Section 5, Lecture 1:

Thermodynamics and Compressible Flow Review

Summary of Fundamental Properties and Relationships

Sutton and Biblarz, Chapter 3

MAE 5540 - Propulsion Systems
Equation of State for a Perfect Gas

• Relationship Between pressure, temperature, and density derived empirically in Modern form by John Dalton

• Theoretically derived by Ludwig Boltzmann using statistical Thermodynamics

• In perfect gas … intermolecular (van der Waals) forces are neglected

\[ p \, v = n \, R_u \, T \]

- \( p \) - pressure acting on gas
- \( v \) - volume of gas in system
- \( n \) - Number of moles of gas in system
- \( R_u \) - Universal gas constant
- \( T \) - Temperature of gas

John Dalton

\[ 1 \text{-mole} \rightarrow 6.02 \times 10^{23} \]

Avagadro's number
Equation of State for a Perfect Gas (cont’d)

• Re organizing the equation of state

\[ p = \frac{M}{V} \frac{n}{M} R_u T = \rho \frac{R_u}{M/n} T = \rho \frac{R_u}{M_w} T = \rho R_g T \]

• \( p \) - pressure acting on gas
• \( v \) - volume of gas in system
• \( n \) - Number of moles of gas in system
• \( R_u \) - Universal gas constant
• \( T \) - Temperature of gas
• \( M_w \) - Molecular weight of gas
• \( R_g \) - Gas Specific Constant
• \( M \) - Mass of gas contained in volume

• Useful working form for Gas Dynamics

\[ p = \rho \frac{R_u}{M_w} T = \rho R_g T \Rightarrow R_g = \frac{R_u}{M_w} \]
Equation of State for a Perfect Gas (cont’d)

• Numerical Values for Universal Gas Constant

\[ R_u = 1545.40 \text{ ft-lbf/}^\circ\text{R-(lbm-mole)} \]
\[ R_u = 49722.01 \text{ ft-lbf/}^\circ\text{R-(slug-mole)} \]
\[ R_u = 8314.4126 \text{ J/}^\circ\text{K-(kg-mole)} \]

• Molecular weight of various gases

<table>
<thead>
<tr>
<th>Gas</th>
<th>Molecular Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>2.0</td>
</tr>
<tr>
<td>Helium</td>
<td>4.0</td>
</tr>
<tr>
<td>Methane</td>
<td>16.0</td>
</tr>
<tr>
<td>Ammonia</td>
<td>17.0</td>
</tr>
<tr>
<td>Water</td>
<td>18.0</td>
</tr>
<tr>
<td>Neon</td>
<td>20.2</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>28.0</td>
</tr>
<tr>
<td>Air</td>
<td>28.9</td>
</tr>
<tr>
<td>Oxygen</td>
<td>32.0</td>
</tr>
<tr>
<td>Argon</td>
<td>39.9</td>
</tr>
<tr>
<td>CO2</td>
<td>44.0</td>
</tr>
</tbody>
</table>

• Gas Specific constant is Universal constant divided by the average molecular weight of the gas
Equation of State for a Perfect Gas (concluded)

• Molecular weight of Air

Average molecular weight of the gases in the atmosphere.
Air on earth at sea level is a mixture of approximately 78% nitrogen, 21% oxygen, with the remaining one percent a mix of argon, carbon dioxide, neon, helium and other rare gases,

~ 28.96443 kg/kg-mole

• Numerical Values for Air Specific Gas Constant

\[ R_g = 53.355 \text{ ft-lbf/°R-(lbm)} \]
\[ R_g = 1716.658 \text{ ft-lbf/°R-(slug)} \]
\[ R_g = 287.056 \text{ J/°K-(kg)} \]
Specific Heats, Internal Energy, and Enthalpy
(cont’d)

• \( e \) is the internal energy per unit mass
  The total kinetic and potential energy associated with the motions and relative
  positions of the molecules a gas, solid, or liquid. An increase in internal
  energy results in a rise in temperature or a change in phase

• \( h \) is the enthalpy per unit mass
  The measure of the heat content of a chemical or physical system;
  equivalent to the sum of the internal energy of the system plus the
  product of its volume multiplied by the pressure exerted on it by its
  Surroundings … i.e.

\[
h = e + p \, v
\]

\( h \) is the specific enthalpy, (SI - joules/kg)
\( e \) is the specific internal energy, (SI - joules.kg)
\( P \) is the pressure of the system, (SI - pascals)
\( v \) is the specific volume, (SI cubic metres/kg) = 1/\( \rho \)
Specific Heats, Internal Energy, and Enthalpy

• Specific heat definitions: (cont’d)
  - Constant Volume:
    \[ c_v = \left( \frac{de}{dT} \right)_v \]
  - Constant Pressure:
    \[ c_p = \left( \frac{dh}{dT} \right)_p = \frac{d}{dT}(e + pv) = \]
    \[ c_p = \left( \frac{dh}{dt} \right)_p = \frac{d}{dT}(e + pv) = \]
    \[ c_p = \left( \frac{dh}{dT} \right)_p + \frac{d}{dT}(R_g T) = c_v + R_g \]

Universal relationship applies to all perfect gases
First Law of Thermodynamics

- Change in internal Energy = heat added + work performed + Energy dissipated

\[ de = dq + dw_{rev} - dw_{irrev} \]
\[ dw_{rev} = Fdx = pA_c dx = -p\,dv \]
\[ dW_{irrev} = F_f dx \]
First Law of Thermodynamics (continued)

• Recall from earlier lecture

1) Adiabatic Process… no heat loss or addition
2) Reversible Process … no dissipative phenomena occur
3) Reversible adiabatic process is referred to as isentropic

\[ dq = 0 \text{ (adiabatic process)} \]

\[ de = -(pdv + F_f dx) \]

\[ F_f = 0 \text{ (isentropic process)} \]

\[ de = -pdv \]
First Law of Thermodynamics (concluded)

• In terms of Enthalpy
  - reversible process

\[ de = dq - pdv \]

\[ h = e + pv \implies dh = de + vdp + pdv \]

\[ dh = dq - pdv + vdp + pdv = dq + vdp \]

  - isentropic process (reversible, adiabatic)

\[ dh = vdp \]
Second Law of Thermodynamics (cont’d)

• Thermodynamic entropy, $s$, is a measure of the amount of energy in a physical system that cannot be used to do work.

• It is a relative measure of the disorder present in a system. (SI unit of entropy is J/°K)

\[ ds = \frac{dq}{T} + ds_{irrev} \]

Reversible heat addition  Effect of dissipative processes

• Calculate change in entropy:

First Law: \[ de = dq - p \, dv - dw_{irrev} \]

Second Law: \[ Tds = \delta q + ds_{irrev} \]
Second Law of Thermodynamics (cont’d)

For a reversible process

\[ de = dq - pdv \]

\[ Tds = \delta q \]

Writing \( e \) in terms of enthalpy \((h = e + p\, v)\)

\[ dh = de + pdv + vdp \]

and

\[ Tds = de + pdv \]

\[ dh = Tds - pdv + pdv + vdp = \]

\[ dh = Tds + vdp \]
Solving for $ds$ and integrating between state 1 and state 2

$$s_2 - s_1 = \int_1^2 \left[ \frac{dh}{T} - \frac{\nu dp}{T} \right]$$

But from definition

$$C_p = \left( \frac{dh}{dT} \right)_p \quad \Rightarrow \quad s_2 - s_1 = \int_1^2 \left[ \frac{C_p dT}{T} - \frac{\nu dp}{T} \right]$$

From the equation of state

$$p = \rho R_g T = \frac{1}{\nu} R_g T$$

$$\nu = \frac{R_g T}{p}$$
Second Law of Thermodynamics (concluded)

- Substituting equation of state into entropy integral

\[
S_2 - S_1 = \int_1^2 \left[ \frac{c_p dT}{T} - \frac{v dp}{T} \right] = \int_1^2 \left[ \frac{c_p dT}{T} - \frac{R_g T}{p} dp \right] = \\
\int_1^2 \left[ \frac{c_p dT}{T} - R_g \frac{dp}{p} \right] \text{ Valid for thermally perfect gas}
\]

- For a thermally perfect gas, \( c_p \sim \text{constant} \)

\[
S_2 - S_1 = c_p \ln \left[ \frac{T_2}{T_1} \right] - R_g \ln \left[ \frac{p_2}{p_1} \right] \text{ Valid for adiabatic process}
\]
Isentropic Process Relationships

• For an isentropic process, (adiabatic, reversible) \( dq = 0 \)

\[
ds = \frac{dq}{T} + ds_{irrev} = 0
\]

and

\[
c_p \ln_2 \left( \frac{T_2}{T_1} \right) - R_g \ln \left( \frac{p_2}{p_1} \right) = \left[ \frac{T_2}{T_1} \right]^{c_p} = \left[ \frac{p_2}{p_1} \right]^{R_g}
\]

Solving for \( p_2 \) in terms of \( T_2 \)

\[
\frac{p_2}{p_1} = \left[ \frac{T_2}{T_1} \right]^{c_p} = \left[ \frac{T_2}{T_1} \right]^{\frac{c_p}{c_p-c_v}}
\]
Isentropic Process Relationships (cont’d)

• Defining a ratio of specific heats, $\gamma = \frac{c_p}{c_v}$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma - 1}}$$

• Applying the equation of state

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \frac{\rho_1}{\rho_2}$$

• and

$$\left[\frac{p_2}{p_1}\right]^{1 - \frac{\gamma}{\gamma - 1}} = \left[\frac{p_2}{p_1}\right]^{-\frac{1}{\gamma - 1}} = \left[\frac{\rho_2}{\rho_1}\right]^{-\frac{\gamma}{\gamma - 1}} \Rightarrow \left[\frac{p_2}{p_1}\right] = \left[\frac{\rho_2}{\rho_1}\right]^{\gamma}$$
Ratio of Specific Heats

• $\gamma = c_p/c_v$ is a critical parameter for compressible flow analysis

Approximate Specific Heat Ratio for Various Gases, at moderate temperatures

<table>
<thead>
<tr>
<th>Gas</th>
<th>Ratio of Specific Heats</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbon Dioxide</td>
<td>1.3</td>
</tr>
<tr>
<td>Helium</td>
<td>1.66</td>
</tr>
<tr>
<td>Hydrogen</td>
<td>1.41</td>
</tr>
<tr>
<td>Methane or Natural Gas</td>
<td>1.31</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>1.4</td>
</tr>
<tr>
<td>Oxygen</td>
<td>1.4</td>
</tr>
<tr>
<td>Standard Air</td>
<td>1.4</td>
</tr>
</tbody>
</table>
Ratio of Specific Heats (cont’d)

• Air at Room Temperature -->

\[
R_g = c_p - c_v = c_p \left(1 - \frac{1}{\gamma}\right) \rightarrow c_p = \frac{\gamma}{\gamma - 1} R_g
\]

\[
c_v = \frac{1}{\gamma} c_p = \frac{1}{\gamma - 1} R_g
\]

\[
c_p = 1004.696 \text{ J/}^\circ\text{K-(kg-mole)}
\]

\[
c_v = c_p - R_g = 1004.696 - 287.056 = 717.64 \text{ J/}^\circ\text{K-(kg-mole)}
\]

\[
\gamma = \frac{c_p}{c_v} = 1007.696/717.64 = 1.400
\]
Thermodynamics Summary

• Equation of State: \( p = \rho R_g T \rightarrow R_g = \frac{R_u}{M_w} \)

- \( R_u = 8314.4126 \) J/°K-(kg-mole)
- \( R_g \) (air) = 287.056 J/°K-(kg-mole)

• Relationship of \( R_g \) to specific heats, \( \gamma \)

\[ C_p = C_v + R_g \]

• Internal Energy and Enthalpy

\[ h = e + P_v \]
\[ C_v = \left( \frac{de}{dT} \right)_v \]
\[ C_p = \left( \frac{dh}{dT} \right)_p \]
Thermodynamics Summary (cont’d)

• First Law of Thermodynamics, *reversible process*

\[ de = dq - pdv \quad dh = dq + vdp \]

• First Law of Thermodynamics, *isentropic process*  
  (adiabatic, reversible)

\[ de = -pdv \quad dh = vdp \]
Thermodynamics Summary (cont’d)

• Second Law of Thermodynamics, *reversible process*

\[
S_2 - S_1 = c_p \ln_2 \left( \frac{T_2}{T_1} \right) - R_g \ln \left( \frac{p_2}{p_1} \right)
\]

\[
T ds = dh - v dp
\]

• Second Law of Thermodynamics, *isentropic process*  
  (adiabatic, reversible)  \[\Rightarrow\] \[S_2 - S_1 = 0\]

\[
\frac{p_2}{p_1} = \left[ \frac{T_2}{T_1} \right]^{\gamma - 1}
\]

\[
\left[ \frac{p_2}{p_1} \right] = \left[ \frac{\rho_2}{\rho_1} \right]^{\gamma}
\]
• From conservation of mass and momentum...

**sonic velocity is**

\[
c = \sqrt{\left[ \frac{\partial p}{\partial \rho} \right]_{\Delta s = 0}}
\]

\[
\begin{bmatrix}
p_2 \\
p_1
\end{bmatrix} = \begin{bmatrix}
\rho_2 \\
\rho_1
\end{bmatrix}^\gamma \rightarrow p_2 = \left[ p_1 \rho_1^\gamma \right] \rho_2^\gamma \Rightarrow p = k \rho^\gamma \Rightarrow \frac{\partial p}{\partial \rho} = \gamma k \rho^{\gamma-1}
\]

\[
\frac{\partial p}{\partial \rho} = \gamma \frac{1}{\rho} \left[ k \rho^\gamma \right] = \gamma \frac{p}{\rho} = \gamma R_g T \Rightarrow c = \sqrt{\gamma R_g T}
\]
Thermodynamics Summary (concluded)

• Speed of Sound for calorically Perfect gas

\[ c = \sqrt{\gamma R_g T} \]

• Mathematic definition of Mach Number

\[ M = \frac{V}{\sqrt{\gamma R_g T}} \]
Review, General Integral Form of Equations of Compressible Flow

- **Continuity** (conservation of mass)

  \[ -\iiint_{C.V} (\rho \vec{V} \cdot \vec{d}s) = \frac{\partial}{\partial t} \left( \iiint_{C.V.} \rho \vec{d}v \right) \]

- **Newton’s Second law** (momentum)-- Time rate of change of momentum equals integral of external forces

  \[ \iiint_{C.V.} \rho \vec{f}_b \, d\vec{v} - \iiint_{C.S.} (p) \vec{d}s + \iiint_{C.S.} \left( \vec{f} \times \vec{d}s \right) \, \left| \frac{\vec{d}s}{S_{C.S.}} \right| = \]

  \[ \iiint_{C.S.} \left( \rho \vec{V} \cdot \vec{d}s \right) \vec{V} + \iiint_{C.V.} \frac{\partial}{\partial t} \left( \rho \vec{V} \right) \, d\vec{v} \]
Review (concluded)

• Conservation of Energy--

\[
\frac{\partial}{\partial t} \iiint_{C.V.} \left( \rho \left( e + \frac{\|V^2\|}{2} \right) \, dv \right) + \iint_{C.S.} \rho \vec{V} \cdot d\vec{S} \left( e + \frac{\|V^2\|}{2} \right) = \\
\iiint_{C.V.} \left( \rho \vec{f} \, dv \right) \cdot \vec{V} - \iint_{C.S.} \left( pd \, d\vec{S} \right) \cdot \vec{V} + \iiint_{C.V.} \left( \rho \vec{q} \, dv \right)
\]
One Dimensional Flow Approximations

• Many Useful and practical Flow Situations can be Approximated by one-dimensional flow analyses

“Air-goes-in, Air-goes-out, or both”
Control Volume for 1-D Flow

- Between C.V. entry (1) and exit (2), there could be
  1) Normal Shock wave (supersonic engine duct),
  2) Heat could be added or subtracted, (heat exchanger), or
  3) There could we work performed (turbine element)

Flow Characterized by motion only along longitudinal axis
1-D, Steady, Flow: Collected Equations

• Continuity

\[ \rho_i V_i = \rho_e V_e \]

• Momentum

\[ p_i + \rho_i V_i^2 = p_e + \rho_e V_e^2 \]

• Energy

\[ q + h_i + \frac{V_i^2}{2} = h_e + \frac{V_e^2}{2} \]
Distinction Between True 1-D Flow and Quasi 1-D Flow

- In “true” 1-D flow, cross sectional area is strictly constant.

- In quasi-1-D flow, cross section varies as a function of the longitudinal coordinate, $x$.

- Flow properties are assumed constant across any cross-section.

- Analytical simplification very useful for evaluating flow properties in nozzles, tubes, ducts, and diffusers where the cross sectional area is large when compared to length.
Distinction between True 1-D Flow and Quasi 1-D Flow (cont’d)

- One-D simplification requires axial symmetry in z and y directions.
  To insure that flow properties are constant across cross section.

True Axi-Symmetric Duct (Circular Cross-section)

Quasi-1-D Rectangular Duct
Quasi 1-D Flow: Governing Equations

- Inviscid, No Body Forces, Steady Flow

- Continuity
  \[ \iint_{C.S.} (\rho \vec{V} \cdot d\vec{s}) = 0 \]

- Momentum
  \[ \iint_{C.S.} (\rho \vec{V} \cdot d\vec{s}) \vec{V} = - \iint_{C.S.} (p) d\vec{S} \]

- Energy
  \[ \Delta Q - \iint_{C.S.} (pd\vec{S}) \cdot \vec{V} = \iint_{C.S.} \rho \left( e + \frac{\|\vec{V}^2\|}{2} \right) \vec{V} \cdot d\vec{S} \]
Review: Continuity Equation for Quasi 1-D Control Volume

\[ \oint\oint (\rho \vec{V} \cdot ds) = 0 \]

- Upper and Lower Surfaces …
  no flow across boundary

\[ \vec{V} \cdot ds = 0 \]

- Inlet (properties constant across Cross section) -->

\[ \oint\oint (\rho \vec{V} \cdot ds) = \rho_1 \vec{V}_1 \cdot \oint\oint ds = \rho_1 V_1 \cos(180^\circ) \oint ds = -\rho_1 V_1 A_1 \]

- Inlet (properties constant across Cross section) -->

\[ \oint\oint (\rho \vec{V} \cdot ds) = \rho_2 \vec{V}_2 \cdot \oint\oint ds = \rho_2 V_2 \cos(0^\circ) \oint ds = \rho_2 V_2 A_2 \]

\[ \rho_1 V_1 A_1 = \rho_2 V_2 A_2 \]
Review: Momentum Equation for Quasi 1-D Control Volume

• Similar Analysis for Momentum Equation Yields

• Newton’s Second law--
  Time rate of change of momentum
  Equals integral of external forces

\[
\iint (\rho \vec{V} \cdot d\vec{s}) V = -\iint (p) d\vec{S} \rightarrow
data
\]

\[
p_1 A_1 V_1 + \rho_1 V_1^2 A_1 + \int_1^2 p ds \cdot i_x = p_2 A_2 V_2 + \rho_2 V_2^2 A_2
\]

- Because of duct symmetry the “Z-axis”
  Component of pressure integrated to zero

\[
\rightarrow 
\]

\[
i_x \quad \text{“Unit vector” x-direction}
\]
Review: Momentum Equation for Quasi 1-D Control Volume

- Similar Analysis for Momentum Equation Yields

- Newton’s Second law --
  Time rate of change of momentum
  Equals integral of external forces

\[
\iiint \left( \rho \vec{V} \cdot ds \right) \vec{V} = -\iiint (p) ds \rightarrow c.s.
\]

\[
p_1 A_1 V_1 + \rho_1 V_1^2 A_1 + \int_{1}^{2} p ds \cdot i_x = p_2 A_2 V_2 + \rho_2 V_2^2 A_2
\]

- Because of duct symmetry the “Z-axis”
  Component of pressure integrated to zero
Review: Energy Equation for Quasi 1-D Control Volume

\[ \dot{Q} - \oint_{c.s.} (p \vec{d} S) \cdot \vec{V} = \iiint_{c.s.} \rho \left( e + \frac{V^2}{2} \right) \vec{V} \cdot d \vec{S} \]

\[ \Delta q + h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \]
Example Usage: Engine Thrust Model

- Steady, Inviscid, One-Dimensional Flow Through Ramjet

\[ - \int \int (p) dS = \int \int (\rho V \cdot ds) V \]

\[ \int_{wall} P_{wall} dA_{wall} + p_\infty A_i - p_e A_e = \rho_e V_e^2 A_e - \rho_i V_i^2 A_i = m_e V_e - m_i V_i \]
Example Usage: Engine Thrust Model (cont’d)

• Adding \( (p_e A_e - p_\infty A_e) \) to Both Sides, and collecting terms

\[
\int P_{\text{wall}} dA_{\text{wall}} + p_\infty A_i - p_e A_e + (p_e A_e - p_\infty A_e) = m_e V_e - m_i V_i + (p_e A_e - p_\infty A_e) \rightarrow
\]

\[
\int P_{\text{wall}} dA_{\text{wall}} + p_\infty (A_i - A_e) = m_e V_e - m_i V_i + (p_e A_e - p_\infty A_e)
\]

**Integrated Pressure Forces Acting on External + Internal Surface of Engine Wall = Thrust**

**Thrust** = \( m_e V_e - m_i V_i + (p_e A_e - p_\infty A_e) \)
Rocket Thrust Equation

\[ Thrust = m_e V_e + (p_e A_e - p_\infty A_e) \]

- \( m_i = 0 \)

- Thrust + Oxidizer enters combustion Chamber at \( \sim 0 \) velocity, combustion
- Adds energy … High Chamber pressure
- Accelerates flow through Nozzle
- Resultant pressure forces produce thrust
Mach Number: Revisited

• From the definition of $c_v$

$$c_v = c_p - R_g \Rightarrow$$

$$c_v = R_g \left( \frac{c_p}{R_g} - 1 \right) = R_g \left( \frac{c_p}{c_p - c_v} - 1 \right) =$$

$$R_g \left( \frac{c_p - c_p + c_v}{c_p - c_v} \right) = R_g \left( \frac{c_v}{c_p - c_v} \right) = \left( \frac{R_g}{\gamma - 1} \right)$$
Mach Number: Revisited (cont’d)

• Recall from the fundamental definition

\[ M = \frac{V}{c} = \frac{V}{\sqrt{\gamma R_g T}} \]

• But if we take the ratio or kinetic to internal energy
  Of a fluid *element*

\[ \frac{V^2}{2e} = \frac{V^2}{2c_v T} \]

• calorically perfect gas
Mach Number: Revisited (cont’d)

- Then

\[
\frac{V^2}{2} = \frac{V^2}{2c_vT} = \frac{V^2}{2} \left( \frac{R_g}{\gamma - 1} \right) T
\]

\[
\frac{\gamma}{\gamma - 1} \frac{V^2}{2} \left( \frac{R_g}{\gamma - 1} \right) T = \frac{\gamma (\gamma - 1)}{2} V^2 \frac{2}{\gamma R_g T} = \frac{\gamma (\gamma - 1)}{2} M^2
\]

i.e. Mach number is a measure of the ratio of the fluid Kinetic energy to the fluid internal energy (direct motion To random thermal motion of gas molecules)
Mach Number: Revisited (cont’d)

• Look at the Steady Flow Energy equation With no heat addition

\[ h_i + \frac{V_i^2}{2} = h_e + \frac{V_e^2}{2} \rightarrow h + \frac{V^2}{2} = \text{const} \]

• Look at Differential form

\[ d\left(h + \frac{V^2}{2}\right) = 0 \rightarrow dh + VdV = 0 \]
Mach Number: Revisited (cont’d)

• For a reversible process

\[ Tds = dh - vdp \]

• With no heat addition, \( ds=0 \)

\[ dh = vdp = \frac{dp}{\rho} \]

• Subbing into previous

\[ dp = -\rho VdV \quad ("euler’s equation") \]
Mach Number: Revisited (cont’d)

• Now look at Continuity equation

\[ \rho_i V_i = \rho_e V_e \rightarrow \rho A V = \text{const} \]

\[
\frac{d(\rho AV)}{\rho AV} = 0 = \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V}
\]

• Substituting in Euler’s Equation \( dp = -\rho V dV \)

\[
\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{- dp}{\rho V} = 0
\]
Mach Number: Revisited (cont’d)

• Solving for \( \frac{dA}{A} \)

\[
\frac{dA}{A} = \frac{dp}{\rho V^2} - \frac{d\rho}{\rho} = \frac{dp}{\rho} \left( \frac{1}{V^2} - \frac{d\rho}{dp} \right)
\]

• But since we are considering a process with \( ds=0 \)

\[
\frac{d\rho}{dp} = \frac{1}{c^2} \quad \text{and} \quad \frac{dA}{A} = \frac{dp}{\rho} \left( \frac{1}{V^2} - \frac{1}{c^2} \right) = \frac{(1 - M^2)}{\rho V^2} \ dp
\]

• Substituting in Euler’s Equation \( dp = -\rho V dV \)

\[
\frac{dA}{A} = \frac{(1 - M^2)}{\rho V^2} (-\rho V dV) = (M^2 - 1) \left( \frac{dV}{V} \right)
\]
Mach Number: Revisited (cont’d)

- Rearranging gives the TWO relationships

Whose ramifications are fundamental to this class

\[
\left( \frac{dV}{dA} \right) = \frac{1}{\left( M^2 - 1 \right)} \frac{V}{A}
\]

\[
\frac{dp}{dA} = \frac{-1}{\left( M^2 - 1 \right)} \frac{\rho V^2}{A}
\]
Mach Number: Revisited (cont’d)

\[
\left( \frac{dV}{dA} \right) = \frac{1}{\left( M^2 - 1 \right)} \frac{V}{A}
\]

\[
\frac{dp}{dA} = \frac{-1}{\left( M^2 - 1 \right)} \frac{\rho V^2}{A}
\]

\[
M < 1 \rightarrow \left( \frac{dV}{dA} \right) < 0 \rightarrow \left( \frac{dp}{dA} \right) > 0
\]

\[
M > 1 \rightarrow \left( \frac{dV}{dA} \right) > 0 \rightarrow \left( \frac{dp}{dA} \right) < 0
\]
Fundamental Properties of Supersonic and Supersonic Flow

- **Subsonic Diffuser**
  - Flow (M<1)
  - \( p \) decreases
  - \( V \) increases

- **Supersonic Nozzle**
  - Flow (M>1)
  - \( p \) decreases
  - \( V \) increases

- **Subsonic Nozzle**
  - Flow (M<1)
  - \( p \) decreases
  - \( V \) increases

- **Supersonic Diffuser**
  - Flow (M>1)
  - \( p \) increases
  - \( V \) decreases
… Hence the shape of the rocket Nozzle