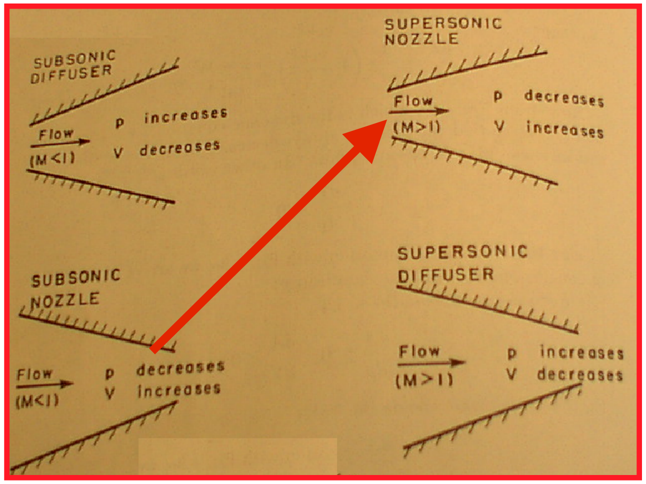
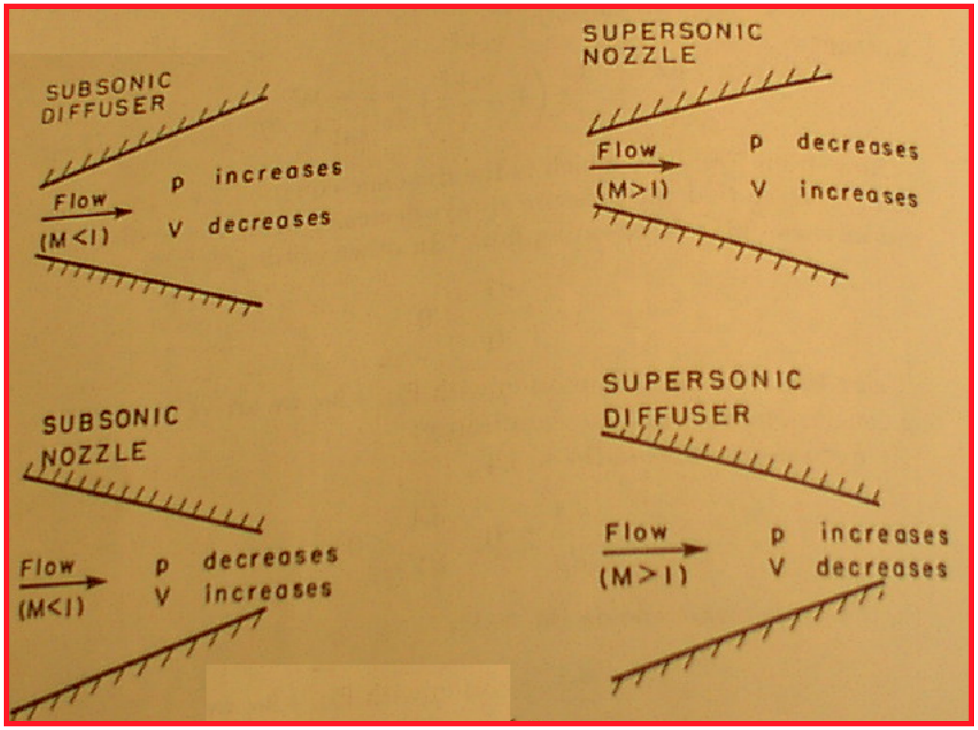
Section 5, Lecture 1:
Review of Idealized Nozzle Theory

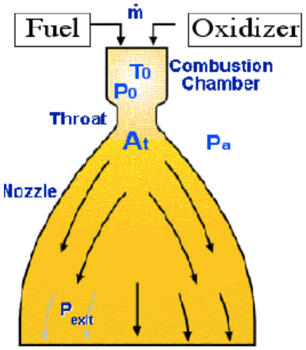


Summary of Fundamental Properties and Relationships

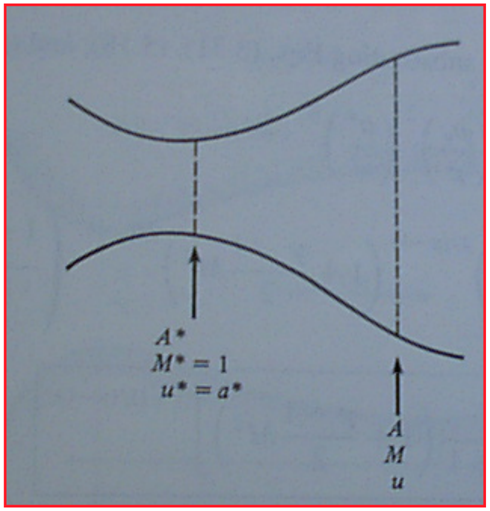
Sutton and Biblarz, Chapter 3

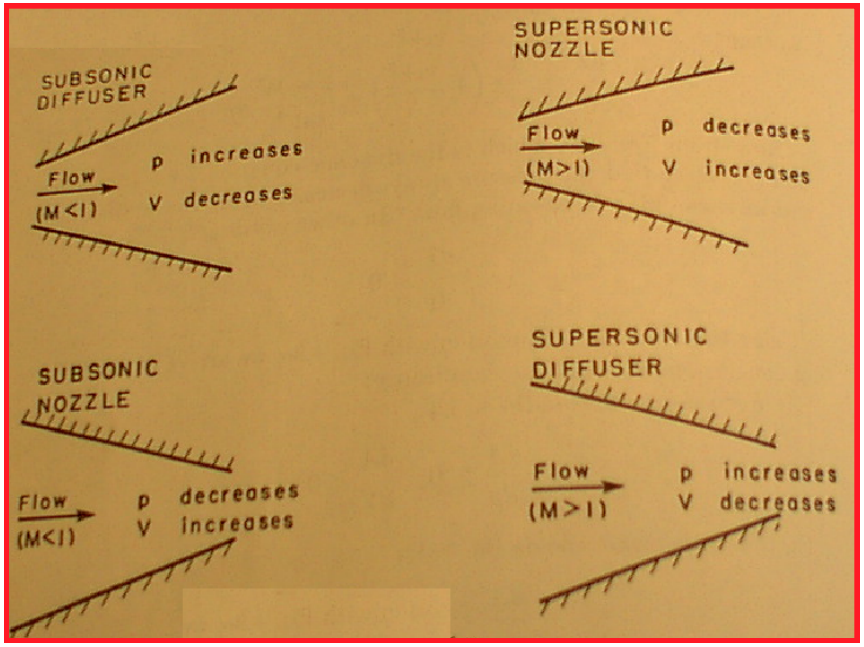
Fundamental Properties of Supersonic and Supersonic Flow





… Hence the shape of the rocket Nozzle





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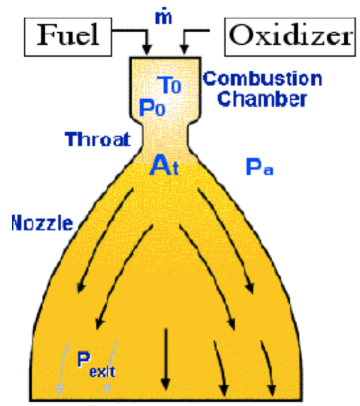
What is a NOZZLE



• FUNCTION of rocket nozzle is to convert thermal energy in propellants into kinetic energy as efficiently as possible

• Nozzle is substantial part of the total engine mass.

• Many of the historical data suggest that 50% of solid rocket failures stemmed from nozzle problems.

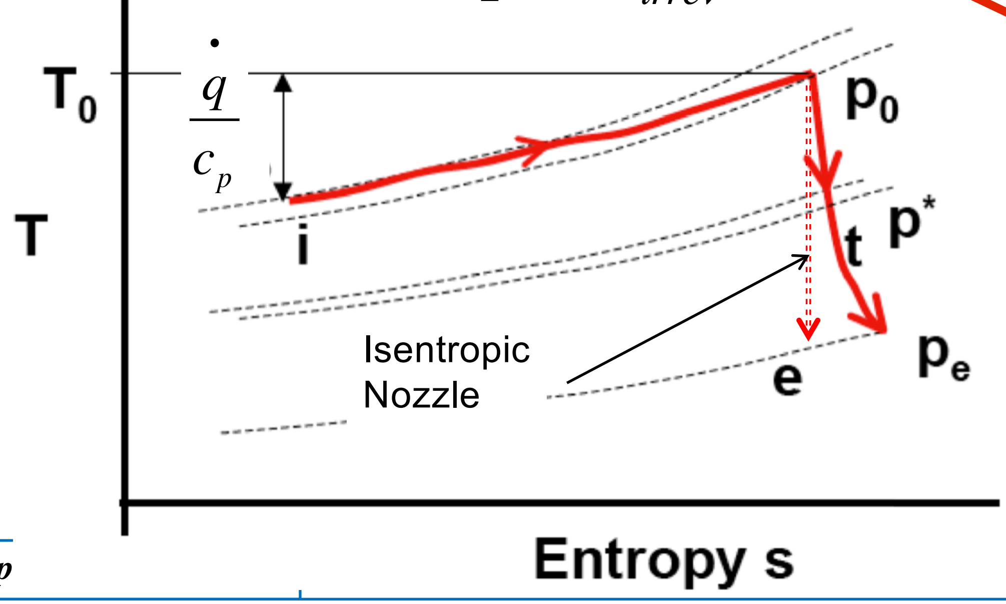
The design of the nozzle must trade off:
1. Nozzle size (needed to get better performance) against nozzle weight penalty.

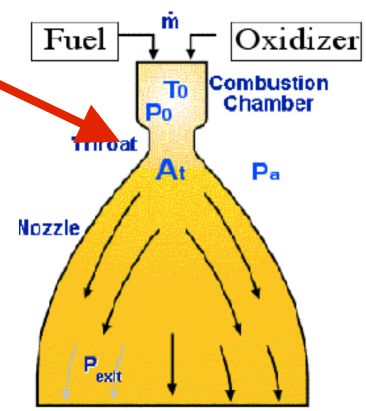
2. Complexity of the shape for shock-free performance vs. cost of fabrication

Temperature/Entropy Diagram for a Typical Nozzle

$$\dot{q} + h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \qquad c_p = \left(\frac{dh}{dT}\right)_p$$

$$Tds = \delta q + ds_{irrev}$$





Stagnation Temperature, Pressure for Adiabatic, Isentropic Flow of a Calorically Perfect Gas

- Stagnation temperature is a measure of the Kinetic Energy of the flow Field.

$$\frac{T_0}{T} = 1 + \frac{(\gamma - 1)}{2} M^2$$

- Stagnation *(total)* pressure: Constant throughout Isentropic flow field

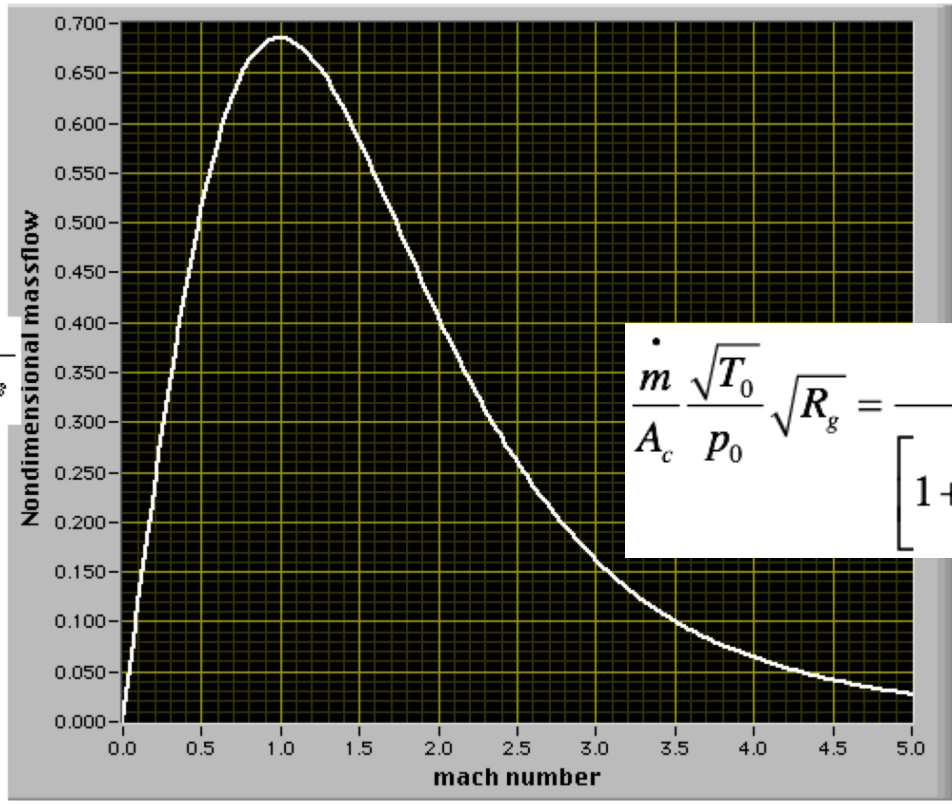
$$\frac{p_0}{p} = \left[\frac{T_0}{T}\right]^{\frac{\gamma}{\gamma-1}} = \left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{\gamma}{\gamma-1}}$$

- In Isentropic Nozzle, T_0, P_0 are constant

$$T(x) = \frac{T_0}{1 + \frac{\gamma - 1}{2} M(x)^2}$$

$$P(x) = \frac{P_0}{\left(1 + \frac{\gamma - 1}{2} M(x)^2\right)^{\frac{\gamma}{\gamma-1}}}$$

Nozzle Mass Flow per Unit Area



$$\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}$$

$$\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g} = \frac{\sqrt{\gamma}M}{\left[1+\frac{(\gamma-1)}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

- maximum Massflow/area Occurs when When M=1

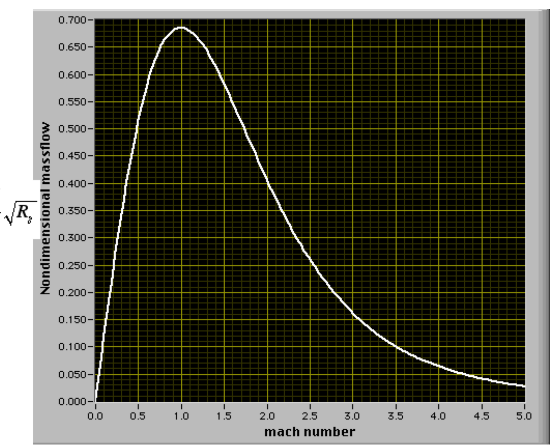
Choking Massflow Equation

- maximum Massflow/area Occurs when When M=1
- Effect known as *Choking* in a Duct or Nozzle
- i.e. nozzle will Have a mach 1 throat

$$\left(\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}\right)_{\max} = \left(\frac{\dot{m}}{A^*}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}\right) =$$

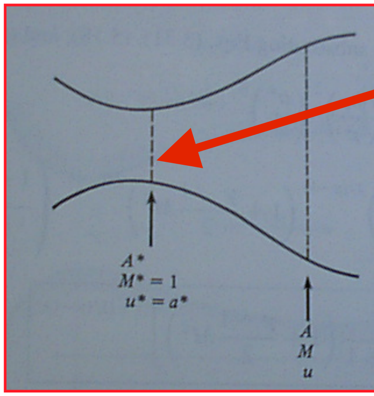
$$\frac{\sqrt{\gamma}}{\left[1+\frac{(\gamma-1)}{2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} = \sqrt{\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \rightarrow$$

$$\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}$$



$$\boxed{\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\frac{p_0}{\sqrt{T_0}}}$$

Isentropic Nozzle Flow: Area Mach Relationship

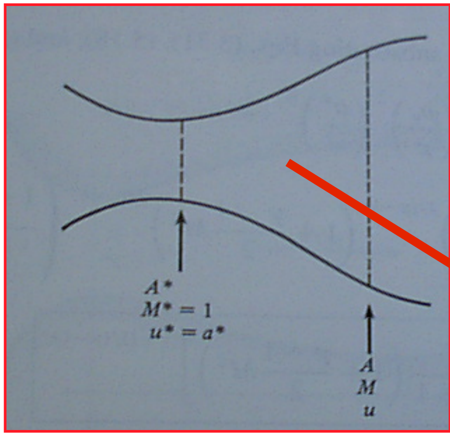


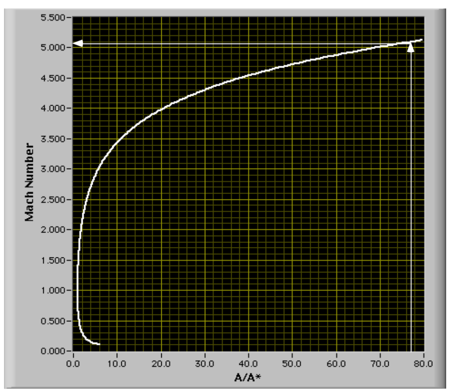
- Consider a "choked-flow" Nozzle … (I.e. M=1 at Throat)

- Then comparing the massflow /unit area at throat to some Downstream station

$$\frac{\dfrac{\dot{m}}{A^*}\dfrac{\sqrt{T_0}}{p_0}\sqrt{R_g}}{\dfrac{\dot{m}}{A}\dfrac{\sqrt{T_0}}{p_0}\sqrt{R_g}} = \frac{A}{A^*} = \frac{\sqrt{\left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}{\dfrac{1}{\left[1+\dfrac{(\gamma-1)}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Isentropic Nozzle Flow: Area Mach Relationship





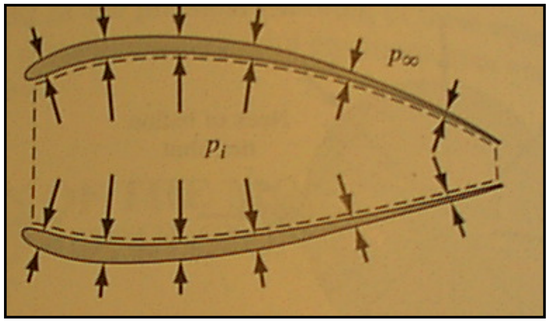
- A/A^* Directly related to Mach number
- "Two-Branch solution: Subsonic, Supersonic

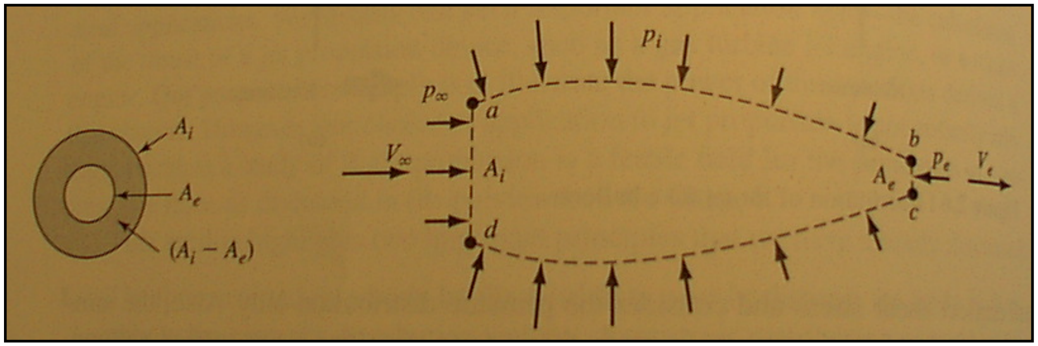
$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

- Nonlinear Equation requires Numerical Solution
- "Newton's Method"

Engine Thrust Model (revisited)

• Steady, Inviscid, One-Dimensional Flow Through Ramjet





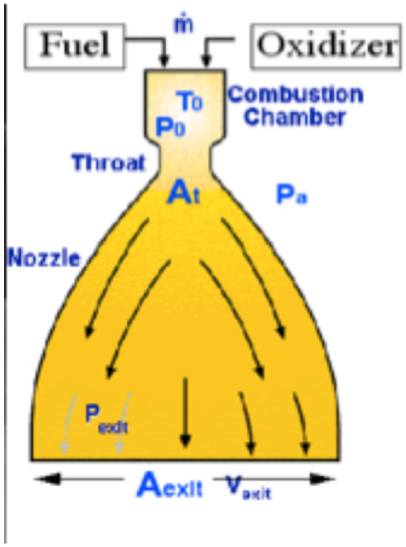
$$\boxed{-\iint_{C.S.} (p)\vec{dS} = \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds}\right)\vec{V}} \longrightarrow$$

$$Thrust = \dot{m}_e V_e - \dot{m}_i V_i + \left(p_e A_e - p_\infty A_e\right)$$

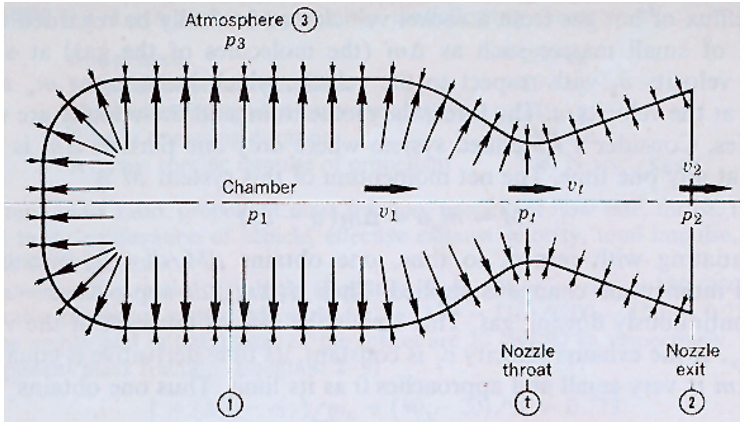
Rocket Thrust Equation, revisited

$$\dot{m}_i = 0$$

$$\boxed{Thrust = \dot{m}_e V_e + \left(p_e A_e - p_\infty A_e\right)}$$



• Thrust + Oxidizer enters combustion Chamber at ~0 velocity, combustion Adds energy … High Chamber pressure Accelerates flow through Nozzle *Resultant pressure forces produce thrust*



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Thrust Coefficient

For an isentropic nozzle $P_{0_{exit}} = P_0$

$$C_F \equiv \frac{Thrust}{P_0 A^*} = \gamma \sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0}$$

$$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}}\sqrt{\frac{\left(\frac{P_0}{p_{exit}}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}}-1\right]}}$$

$$C_F = \gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\cdot\left\{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + \frac{\gamma-1}{2\gamma}\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\left[\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right]\right\}$$

- **C_F is a function of Nozzle pressure ratio and back pressure only**

Thrust Coefficient Summary

Ideal Thrust Coefficient

$$C_F = \frac{F_{thrust}}{P_0 \cdot A^*} = \gamma \cdot \sqrt{\frac{2}{\gamma-1} \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}} \cdot \left(1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right)} + \frac{A_{exit}}{A^*} \cdot \left(\frac{p_{exit}-p_\infty}{P_0}\right)$$

Optimal Thrust Coefficient $\rightarrow p_{exit} = p_\infty$

$$C_F = \frac{F_{thrust}}{P_0 \cdot A^*} = \gamma \cdot \sqrt{\frac{2}{\gamma-1} \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}} \cdot \left(1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right)}$$

Maximum Thrust Coefficient $\rightarrow$ *Expand Nozzle Until* $P_{exit} \sim 0$

$$C_{F\max} = \gamma \cdot \sqrt{\frac{2}{\gamma-1} \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

Maximum Possible Thrust Coefficient for a Given Combination of Propellants

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Characteristic Velocity, C*

Define ….

$$C^* = \left(\frac{P_0 A^*}{\dot{m}_{exit}}\right) = \left(\frac{C_e}{\frac{\dot{m}_{exit}}{P_0 A^*}\cdot C_e}\right) = \left(\frac{C_e}{\frac{\dot{m}_{exit}}{P_0 A^*}\cdot\frac{Thrust}{\dot{m}_{exit}}}\right) = \frac{C_e}{C_F} = \frac{V_{exit} + \frac{A_{exit}}{\dot{m}_{exit}}(p_{exit} - p_\infty)}{\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\cdot\sqrt{\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \frac{A_{exit}}{A^*}\left(\frac{p_{exit}-p_\infty}{P_0}\right)}$$

- Let nozzle expand infinitely in Vacuum….

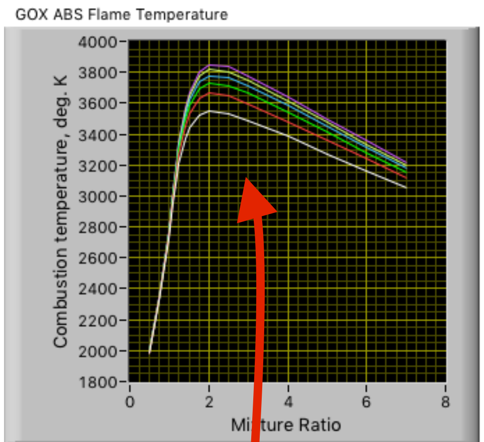
$$\frac{V_{exit}^2}{2\cdot c_p} >> T_{exit} \quad \cdots \quad p_{exit}, T_{exit}, p_\infty \to 0 \to V_{exit} \to \sqrt{2\cdot c_p\cdot T_0}$$

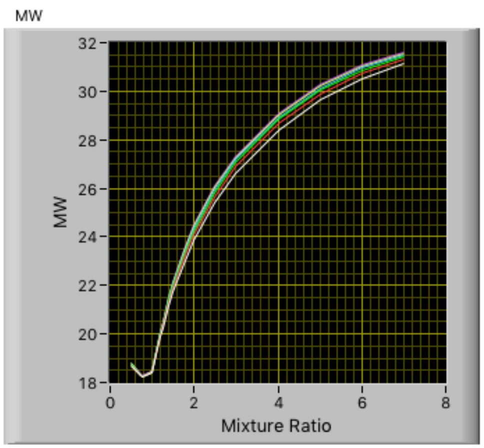
$$\boxed{\to C^* = \frac{\sqrt{\gamma R_g T_0}}{\gamma\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}} = \frac{\sqrt{\gamma R_u}}{\gamma\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}\sqrt{\frac{T_o}{M_W}}}$$

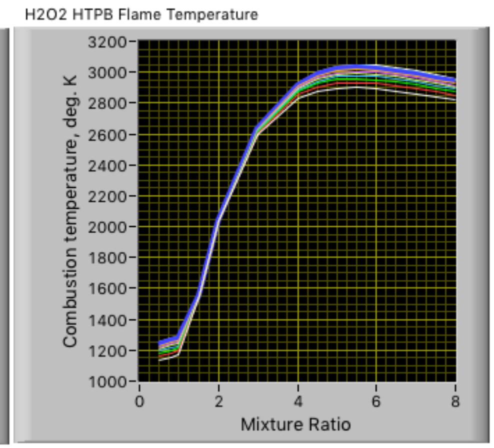
- The *characteristic velocity* is a figure of thermo-chemical merit for a particular propellant and may be considered to be Indicative of the *combustion performance of propellants*.

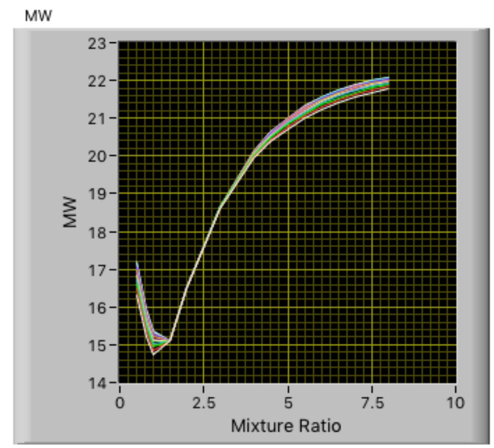
- Propellants that burn Hot and have a low product Molecular weight … best C*

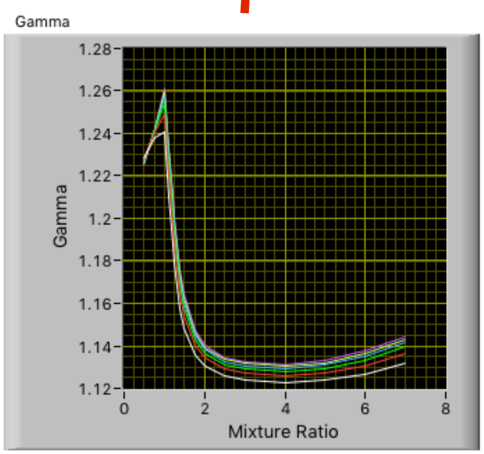
C* of Given Propellants

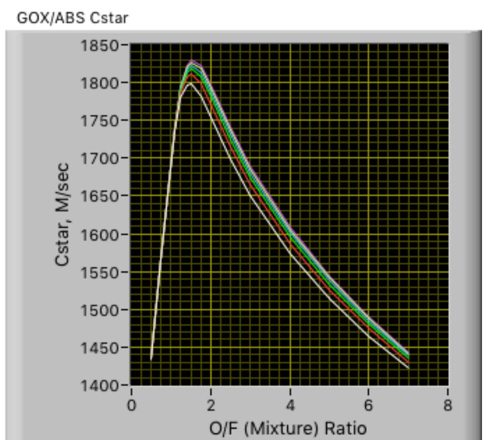


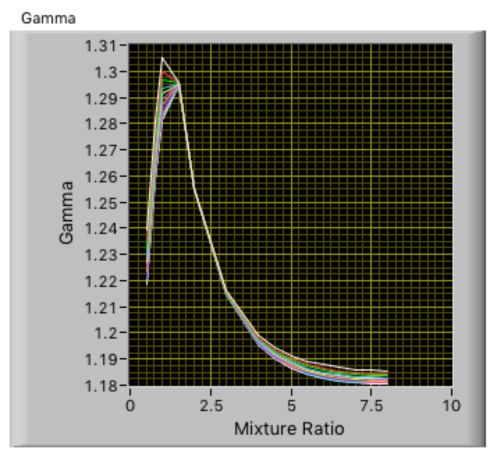


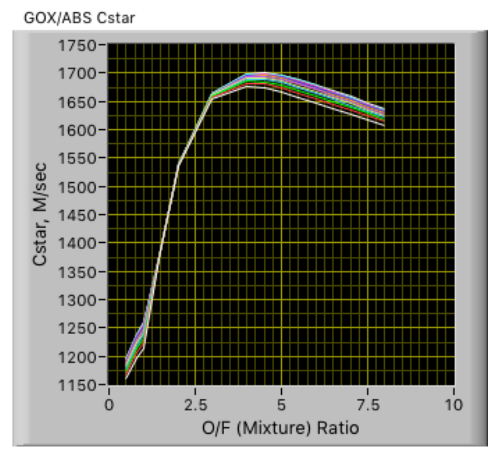












Maximum I_{sp} of a Combustion Process

• from Earlier

$$I_{sp} = \frac{Thrust}{g_o \dot{m}} = \frac{Thrust / P_0 A^*}{g_o \dot{m} / P_0 A^*} = \frac{C_F \cdot C^*}{g_o}$$

• Assuming an infinitely expanded nozzle in a vacuum, Maximum Achievable Specific Impulse for Selected propellants is

$$Isp_{\ Max} = \frac{C_F \cdot C^*}{g_o} = \frac{\gamma}{g_o}\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \cdot \frac{\sqrt{\gamma R_u}}{\gamma\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}\sqrt{\frac{T_o}{M_W}} = \frac{1}{g_o}\frac{\sqrt{2\cdot\gamma\cdot R_u}}{\sqrt{\gamma-1}}\sqrt{\frac{T_o}{M_W}}$$

Performance Parameter Summary

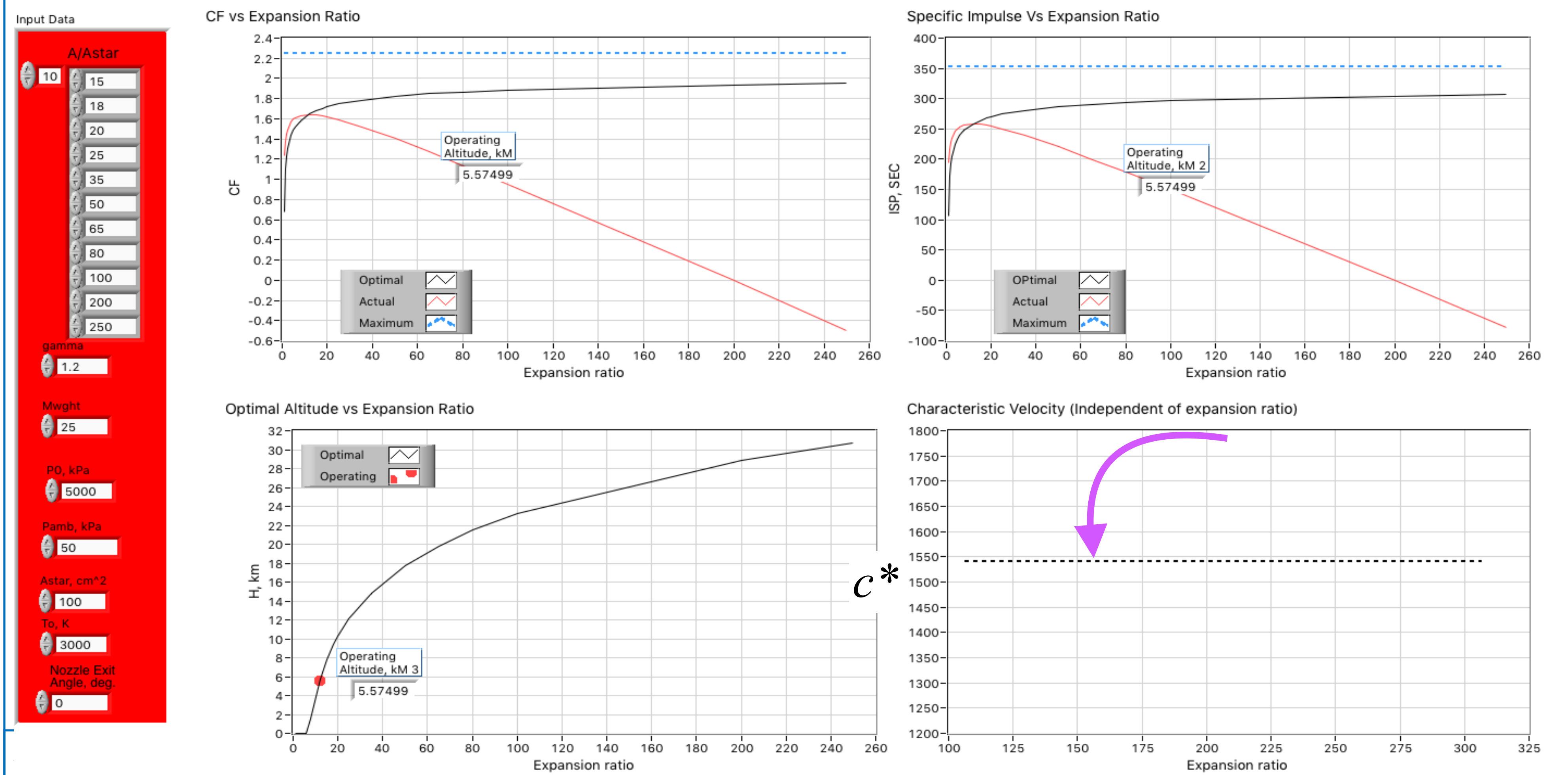
$$\rightarrow C_F \equiv \frac{Thrust}{P_0 A^*} = \gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \cdot \left\{ \left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + \frac{\gamma-1}{2\gamma}\sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}}-1\right]}}\left[\frac{p_{exit}}{P_0}-\frac{p_\infty}{P_0}\right] \right\}$$

$$\rightarrow C^* \equiv \frac{P_0 A^*}{\dot{m}} \rightarrow \left(C^*\right)_{ideal} = \frac{\sqrt{\gamma R_g T_0}}{\gamma\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}} = \frac{\sqrt{\gamma R_u}}{\gamma\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}\sqrt{\frac{T_o}{M_W}}$$

$$\rightarrow I_{sp} = \frac{Thrust}{g_o \dot{m}} = \frac{Thrust/P_0 A^*}{g_o \dot{m}/P_0 A^*} = \frac{C_F \cdot C^*}{g_o}$$

$$\rightarrow Isp_{\ Max} = \frac{C^*}{g_o}\left[\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right] = \frac{1}{g_o}\sqrt{\frac{2\gamma R_u}{\gamma-1}}\sqrt{\frac{T_o}{M_W}}$$

Example Performance Calculations



Real Rocket Loss Coefficients

1. Combustor /Nozzle efficiency correction coefficient → η^*
2. Nozzle divergence correction coefficient → λ
3. Chamber pressure correction coefficient → ξ_p
4. Nozzle discharge correction coefficient → ξ_d

• Manufacturers often use empirically determined "fudge factors" to model engine/rocket motor losses

"adjustments" to de Laval Flow Equations

Combustor Efficiency Correction Coefficient

-- Combustion inefficiency and heat losses through the chamber wall both tend to produce a lower chamber pressure than predicted by theory.

$$\eta^* = \frac{C^*_{actual}}{C^*_{theoretical}} = \frac{\left(P_0 \cdot A^* / \dot{m}\right)}{C^*_{theoretical}} = \frac{\left(\frac{1}{\gamma}\sqrt{\gamma \cdot R_g \cdot T_0 \cdot \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}}\right)_{actual}}{\left(\frac{1}{\gamma}\sqrt{\gamma \cdot R_g \cdot T_0 \cdot \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}}\right)_{theoretical}} \rightarrow \boxed{T_{0_{actual}} \approx \left(\eta^*\right)^2 \cdot T_{0_{theoretical}}}$$

• Squared Efficiency Proportional to ratio of True-to-Actual Flame Temperature

Combustor Efficiency Correction Coefficient (cont'd)

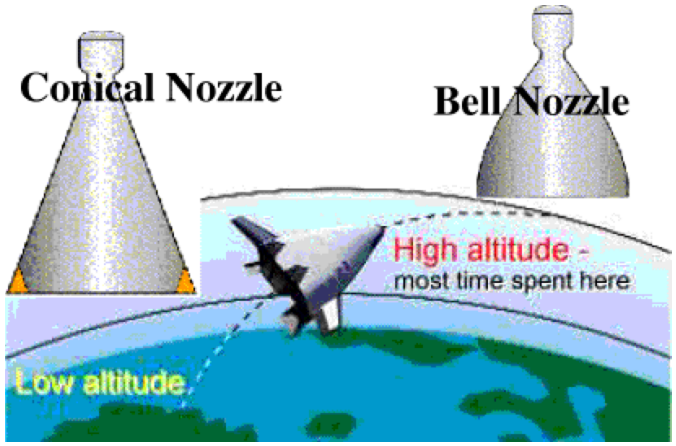
$$\eta^* = \frac{C^*_{actual}}{C^*_{theoretical}} = \frac{\left(P_0 \cdot A^* / \dot{m}\right)}{C^*_{theoretical}} = \frac{\left(\frac{1}{\gamma}\sqrt{\gamma \cdot R_g \cdot T_0 \cdot \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}}\right)_{actual}}{\left(\frac{1}{\gamma}\sqrt{\gamma \cdot R_g \cdot T_0 \cdot \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}}\right)_{theoretical}} \rightarrow \boxed{T_{0_{actual}} \approx \left(\eta^*\right)^2 \cdot T_{0_{theoretical}}}$$

… What factors can cause T_0 to Drop in Combustor

- Combustion
- Transport

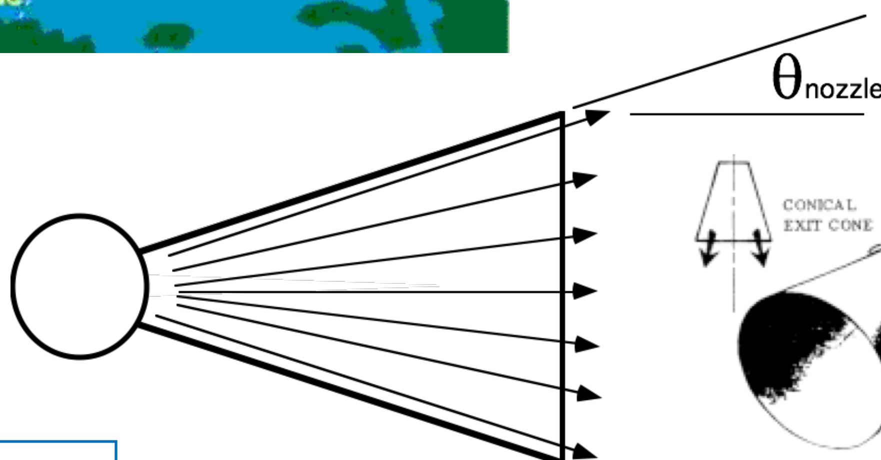
1) Heat Loss thru Combustor walls
2) Friction (Very Small in Combustor)
3) Combustion Process Efficiency / mixture ratio
4) **Typical Values … 85-99%**

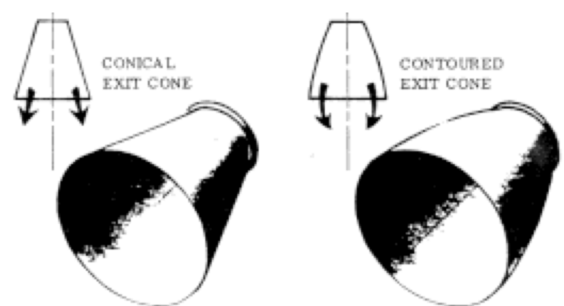
Nozzle Divergence Correction Coefficient



- Quasi-1-D analysis assumes exit flow leaves parallel to longitudinal axis of the nozzle

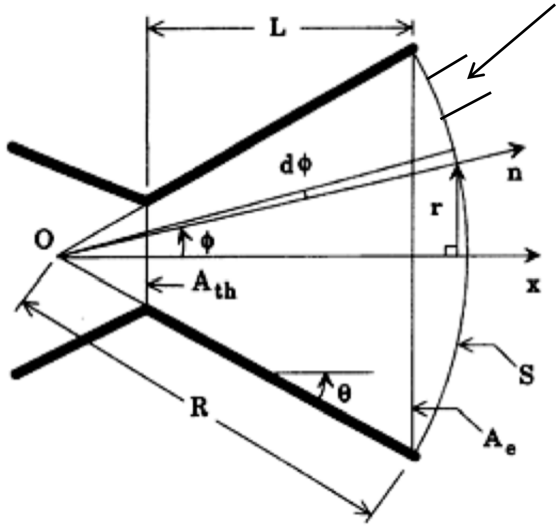
- In reality … this rarely happens

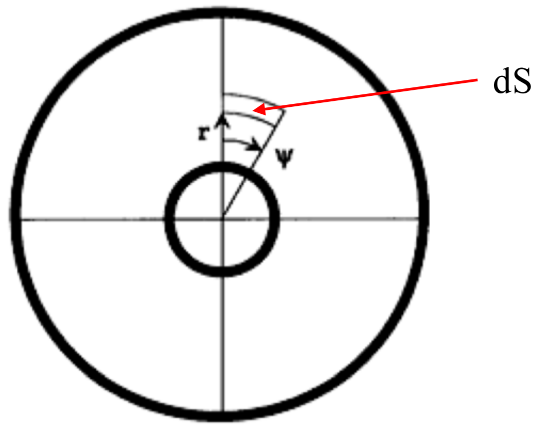




Nozzle Divergence Correction Coefficient (cont'd)

$$dS = r\sin[\phi]d\psi \times Rd\phi = R^2\sin[\phi]d\phi d\psi$$

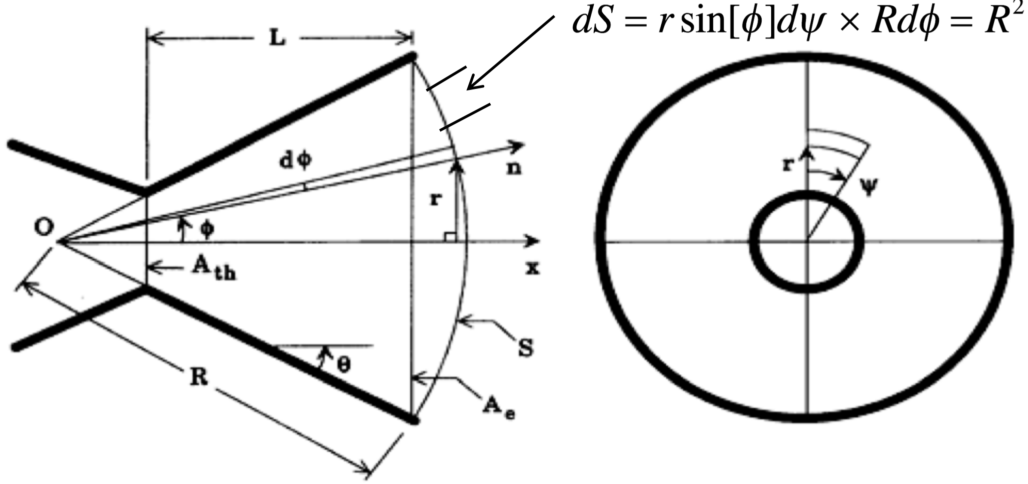




- Look at mass flow across spherical exit surface

$$\dot{m} = \int_0^{2\pi}\int_0^{\theta}\rho_{exit}V_{exit}dS = 2\int_0^{2\pi}\int_0^{\theta}\rho_{exit}V_{exit}R^2\sin[\phi]d\phi d\psi = 2\pi\rho_{exit}V_{exit}R^2\left\{1-\cos[\theta]\right\}$$

Nozzle Divergence Correction Coefficient (cont'd)



$$dS = r\sin[\phi]d\psi \times R d\phi = R^2 \sin[\phi]d\phi d\psi$$

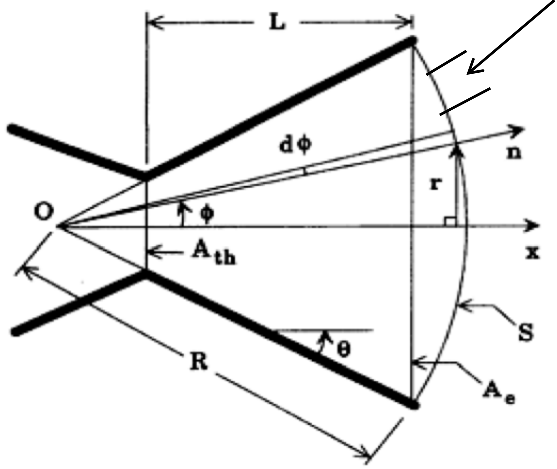
- Look at axial momentum flow across spherical exit surface

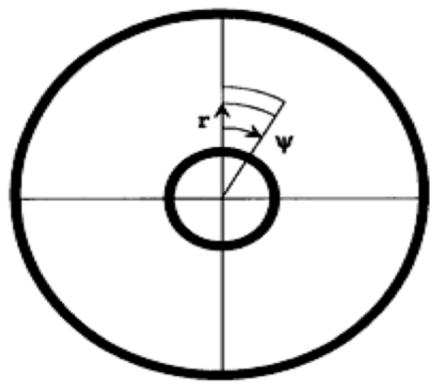
$$F_{axial} = \dot{M}_{axial} = \int_0^{2\pi}\int_0^{\theta} \rho_{exit} V_{exit}{}^2 \cos[\phi] dS = \int_0^{2\pi}\int_0^{\theta} \rho_{exit} V_{exit}{}^2 R^2 \sin[\phi]\cos[\phi] d\phi d\psi =$$

$$2\pi\rho_{exit} V_{exit}{}^2 R^2 \frac{\{1-\cos^2[\theta]\}}{2} = \pi\rho_{exit} V_{exit}{}^2 R^2 \sin^2[\theta]$$

Nozzle Divergence Correction Coefficient (cont'd)

$$dS = r\sin[\phi]d\psi \times Rd\phi = R^2\sin[\phi]d\phi d\psi$$

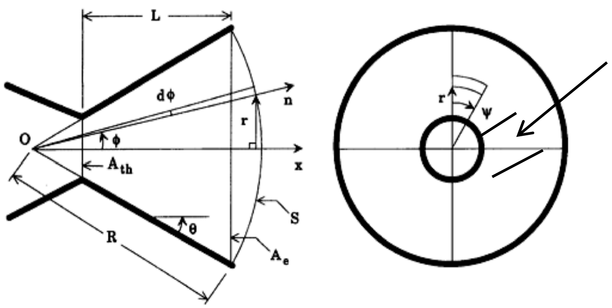


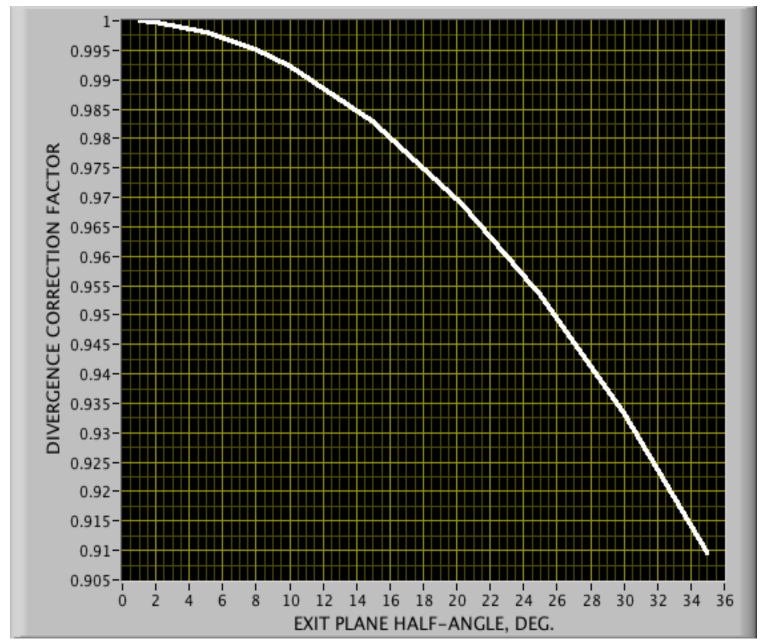


$$\frac{F_{axial}}{\dot{m}V_{exit}} = \frac{\pi\rho_{exit}{V_{exit}}^2R^2\sin^2[\theta]}{\left[2\pi\rho_{exit}V_{exit}R^2\left\{1-\cos[\theta]\right\}\right]V_{exit}} = \frac{\pi\rho_{exit}{V_{exit}}^2R^2\sin^2[\theta]}{2\pi\rho_{exit}{V_{exit}}^2R^2\left\{1-\cos[\theta]\right\}} = \frac{\sin^2[\theta]}{2\left\{1-\cos[\theta]\right\}} =$$

$$\frac{\sin^2[\theta]\left\{1+\cos[\theta]\right\}}{2\left\{1-\cos[\theta]\right\}\left\{1+\cos[\theta]\right\}} = \frac{\sin^2[\theta]\left\{1+\cos[\theta]\right\}}{2\left\{1-\cos^2[\theta]\right\}} = \frac{\sin^2[\theta]\left\{1+\cos[\theta]\right\}}{2\left\{\sin^2[\theta]\right\}} = \frac{1}{2}\left\{1+\cos[\theta]\right\}$$

Nozzle Divergence Correction Coefficient (cont'd)





Actual Momentum Thrust

Momentum Thrust of idealized Nozzle

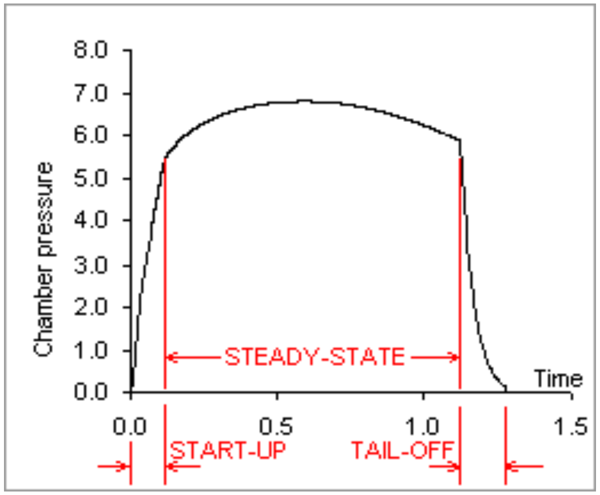
$$\frac{F_{axial}}{\dot{m} V_{exit}} = \frac{1}{2}\left\{1 + \cos[\theta]\right\} \equiv \lambda$$

Application of Correction Factor

$$T_{hrust} = \lambda \dot{m} V_{exit} + A_{exit}\left[P_{exit} - P_{\infty}\right]$$

Chamber Pressure Correction Coefficient

--Models effects of transient
startup, stagnation pressure loss
due to non-zero Chamber Mach Number



- Rocket Engines with short burn times typically have a significant portion of the total impulse resulting from the pressure *start-up* or *tail-off* phases of the burn, when the chamber pressure is well below the steady-state operating pressure level.

- Total *delivered* impulse is less than impulse based on steady-state calculations.

$$\overline{P}_0 = \frac{1}{T_{burn}} \int_0^{T_{burn}} P_0(t)dt$$

- Use mean Stagnation pressure through Burn as correction factor

Chamber Pressure Correction Coefficient(cont'd)

$$\bar{P}_0 = \frac{1}{T_{burn}} \int_0^{T_{burn}} P_0(t)dt$$

- Define

$$\xi_p = \frac{\bar{C}_F}{C_{F_{ideal}}} = \frac{\left[\dot{m} C_e\right]_{actual}}{\bar{P}_0 A^*} \times \frac{P_0 A^*}{\left[\dot{m} C_e\right]_{ideal}} = \frac{\dot{m}_{actual}}{\bar{P}_0} \times \frac{P_0}{\dot{m}_{ideal}} \times \frac{\left[C_e\right]_{actual}}{\left[C_e\right]_{ideal}}$$

$$C_F = \frac{Thrust}{P_0 A^*}$$

- Look At

$$\frac{\dot{m}}{P_0} = \frac{A^*}{\sqrt{T_{0_{actial}}}} \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \rightarrow \frac{\dot{m}_{actual}}{\bar{P}_0} \times \frac{P_0}{\dot{m}_{ideal}} \approx 1$$

$$\boxed{\rightarrow \xi_p = \frac{\bar{C}_F}{C_{F_{ideal}}} = \frac{\left[C_e\right]_{actual}}{\left[C_e\right]_{ideal}} = \frac{V_e + \frac{A_e}{A^*}\frac{(p_{exit} - p_\infty)}{\dot{m}_{actual}}}{V_e + \frac{A_e}{A^*}\frac{(p_{exit} - p_\infty)}{\dot{m}_{ideal}}}}$$

Chamber Pressure Correction Coefficient(cont'd)

- Alternate Formulation

$$\frac{\dot{m}}{A^* P_0} = \frac{1}{\sqrt{T_{0_{actial}}}} \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \rightarrow$$

$$C_F = \frac{Thrust}{P_0 A^*} = \frac{V_{exit}\,\dot{m}}{P_0 A^*} + \frac{A_e}{A^*}\left[\frac{p_{exit}}{P_0} - \frac{p_\infty}{P_0}\right] = \frac{V_{exit}}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} + \frac{A_e}{A^*}\left[\frac{p_{exit}}{P_0} - \frac{p_\infty}{P_0}\right]$$

Chamber Pressure Correction Coefficient (cont'd)

- But V_e, $\frac{p_{exit}}{P_0}$ are independent of P_0 (for isentropic Nozzle)

$$\overline{C}_F - C_{F_{ideal}} \approx \frac{A_{exit}}{A^*}\left[\frac{p_\infty}{P_0} - \frac{p_\infty}{\overline{P}_0}\right] = \frac{A_{exit}}{A^*} p_\infty \left[\frac{1}{P_0} - \frac{1}{\overline{P}_0}\right] \rightarrow$$

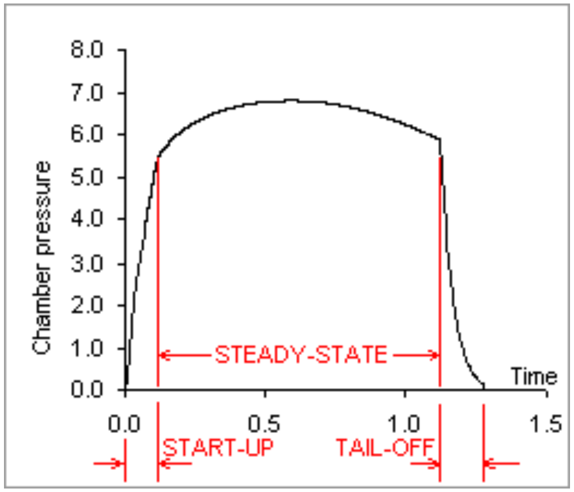
$$\boxed{\xi_p = \frac{\overline{C}_F}{C_{F_{ideal}}} = 1 - \frac{\frac{A_{exit}}{A^*} p_\infty \left[\frac{P_0 - \overline{P}_0}{\overline{P}_0 P_0}\right]}{C_{F_{ideal}}}}$$

- Mathematically identically to previous Formula … but easier to use

$$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}} \sqrt{\frac{\left(\frac{P_0}{p_{exit}}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]}}$$

- **Typical Values … 95-99.5%**

$P_{chamber}$ Correction, SSME Example, revisited



- Assume

$$\overline{P}_0 = \frac{1}{T_{burn}} \int_0^{T_{burn}} P_0(t)dt \approx 18.6mPa$$

$$C_{F_{ideal}} = \frac{Thrust}{P_0 A^*} = \frac{1.671 \cdot 10^6}{18.94 \cdot 10^6 0.05785} = 1.525$$

$$\xi_p = \frac{\overline{C}_F}{C_{F_{ideal}}} = 1 - \frac{\dfrac{77.5 \cdot 101325\,(18.94 - 18.6)\,10^6}{18.94\,(18.6)\,10^6 10^6}}{1.525}$$

Shuttle has A very long Burn time

= 0.995

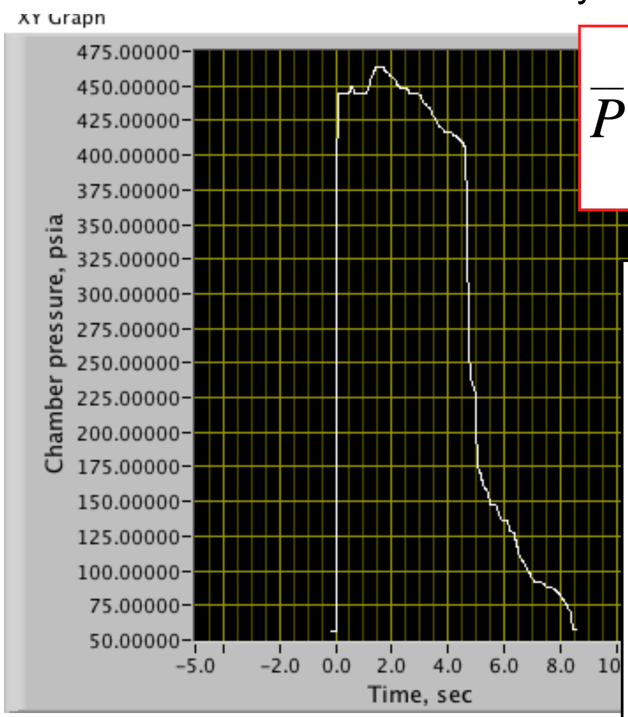
$P_{chamber}$ Correction, Chimaera Rocket Example

- Expected Thrust ~ 2300 lbf
- Expected $P_{chamber}$ ~ 450 psia
- A^* ~ 1.25 in

$$C_{F_{ideal}} = \frac{2200_{lbf}}{450_{\frac{lbf}{in^2}} \times \frac{\pi\left(1.25^2{}_{in}{}^2\right)}{4}} = 3.98$$

Chimaera Rocket Example (cont'd)

- Chamber Pressure Time History



$$\overline{P}_0 = \frac{1}{T_{burn}} \int_0^{T_{burn}} P_0(t)dt \approx 289.63_{psia}$$

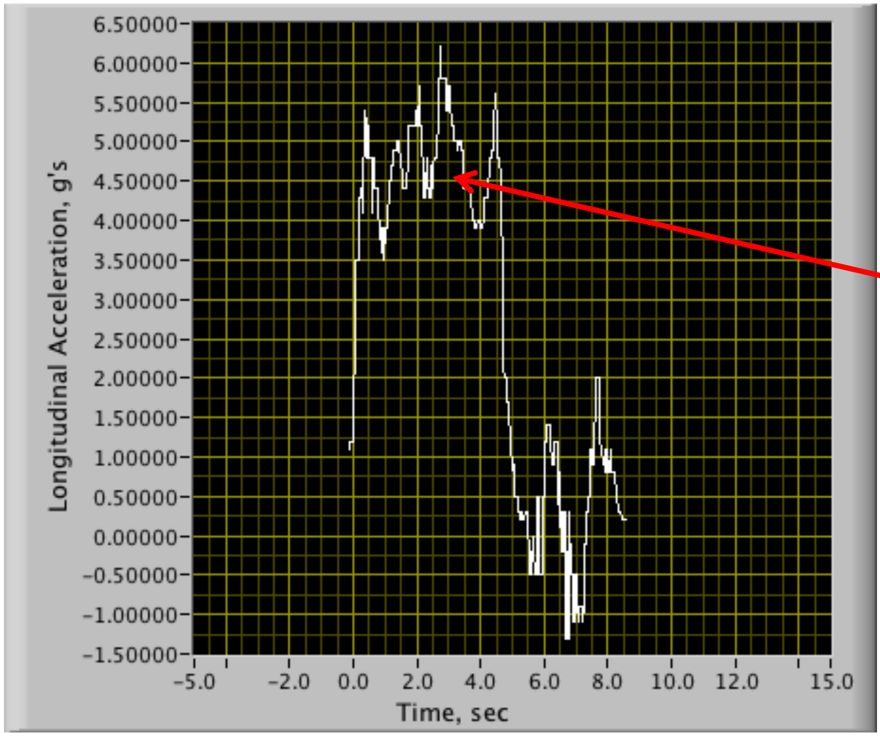
$$\xi_p = \frac{\overline{C}_F}{C_{F_{ideal}}} = 1 - \frac{\frac{A_{exit}}{A^*} p_\infty \left[\frac{P_0 - \overline{P}_0}{\overline{P}_0 P_0} \right]}{C_{F_{ideal}}} =$$

$$1 - \frac{\frac{8\left(\frac{1800}{144}\right)(450 - 289.63)}{450 \cdot 289.63}}{3.98}$$

=0.969

Chimaera Rocket Example (cont'd)

- Thrust Time History (Ax) curve



Total Impulse:
23.26 g-seconds

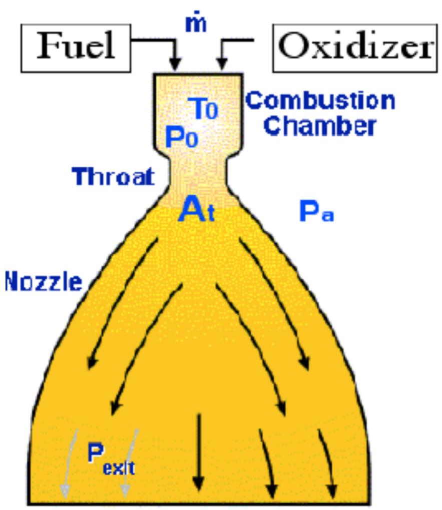
"Steady State"
Impulse:
22.10 g-seconds

~95%

… so we are
Pretty consistent

Nozzle Discharge Correction Coefficient

• Once the flow clears the throat and enters the Nozzle a variety of losses can occur



• The *discharge correction factor* is used to express how well the nozzle design permits the mass flow rate through the throat to approach the theoretical rate, and is given by the ratio of delivered mass flow rate to ideal mass flow rate:

Nozzle Discharge Correction Coefficient (cont'd)

- $$\xi_d = \frac{\overline{\dot{m}}}{\dot{m}} \approx \frac{\overline{\dot{m}}}{A^* \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \frac{p_0}{\sqrt{T_0}}}$$

- Value of the Discharge Correction Coefficient is typically > 1
  1) MW increases due to reactions within nozzle
  2) Heat transfer to Nozzle wall Lowers Gas density

- ***Typical values 1.0 to 1.15***

Correction Coefficient Summary

- Basic Definitions

$$\overline{C^*} = \eta^* C^*_{ideal}$$

$$\overline{C_F} = \xi_p C_{F_{ideal}}$$

$$\overline{\dot{m}} = \xi_d \dot{m}_{ideal}$$

$$T_{hrust} = \lambda \dot{m} V_{exit} + A_{exit}\left[P_{exit} - P_\infty\right]$$

Correction Coefficient Summary
(cont'd)

• Thrust Coefficient

$$\overline{C}_F = \xi_p \left[\frac{\lambda \times \xi_d \left[\dot{m} V_{exit} \right]_{ideal} + A_{exit} \left[P_{exit} - P_\infty \right]}{P_0 A^*} \right] \rightarrow \left[\xi_p \lambda \xi_d \right] C_{F_{ideal}}$$

• Specific Impulse

$$I_{sp} = \frac{C_F \times C^*}{g_0} = \eta^* \xi_p \lambda \xi_d I_{sp\ ideal} \text{ (@ design point)}$$

Appendix 5.2 SSME Computational Example

SSME Computational Example

• Space Shuttle Main Engine …

• **Unlike other propellants, the optimum mixture ratio for liquid oxygen and liquid hydrogen is not necessarily that which will produce the maximum specific impulse. Because of the extremely low density of liquid hydrogen, the propellant volume decreases significantly at higher mixture ratios.**

• **Maximum specific impulse typically occurs at a mixture ratio of around 3.5, however by increasing the mixture ratio to, say, 5.5 the storage volume is reduced by one-fourth. This results in smaller propellant tanks, lower vehicle mass, and less drag, which generally offsets the loss in performance that comes with using the higher mixture ratio. In practice, most liquid oxygen/liquid hydrogen engines typically operate at mixture ratios from about 5 to 6.**

What is the Stoichiometric Mixture Ratio of LOX/LH_2?

$$2H_2 + O_2 \rightleftharpoons 2H_2O$$

$$M_w LH_2 \rightarrow 2.016_{kg/kg-mol}$$

$$M_w LO_2 \rightarrow 31.999_{kg/kg-mol}$$

$$MR = \frac{1_{mol} LO_2 \times M_w LO_2}{2_{mol} LH_2 \times M_w LH_2} = \frac{31.999}{2 \times 2.016} = 7.936$$

MR=6.0 (What the shuttle operates at) --> “Rich Mixture”

Compare Tank Volumes

- Space Shuttle has the following mass fraction characteristics

Weight (lb)	
Gross lift-off	4,500,000
External Tank (full)	1,655,600
External Tank (Inert)	66,000
SRBs (2) each at launch . . .	1,292,000
SRB inert weight, each	192,000

- Shuttle has 721,000 kg of propellant in main tank on pad

Compare Tank Volumes (cont'd)

$$O/F = 7.936 \rightarrow 721{,}000_{kg}\left[\frac{7.396}{1140_{kg/m^3}} + \frac{1}{67.78_{kg/m^3}}\right] = 15315 m^3$$

"best compromise"

$$\boxed{O/F = 6.000 \rightarrow 721{,}000_{kg}\left[\frac{6.000}{1140_{kg/m^3}} + \frac{1}{67.78_{kg/m^3}}\right] = 14432 m^3}$$

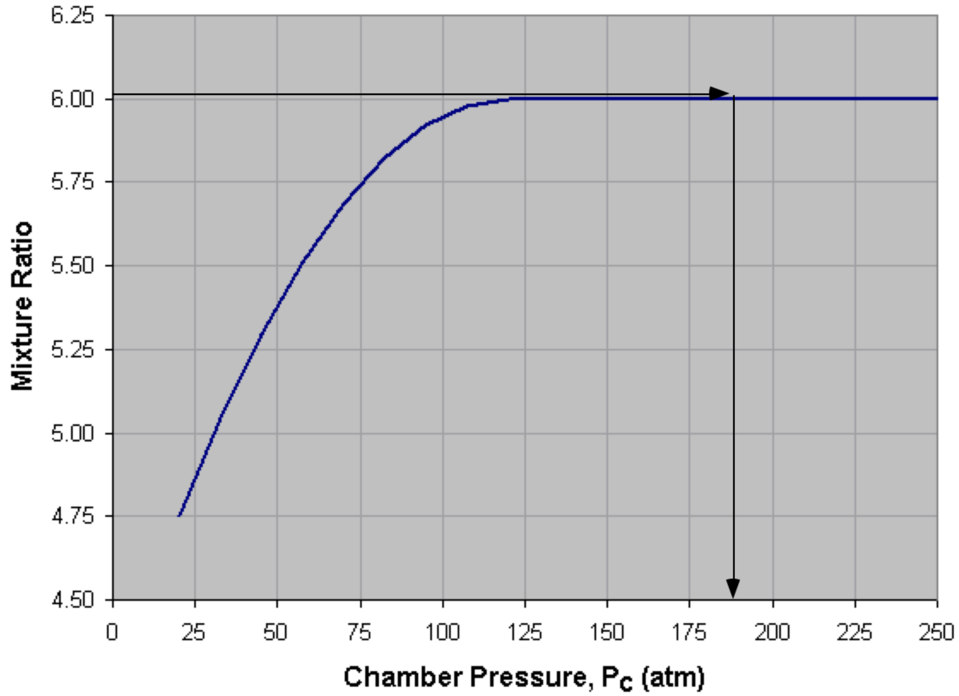
$$O/F = 3.5 \rightarrow 721{,}000_{kg}\left[\frac{6.000}{1140_{kg/m^3}} + \frac{1}{67.78_{kg/m^3}}\right] = 12850 m^3$$

SSME Computational Example

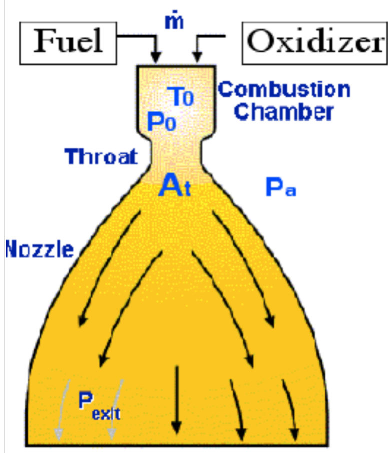
(cont'd)

- Space Shuttle Main Engine …

- LOX/LH2 Propellants, 6.03: 1 Mixture ratio



P_0=186.92 atm
=18940 Kpa

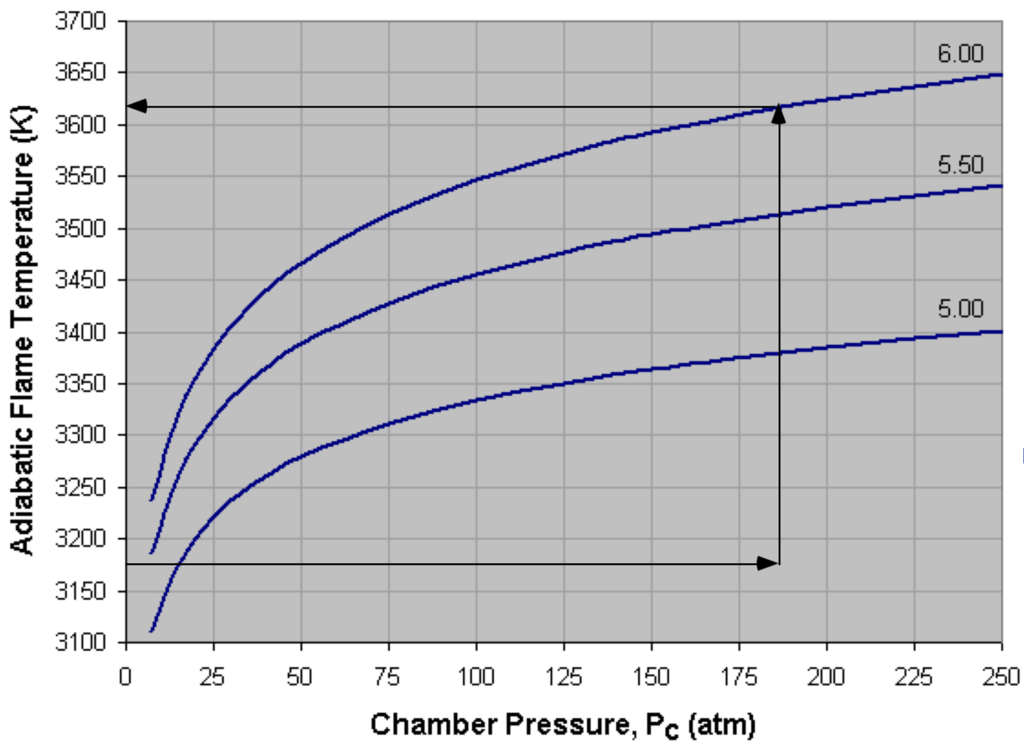


Ref: http://www.braeunig.us/space/

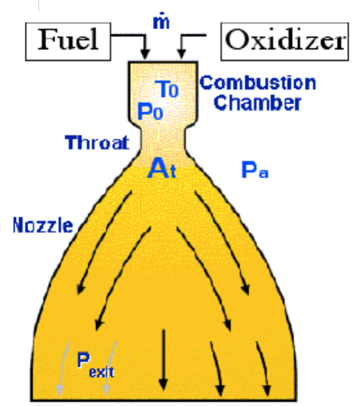
SSME Computational Example

(cont'd)

- Space Shuttle Main Engine …

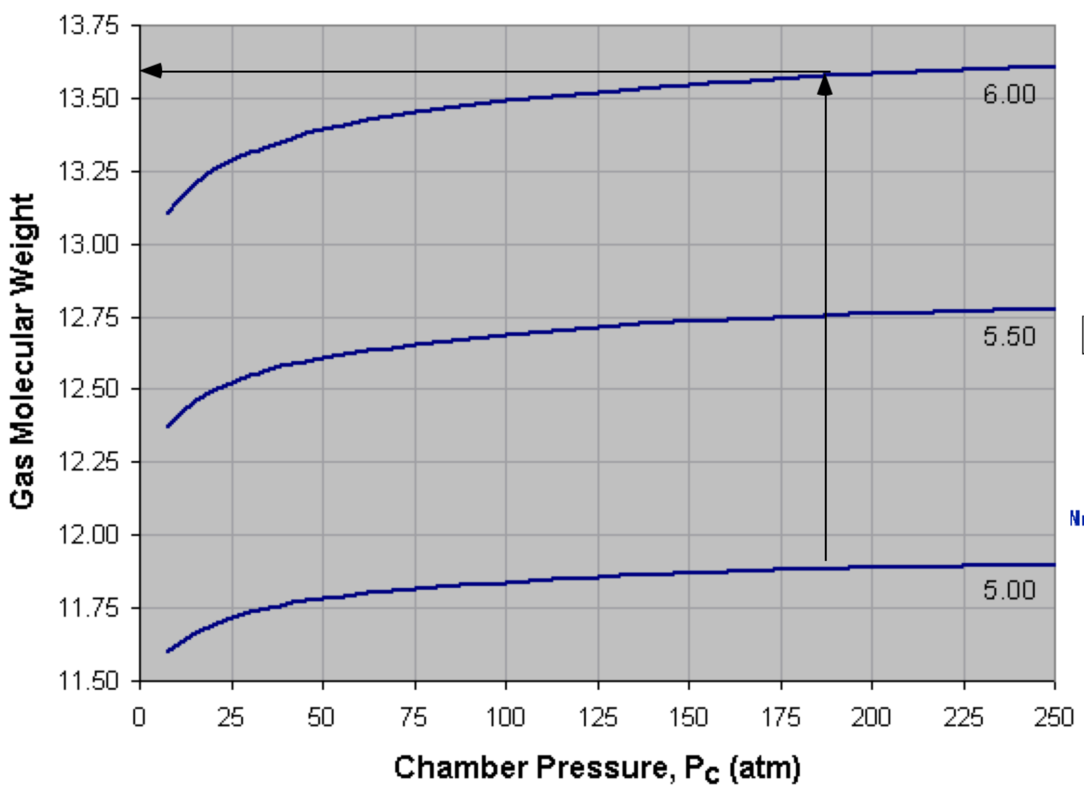


T_0 ~ 3615°K

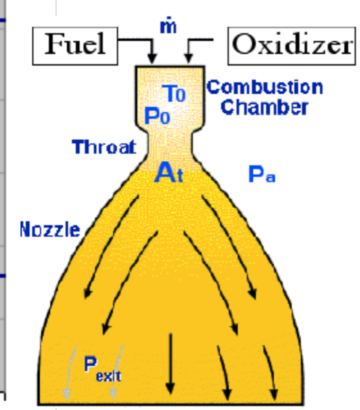


SSME Computational Example

(cont'd)



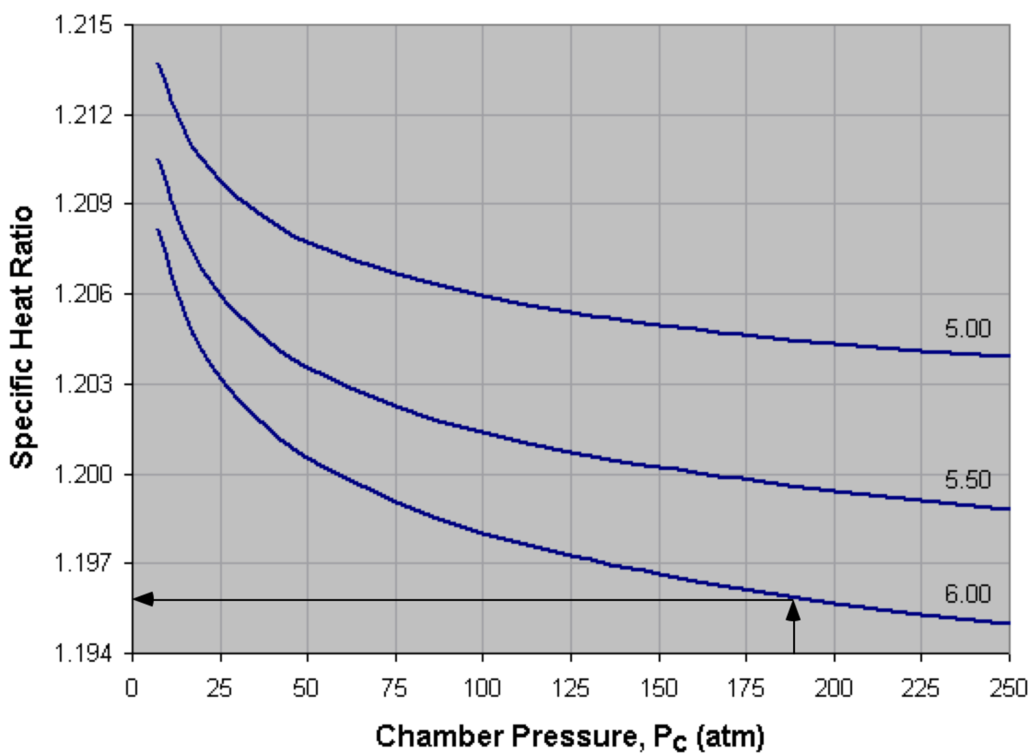
M_w ~ 13.6 kg/kg-mol



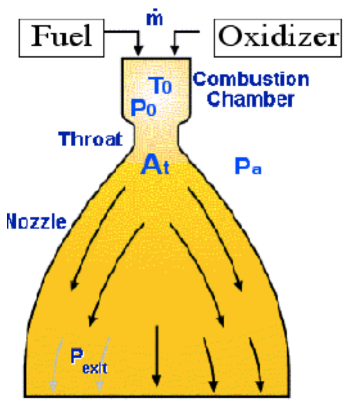
SSME Computational Example

(cont'd)

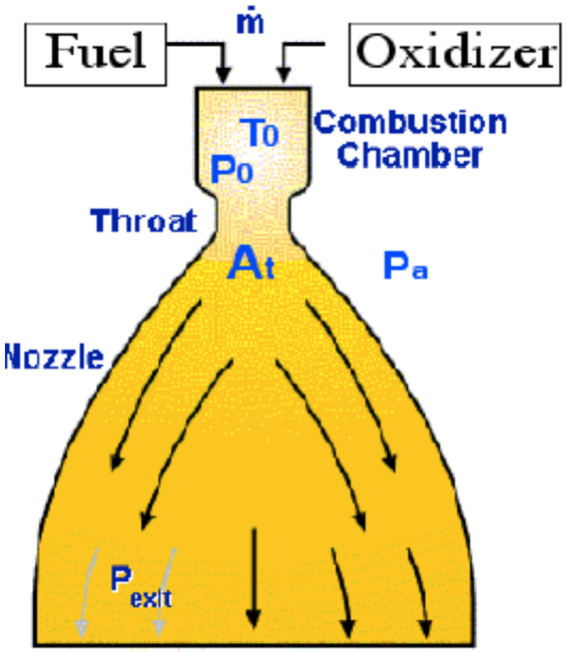
- Space Shuttle Main Engine …



$\gamma \sim$
1.196



Example: SSME Rocket Engine



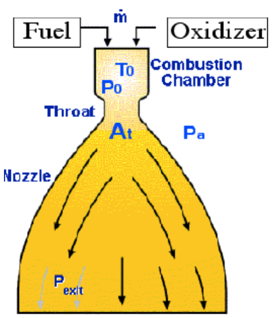
- The Space Shuttle Main Engines Burn LOX/LH2 for Propellants with A ratio of LOX:LH2 =6:1

- The Combustor Pressure, p_0 Is 18.94 Mpa, combustor temperature, T_0 is 3615°K, throat diameter is 26.0 cm

- What propellant mass flow rate is required for choked flow in the Nozzle?

- Assume no heat transfer Thru Nozzle no frictional losses, γ=1.196

Example: SSME Rocket Engine (cont'd)



-- By product ~ Burns rich, byproduct is water vapor + GH_2

$M_W \sim 13.6$ kg/kg-mole

-- $R_g = 8314.4612 / 13.6 = 611.35$ J/°K-kg

$$\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\,\frac{p_0}{\sqrt{T_0}} =$$

$$\left(\frac{1.196}{611.35}\left(\frac{2}{1.196+1}\right)^{\frac{(1.196+1)}{1.196-1}}\right)^{0.5}\frac{18.94\cdot 10^6}{(3615)^{0.5}}$$

= 8252.59 kg/sec-m^2

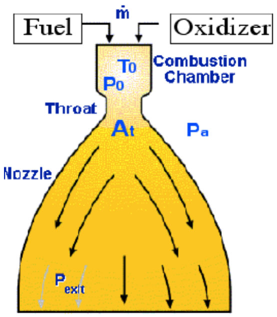
- Compute Throat Area

$$\left(\frac{26}{100}\right)^2\frac{\pi}{4} = 0.05309 \quad \text{m}^2$$

- Mass flow =

$$\left(\frac{\dot{m}}{A^*}\right)\times A^* = 8252.59\cdot 0.05297 = 438.15 \text{ kg/sec}$$

Example: SSME Rocket Engine (continued)



- The nozzle expansion ratio is 77.5 -- what is the exit Mach number

$$\frac{A}{A^*} = 77.5 = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} =$$

$$\frac{\left(\left(\frac{2}{1.196+1}\right)\left(1+\frac{1.196-1}{2}\,(4.677084^2)\right)\right)^{\frac{1.196+1}{2\,(1.196-1)}}}{4.677084}$$

$= 77.49998$ ----> $M_{exit} = 4.677084$

Newton Solver comes in handy here!

Example: SSME Rocket Engine (cont'd)

M_{exit} = 4.677084

Compute Exit Temperature

$$T_{exit} = \frac{T_0}{1 + \frac{(\gamma - 1)}{2} {M_{exit}}^2} =$$

$$3615 \left(1 + \frac{1.196 - 1}{2} (4.677084^2)\right)^{-1}$$

=1149.90 °K

Compute Exit Velocity

$$V_{exit} = M_{exit} \sqrt{\gamma R_g T_{exit}} =$$

$$4.677084\,(1.196 \cdot 611.35 \cdot 1149.9)^{0.5}$$

= 4288.61 m/sec

Compute Exit Pressure

$$P_{exit} = \frac{P_0}{\left(1 + \frac{(\gamma - 1)}{2} {M_{exit}}^2\right)^{\frac{\gamma}{\gamma - 1}}} =$$

$$\frac{18.94 \cdot 10^6}{\left(1 + \frac{1.196 - 1}{2} (4.677084^2)\right)^{\left(\frac{1.196}{1.196 - 1}\right)}}$$

=17.45511 kPa

Compute Thrust Coefficients at Sea Level and In Vacuum

- Sea Level

$$C_F = \frac{Thrust}{P_0 A^*} = \gamma \sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0} =$$

$$1.196\left(\left(\frac{2}{1.196-1}\left(\frac{2}{1.196+1}\right)^{\frac{1.196+1}{1.196-1}}\right)\left(1-\left(\frac{17.4551}{18.94\cdot 10^3}\right)^{\frac{1.196-1}{1.196}}\right)\right)^{0.5} + 77.5\left(\frac{17.4551-101.325}{18.94\cdot 10^3}\right) = 1.52546$$

- Vacuum

$$C_F = \frac{Thrust}{P_0 A^*} = \gamma \sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-\cancel{p_\infty})}{P_0} =$$

$$1.196\left(\left(\frac{2}{1.196-1}\left(\frac{2}{1.196+1}\right)^{\frac{1.196+1}{1.196-1}}\right)\left(1-\left(\frac{17.4551}{18.94\cdot 10^3}\right)^{\frac{1.196-1}{1.196}}\right)\right)^{0.5} + 77.5\left(\frac{17.4551}{18.94\cdot 10^3}\right) = 1.94006$$

Compute Thrust Sea Level, Vacuum, and Optimal Altitude?

- Sea Level C_F =1.341

$$Thrust = C_F \cdot \frac{Thrust}{P_0 A^*} = 1.52546 \cdot 18.94 \cdot 10^6 \left(\frac{26}{100}\right)^2 \frac{\pi}{4}$$ = 1.53397 x 10^6 N

- Vacuum

$$Thrust = C_F \cdot \frac{Thrust}{P_0 A^*} = 1.94006 \cdot 18.94 \cdot 10^6 \left(\frac{26}{100}\right)^2 \frac{\pi}{4}$$ = 1.95089 x 10^6 N

- @ Optimal Altitude?

$$Thrust = C_F \cdot \frac{Thrust}{P_0 A^*} = 1.196 \left(\left(\frac{2}{1.196 - 1} \left(\frac{2}{1.196 + 1} \right)^{\frac{1.196 + 1}{1.196 - 1}} \right) \left(1 - \left(\frac{17.4551}{18.94 \cdot 10^3} \right)^{\frac{1.196 - 1}{1.196}} \right) \right)^{0.5} 18.94 \cdot 10^6 \left(\frac{26}{100}\right)^2 \frac{\pi}{4}$$

= 1.87907 x 10^6 N

Example: SSME Rocket Engine (cont'd)

Compute Characteristic Velocity C^* in two ways

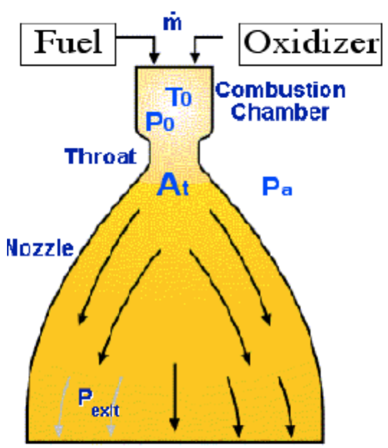
$$C^* = \frac{P_0 A^*}{\dot{m}} = \frac{18.95 \cdot 10^6 \left(\frac{26}{100}\right)^2 \frac{\pi}{4}}{438.15}$$ =2296.27 m/sec

Close enough!

$$C^* = \frac{\sqrt{\gamma R_u}}{\gamma \sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}} \sqrt{\frac{T_o}{M_W}} =$$ =2296.25 m/sec

Example: SSME Rocket Engine
(cont'd)

Compute Max I_{sp}



$$I_{sp_{max}} = \frac{C^*}{g_o}\left[\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right] =$$

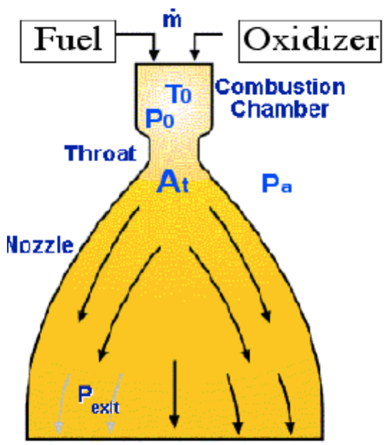
$$\frac{2296\cdot 1.196\left(\frac{2}{1.196-1}\left(\frac{2}{1.196+1}\right)^{\frac{1.196+1}{1.196-1}}\right)^{0.5}}{9.8067}$$

= 529.80 sec

Example: SSME Rocket Engine
(cont'd)

Compute Maximum Thrust.. Best we could ever get under perfect conditions

$$Thrust_{ideal} = \dot{m} I_{sp_{max}} g_0 =$$



$$\frac{2296 \cdot 1.196 \left(\frac{2}{1.196-1} \left(\frac{2}{1.196+1} \right)^{\frac{1.196+1}{1.196-1}} \right)^{0.5}}{9.8067} 438.15 \cdot 9.8067$$

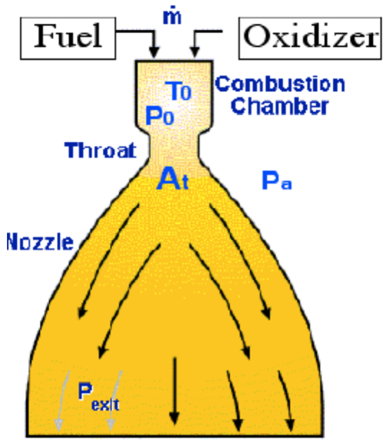
= 2.276 mNt

Example: SSME Rocket Engine

(cont'd)

Compute Effective Exhaust Velocity (Vacuum)

$$\mathrm{C_e} = \frac{Thrust}{\dot{m}} = V_{exit} + \frac{A_{exit}}{A^*} A^* \frac{(p_{exit} - p_\infty)}{\dot{m}} =$$



$$4288.61 + \frac{77.5 \cdot 0.0529708\,(17.455 \cdot 1000)}{437.14}$$

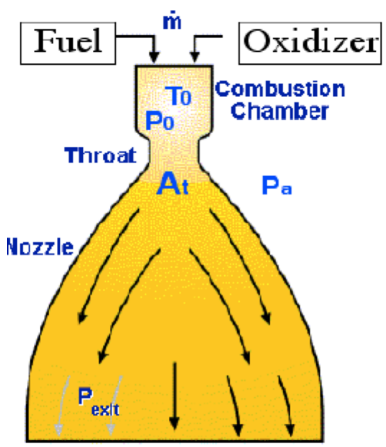
= 4452.53 m/sec

Example: SSME Rocket Engine

(cont'd)

Compute Effective Exhaust Velocity (Sea level)

$$C_e = \frac{Thrust}{\dot{m}} = V_{exit} + \frac{A_{exit}}{A^*} A^* \frac{(p_{exit} - p_\infty)}{\dot{m}} =$$



$$4288.61 + \frac{77.5 \cdot 0.0529708\,(17.455 \cdot 1000 - 101325)}{437.14}$$

$= 3500.98$ m/sec

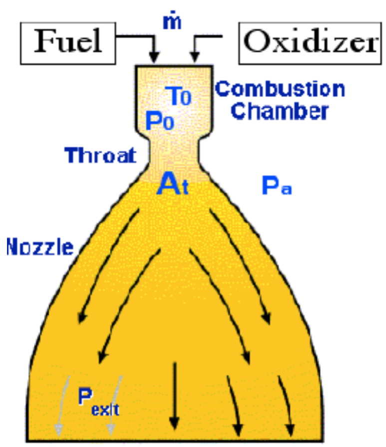
P sea Level =101.325 kPa

Example: SSME Rocket Engine

(cont'd)

Compute True I_{sp} (Seal level) (ignore nozzle Losses)

$$I_{sp} = \frac{C_e}{g_0} =$$



$$\frac{3500.976}{9.806}$$ = 357.024 sec

Example: SSME Rocket Engine

(cont'd)

Summary:

	Ideal	Calc. Vac.	Calc. S.L.	Actual Vac.	Actual S.L
I_{sp} (sec):	529.69	454.06	357.03	452.5	363
Thrust: (mNt)	2.271	1946.37	1530.42	2.10	1.67

• Obviously Our estimate of throat area is a bit small ….
… but you get the point …