Numerical Solution for Mach Number in Isentropic Nozzle

- Graphical Solutions are good for "sanity check" but really Need automated solver to allow for iterative design, trade studies, sensitivity analyses, etc.

- Use "Newton's Method" to extract numerical solution

- Define: $$F(M) \equiv \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} - \frac{A}{A^*}$$

- At correct Mach number (for given A/A*) … $F(M) = 0$

Numerical Solution for Mach (cont'd)

- Expand F(M) is Taylor's series about some arbitrary Mach number $M_{(j)}$

$$F(M) = F(M_{(j)}) + \left(\frac{\partial F}{\partial M}\right)_{(j)} \left(M - M_{(j)}\right) + \frac{\left(\frac{\partial^2 F}{\partial M^2}\right)_{(j)} \left(M - M_{(j)}\right)^2}{2} + \ldots O\left(M - M_{(j)}\right)^3$$

- Solve for M

$$M = M_{(j)} + \frac{F(M) - F(M_{(j)}) - \left[\frac{\left(\frac{\partial^2 F}{\partial M^2}\right)_{(j)} \left(M - M_{(j)}\right)^2}{2} + \ldots O\left(M - M_{(j)}\right)^3\right]}{\left(\frac{\partial F}{\partial M}\right)_{(j)}}$$

Numerical Solution for Mach (cont'd)

- From Earlier Definition $F(M) = 0$, thus

$$M = M_{(j)} - \frac{F(M_{(j)}) + \left[\frac{\left(\frac{\partial^2 F}{\partial M^2}\right)_{(j)} \left(M - M_{(j)}\right)^2}{2} + \ldots O\left(M - M_{(j)}\right)^3 \right]}{\left(\frac{\partial F}{\partial M}\right)_{(j)}}$$

Still exact expression

- if $M_{(j)}$ is chosen to be "close" to M $\left(M - M_{(j)}\right)^2 \ll \left(M - M_{(j)}\right)$

And we can truncate after the first order terms with "little" Loss of accuracy

Numerical Solution for Mach (cont'd)

- First Order approximation of solution for M

$$\hat{M} = M_{(j)} - \frac{F(M_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{(j)}}$$

"Hat" indicates that solution is no longer exact

- However; one would anticipate that $\left\| M - \hat{M} \right\| < \left\| M - M_{(j)} \right\|$

"estimate is closer than original guess"

Numerical Solution for Mach (cont'd)

- If we substitute $\hat{M}$ back into the approximate expression

$$\hat{\hat{M}} = \hat{M} - \frac{F(\hat{M})}{\left(\frac{\partial F}{\partial M}\right)_{|\hat{M}}}$$

- And we would anticipate that $\left\| M - \hat{\hat{M}} \right\| < \left\| M - \hat{M} \right\|$

"refined estimate" …. Iteration 1

Numerical Solution for Mach

- Abstracting to a "j^th" iteration

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$

Iterate until convergence
j={0,1,….}

- Drop from loop when

$$\frac{\left\| \frac{1}{\hat{M}_{(j+1)}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\hat{M}_{(j+1)}^{\;2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} - \frac{A}{A^*}\right\|}{\frac{A}{A^*}} < \varepsilon$$

Numerical Solution for Mach (cont'd)

$$F\left(\hat{M}_{(j)}\right)=\frac{1}{\hat{M}_{(j+1)}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\hat{M}_{(j+1)}^{\;2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}-\frac{A}{A^*}$$

$$\left(\frac{\partial F}{\partial M}\right)_{|(j)}=\frac{\partial}{\partial \hat{M}_{(j)}}\left(\frac{1}{\hat{M}_{(j)}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\hat{M}_{(j)}^{\;2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right)=$$

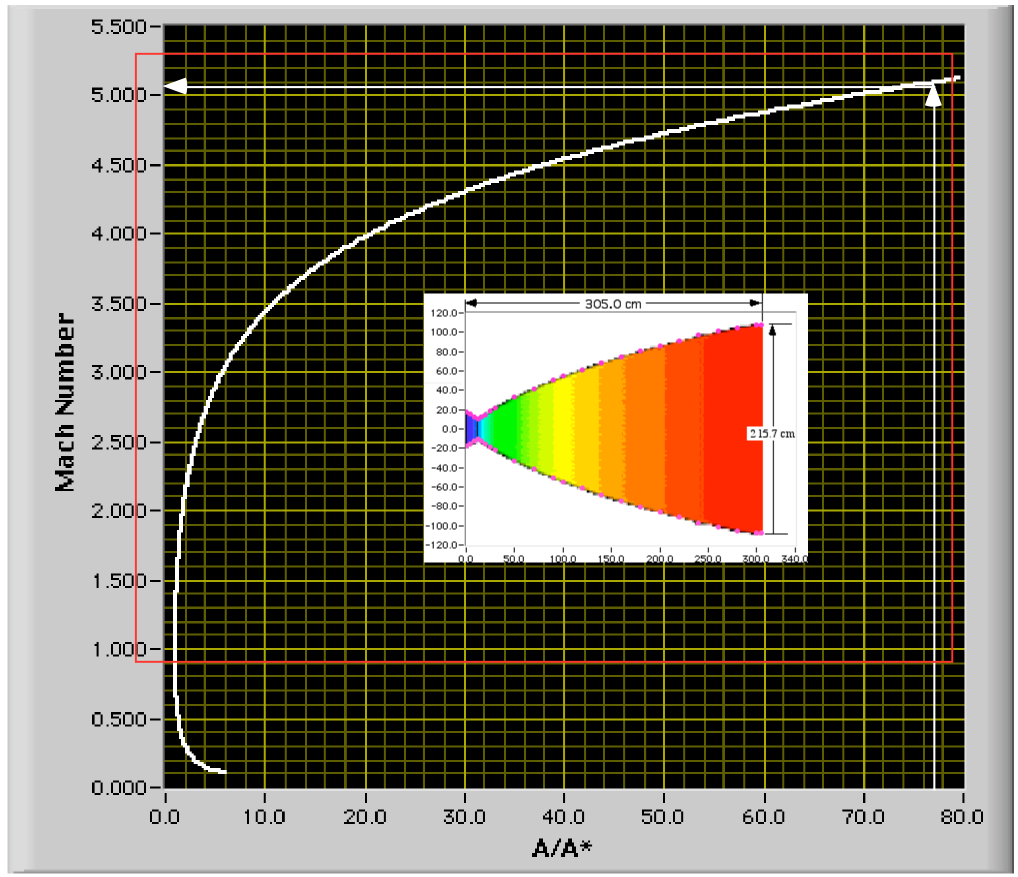
$$\boxed{\left(2^{\left(\frac{1-3\gamma}{2-2\gamma}\right)}\right)\frac{\left(\hat{M}_{(j)}^{\;2}-1\right)}{\hat{M}_{(j)}^{\;2}\left[2+\hat{M}_{(j)}^{\;2}(\gamma-1)\right]}\left(\frac{1+\frac{(\gamma-1)}{2}\hat{M}_{(j)}^{\;2}}{\gamma+1}\right)^{\left(\frac{\gamma+1}{2(\gamma-1)}\right)}}$$

Numerical Solution for Mach (cont'd)

- Numerical Solution (Newton's Method)

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$

- Solves for Supersonic Branch of Curve



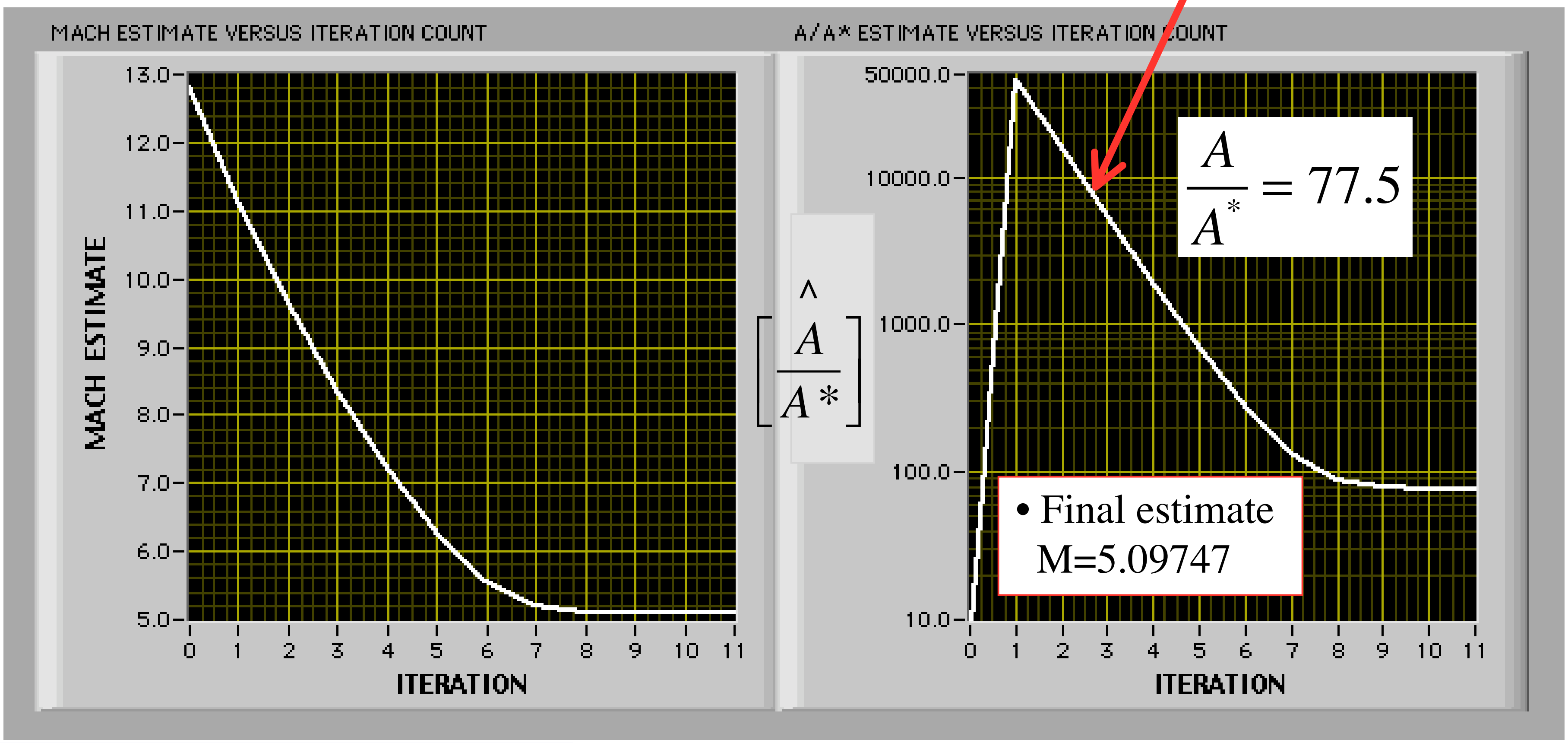
- Example $\frac{A}{A^*} = 77.5$
- Starting mach ->3.0
- Allowable Error, 0.001%

¥ γ=1.25

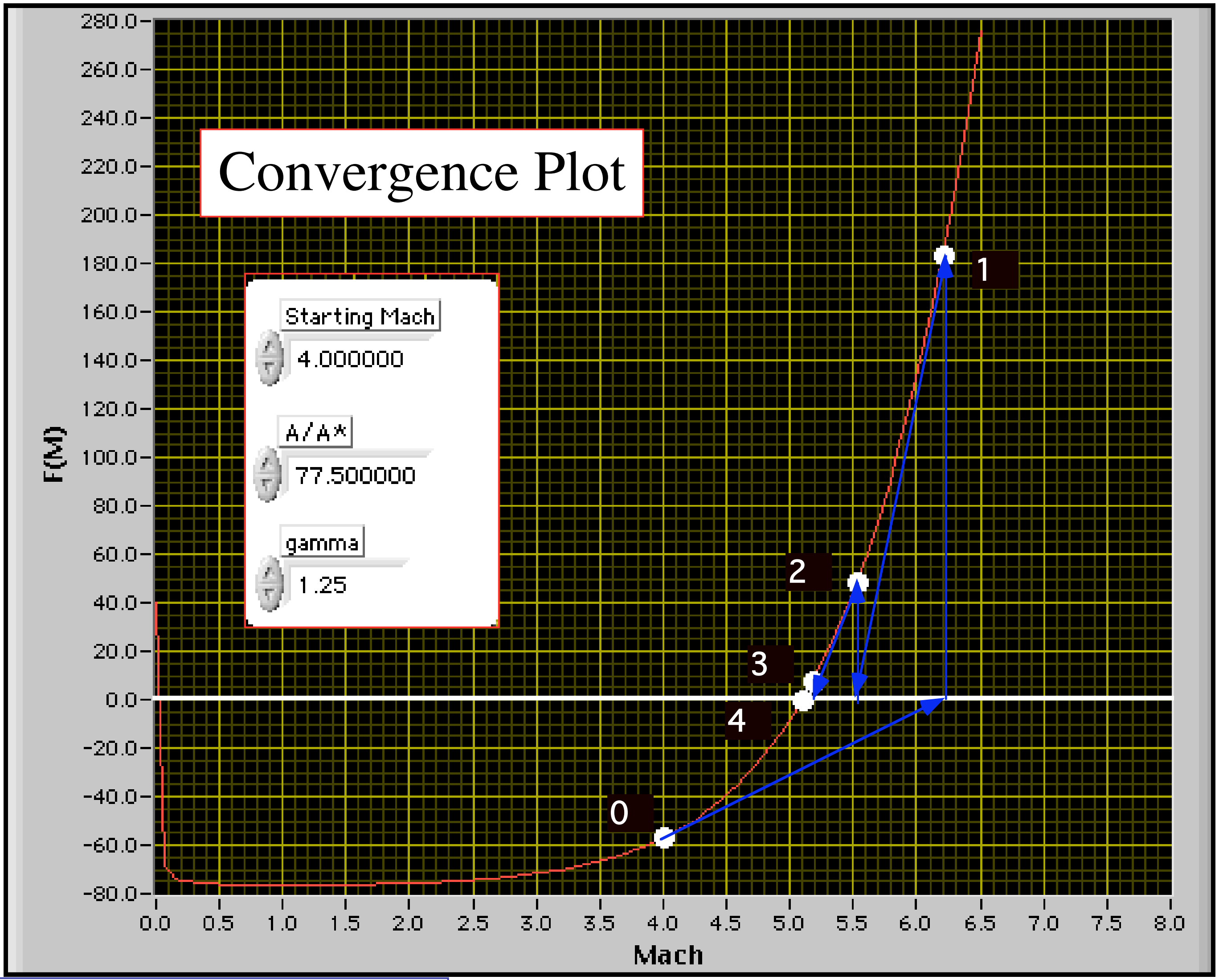
Numerical Solution for Mach (concluded)

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$

$$\frac{1}{\hat{M}_{(j)}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\hat{M}_{(j)}^{\;2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$



Numerical Solution for Mach (concluded)

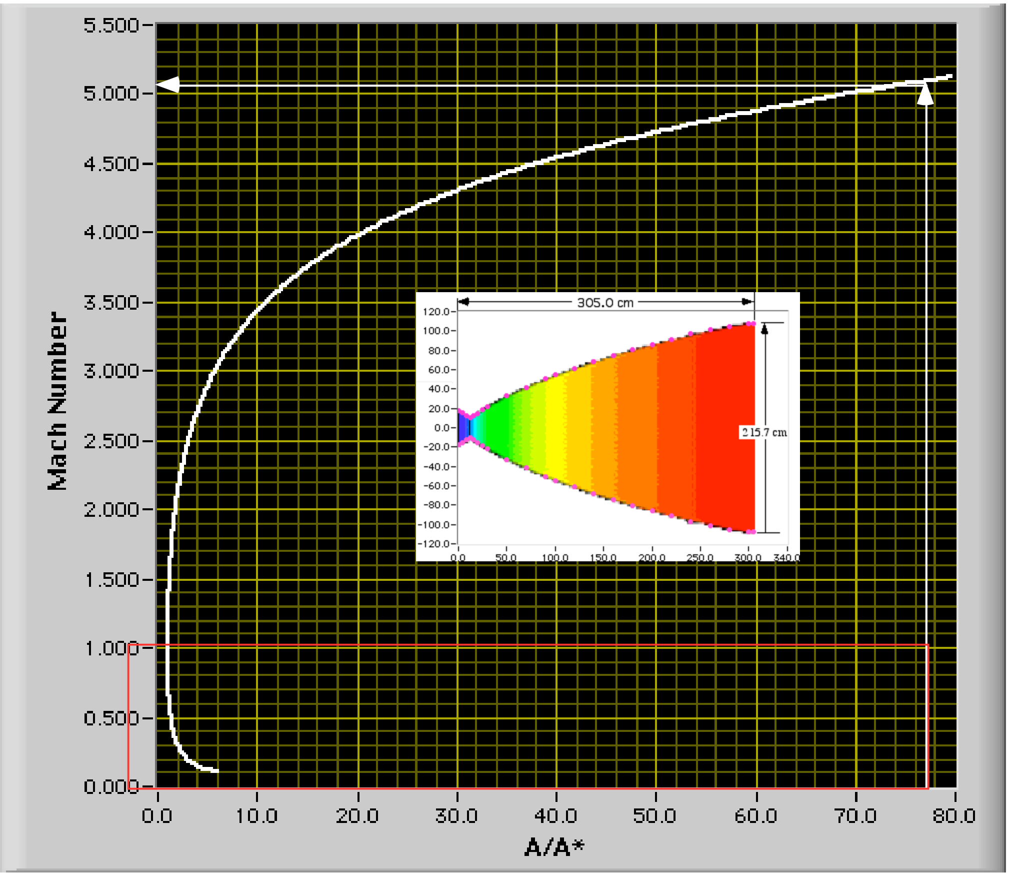


Effect of Startup Condition

- Example $\frac{A}{A^*} = 77.5$
- Starting mach ->0.01
- Allowable Error, 0.001%

¥ γ=1.25

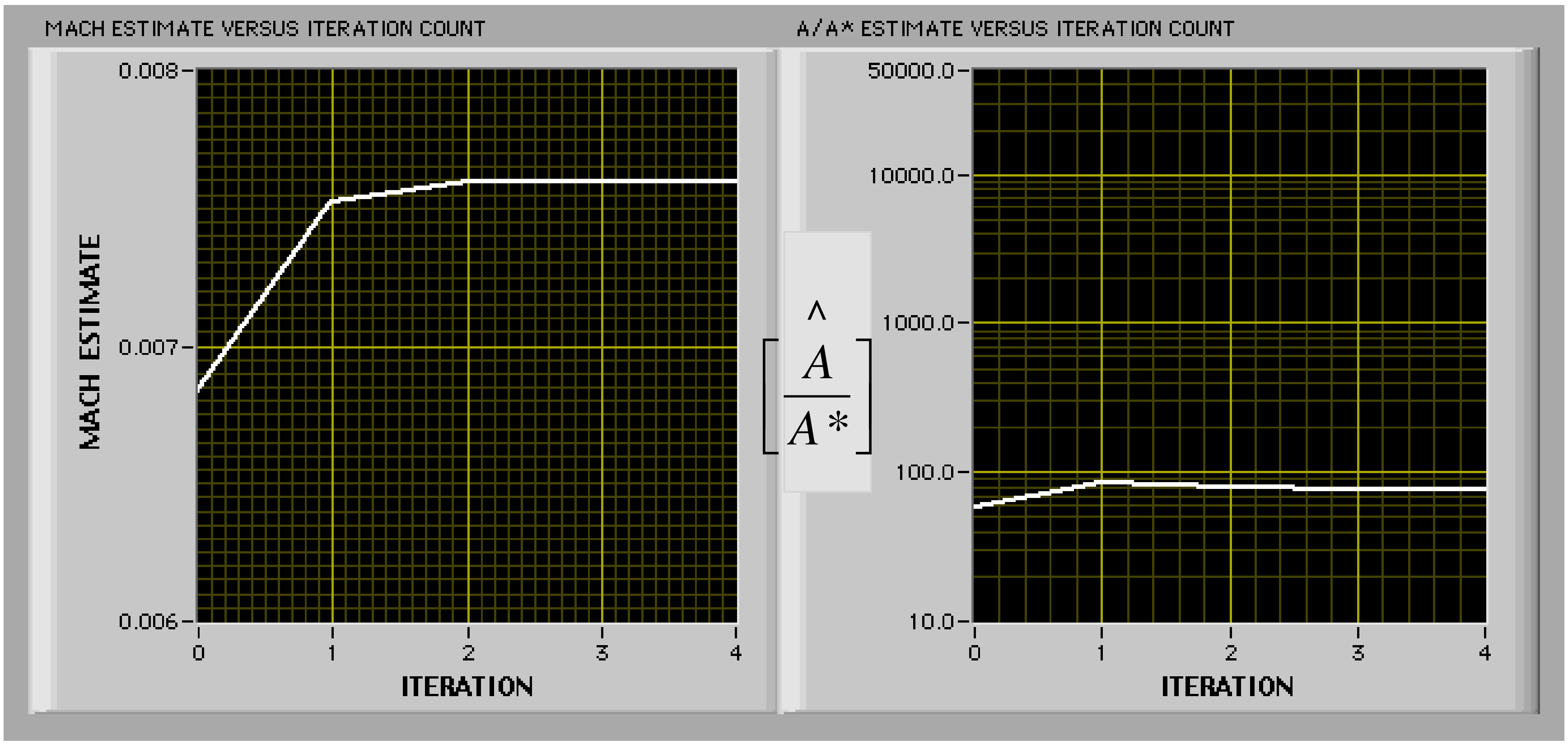
- Solves for Subsonic Branch Of Curve



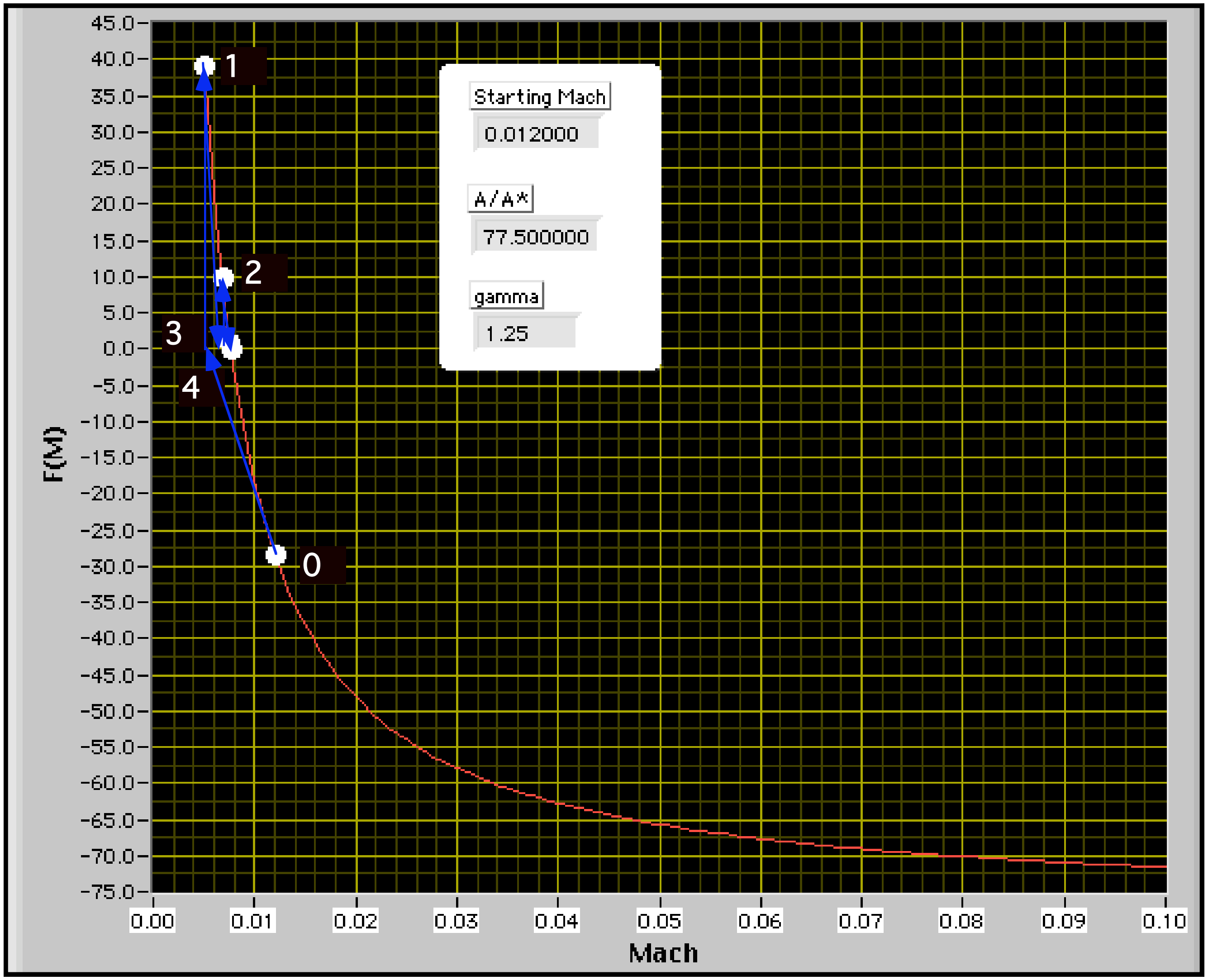
Effect of Startup Condition (concluded)

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$

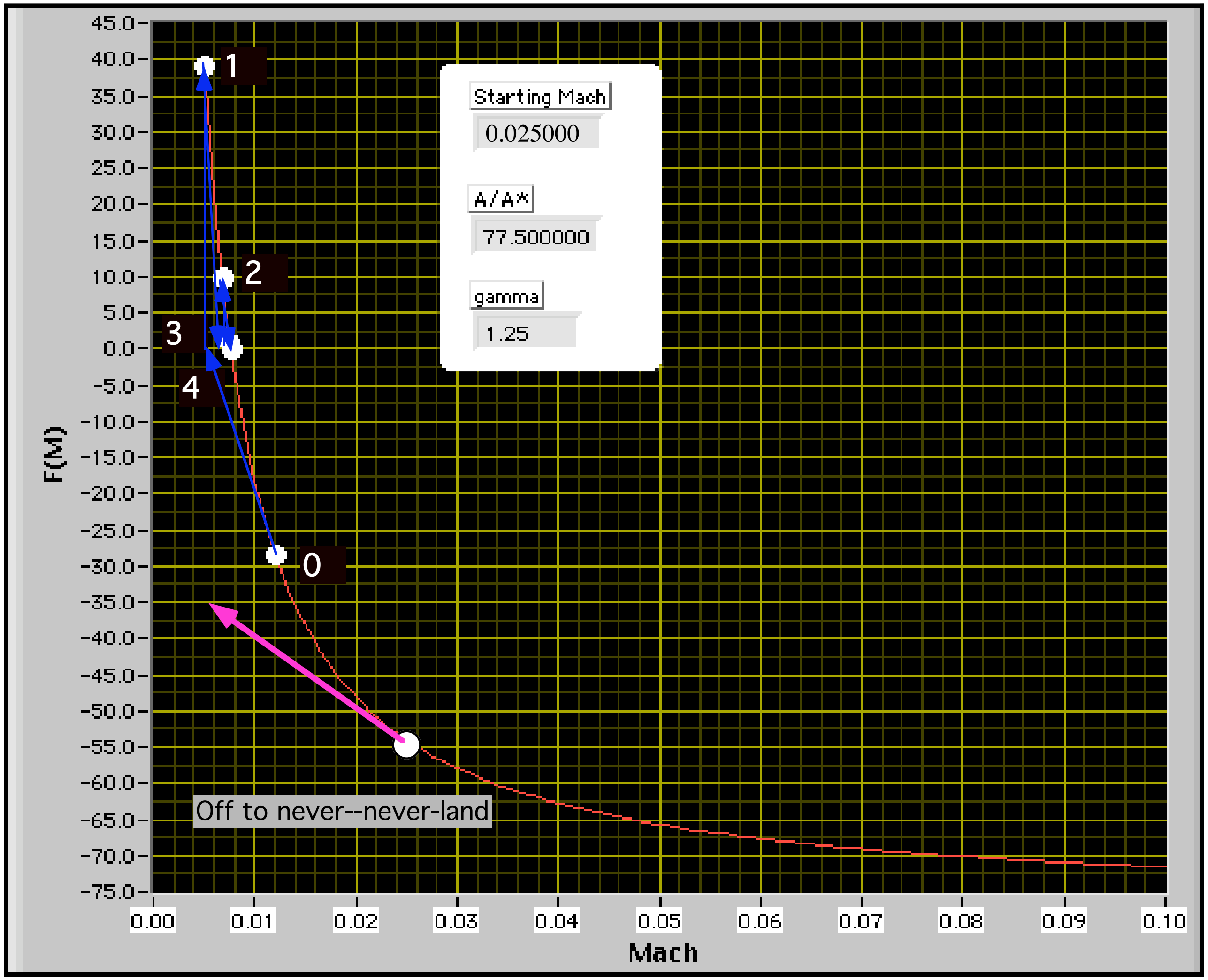
- Final estimate M=0.00759



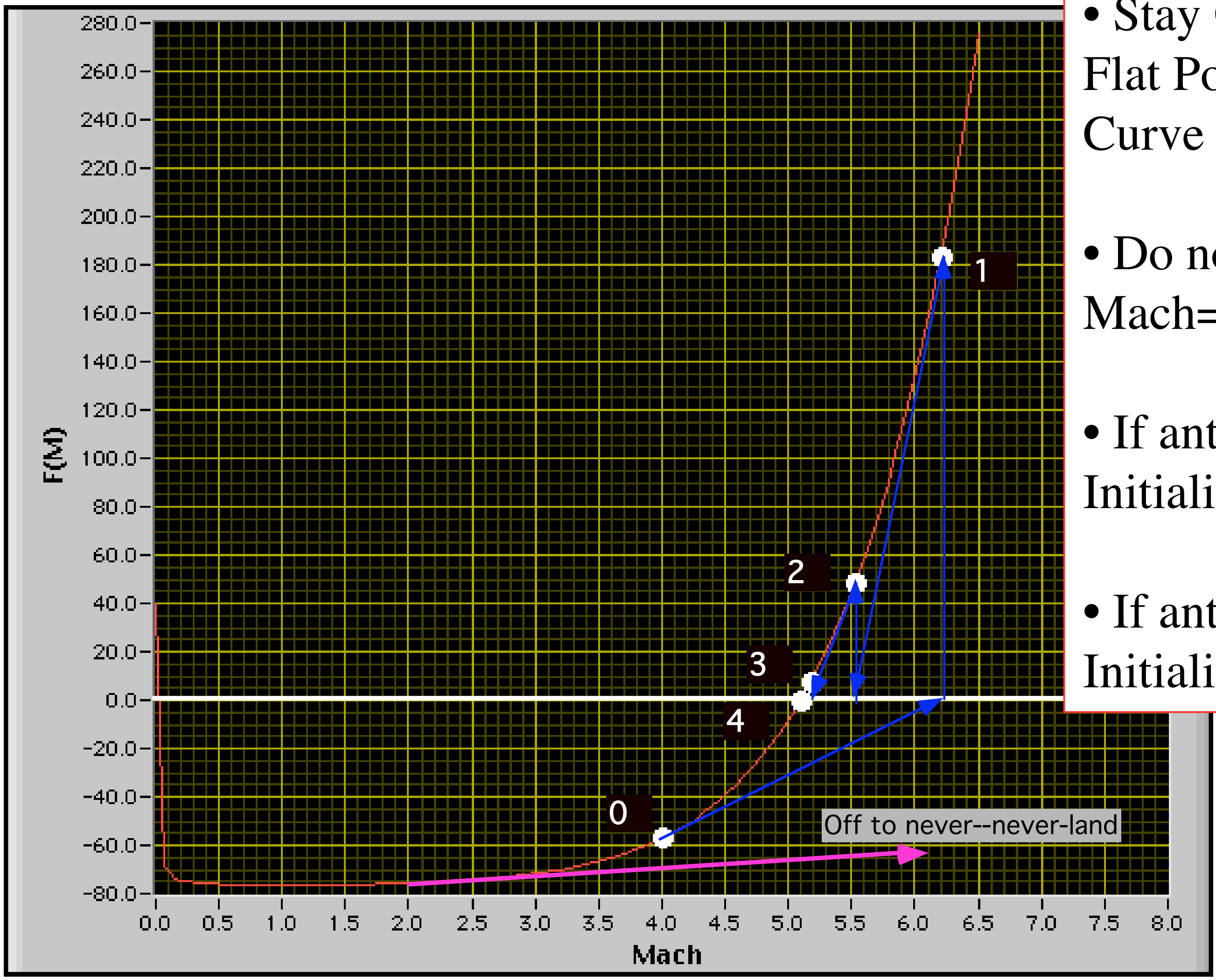
Convergence Plot



Be Careful About Startup Condition



Be Careful About Startup Condition (cont'd)



- Stay Off of Flat Portion of Curve at Startup
- Do not start at Mach=0, Mach 1
- If anticipate Mach < 1 Initialize at M slightly > 0
- If anticipated Mach > 1 Initialize at Mach > 3