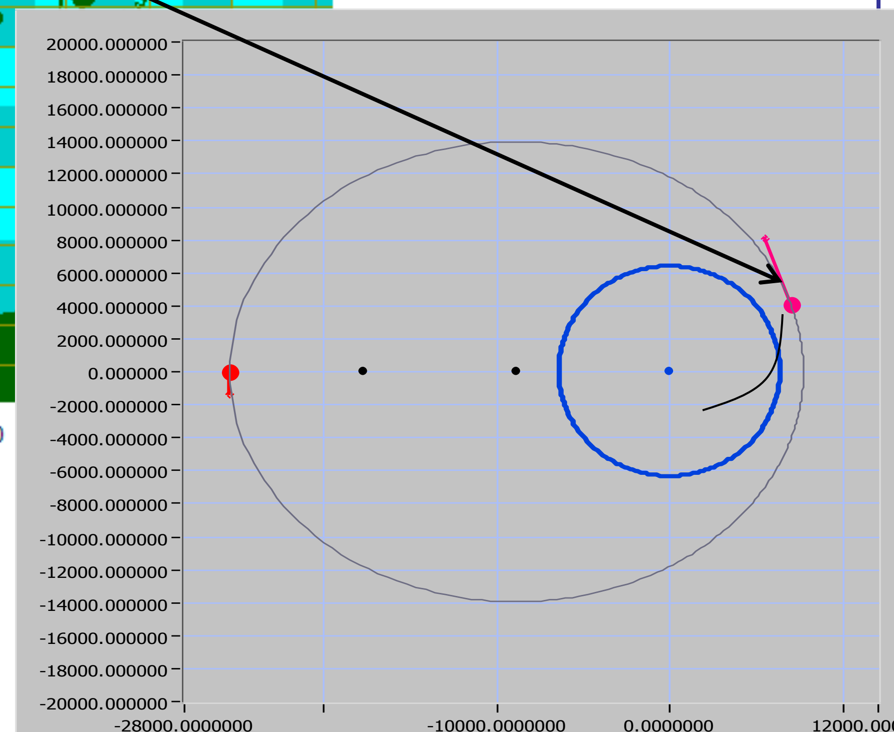
Mission CONOPS/Timeline



No.	Event	Time (sec)	Altitude (km)
1	Stage 1 Ignition/Liftoff	0	0
2	S1 Separation/S2 Ignition	56.6	29.1
3	S2 Separation/S3 Ignition	116	95
4	Fairing Separation	126	113.4
5	Stage 3 Burnout	184.9	235.8
6	S3 Separation	1005	1590
7	S4 Ignition	1016	1591
8	S4 Burnout/Orbit Insertion	1089	1680

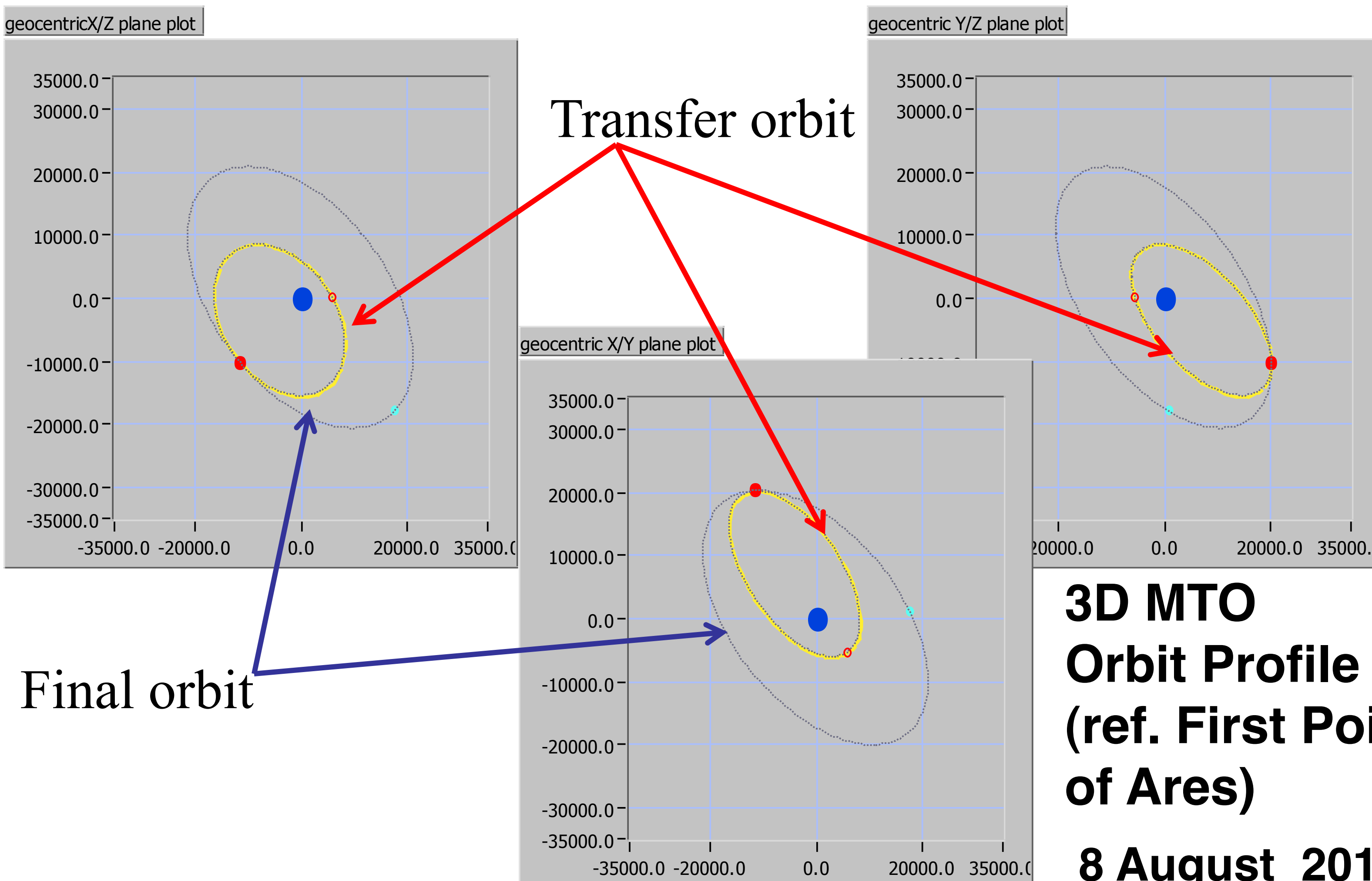
Launch Mission Plan





- Direct insertion into MTO orbit

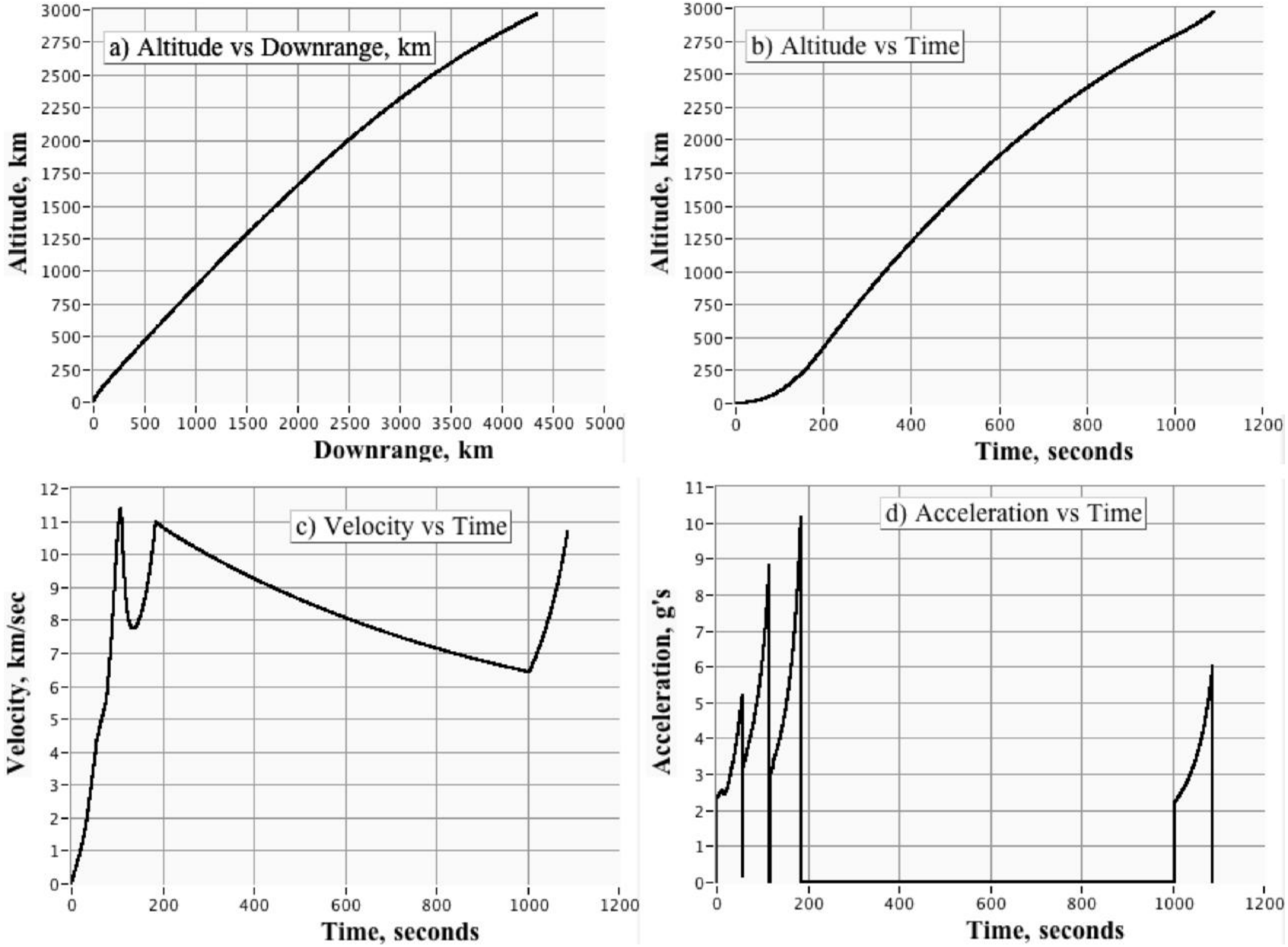
Launch Mission Plan



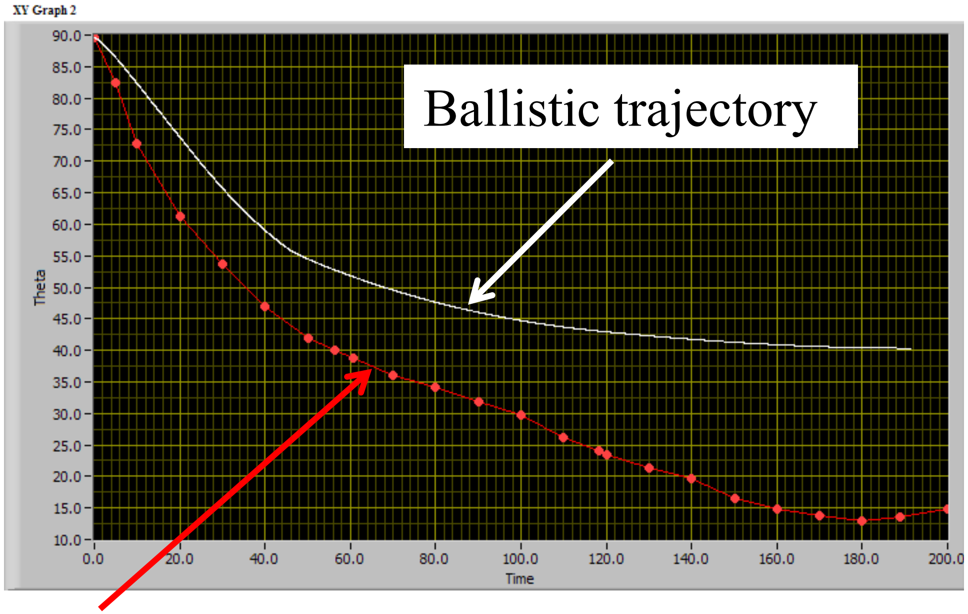
3D MTO Orbit Profile (ref. First Point of Ares)

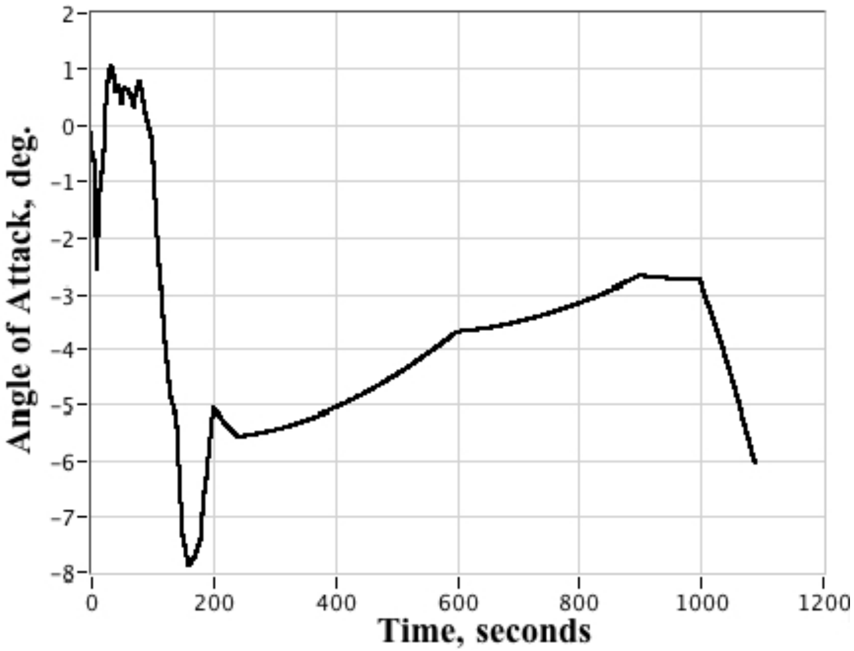
8 August 2010 launch

Ballistic Launch Profile



Pitch Profile Optimization





Optimized Pitch Profile

- 3-Degree of freedom Launch simulation used to optimize pitch profile for maximum stage IV mass to MTO

- Negative lift used to "turn the corner" during stage 2 burn.

Optimized (Non-Ballistic) Launch Profile