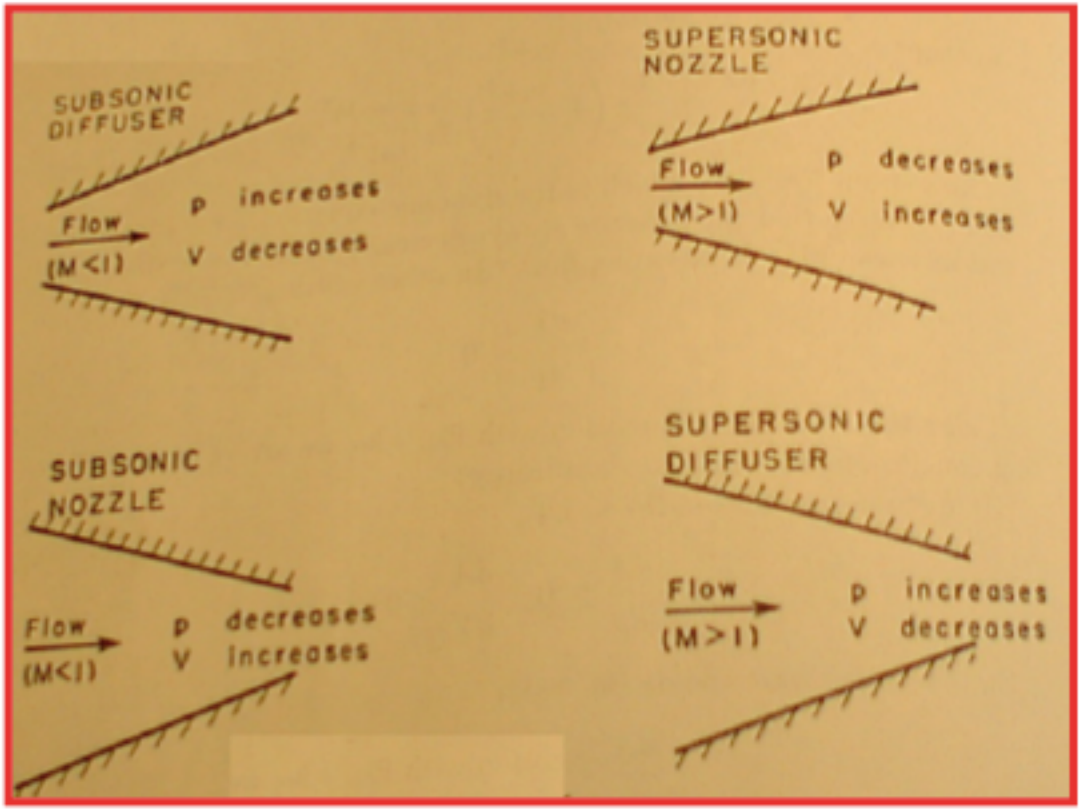
Section 5, Lecture 1: Review of Idealized Nozzle Theory



Taylor Chapter 4, Material also taken from Sutton and Biblarz, Chapter 3

Thermodynamics Summary

- Equation of State: $p = \rho R_g T \rightarrow R_g = \frac{R_u}{M_w}$
  - R_u = 8314.4126 J/°K-(kg-mole)
  - $R_{g\,(air)}$ = 287.056 J/°K-(kg-mole)

- Relationship of R_g to specific heats

$$c_p = c_v + R_g$$

- Internal Energy and Enthalpy

$$h = e + Pv$$

$$c_v = \left(\frac{de}{dT}\right)_v$$

$$c_p = \left(\frac{dh}{dT}\right)_p$$

Thermodynamics Summary (2)

Ratio of Specific Heats

- $\gamma = c_p/c_v$ is a critical parameter for compressible flow analysis

Approximate Specific Heat Ratio for Various Gases, at moderate temperatures

Gas	Ratio of Specific Heats
Carbon Dioxide	1.3
Helium	1.66
Hydrogen	1.41
Methane or Natural Gas	1.31
Nitrogen	1.4
Oxygen	1.4
Standard Air	1.4

Thermodynamics Summary (3)

- *Useful relationships*

$$R_g = c_p - c_v = c_p\left(1 - \frac{1}{\gamma}\right) \rightarrow c_p = \frac{\gamma}{\gamma - 1} R_g$$

$$c_v = \frac{1}{\gamma} c_p = \frac{1}{\gamma - 1} R_g$$

- Air at Room Temperature -->

$c_p =$	1004.696 J/°K-(kg-mole)	
$c_v = c_p - R_g =$	1004.696 - 287.056	= 717.64 J/°K-(kg-mole)
$\gamma = c_p/c_v =$	1007.696/717.64	= 1.400

Thermodynamics Summary (4)

- Speed of Sound for calorically Perfect gas

$$c = \sqrt{\gamma R_g T}$$

- Mathematic definition of Mach Number

$$M = \frac{V}{\sqrt{\gamma R_g T}}$$

Thermodynamics Summary (5)

- First Law of Thermodynamics, *reversible process*

$$de = dq - pdv \qquad dh = dq + vdp$$

- First Law of Thermodynamics, *isentropic process*
(adiabatic, reversible)

$$de = -pdv \qquad dh = vdp$$

Thermodynamics Summary (6)

- Second Law of Thermodynamics, *reversible process*

$$s_2 - s_1 = c_p \ln_2 \left[\frac{T_2}{T_1}\right] - R_g \ln\left[\frac{p_2}{p_1}\right]$$

$$Tds = dh - vdp$$

- Second Law of Thermodynamics, *isentropic process*

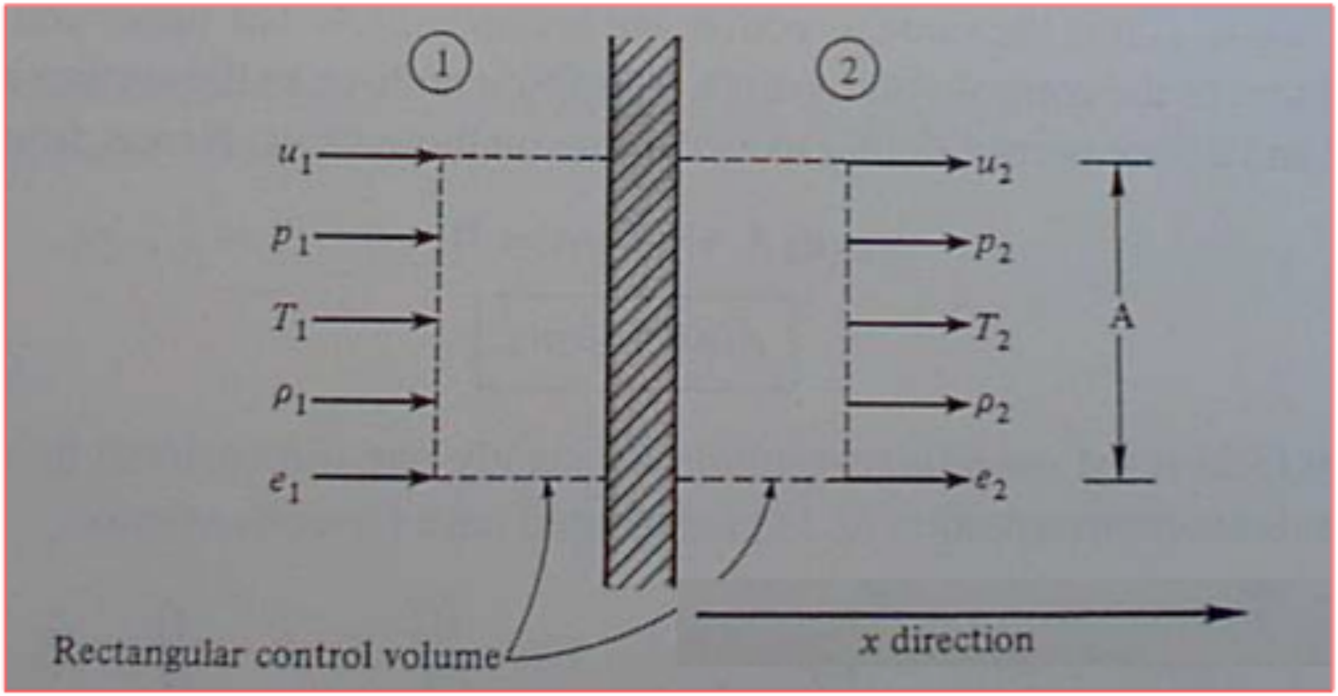
(adiabatic, reversible) ------> $s_2 - s_1 = 0$

$$\frac{p_2}{p_1} = \left[\frac{T_2}{T_1}\right]^{\frac{\gamma}{\gamma - 1}}$$

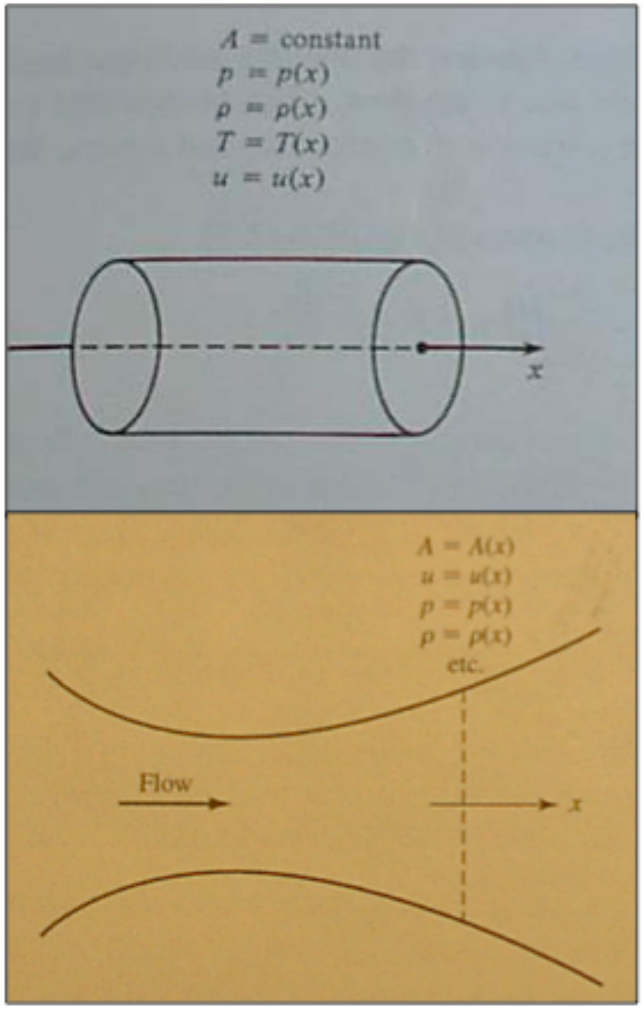
$$\left[\frac{p_2}{p_1}\right] = \left[\frac{\rho_2}{\rho_1}\right]^{\gamma}$$

One Dimensional Compressible Flow Approximations

- Many Useful and practical Flow Situations can be Approximated by one-dimensional flow analyses
- Flow Characterized by motion only along longitudinal axis



Distinction Between True 1-D Flow and Quasi 1-D Flow

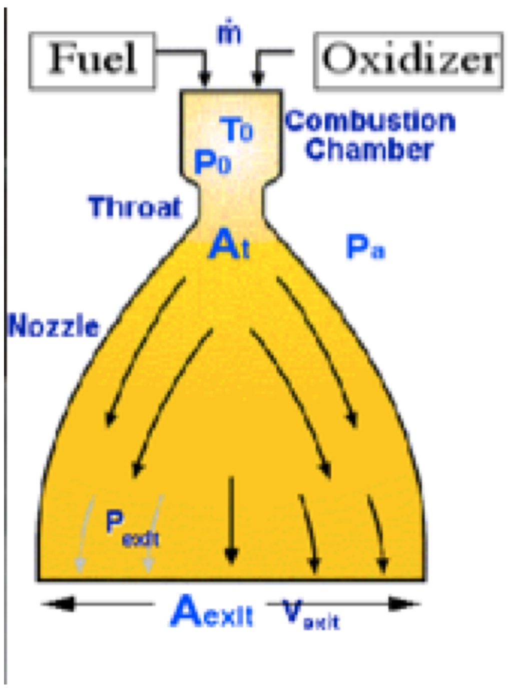


- In "true" 1-D flow Cross sectional area is strictly constant
- In quasi-1-D flow, cross section varies as a Function of the longitudinal coordinate, x
- Flow Properties are assumed constant across any cross-section
- Analytical simplification very useful for evaluating Flow properties in Nozzles, tubes, ducts, and diffusers Where the cross sectional area is large when compared to length

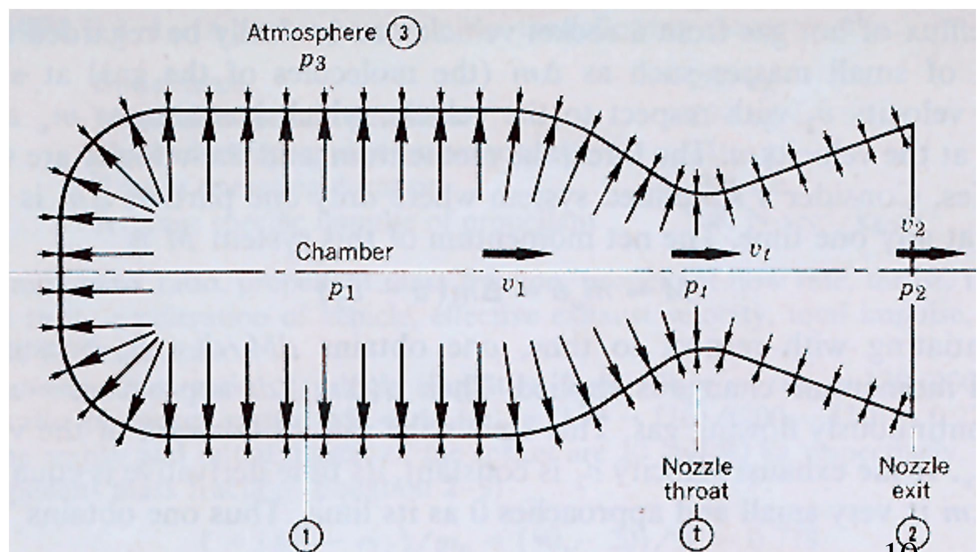
Rocket Thrust Equation

$$Thrust = \dot{m}_e V_e + \left(p_e A_e - p_\infty A_e \right)$$

$$\dot{m}_i = 0$$

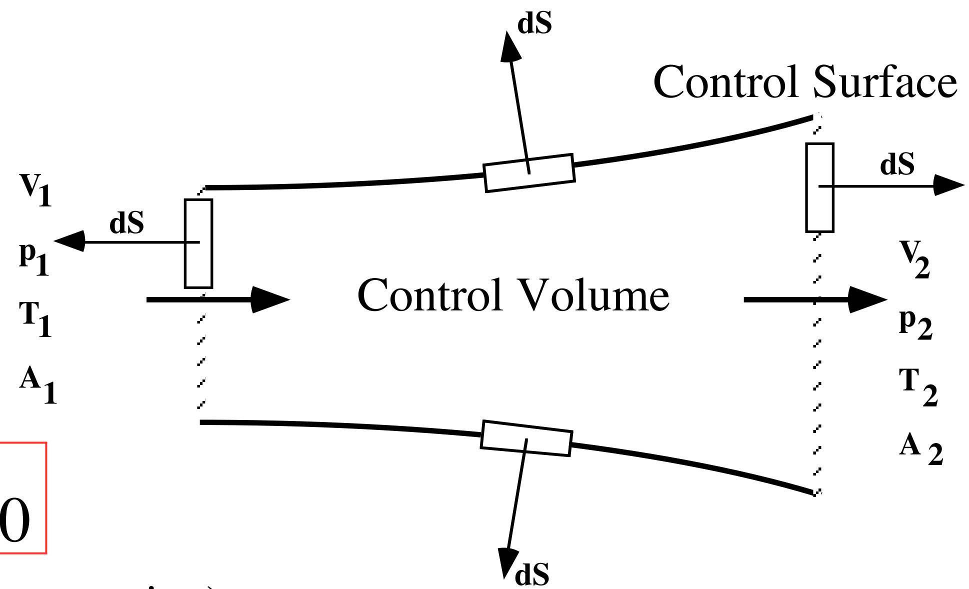


- Thrust + Oxidizer enters combustion Chamber at ~0 velocity, combustion Adds energy … High Chamber pressure Accelerates flow through Nozzle *Resultant pressure forces produce thrust*



Review: Continuity Equation for Quasi 1-D Control Volume

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) = 0$$



- Upper and Lower Surfaces … no flow across boundary

$$\vec{V} \bullet \vec{ds} = 0$$

- Inlet (properties constant across Cross section) -->

$$\iint_1 \left(\rho \vec{V} \bullet \vec{ds} \right) = \rho_1 \vec{V_1} \bullet \iint_1 \vec{ds} = \rho_1 V_1 \cos(180^o) \iint_1 ds = -\rho_1 V_1 A_1$$

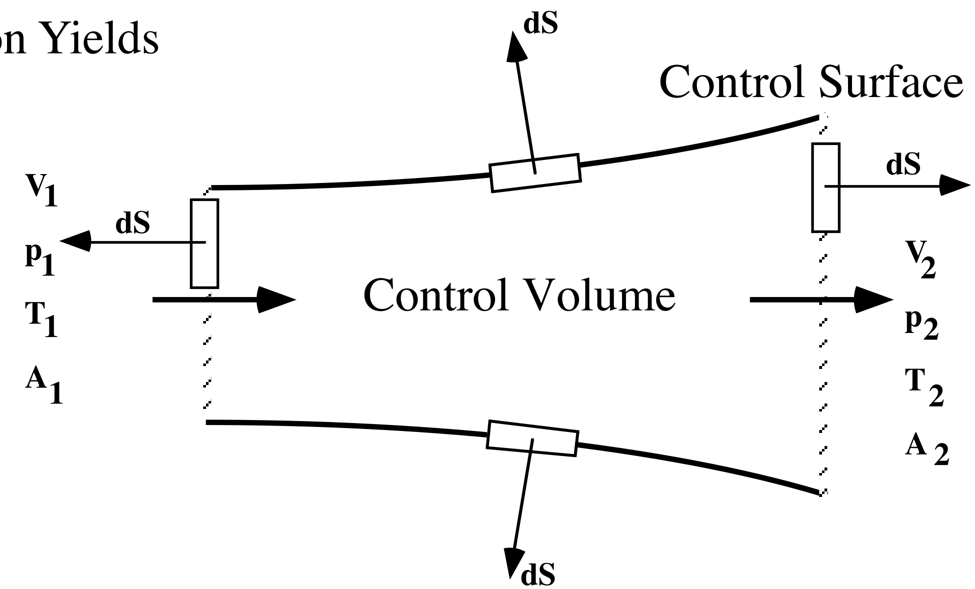
- Inlet (properties constant across Cross section) -->

$$\iint_2 \left(\rho \vec{V} \bullet \vec{ds} \right) = \rho_2 \vec{V_2} \bullet \iint_2 \vec{ds} = \rho_2 V_2 \cos(0^o) \iint_2 ds = \rho_2 V_2 A_2$$

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

Review: Momentum Equation for Quasi 1-D Control Volume

• Similar Analysis for Momentum Equation Yields



• Newton's Second law--
Time rate of change of momentum
Equals integral of external forces

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} = -\iint_{C.S.} (p) \vec{dS} \rightarrow$$

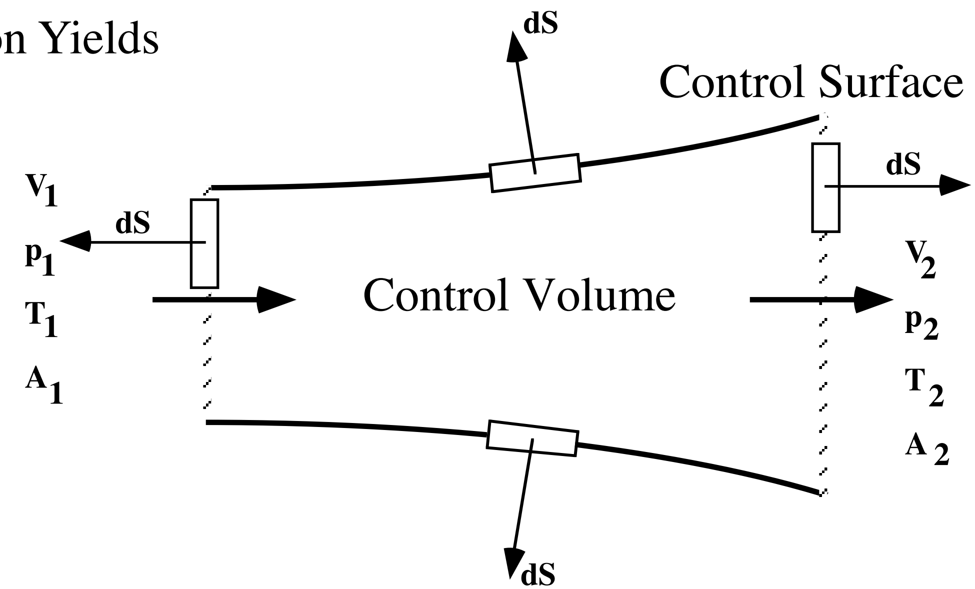
$$p_1 A_1 V_1 + \rho_1 V_1^{\,2} A_1 + \int_1^2 p\, \vec{ds} \bullet \vec{i_x} = p_2 A_2 V_2 + \rho_2 V_2^{\,2} A_2$$

• Because of duct symmetry the "Z-axis"
Component of pressure integrated to zero

$\vec{i_x}$	"Unit vector" x-direction

Review: Momentum Equation for Quasi 1-D Control Volume

• Similar Analysis for Momentum Equation Yields



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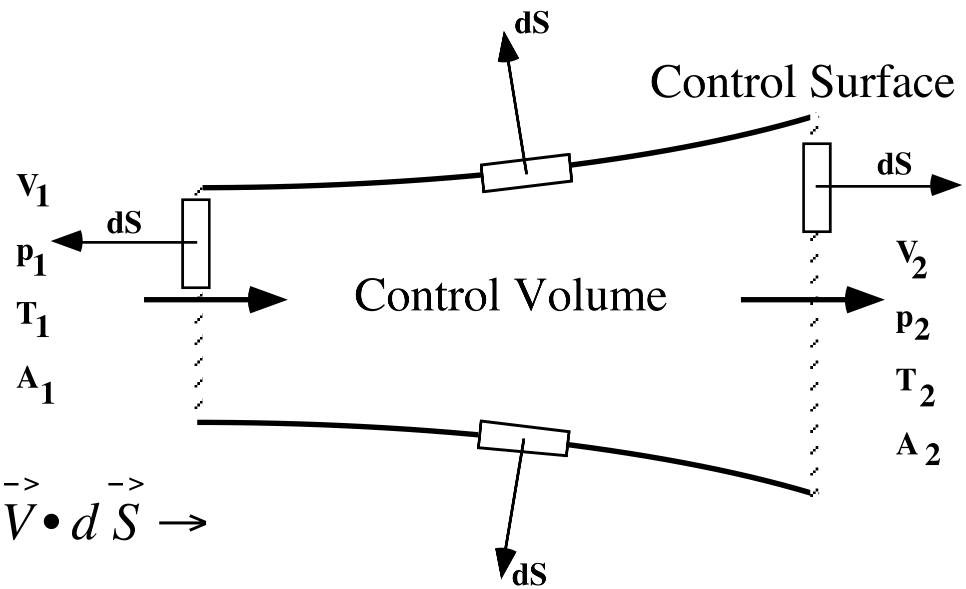
• Because of duct symmetry the "Z-axis"
Component of pressure integrated to zero

Review: Energy Equation for Quasi 1-D Control Volume



$$\dot{Q} - \iint_{C.S.} (p\,d\vec{S}) \bullet \vec{V} = \iint_{C.S.} \rho \left(e + \frac{\|V^2\|}{2} \right) \vec{V} \bullet d\vec{S} \rightarrow$$

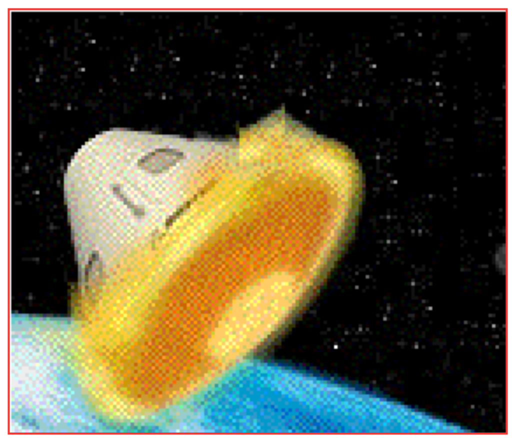
$$\Delta q + h_1 + \frac{V_1^{\,2}}{2} = h_2 + \frac{V_2^{\,2}}{2}$$

Fundamental Results of 1-D Compressible Flow

• ***Energy Equation for Adiabatic Flow:*** Stagnation temperature is a measure of the Kinetic Energy of the flow Field.

• Largely responsible for the high Level of heating that occurs on high speed aircraft or reentering space Vehicles …

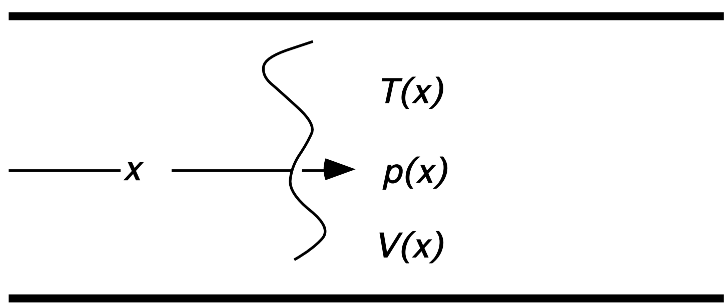
$$\frac{T_0}{T} = 1 + \frac{(\gamma - 1)}{2} M^2$$



Stagnation Pressure for the Isentropic Flow of a Calorically Perfect Gas

• ***Energy Equation for Isentropic Flow:***

Temperature *T(x)*, pressure *p(x)*, and a velocity *V(x)*

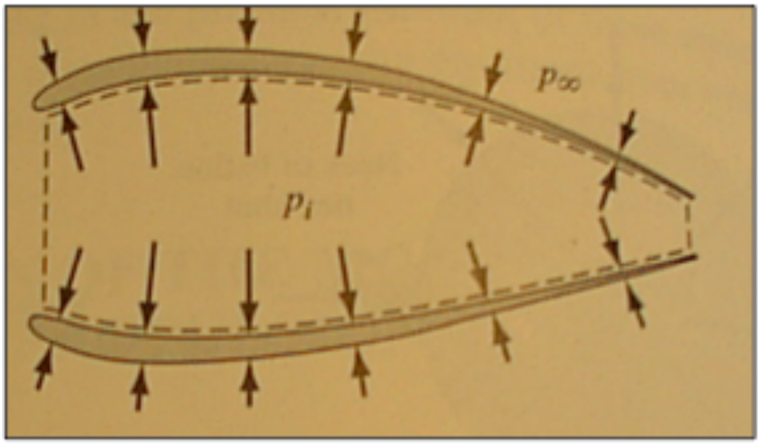


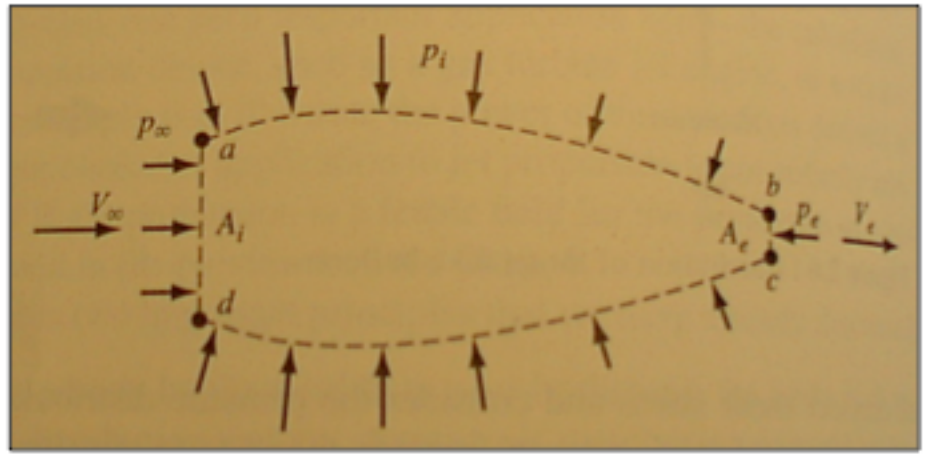
$$\frac{p_0}{p} = \left[\frac{T_0}{T}\right]^{\frac{\gamma}{\gamma-1}} = \left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{\gamma}{\gamma-1}}$$

"stagnation" *(total)* pressure: Constant throughout Isentropic flow field

Thrust Equation

- Steady, Inviscid, One-Dimensional Flow Through Ramjet

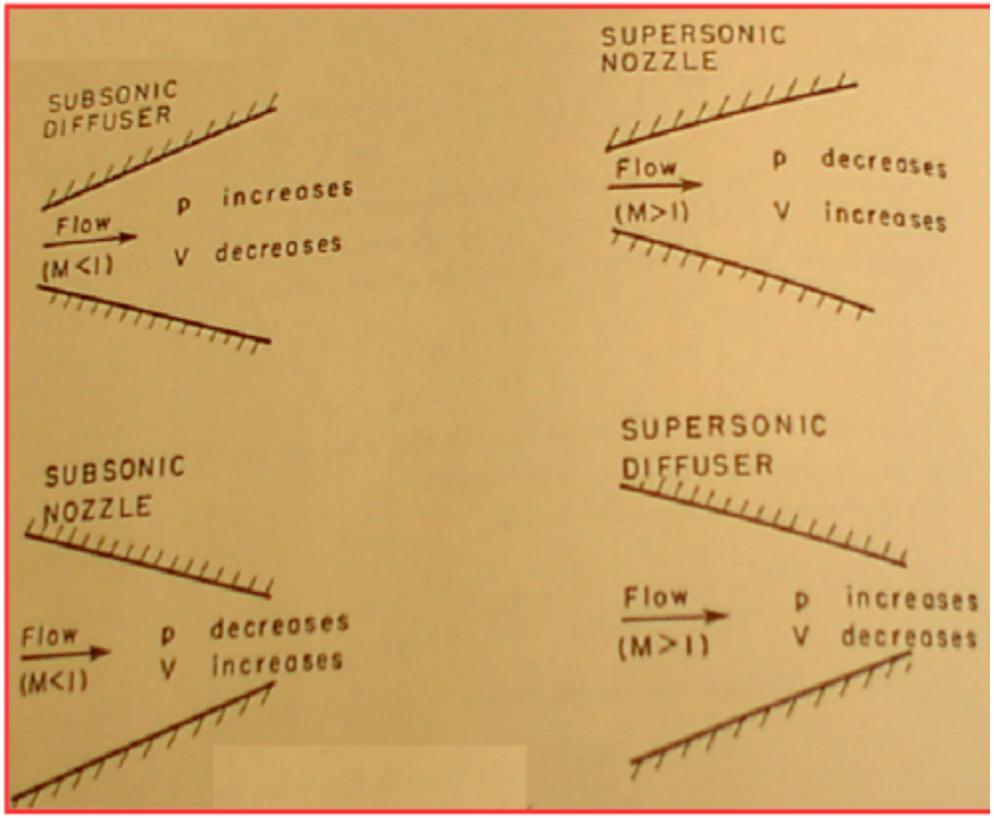




$$-\iint_{C.S.} (p)\,\vec{dS} = \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds}\right) \vec{V} \longrightarrow$$

$$Thrust = \dot{m}_e V_e - \dot{m}_i V_i + \left(p_e A_e - p_\infty A_e\right)$$

Fundamental Properties of Supersonic and Supersonic Flow



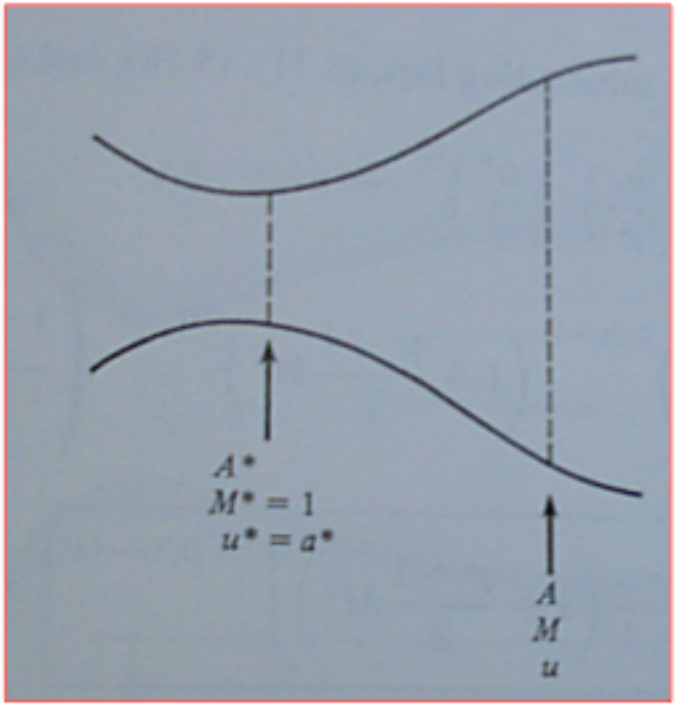
$$\left(\frac{dV}{dA}\right) = \frac{1}{\left(M^2-1\right)}\frac{V}{A}$$

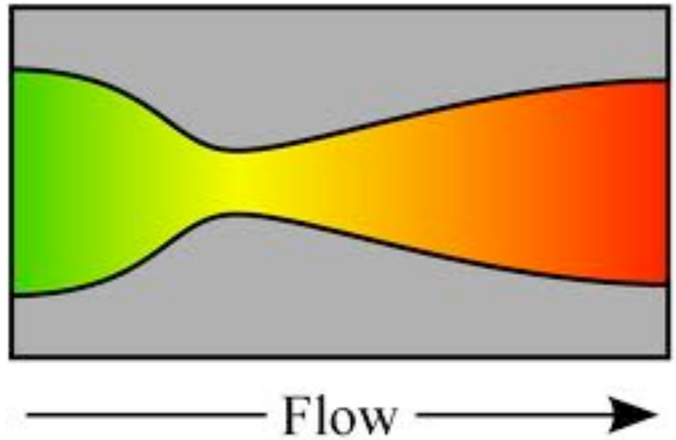
$$\frac{dp}{dA} = \frac{-1}{\left(M^2-1\right)}\frac{\rho V^2}{A}$$

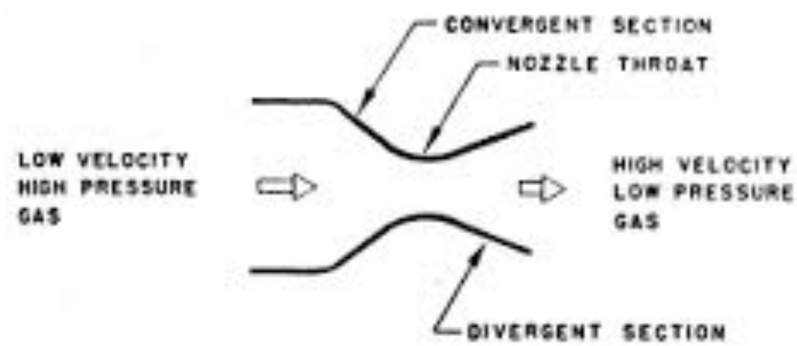
$$M<1 \rightarrow \left(\frac{dV}{dA}\right)<0 \rightarrow \left(\frac{dp}{dA}\right)>0$$

$$M>1 \rightarrow \left(\frac{dV}{dA}\right)>0 \rightarrow \left(\frac{dp}{dA}\right)<0$$

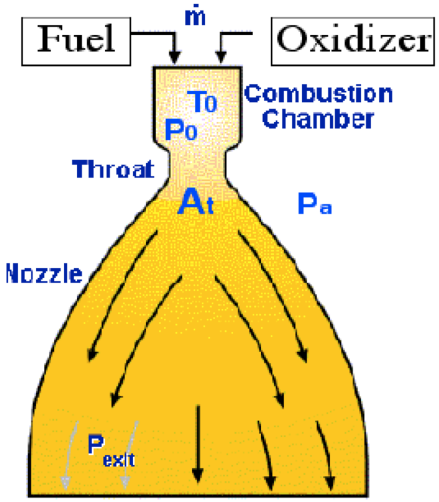
… Hence the Shape of the De-Laval Rocket Nozzle







What is a NOZZLE



• **FUNCTION of rocket nozzle is to convert thermal energy in propellants into kinetic energy as efficiently as possible**

• **Nozzle is substantial part of the total engine mass.**

• **Many of the historical data suggest that 50% of solid rocket failures stemmed from nozzle problems.**

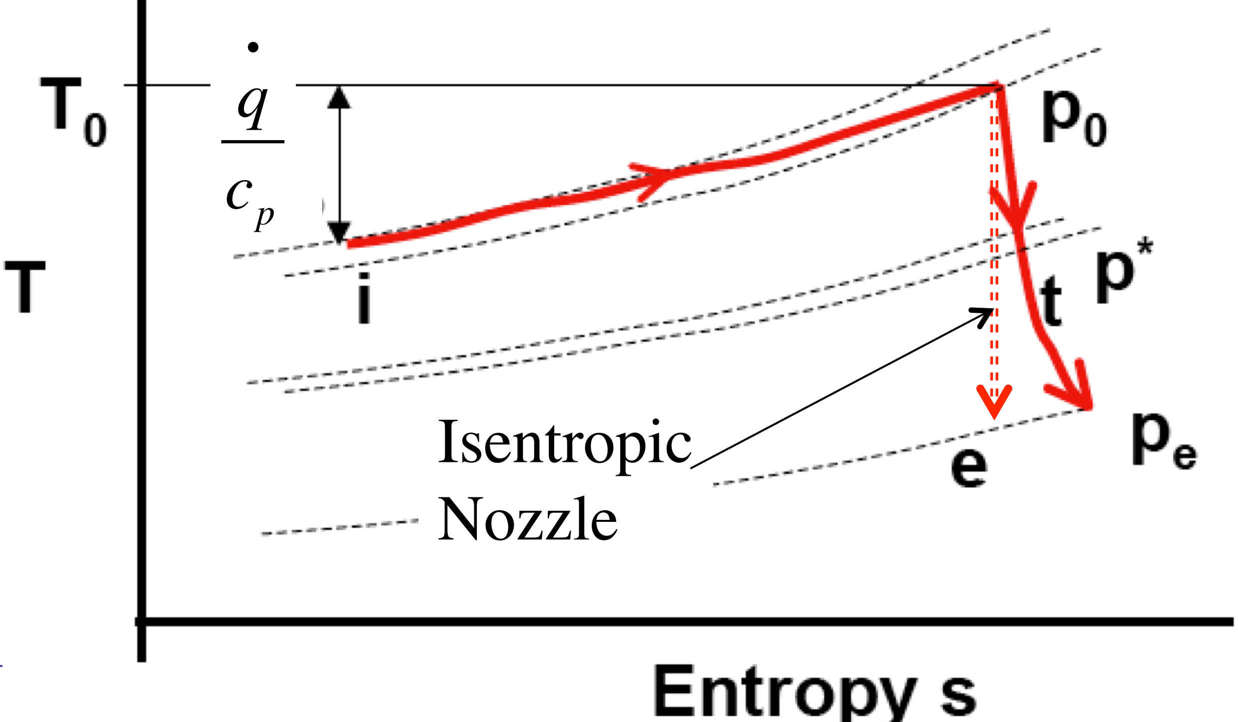
The design of the nozzle must trade off:
1. Nozzle size (needed to get better performance) against nozzle weight penalty.

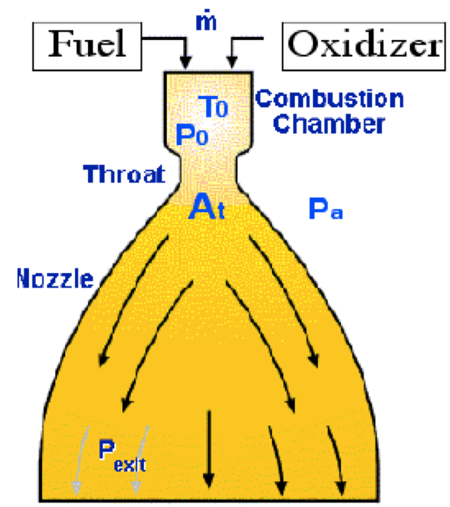
2. Complexity of the shape for shock-free performance vs. cost of fabrication

Temperature/Entropy Diagram for a Typical Nozzle

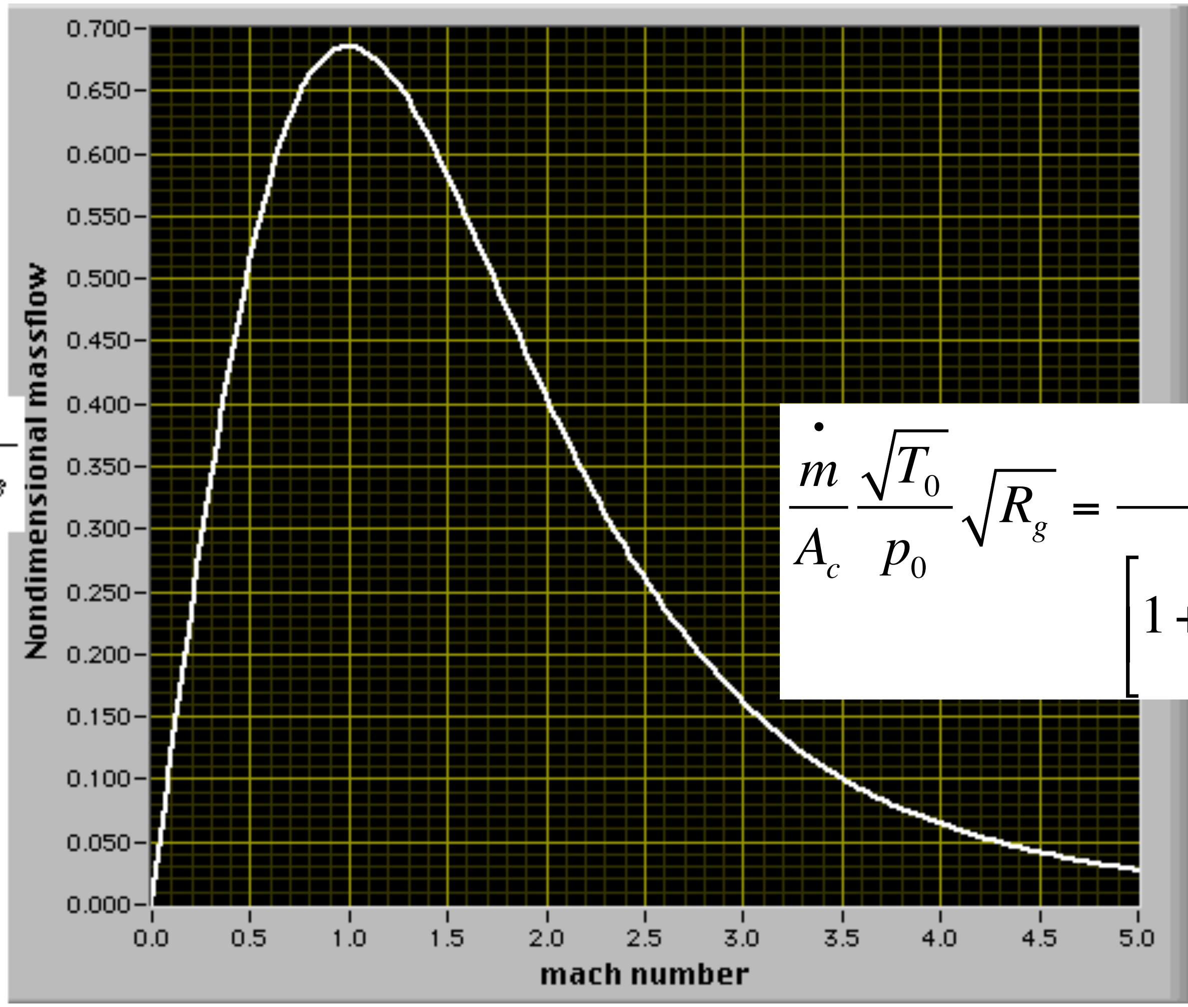
$$\dot{q} + h_1 + \frac{V_1^{\,2}}{2} = h_2 + \frac{V_2^{\,2}}{2} \qquad c_p = \left(\frac{dh}{dT}\right)_p$$

$$Tds = \delta q + ds_{irrev}$$





Nozzle Mass Flow per Unit Area



$$\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}$$

$$\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g} = \frac{\sqrt{\gamma}\,M}{\left[1+\frac{(\gamma-1)}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

- maximum Massflow/area Occurs when When M=1

Nozzle Mass Flow per Unit Area (concluded)

- maximum Massflow/area Occurs when When M=1

- Effect known as *Choking* in a Duct or Nozzle

- i.e. nozzle will Have a mach 1 throat

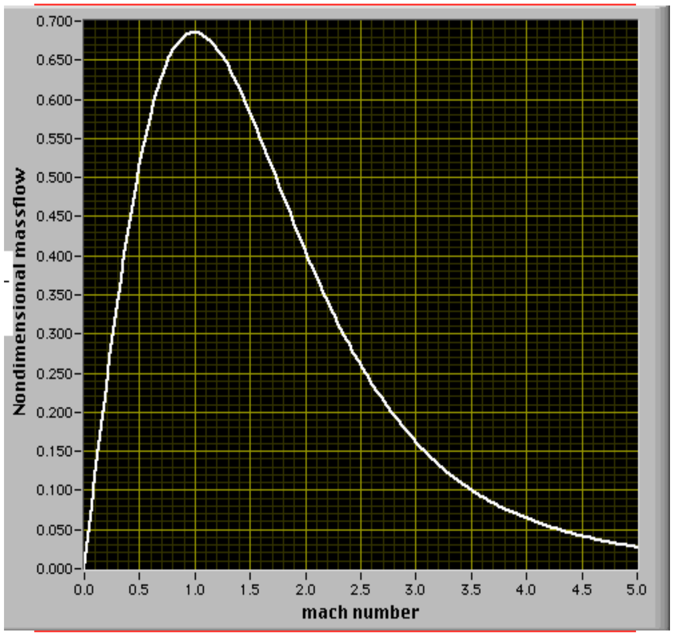
$$\left(\frac{\dot{m}}{A_c}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}\right)_{\max} = \left(\frac{\dot{m}}{A^*}\frac{\sqrt{T_0}}{p_0}\sqrt{R_g}\right) =$$

$$\frac{\sqrt{\gamma}}{\left[1+\frac{(\gamma-1)}{2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} = \sqrt{\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \rightarrow$$

$$\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\frac{p_0}{\sqrt{T_0}}$$

- **Choking Massflow Equation**

Isentropic Nozzle Flow: Area Mach Relationship



- Consider a "choked-flow" Nozzle … (I.e. M=1 at Throat)

- Then comparing the massflow /unit area at throat to some Downstream station

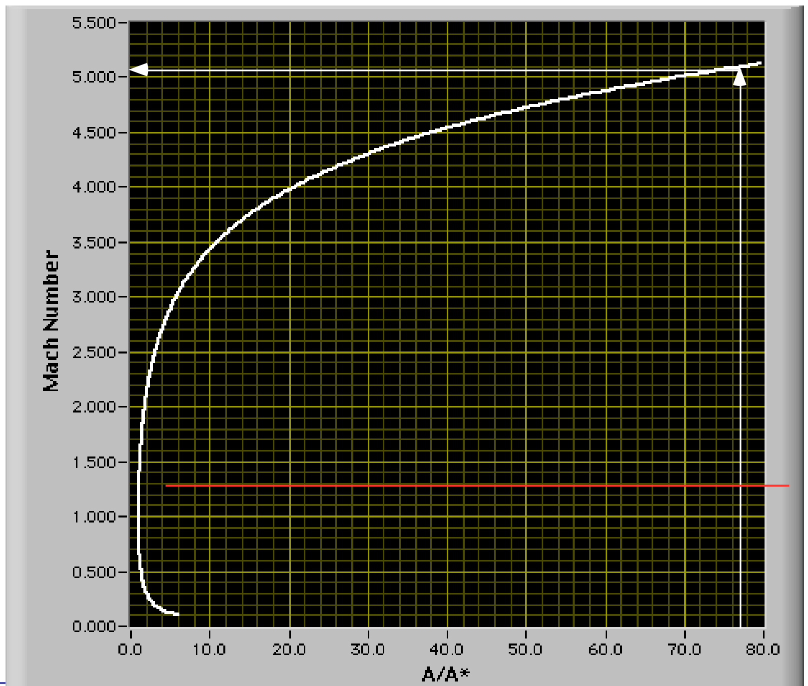
$$\frac{\dfrac{\dot{m}}{A^*}\dfrac{\sqrt{T_0}}{p_0}\sqrt{R_g}}{\dfrac{\dot{m}}{A}\dfrac{\sqrt{T_0}}{p_0}\sqrt{R_g}} = \frac{A}{A^*} = \frac{\sqrt{\left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}{\dfrac{1}{\left[1+\dfrac{(\gamma-1)}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Isentropic Nozzle Flow: Area Mach Relationship (cont'd)

- A/A* Directly related to Mach number

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

- "Two-Branch solution: Subsonic, Supersonic



- Nonlinear Equation requires Numerical Solution

- "Newton's Method"

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$

Numerical Solution for Mach

- Abstracting to a "j^th" iteration

$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$

Iterate until convergence
j={0,1,….}

- Drop from loop when

$$\frac{\left\| \frac{1}{\hat{M}_{(j+1)}} \left[\left(\frac{2}{\gamma+1}\right) \left(1 + \frac{(\gamma-1)}{2} \hat{M}_{(j+1)}^{2}\right) \right]^{\frac{\gamma+1}{2(\gamma-1)}} - \frac{A}{A^*} \right\|}{\frac{A}{A^*}} < \varepsilon$$

Numerical Solution for Mach (cont'd)

$$F(\hat{M}_{(j)}) = \frac{1}{\hat{M}_{(j)}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\hat{M}_{(j)}^{2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} - \frac{A}{A^*}$$

$$\left(\frac{\partial F}{\partial M}\right)_{|(j)} = \frac{\partial}{\partial \hat{M}_{(j)}}\left(\frac{1}{\hat{M}_{(j)}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}\hat{M}_{(j)}^{2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right) =$$

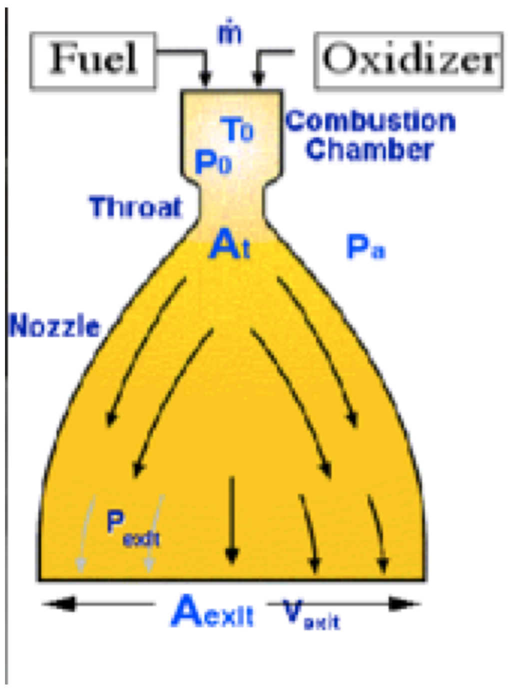
$$\boxed{\left(2^{\left(\frac{1-3\gamma}{2-2\gamma}\right)}\right)\frac{\left(\hat{M}_{(j)}^{2}-1\right)}{\hat{M}_{(j)}^{2}\left[2+\hat{M}_{(j)}^{2}(\gamma-1)\right]}\left(\frac{1+\frac{(\gamma-1)}{2}\hat{M}_{(j)}^{2}}{\gamma+1}\right)^{\left(\frac{\gamma+1}{2(\gamma-1)}\right)}}$$

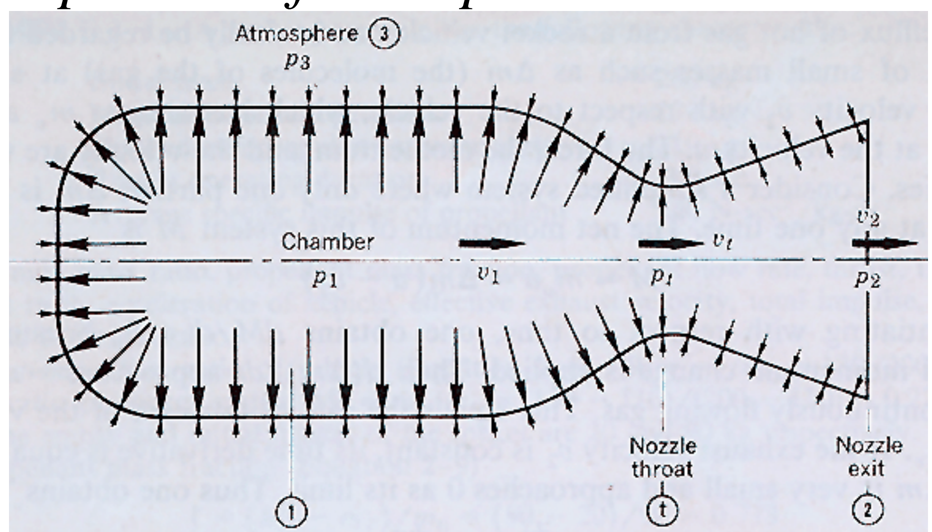
Rocket Thrust Equation, revisited

$$Thrust = \dot{m}_e V_e + \left(p_e A_e - p_\infty A_e \right)$$

$$\dot{m}_i = 0$$

• Thrust + Oxidizer enters combustion Chamber at ~0 velocity, combustion Adds energy … High Chamber pressure Accelerates flow through Nozzle *Resultant pressure forces produce thrust*





Rocket Thrust Equation, revisited

$$Thrust = \dot{m} V_{exit} + A_{exit} \left(p_{exit} - p_{\infty} \right)$$

- Non dimensionalize as

$$\frac{Thrust}{P_0 A_{throat}} = \frac{\dot{m} V_{exit}}{P_0 A_{throat}} + \frac{A_{exit}}{A_{throat}} \frac{\left(p_{exit} - p_{\infty} \right)}{P_0}$$

- For a choked throat

$$\frac{\dot{m}}{A^* P_0} = \frac{1}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R_g} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{(\gamma - 1)}}} \longrightarrow \frac{Thrust}{P_0 A^*} = \frac{V_{exit}}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R_g} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{(\gamma - 1)}}} + \frac{A_e}{A^*} \frac{\left(p_{exit} - p_{\infty} \right)}{P_0}$$

Rocket Thrust Equation, revisited (cont'd)

$$\frac{Thrust}{P_0 A^*} = \frac{V_{exit}}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} + \frac{A_e}{A^*}\frac{(p_{exit} - p_\infty)}{P_0}$$

- For isentropic flow

$$V_{exit} = \sqrt{2c_p\left[T_{0_{exit}} - T_{exit}\right]} = \sqrt{2c_p T_{0_{exit}}}\left[1 - \frac{T_{exit}}{T_{0_{exit}}}\right]^{1/2}$$

- Also for isentropic flow

$$\frac{p_2}{p_1} = \left[\frac{T_2}{T_1}\right]^{\frac{\gamma}{\gamma-1}} \longrightarrow \frac{T_{exit}}{T_{0_{exit}}} = \left(\frac{p_{exit}}{P_{0_{exit}}}\right)^{\frac{\gamma-1}{\gamma}}$$

Rocket Thrust Equation, revisited (cont'd)

- Subbing into velocity equation

$$V_{exit} = \sqrt{2c_p\left[T_{0_{exit}} - T_{exit}\right]} = \sqrt{2c_p T_{0_{exit}}}\left[1-\left(\frac{p_{exit}}{P_{0_{exit}}}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}$$

- Subbing into the thrust equation

$$\frac{Thrust}{p_0 A^*} = \frac{\sqrt{2c_p T_{0_{exit}}}\left[1-\left(\frac{p_{exit}}{P_{0_{exit}}}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0} =$$

$$\left[1-\left(\frac{p_{exit}}{P_{0_{exit}}}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}\sqrt{\frac{2c_p\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0}$$

Thrust Coefficient

- Simplifying $$\frac{2c_p\gamma}{R_g} = \frac{2c_p\gamma}{c_p - c_v} = \frac{2\gamma}{1-\frac{1}{\gamma}} = \frac{2\gamma^2}{\gamma - 1}$$

- Finally, for an isentropic nozzle $P_{0_{exit}} = P_0$

$$C_F = \frac{Thrust}{P_0 A^*} = \gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]}^{1/2} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0}$$

- **Non-dimensionalized thrust is a function of Nozzle pressure ratio and back pressure only …. "Thrust coefficient"**

Thrust Coefficient (cont'd)

$$C_F = \frac{Thrust}{P_0 A^*} = \gamma \sqrt{\frac{2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{(\gamma-1)}} \left[1 - \left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}} + \frac{A_{exit}}{A^*}\frac{(p_{exit} - p_\infty)}{P_0}$$

$$\frac{A_{exit}}{A^*} = \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}{\left(\frac{2}{\gamma-1}\right)}} \sqrt{\frac{\left(\frac{P_0}{p_{exit}}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_0}{p_{exit}}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]}}$$

Thrust Coefficient(cont'd)

- Thrust Coefficient is a function only of Combustion process $\{P_0, \gamma\}$, the Nozzle expansion (p_{exit}), and the back pressure, (p_∞)

$$C_F = \gamma \sqrt{\frac{2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \left\{ \left[1 - \left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}} \right]^{1/2} + \frac{\gamma - 1}{2\gamma} \sqrt{\frac{\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma+1}{-\gamma}}}{\left[\left(\frac{p_{exit}}{P_0}\right)^{\frac{(\gamma-1)}{-\gamma}} - 1\right]}} \left[\frac{p_{exit}}{P_0} - \frac{p_\infty}{P_0}\right] \right\}$$

Characteristic Velocity, C*

- Solving for $\dfrac{\dot{m}_{exit} V_{exit}}{P_0 A^*}$

$$\frac{\dot{m}_{exit} V_{exit}}{P_0 A^*} = \frac{Thrust}{P_0 A^*} - \frac{A_{exit}}{A^*}\frac{(p_{exit} - p_\infty)}{P_0} = \gamma \sqrt{\frac{2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \left[1 - \left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2}$$

- Letting the nozzle expand until

$$1 >> \left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}$$

Characteristic Velocity, C* (cont'd)

$$\left(\frac{\dot{m}_{exit} V_{exit}^{*}}{P_0 A^{*}}\right)_{\text{infinite expansion}} = \gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \rightarrow C^{*} = \left(\frac{P_0 A^{*}}{\dot{m}_{exit}}\right) = \frac{V_{exit}^{*}\left(\frac{\gamma-1}{2}\right)}{\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}$$

- Measure of Combustion performance
  Independent of Nozzle design … function of combustion Only
  See tables 5.5 - 5.6 In Sutton and Biblarz

- From earlier for a choked throat

$$\frac{\dot{m}}{A^{*}P_0} = \frac{1}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}} \rightarrow$$

$$\rightarrow C^{*} = \frac{\sqrt{\gamma R_g T_0}}{\gamma\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}} = \frac{c_0}{\gamma\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}$$

Characteristic Velocity, C* (cont'd)

• The *characteristic velocity* is a figure of thermo-chemical merit for a particular propellant and may be considered to be Indicative of the *combustion efficiency*.

$$\rightarrow C^* = \frac{\sqrt{\gamma R_g T_0}}{\gamma \sqrt{\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}} = \frac{c_0}{\gamma \sqrt{\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}} = \frac{\sqrt{\gamma R_u}}{\gamma \sqrt{\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}} \sqrt{\frac{T_o}{M_W}}$$

• Lower Molecular Weight Propellants Produce Higher C^*

Ideal I_{sp} of a Combustion Process

- from Earlier

$$\frac{Thrust}{P_0 A^*} = \gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0}$$

$$I_{sp} = \frac{Thrust}{g_o \dot{m}} = \frac{P_0 A^*}{g_o \dot{m}}\left[\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\left[1-\left(\frac{p_{exit}}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} + \frac{A_{exit}}{A^*}\frac{(p_{exit}-p_\infty)}{P_0}\right]$$

- Assuming an infinitely expanded nozzle in a vacuum

Ideal I_{sp} of a Combustion Process (cont'd)

$$\left(I_{sp}\right)_{ideal} = \frac{C^*}{g_o}\left[\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right] \Rightarrow$$

maximum possible specific impulse for given propellants

$$\left(I_{sp}\right)_{ideal} = \frac{C^*}{g_o}\left[\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right] \Rightarrow \left(I_{sp}\right)_{ideal} =$$

$$\frac{\dfrac{\sqrt{\gamma R_u}}{\gamma\sqrt{\left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}}\sqrt{\dfrac{T_o}{M_W}}}{g_o}\left[\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right] = \boxed{\frac{1}{g_o}\sqrt{\frac{2\gamma R_u}{\gamma-1}}\sqrt{\frac{T_o}{M_W}}}$$

- Propellants that burn Hot and have a low product Molecular weight … have better I_{sp}

SSME Computational Example

- Space Shuttle Main Engine …

- **Unlike other propellants, the optimum mixture ratio for liquid oxygen and liquid hydrogen is not necessarily that which will produce the maximum specific impulse. Because of the extremely low density of liquid hydrogen, the propellant volume decreases significantly at higher mixture ratios.**

- **Maximum specific impulse typically occurs at a mixture ratio of around 3.5, however by increasing the mixture ratio to, say, 5.5 the storage volume is reduced by one-fourth. This results in smaller propellant tanks, lower vehicle mass, and less drag, which generally offsets the loss in performance that comes with using the higher mixture ratio. In practice, most liquid oxygen/liquid hydrogen engines typically operate at mixture ratios from about 5 to 6.**

What is the Stoichiometric Mixture Ratio of LOX/LH_2?

$$2H_2 + O_2 \rightleftharpoons 2H_2O$$

$$M_w LH_2 \rightarrow 2.016_{kg/kg-mol}$$

$$M_w LO_2 \rightarrow 31.999_{kg/kg-mol}$$

$$MR = \frac{1_{mol} LO_2 \times M_w LO_2}{2_{mol} LH_2 \times M_w LH_2} = \frac{31.999}{2 \times 2.016} = 7.936$$

MR=6.0 (What the shuttle operates at) --> “Rich Mixture”

Compare Tank Volumes

- Space Shuttle has the following mass fraction characteristics

```
Weight (lb)
 Gross lift-off . . . . . . . .  4,500,000
 External Tank (full) . . . .  1,655,600
 External Tank (Inert) . . . .     66,000
 SRBs (2) each at launch . . . 1,292,000
 SRB inert weight, each . . .  . 192,000
```

- Shuttle has 721,000 kg of propellant in main tank on pad

Compare Tank Volumes (cont'd)

$$MR = 7.936 \rightarrow 721{,}000_{kg}\left[\frac{7.396}{1140_{kg/m^3}} + \frac{1}{67.78_{kg/m^3}}\right] = 15315m^3$$

"best compromise"

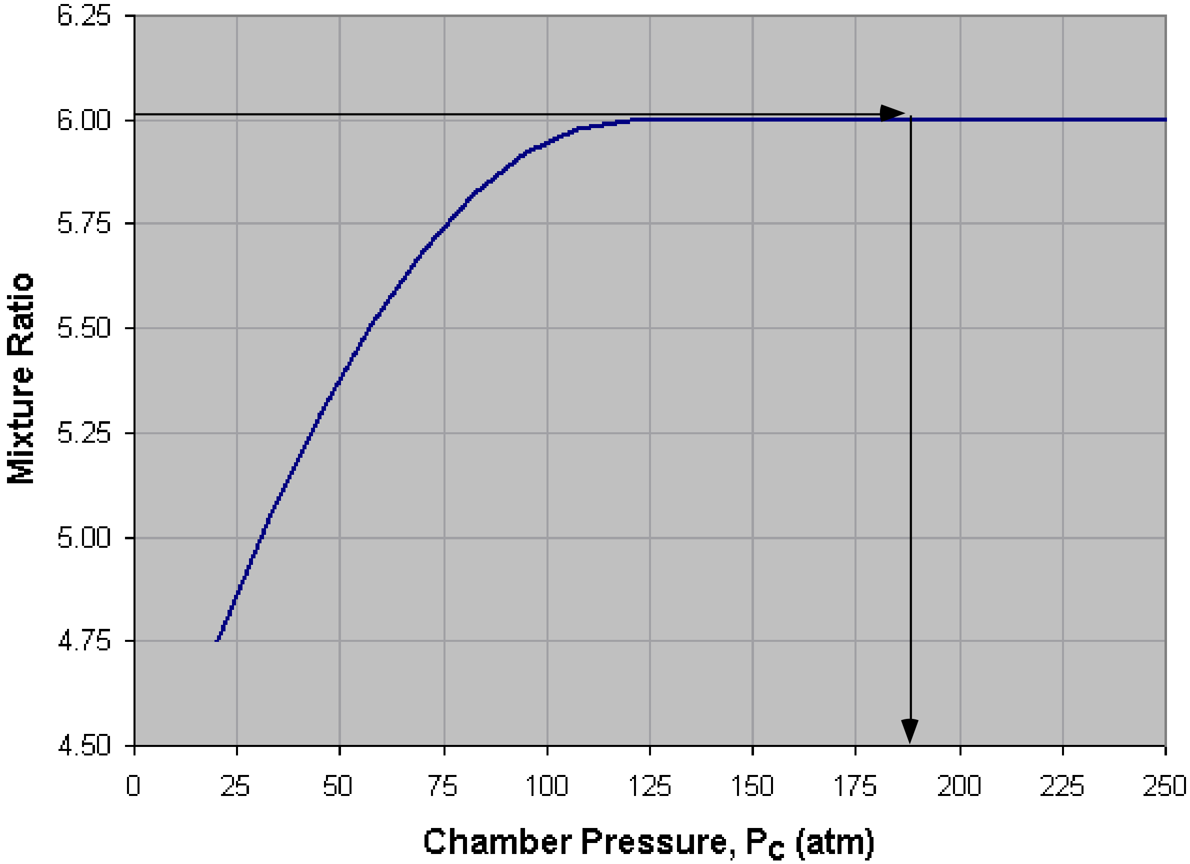
$$MR = 6.000 \rightarrow 721{,}000_{kg}\left[\frac{6.000}{1140_{kg/m^3}} + \frac{1}{67.78_{kg/m^3}}\right] = 14432m^3$$

$$MR = 3.5 \rightarrow 721{,}000_{kg}\left[\frac{6.000}{1140_{kg/m^3}} + \frac{1}{67.78_{kg/m^3}}\right] = 12850m^3$$

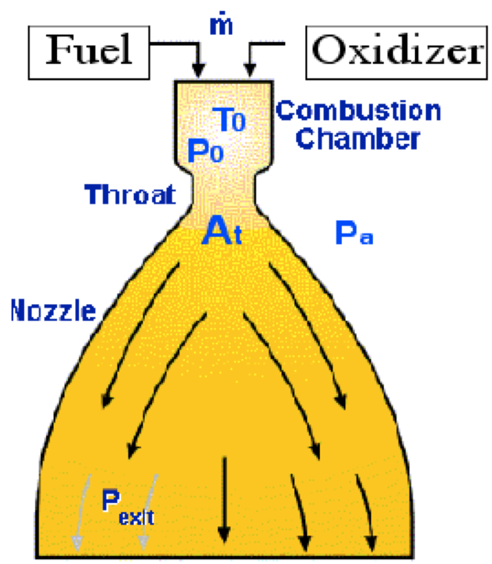
SSME Computational Example (cont'd)

- Space Shuttle Main Engine …

- LOX/LH2 Propellants, 6.03: 1 Mixture ratio



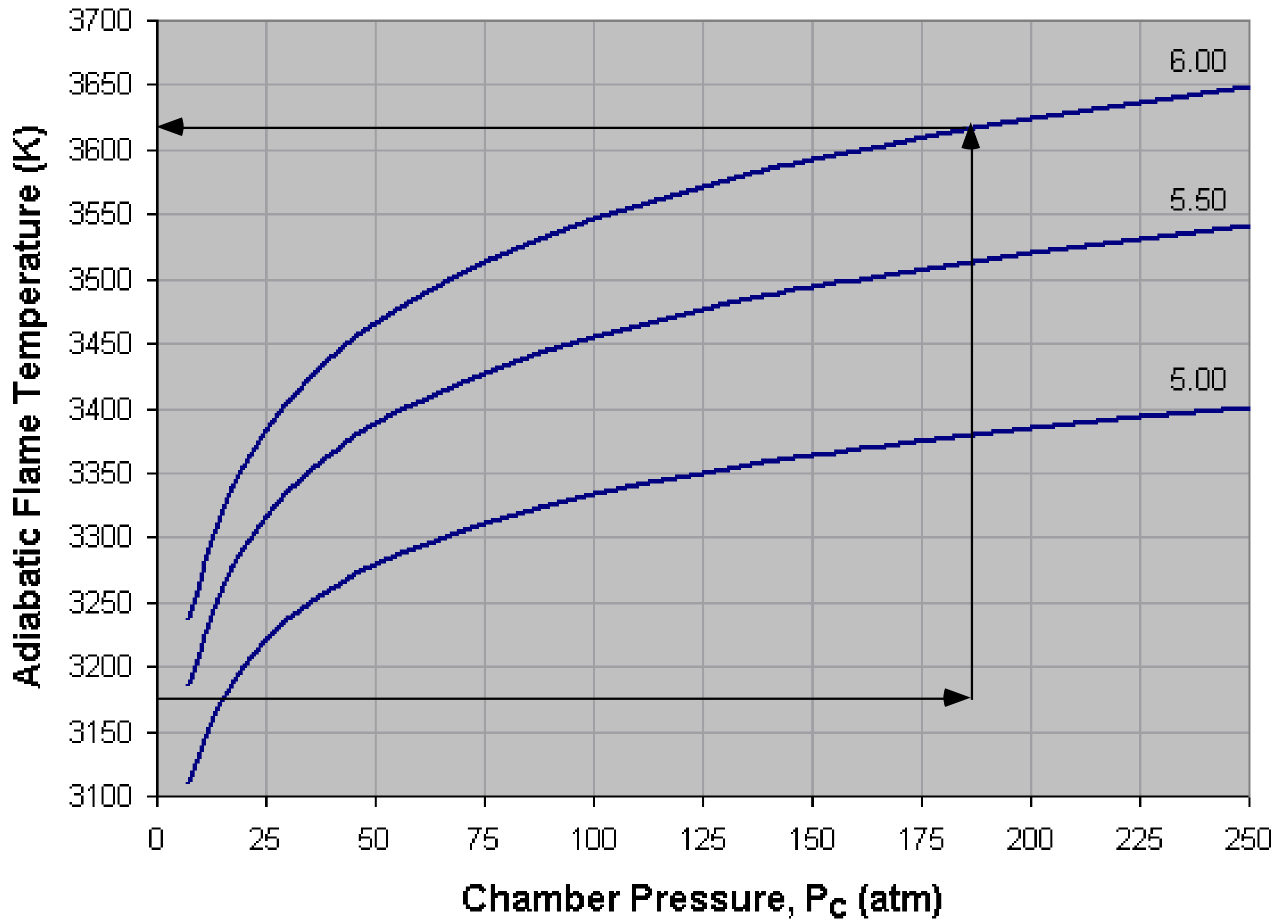
P_0=186.92 atm
=18940 Kpa



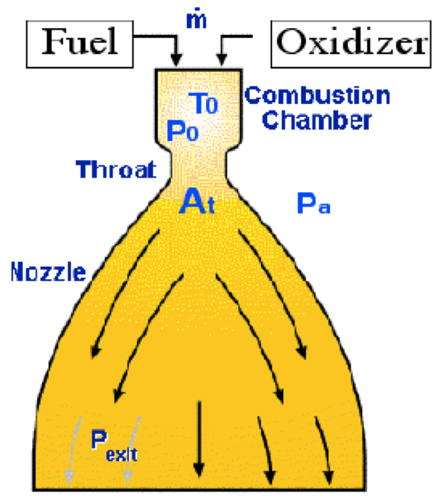
Ref: http://www.braeunig.us/space/

SSME Computational Example (cont'd)

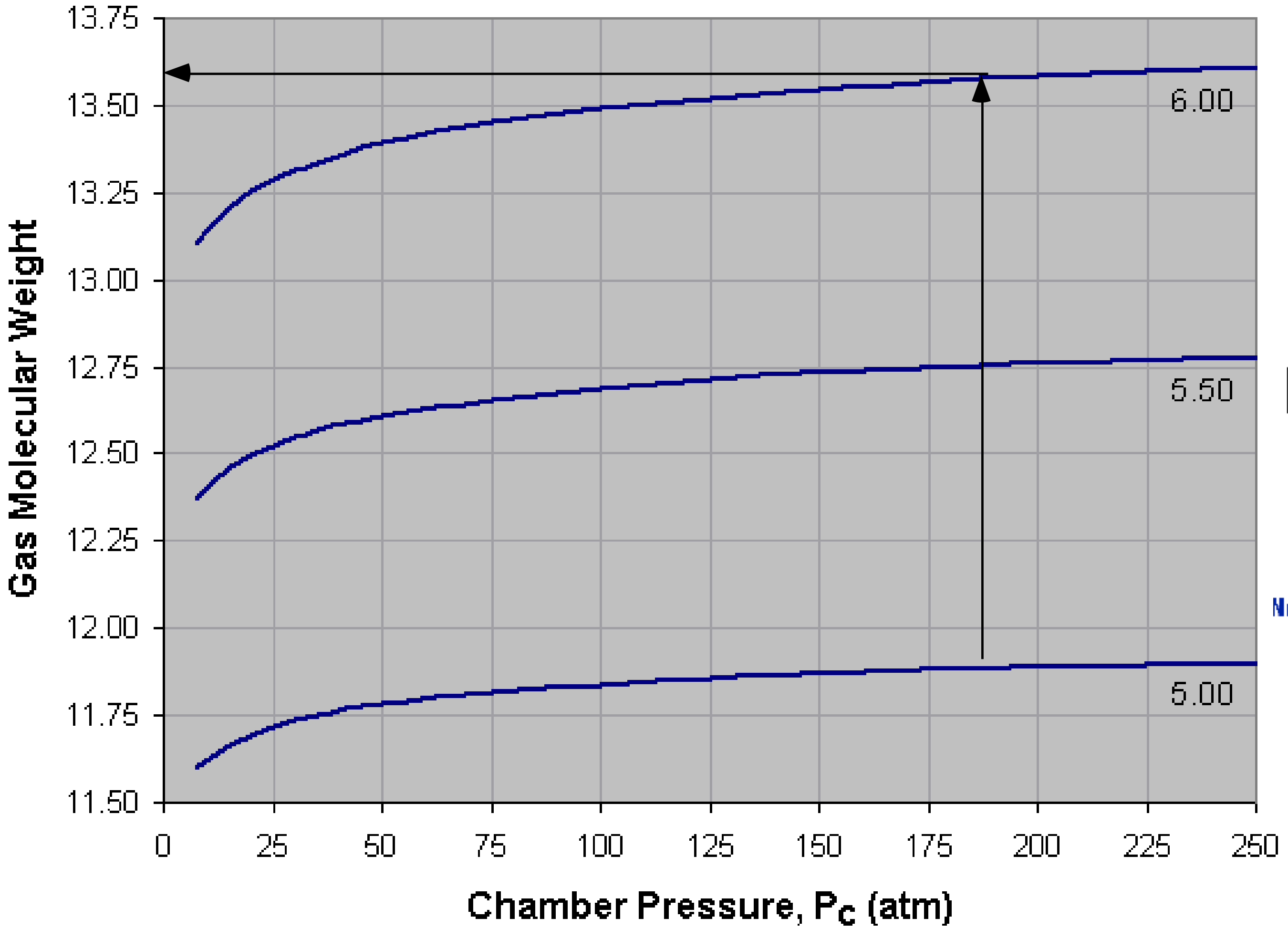
- Space Shuttle Main Engine …



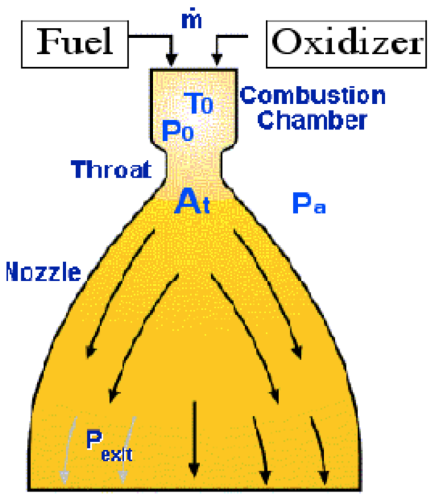
$T_0 \sim$ 3615°K



SSME Computational Example (cont'd)

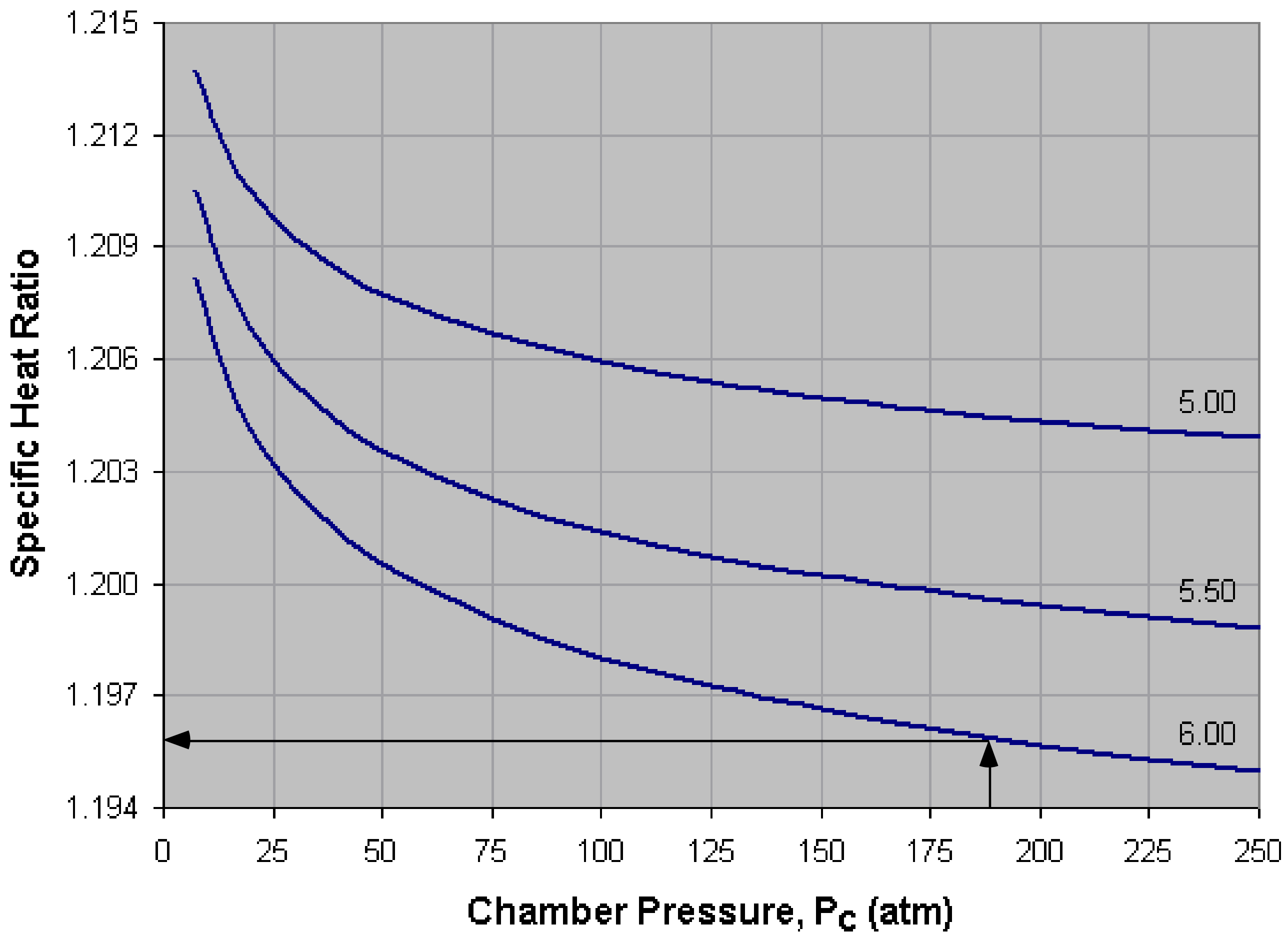


$M_w \sim$ 13.6 kg/kg-mol

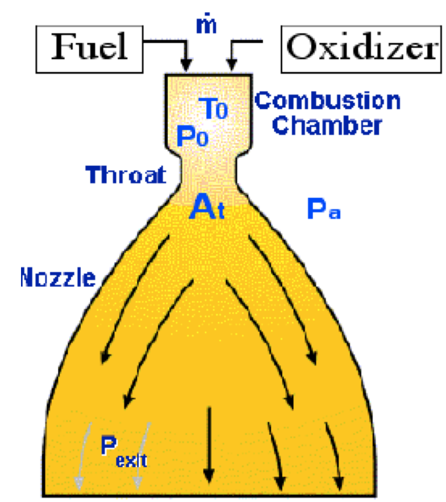


SSME Computational Example (cont'd)

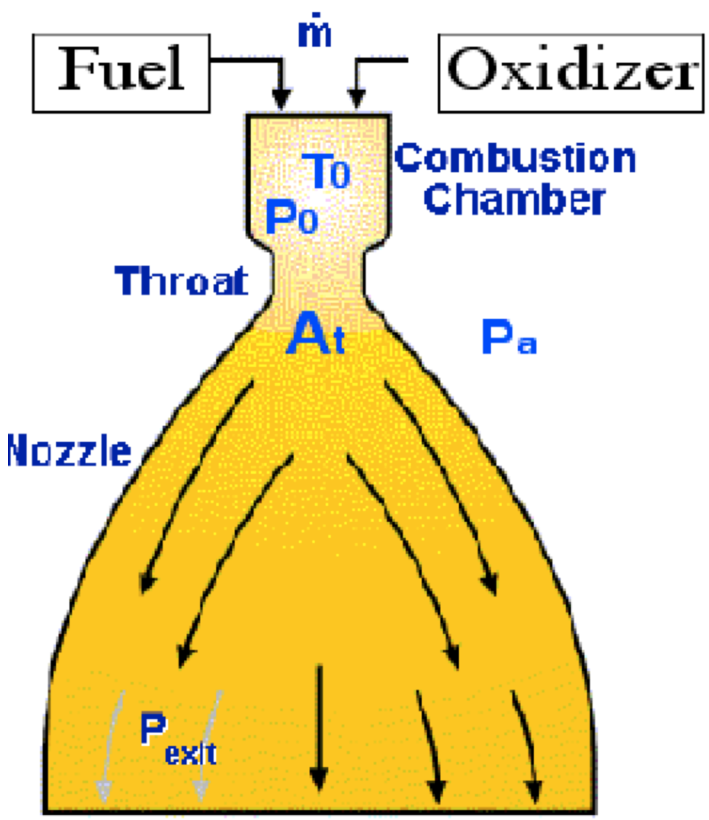
- Space Shuttle Main Engine …



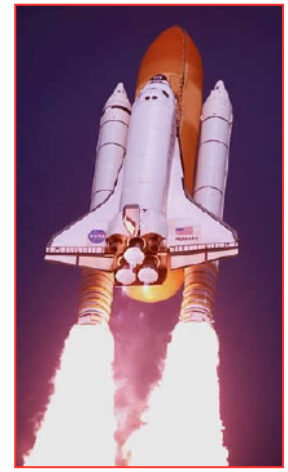
$\gamma \sim$
1.196



Example: SSME Rocket Engine



- The Space Shuttle Main Engines Burn LOX/LH2 for Propellants with A ratio of LOX:LH2 =6:1

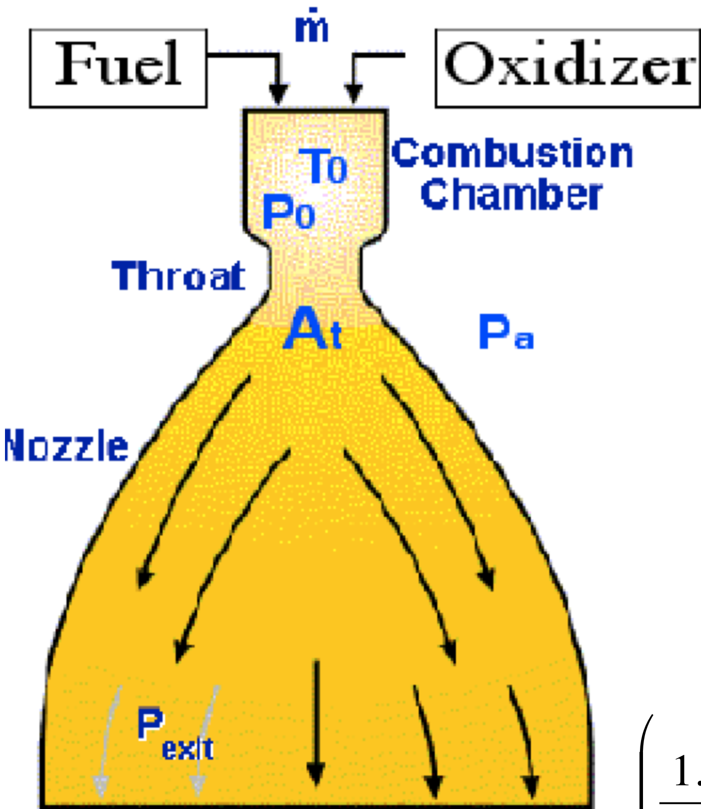


- The Combustor Pressure, p_0 Is 18.94 Mpa, combustor temperature, T_0 is 3615°K, throat diameter is 26.0 cm

- What propellant mass flow rate is required for choked flow in the Nozzle?

- Assume no heat transfer Thru Nozzle no frictional losses, γ=1.196

Example: SSME Rocket Engine (cont'd)



-- By product ~ Burns rich, byproduct is water vapor + GH_2

$M_W \sim 13.6$ kg/kg-mole

-- $R_g = 8314.4126 / 13.6 = 611.35$ J/°K-kg

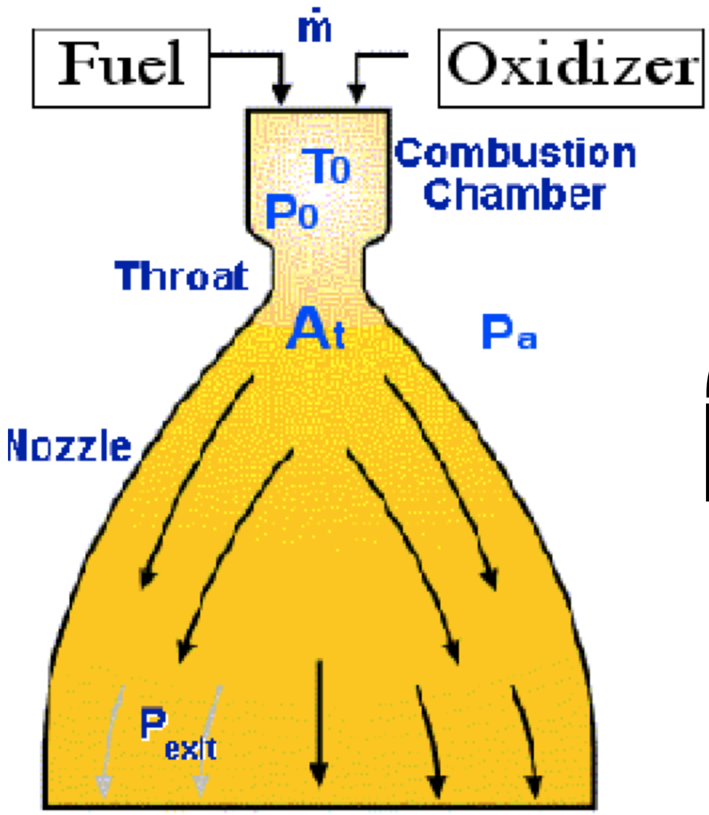
$$\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R_g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\frac{p_0}{\sqrt{T_0}} =$$

$$\left(\frac{1.196}{611.35}\left(\frac{2}{1.196+1}\right)^{\frac{(1.196+1)}{1.196-1}}\right)^{0.5}\frac{18.94\cdot 10^6}{(3615)^{0.5}}$$

$= 8252.59$ kg/sec-m^2

Example: SSME Rocket Engine (cont'd)

Massflow rate



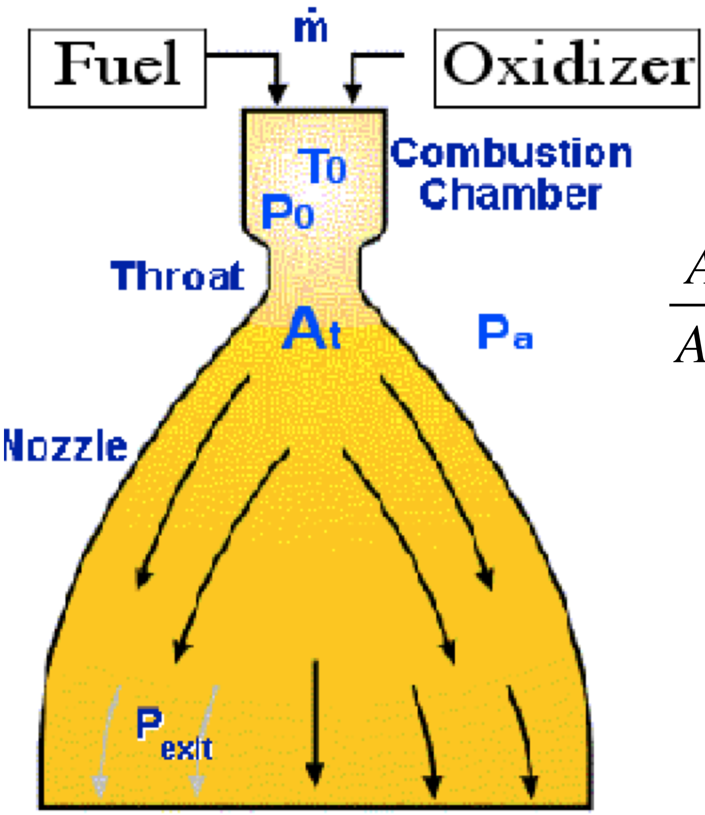
- Compute Throat Area

$$\left(\frac{26}{100}\right)^2 \frac{\pi}{4} = 0.05297 \ m^2$$

- Mass flow =

$$\left(\frac{\dot{m}}{A^*}\right) \times A^* = 8252.59 \cdot 0.05297 = 437.1 \text{ kg/sec}$$

Example: SSME Rocket Engine (concluded)



- The nozzle expansion ratio is 77.5 -- what is the exit mach number

$$\frac{A}{A^*} = 77.5 = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

- Non -linear function of mach number

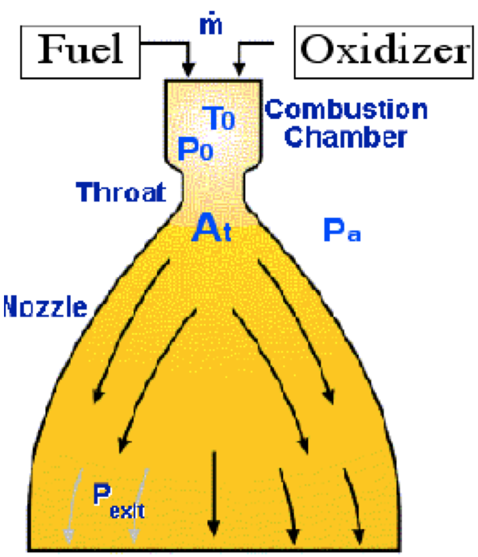
- Solution methods
  - i) Plot A/A^* versus mach
  - ii) Numerical Solution

Example: SSME Rocket Engine (cont'd)

Compute Exit Mach Number

Expansion ratio = 77.5

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} =$$



$$\frac{\left(\left(\frac{2}{1.196+1}\right)\left(1+\frac{1.196-1}{2}\left(4.677084^2\right)\right)\right)^{\frac{1.196+1}{2\,(1.196-1)}}}{4.677084}$$

$= 77.49998$ ----> $M_{exit} = 4.677084$

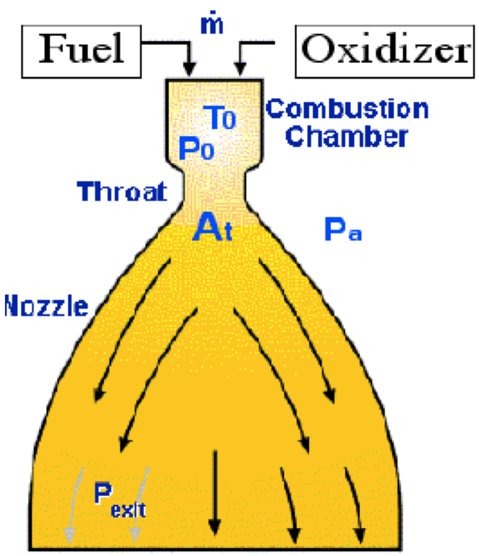
Newton Solver comes in handy here!

Example: SSME Rocket Engine (cont'd)

Compute Exit Temperature

$M_{exit} = 4.677084$

$$\frac{T_0}{T} = 1 + \frac{(\gamma - 1)}{2} M^2 \longrightarrow$$



$$T_{exit} = \frac{T_0}{1 + \frac{(\gamma - 1)}{2} M_{exit}^2} =$$

$$3615\left(1 + \frac{1.196 - 1}{2}\left(4.677084^2\right)\right)^{-1} = 1149.90\ °K$$

Example: SSME Rocket Engine (cont'd)

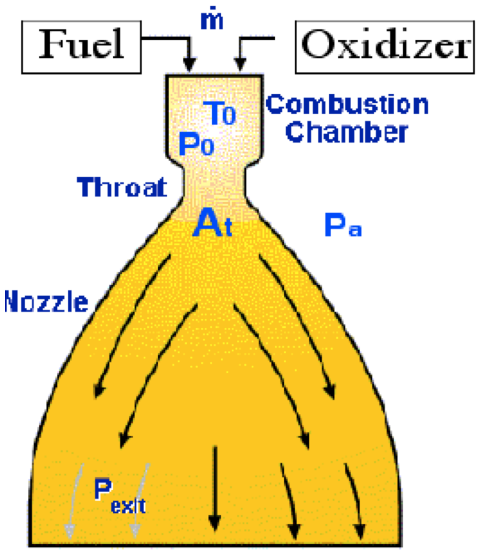
Compute Exit Velocity

$M_{exit} = 4.677084$

$$V_{exit} = M_{exit}\sqrt{\gamma R_g T_{exit}} =$$

$$4.677084\,(1.196 \cdot 611.35 \cdot 1149.9)^{0.5}$$

$$= 4288.61 \text{ m/sec}$$

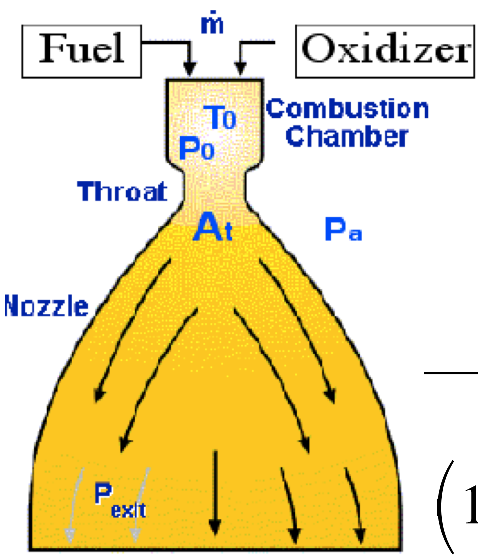


Example: SSME Rocket Engine (cont'd)

Compute Exit Pressure

$M_{exit} = 4.677084$

$$P_{exit} = \frac{P_0}{\left(1 + \frac{(\gamma - 1)}{2} M_{exit}^{2}\right)^{\frac{\gamma}{\gamma - 1}}} =$$

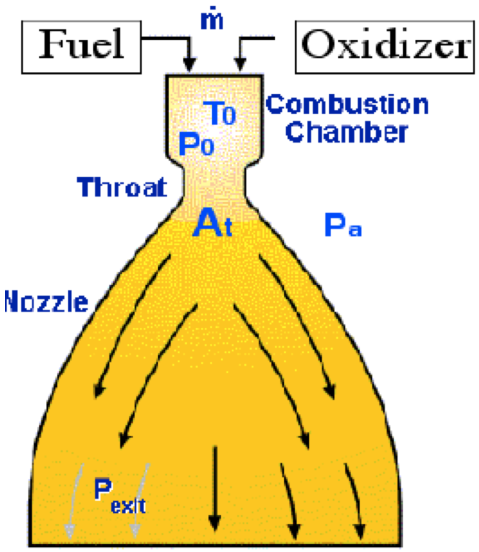


$$\frac{18.94 \cdot 10^{6}}{\left(1 + \frac{1.196 - 1}{2}(4.677084^{2})\right)^{\left(\frac{1.196}{1.196 - 1}\right)}} = 17.45511 \text{ kPa}$$

Example: SSME Rocket Engine (cont'd)

Compute C*

$$\rightarrow C^* = \frac{\sqrt{\gamma R_g T_0}}{\gamma \sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}} =$$

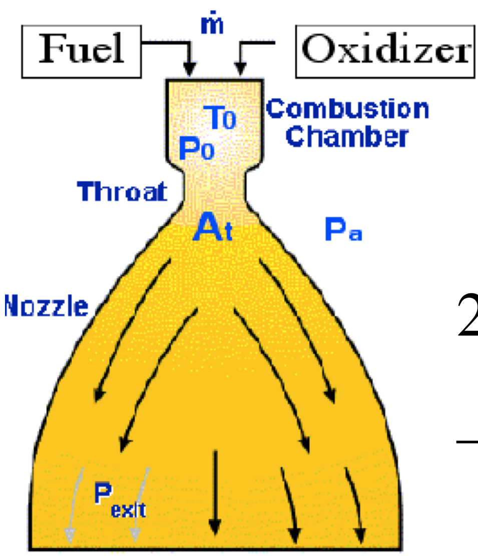


$$\frac{(1.196 \cdot 611.35 \cdot 3616)^{0.5}}{1.196\left(\left(\frac{2}{1.196+1}\right)^{\left(\frac{1.196+1}{1.196-1}\right)}\right)^{0.5}} = 2295.35 \text{ m/sec}$$

Example: SSME Rocket Engine (cont'd)

Compute Idealized I_{sp}

$$\left(I_{sp}\right)_{ideal} = \frac{C^*}{g_o}\left[\gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}}\right] =$$



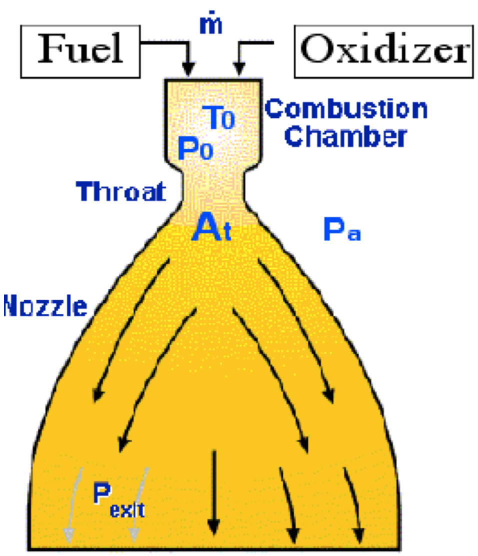
$$\frac{2295.35\cdot 1.196\left(\frac{2}{1.196-1}\left(\frac{2}{1.196+1}\right)^{\left(\frac{1.196+1}{1.196-1}\right)}\right)^{0.5}}{9.806}$$

$$= 529.69 \text{ sec}$$

Example: SSME Rocket Engine (cont'd)

Compute Idealized Thrust

$$Thrust_{ideal} = \dot{m}\left(I_{sp}\right)_{ideal} g_0 =$$

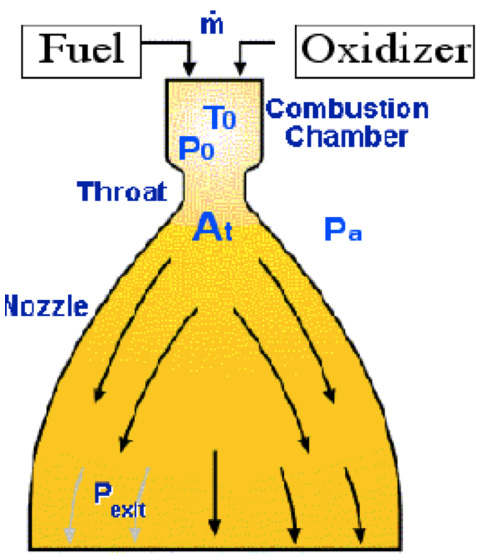


$$\frac{529.69 \cdot 437.14 \cdot 9.806}{10^6} = 2.271 \text{ mNt}$$

Example: SSME Rocket Engine (cont'd)

Compute Effective Exhaust Velocity (Vacuum)

$$C_e = \frac{Thrust}{\dot{m}} = V_{exit} + \frac{A_{exit}}{A^*} A^* \frac{(p_{exit} - p_\infty)}{\dot{m}} =$$



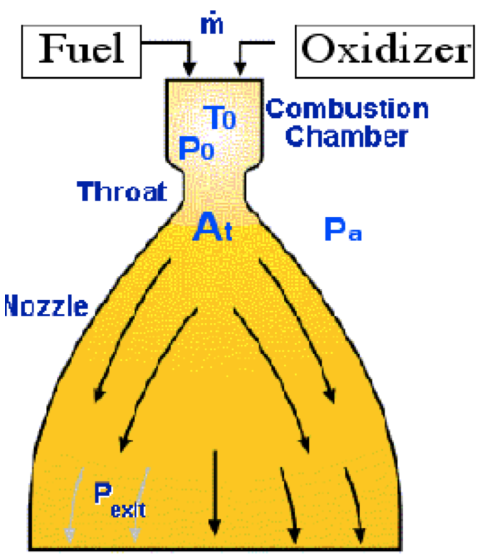
$$4288.61 + \frac{77.5 \cdot 0.0529708\,(17.455 \cdot 1000)}{437.14}$$

$$= 4452.53 \text{ m/sec}$$

Example: SSME Rocket Engine (cont'd)

Compute Thrust (Vacuum)

$$Thrust = \dot{m} C_e =$$

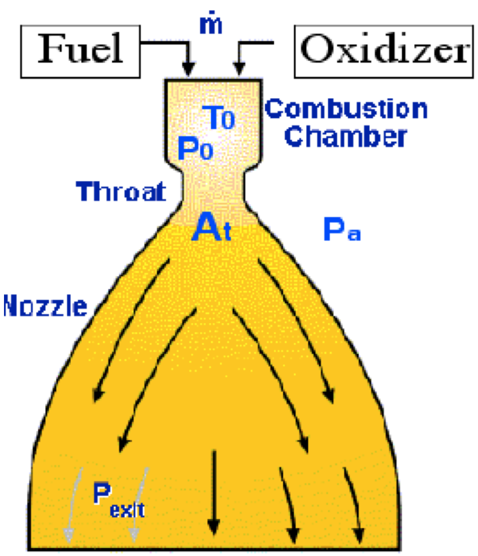


$$\frac{437.14 \cdot 4452.53}{10^6} = 1.9464 \text{ mNt}$$

Example: SSME Rocket Engine (cont'd)

Compute True I_{sp} (Vacuum) (ignore nozzle Losses)

$$I_{sp} = \frac{C_e}{g_0} =$$

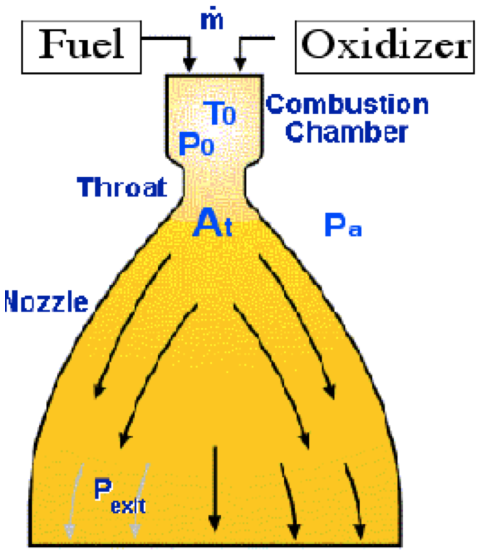


$$\frac{4803.891}{9.806} = 454.06 \text{ sec}$$

Example: SSME Rocket Engine (cont'd)

Compute Effective Exhaust Velocity (Sea level)

$$C_e = \frac{Thrust}{\dot{m}} = V_{exit} + \frac{A_{exit}}{A^*} A^* \frac{(p_{exit} - p_\infty)}{\dot{m}} =$$



$$4288.61 + \frac{77.5 \cdot 0.0529708\,(17.455 \cdot 1000 - 101325)}{437.14}$$

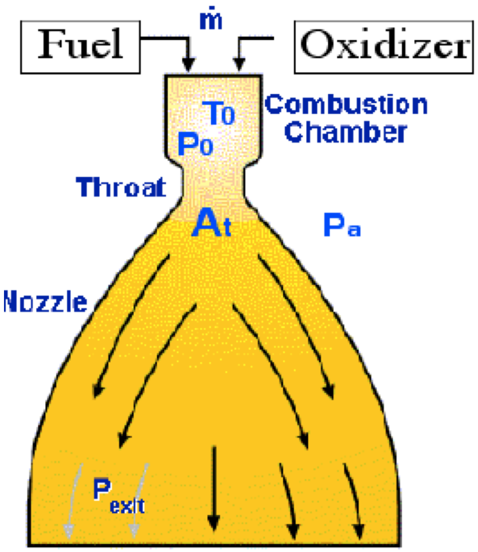
$$= 3500.98 \text{ m/sec}$$

P sea Level =101.325 kPa

Example: SSME Rocket Engine (cont'd)

Compute Thrust (Seal level) (ignore nozzle Losses)

$$Thrust = \dot{m} C_e =$$

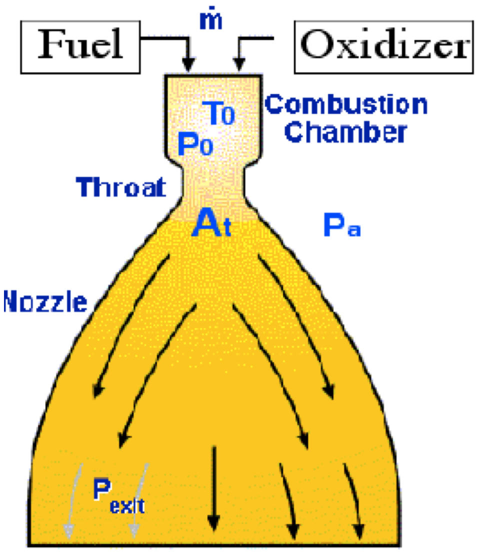


$$\frac{437.14 \cdot 3500.976}{10^6} = 1.540 \text{ mNt}$$

Example: SSME Rocket Engine (cont'd)

Compute True I_{sp} (Seal level) (ignore nozzle Losses)

$$I_{sp} = \frac{C_e}{g_0} =$$



$$\frac{3500.976}{9.806} = 357.024 \text{ sec}$$

Example: SSME Rocket Engine (cont'd)

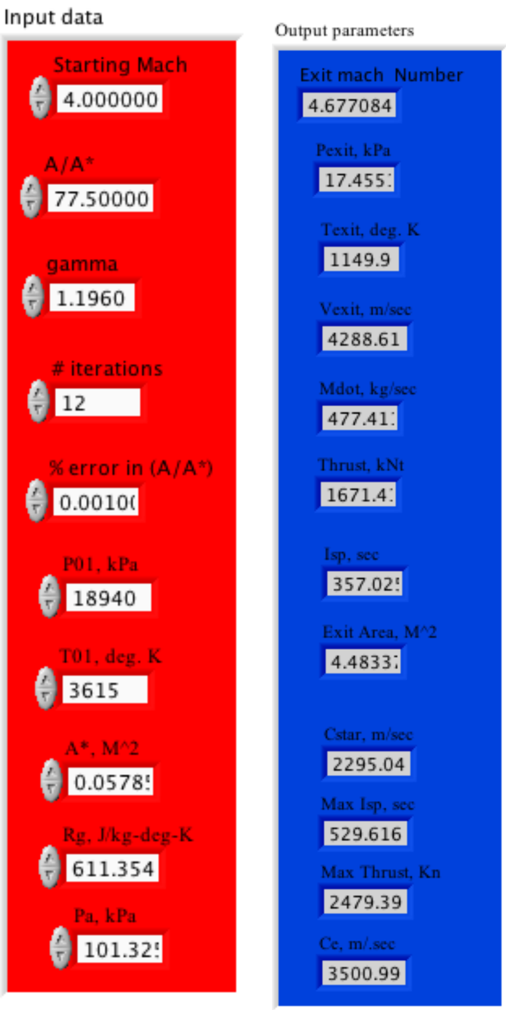
Summary:

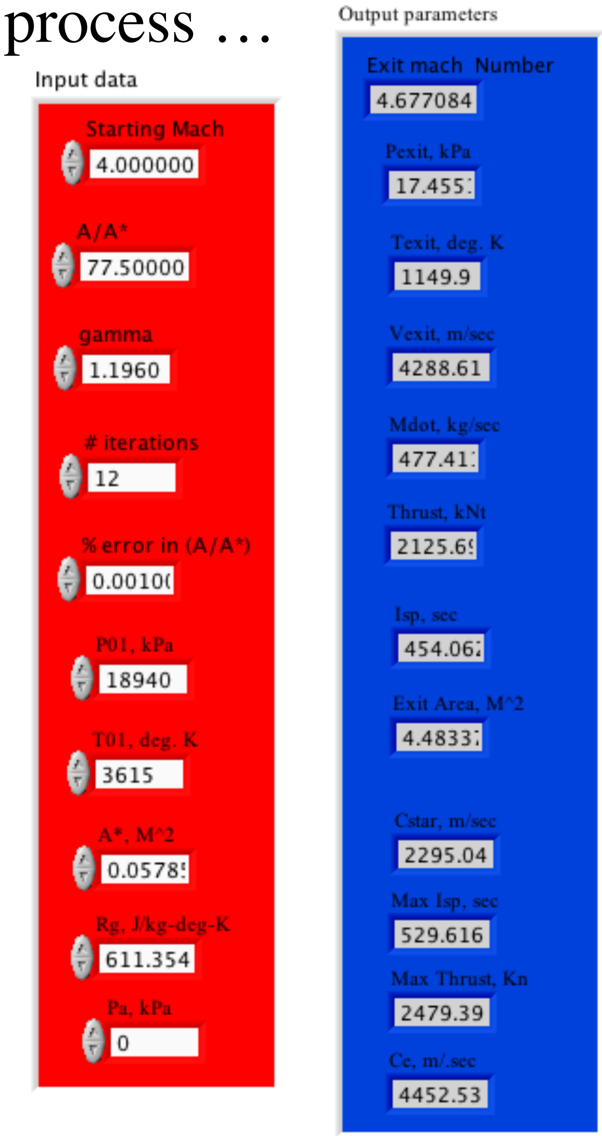
	Ideal	Calc. Vac.	Calc. S.L.	Actual Vac.	Actual S.L
I_{sp} (sec):	529.69	454.06	357.03	452.5	363
Thrust: (mNt)	2.271	1946.37	1530.42	2.10	1.67

• Obviously Out estimate of throat area is a bit small ….
… but you get the point …

Example: SSME Rocket Engine (cont'd)

- When we automate the process …



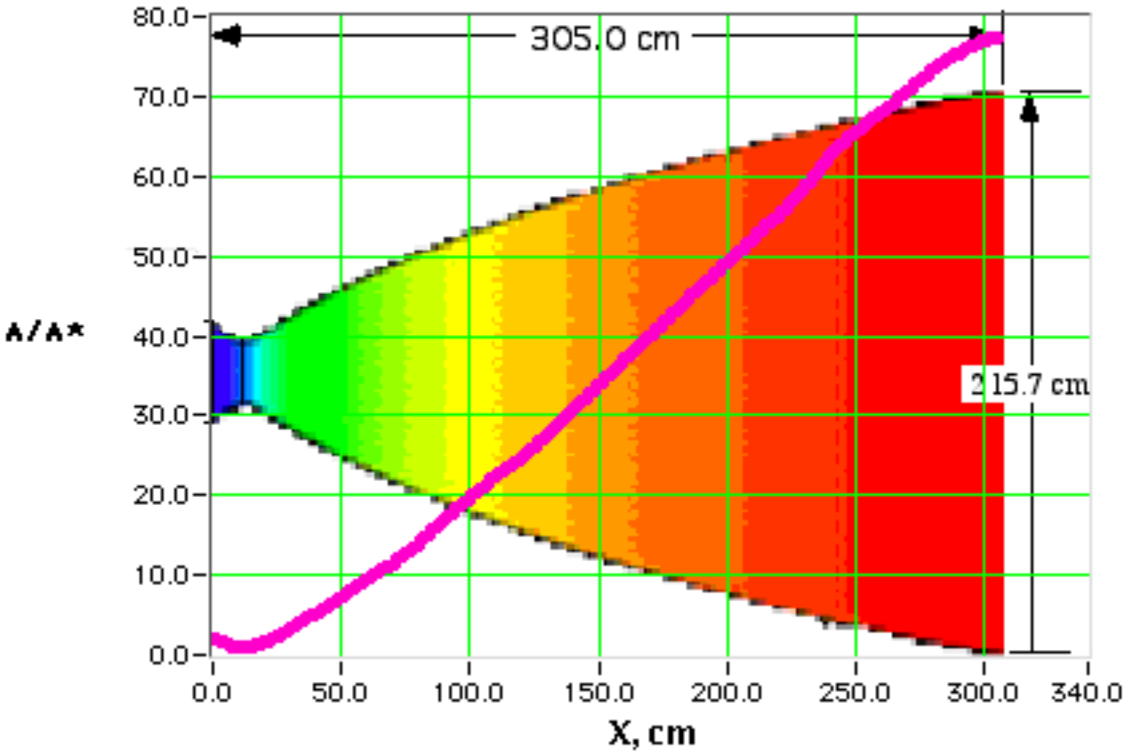


… It appears that $A^* \sim 0.05785$

… or a Throat diameter Of ~ 27.2 cm!

Plot Flow Properties Along SSME Nozzle Length

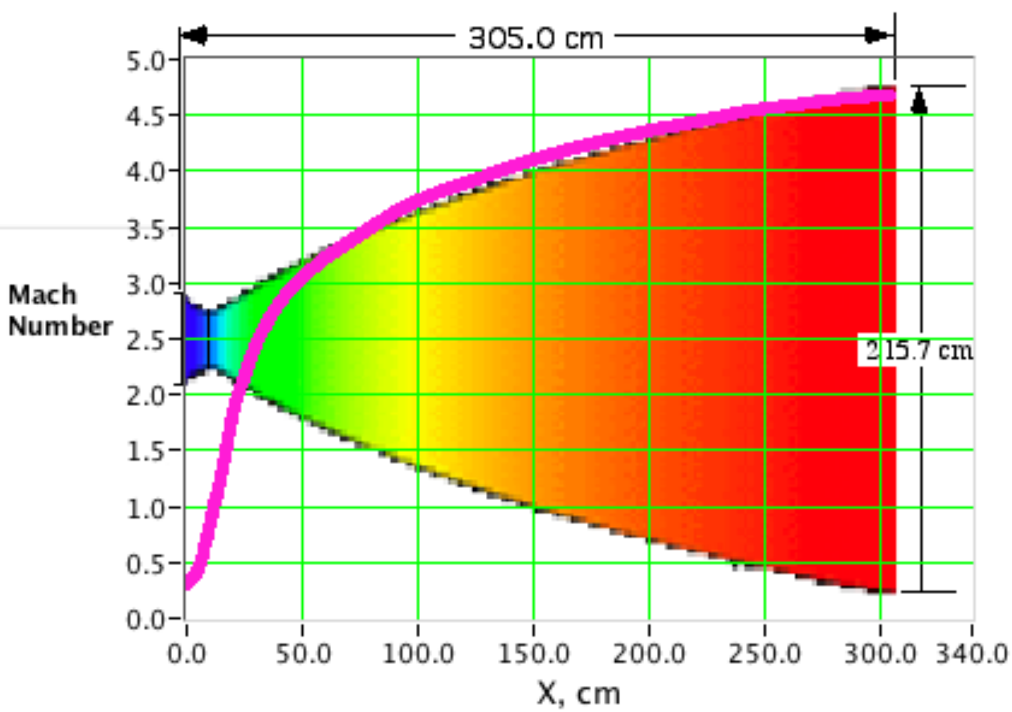
- A/A*



Plot Flow Properties Along SSME Nozzle Length (cont'd)

- Mach Number

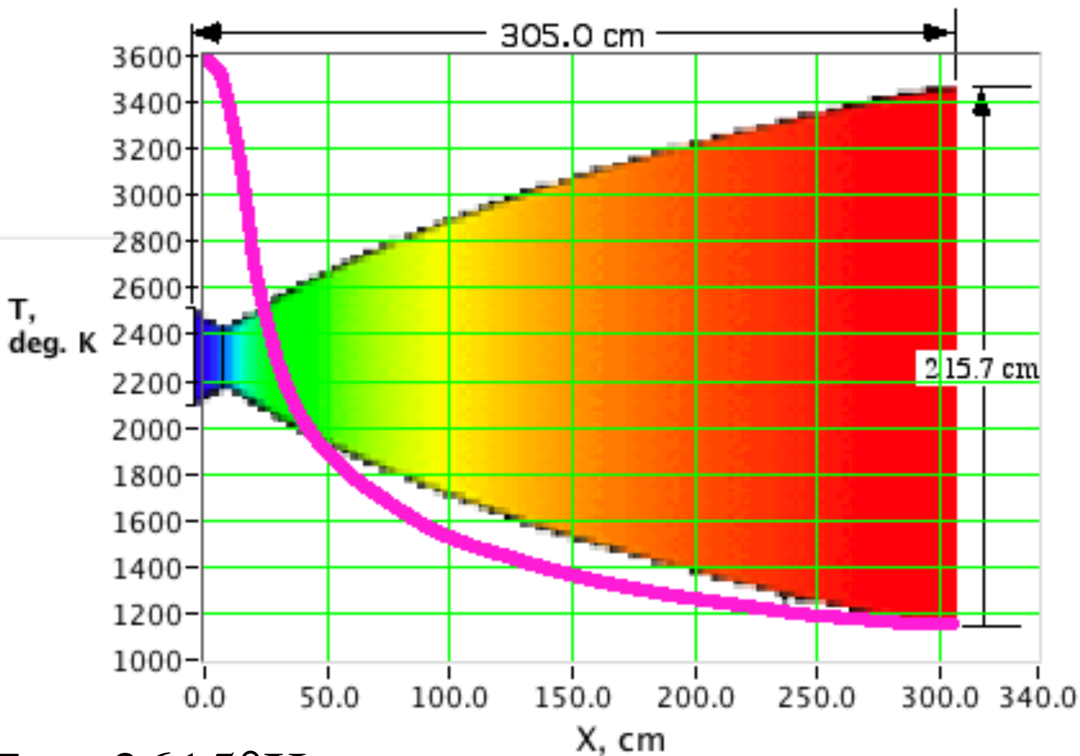
$$\hat{M}_{(j+1)} = \hat{M}_{(j)} - \frac{F(\hat{M}_{(j)})}{\left(\frac{\partial F}{\partial M}\right)_{|(j)}}$$



Plot Flow Properties Along SSME Nozzle Length (cont'd)

- Temperature

$$T(x) = \frac{T_0}{1 + \frac{\gamma - 1}{2} M(x)^2}$$



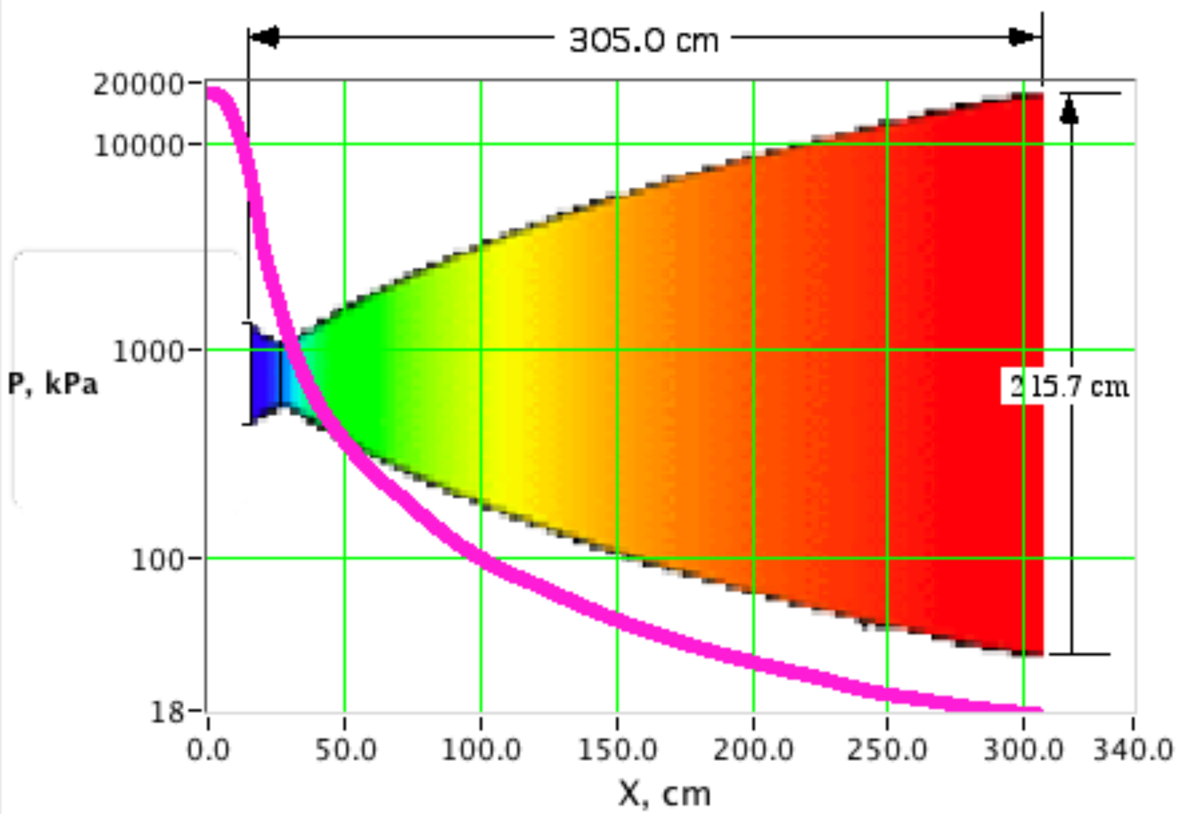
$T_0 = 3615°K$

$T_{throat} = 3292.3 °K$

Plot Flow Properties Along SSME Nozzle Length (concluded)

$$\frac{P_0}{\left(1 + \frac{\gamma - 1}{2} M(x)^2\right)^{\frac{\gamma}{\gamma - 1}}}$$

- Pressure



$P_0 = 18.94$ Mpa

$P_{throat} = 11.71$ MPa