Introduction to the Method of Characteristics and the Minimum Length Nozzle

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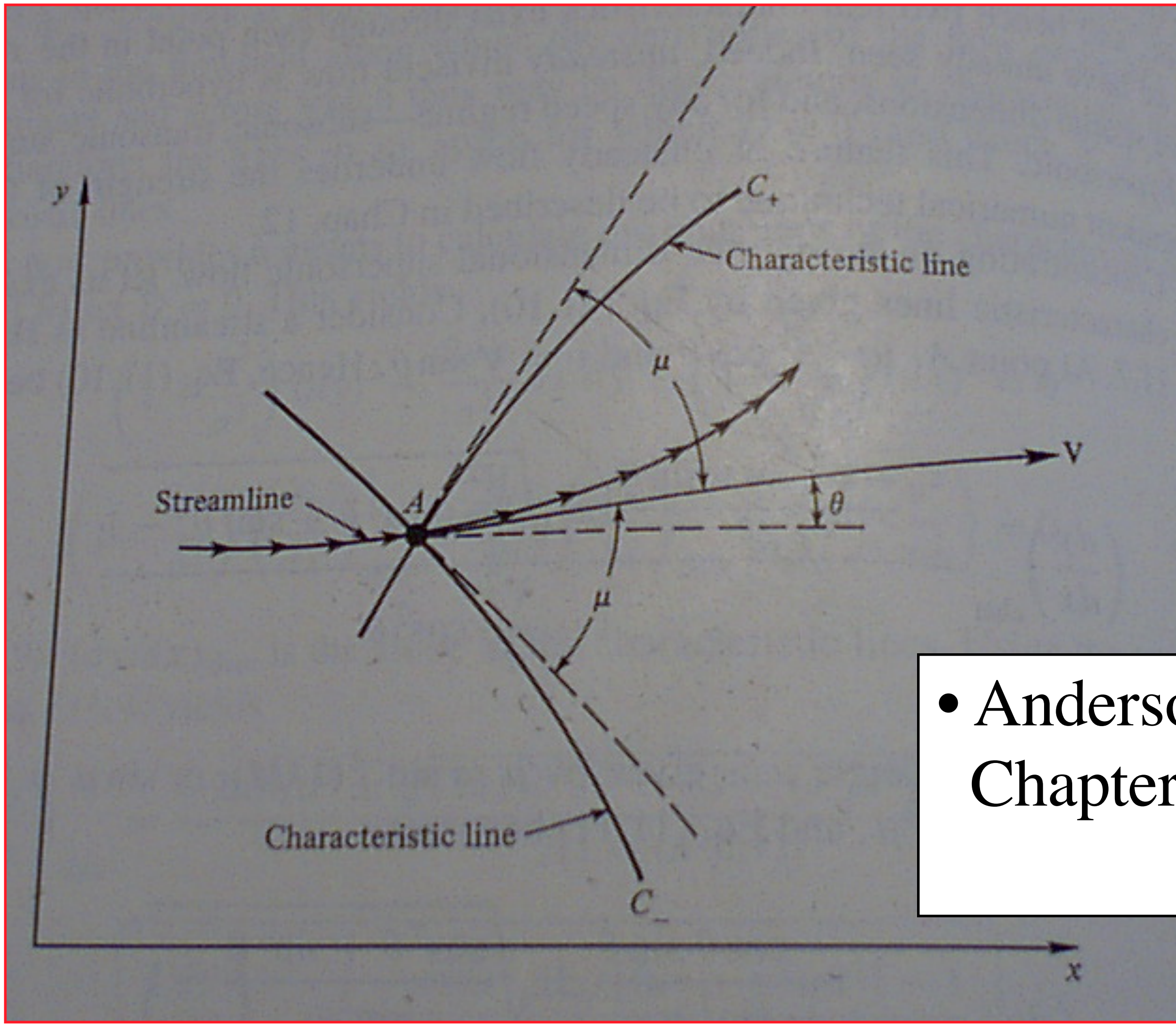
Material Taken from Anderson, Chapter 11, pp. 377-403

An Introduction to the Two-Dimensional Method of Characteristics



- Anderson, Chapter 11 pp. 377-403

"Method of Characteristics"

- **Basic principle of Methods of Characteristics**

-- If supersonic flow properties
are known at two points in a flow field,

-- There is one and only one set of properties
*compatible** with these at ***a*** third point,

-- Determined by the intersection
of *characteristics*, or *mach* ***waves*****,** from the two
original points.

*Root of term *"compatibility equations"*

"Method of Characteristics" (cont'd)

- Compatibility Equations relate the velocity magnitude and direction along the characteristic line.

- In 2-D and quasi 1-D flow, compatibility equations are Independent of spatial position, in 3-D methods, space Becomes a player and complexity goes up considerably

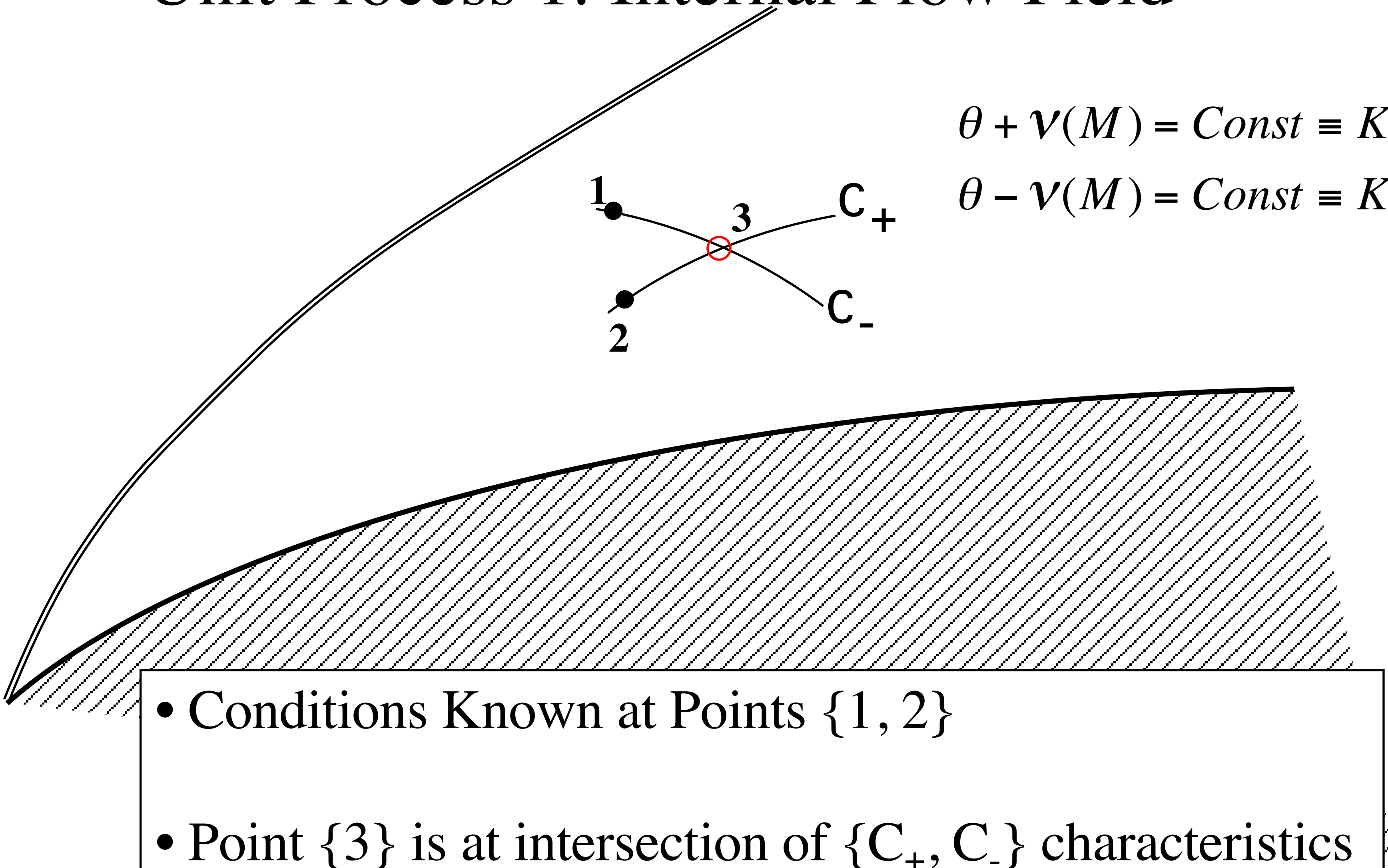
- Computational Machinery for applying the method of Characteristics are the so-called "unit processes"

- By repeated application of unit processes, flow field Can be solved in entirety

Unit Process 1: Internal Flow Field

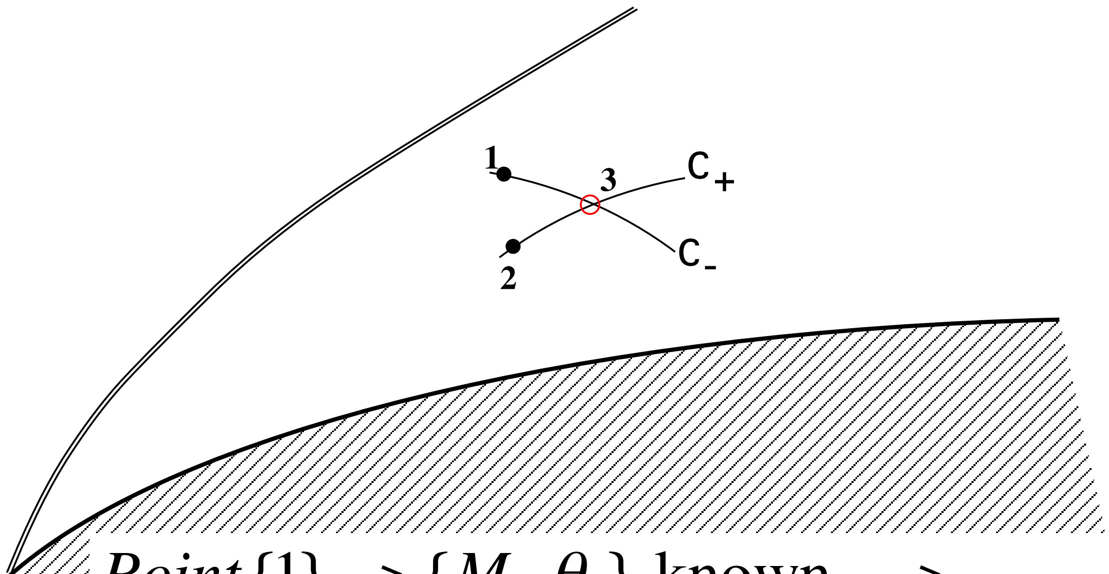
$$\theta + \nu(M) = Const \equiv K_-$$

$$\theta - \nu(M) = Const \equiv K_+$$



- Conditions Known at Points {1, 2}
- Point {3} is at intersection of $\{C_+, C_-\}$ characteristics

Unit Process 1: Internal Flow Field (cont'd)

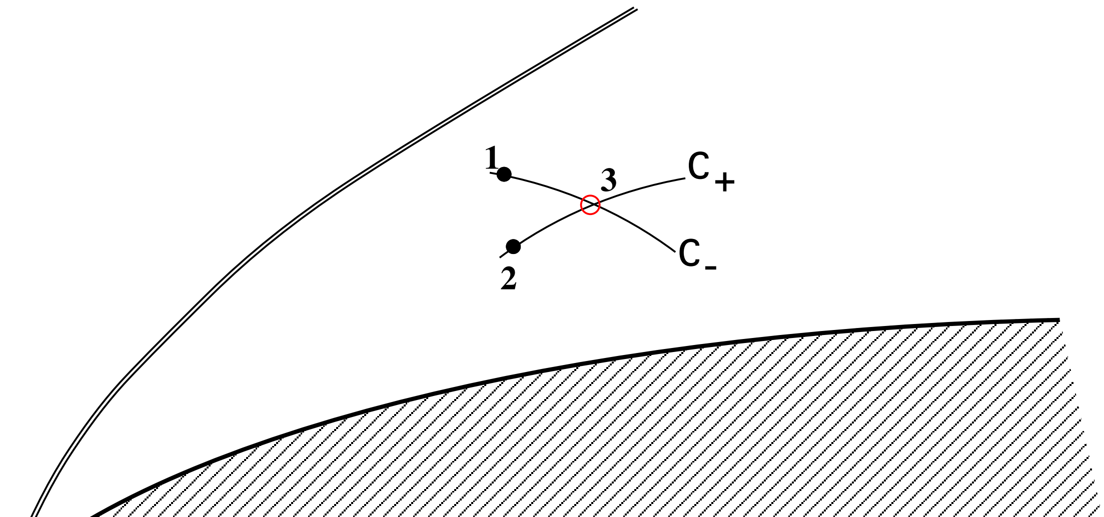


$$Point\{1\} \rightarrow \{M_1, \theta_1\} \text{ known } \rightarrow$$

$$\nu_1 = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_1^{\,2}-1\right)}\right\} - \tan^{-1}\sqrt{M_1^{\,2}-1}$$

$$Along\ \{C_-\} \rightarrow\ \theta_1 + \nu_1 = const = \left(K_-\right)_1$$

Unit Process 1: Internal Flow Field (cont'd)

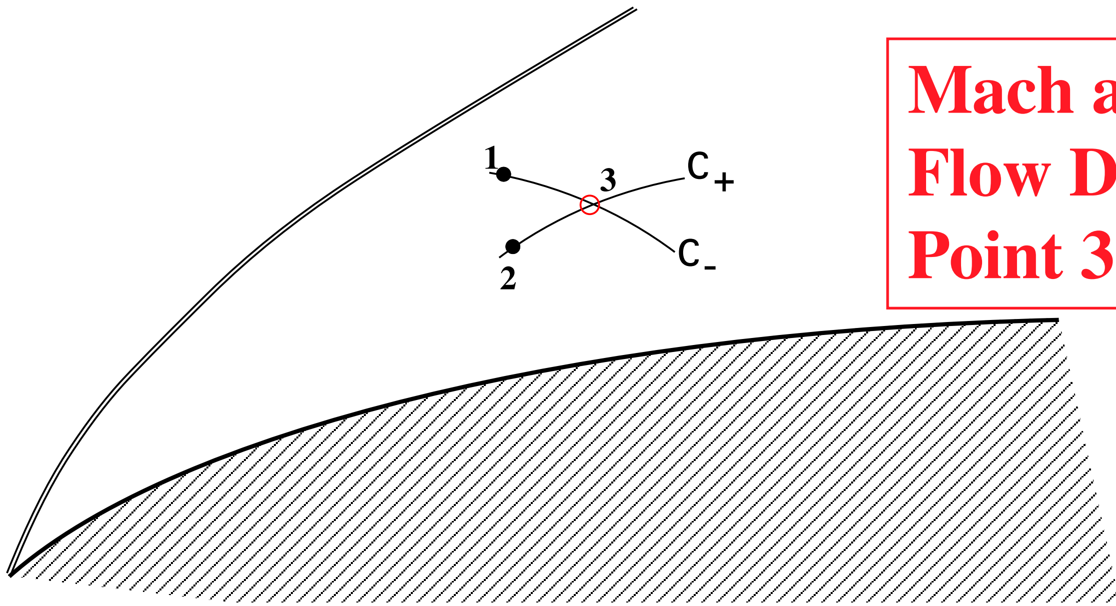


$$Point\{2\} \rightarrow \{M_2, \theta_2\} \text{ known } \rightarrow$$

$$\nu_2 = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_2^{\,2}-1\right)}\right\} - \tan^{-1}\sqrt{M_2^{\,2}-1}$$

$$Along\ \{C_+\} \rightarrow \theta_2 - \nu_2 = const = \left(K_+\right)_2$$

Unit Process 1: Internal Flow Field (cont'd)



Mach and Flow Direction solved for at Point 3

$$\theta + \nu(M) = Const \equiv K_-$$

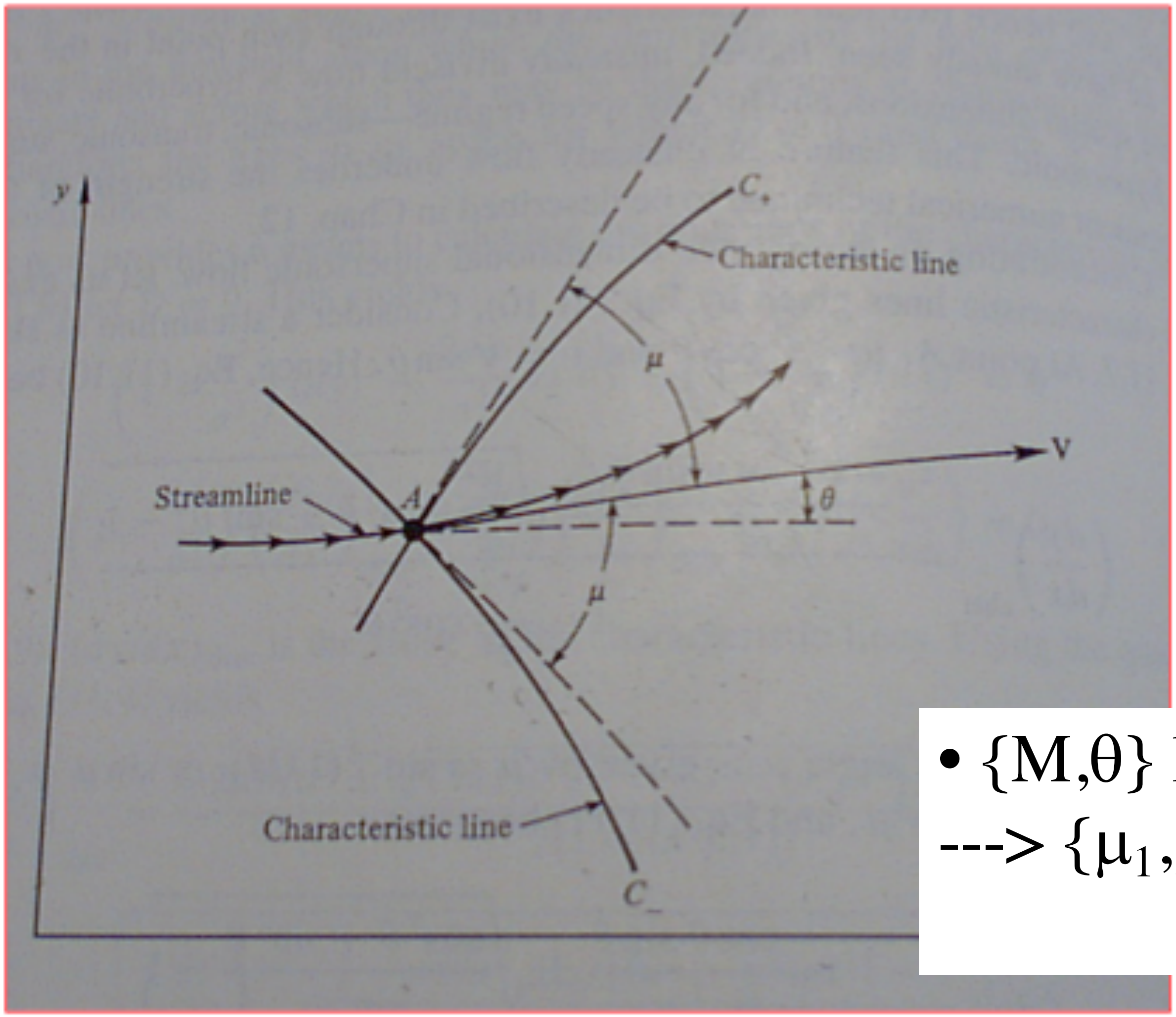
$$\theta - \nu(M) = Const \equiv K_+$$

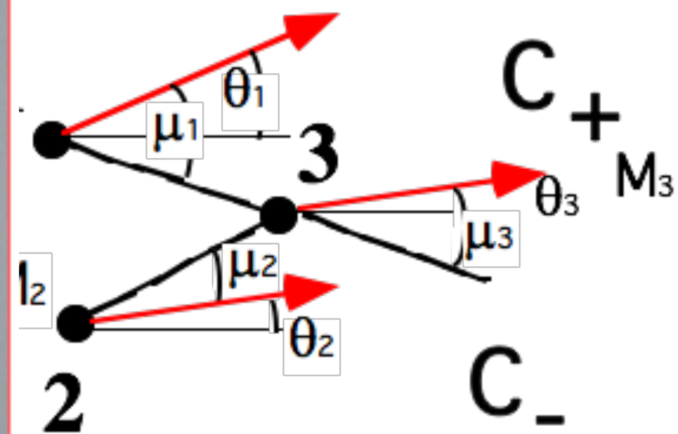
$$Point\{3\} \rightarrow \begin{bmatrix} \theta_1 + \nu_1 = \theta_3 + \nu_3 \\ \theta_2 - \nu_2 = \theta_3 - \nu_3 \end{bmatrix} \rightarrow \begin{bmatrix} \theta_3 = \dfrac{(\theta_1 + \nu_1) + (\theta_2 - \nu_2)}{2} = \dfrac{(K_-)_1 + (K_+)_2}{2} \\ \nu_3 = \dfrac{(\theta_1 + \nu_1) - (\theta_2 - \nu_2)}{2} = \dfrac{(K_-)_1 - (K_+)_2}{2} \end{bmatrix}$$

$$M_3 = Solve\left[\nu_3 = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_3^{\,2} - 1\right)}\right\} - \tan^{-1}\sqrt{M_3^{\,2} - 1}\right]$$

Unit Process 1: Internal Flow Field (cont'd)

But where is Point {3} ?

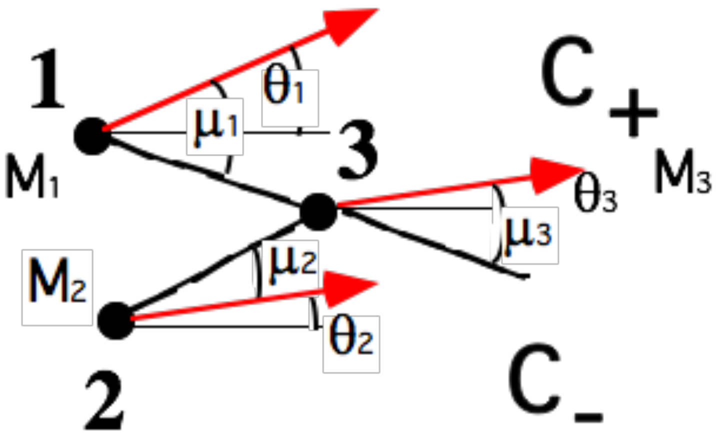




- {M,θ} known at points {1,2,3}

---> $\{\mu_1,\mu_2,\mu_3\}$ known

Unit Process 1: Internal Flow Field (concluded)



But where is Point {3} ?

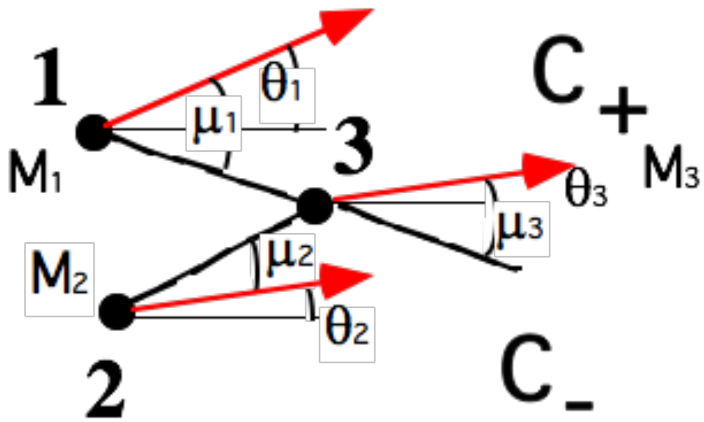
- Slope of characteristics lines approximated by:

$$slope\{C_-\} = \frac{(\theta_1 - \mu_1) + (\theta_3 - \mu_3)}{2}$$

Intersection locates point 3

$$slope\{C_+\} = \frac{(\theta_2 + \mu_2) + (\theta_3 + \mu_3)}{2}$$

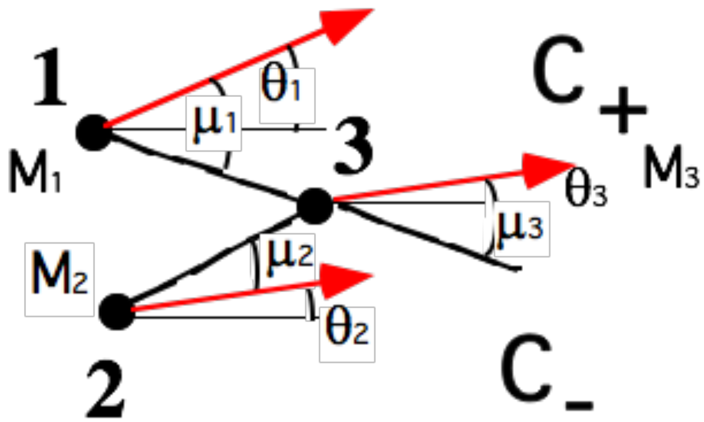
Unit Process 1: Internal Flow Example (cont'd)



- Solve for $\{x_3, y_3\}$

$$\begin{bmatrix} \dfrac{y_3 - y_1}{x_3 - x_1} = \tan\left[slope\left\{C_-\right\}\right] \\ \dfrac{y_3 - y_2}{x_3 - x_2} = \tan\left[slope\left\{C_+\right\}\right] \end{bmatrix} \rightarrow \begin{bmatrix} y_3 = \left(x_3 - x_1\right)\tan\left[slope\left\{C_-\right\}\right] + y_1 \\ y_3 = \left(x_3 - x_2\right)\tan\left[slope\left\{C_+\right\}\right] + y_2 \end{bmatrix}$$

Unit Process 1: Internal Flow Example (cont'd)

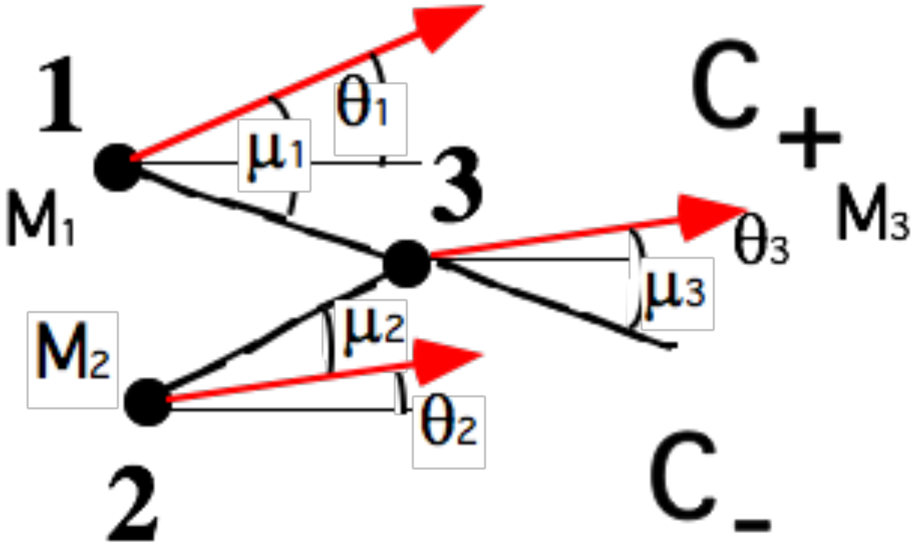


- Solve for $\{x_3, y_3\}$

$$x_3 = -\frac{x_1 \cdot \tan\left[slope\{C_-\}\right] - x_2 \cdot \tan\left[slope\{C_+\}\right] + (y_2 - y_1)}{\tan\left[slope\{C_-\}\right] - \tan\left[slope\{C_+\}\right]}$$

$$y_3 = \frac{\tan\left[slope\{C_-\}\right] \cdot \tan\left[slope\{C_+\}\right] \cdot (x_1 - x_2) + \tan\left[slope\{C_-\}\right] \cdot y_2 - \tan\left[slope\{C_+\}\right] \cdot y_1}{\tan\left[slope\{C_-\}\right] - \tan\left[slope\{C_+\}\right]}$$

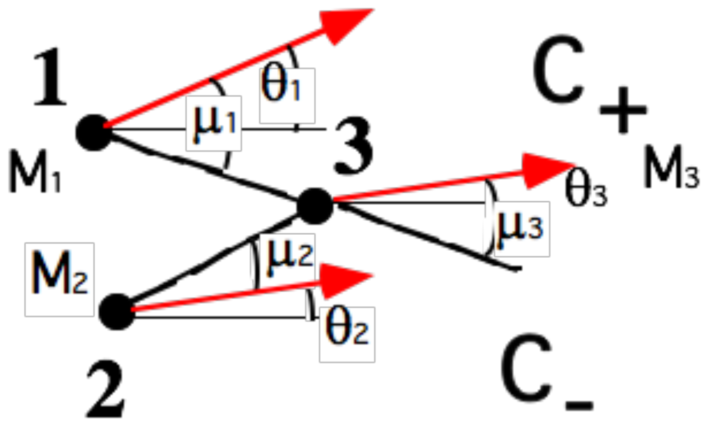
Unit Process 1: Internal Flow Example



$$M_1 = 2.0,\ \theta_1 = 10^o,\ \left\{x_1, y_1\right\} = \left\{1.0, 2.0\right\}$$

$$M_2 = 1.75,\ \theta_2 = 5^o,\ \left\{x_2, y_2\right\} = \left\{1.5, 1.0\right\}$$

Unit Process 1: Internal Flow Example (cont'd)



- Point 1, compute

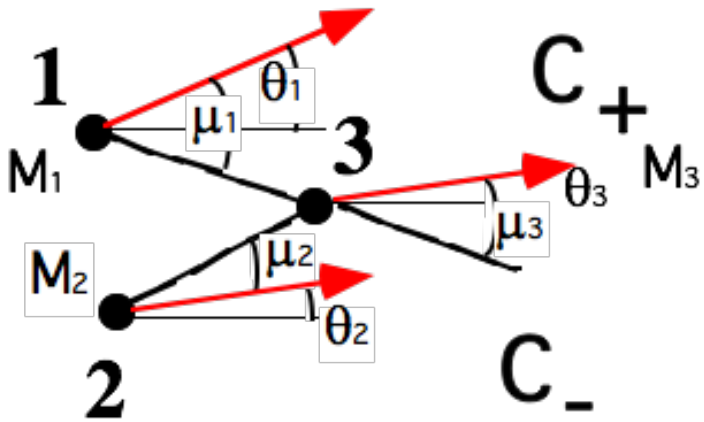
$$\left\{\nu_1, \mu_1, \left(K_-\right)_1\right\}$$

$$\nu_1 = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(2.0^2-1\right)}\right\} - \tan^{-1}\sqrt{2.0^2-1} = 26.37976^{\circ}$$

$$\mu_1 = \frac{180}{\pi}\sin^{-1}\left[\frac{1}{2.0}\right] = 30^{o}$$

$$\left(K_-\right)_1 = \theta_1 + \nu_1 = 10^{o} + 26.37976^{\circ} = 36.37976^{\circ}$$

Unit Process 1: Internal Flow Example (cont'd)



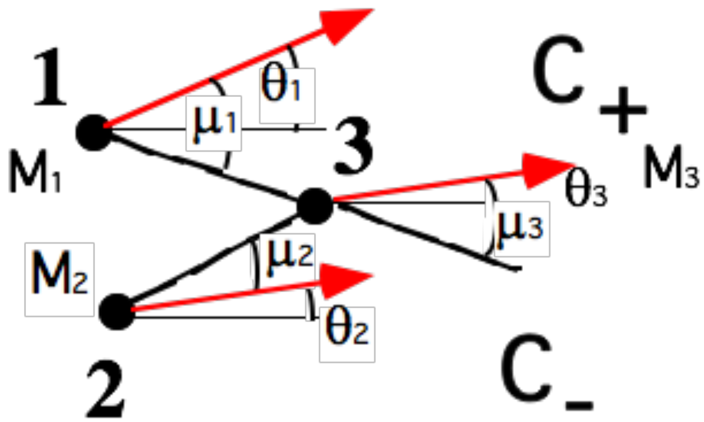
- Point 2, compute
  $$\left\{ \nu_2, \mu_2, \left(K_+\right)_2 \right\}$$

$$\nu_2 = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1}\left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1}\left(1.75^2 - 1\right)} \right\} - \tan^{-1}\sqrt{1.75^2 - 1} = 19.27319^{\circ}$$

$$\mu_2 = \frac{180}{\pi} \sin^{-1}\left[\frac{1}{1.75}\right] = 34.84990^{o}$$

$$\left(K_+\right)_2 = \theta_2 - \nu_2 = 5^{o} - 19.27319^{\circ} = -14.27319^{\circ}$$

Unit Process 1: Internal Flow Example (cont'd)



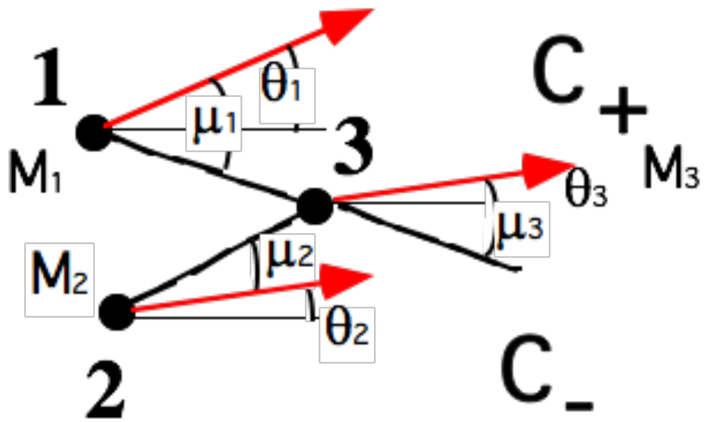
- Point 3 Solve for

$$\left\{\theta_3, \nu_3\right\}$$

$$\theta_3 = \frac{\left(K_-\right)_1 + \left(K_+\right)_2}{2} = \frac{36.37976 + (-14.27319)}{2} = 11.0533 \text{ deg.}$$

$$\nu_3 = \frac{\left(K_-\right)_1 - \left(K_+\right)_2}{2} = \frac{36.37976 - (-14.27319)}{2} = 25.3265 \text{ deg.}$$

Unit Process 1: Internal Flow Example (cont'd)



- Point 3 Solve for

$$\{M_3, \mu_3\}$$

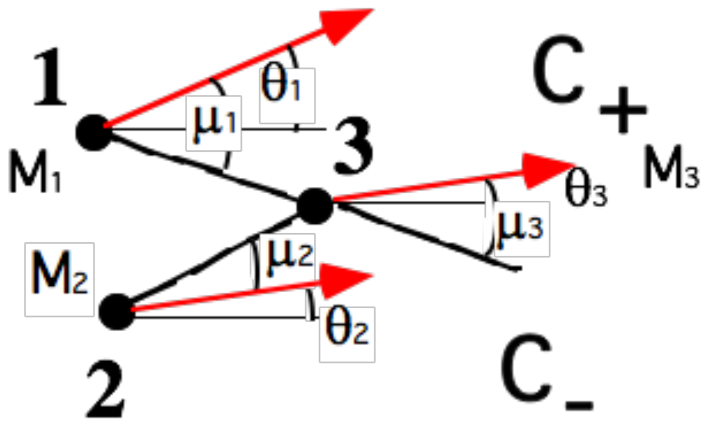
$$M_3 = Solve\left[25.3265\ \frac{\pi}{180} = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_3^{\ 2}-1\right)}\right\} - \tan^{-1}\sqrt{M_3^{\ 2}-1}\right]$$

$$\boxed{\sin(\mu) = \frac{1}{M}}$$

$M_3 = 1.96198$

---> $\mu_3 = 30.6431^\circ$

Unit Process 1: Internal Flow Example (cont'd)



- Locate Point 3

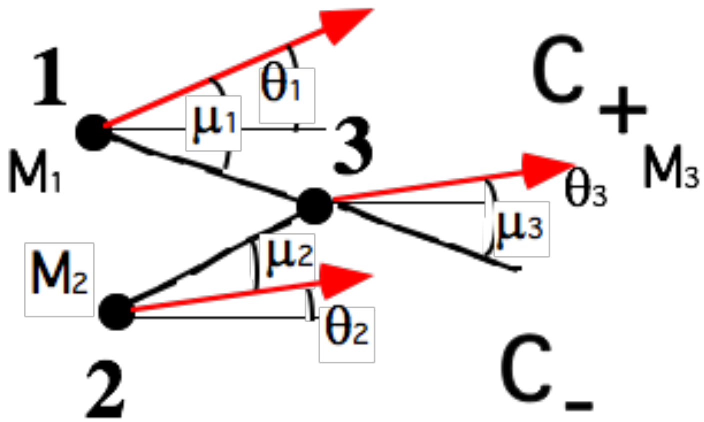
- Line Slope Angles

$$slope\{C_-\} = \frac{(\theta_1 - \mu_1) + (\theta_3 - \mu_3)}{2} = \frac{(10 - 30) + (11.053 - 30.6431)}{2} = -19.795\ \text{deg}$$

$$slope\{C_+\} = \frac{(\theta_2 + \mu_2) + (\theta_3 + \mu_3)}{2} =$$

$$\frac{(5 + 34.8499) + (11.053 + 30.6431)}{2} = 40.773\ \text{deg}$$

Unit Process 1: Internal Flow Example (cont'd)

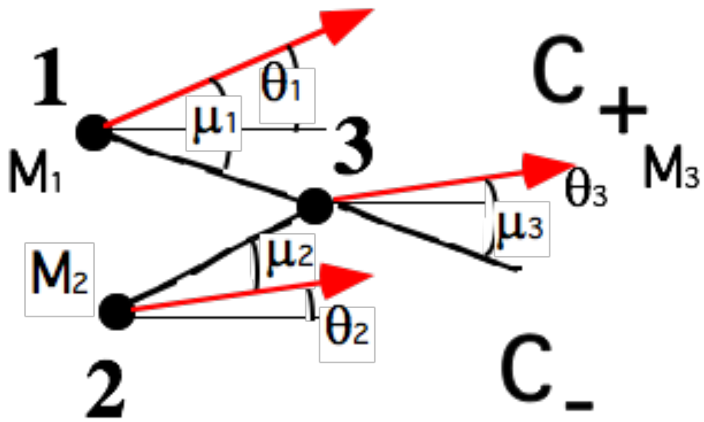


- Solve for $\{x_3, y_3\}$

$$x_3 = \frac{-1\tan\left(\frac{\pi}{180}(-19.794887)\right) + 1.5\tan\left(40.773123\frac{\pi}{180}\right) + 2 - 1}{\tan\left(\frac{\pi}{180}(-19.794887)\right) - \tan\left(\frac{\pi}{180}40.773123\right)}$$

$$= 2.17091$$

Unit Process 1: Internal Flow Example (cont'd)



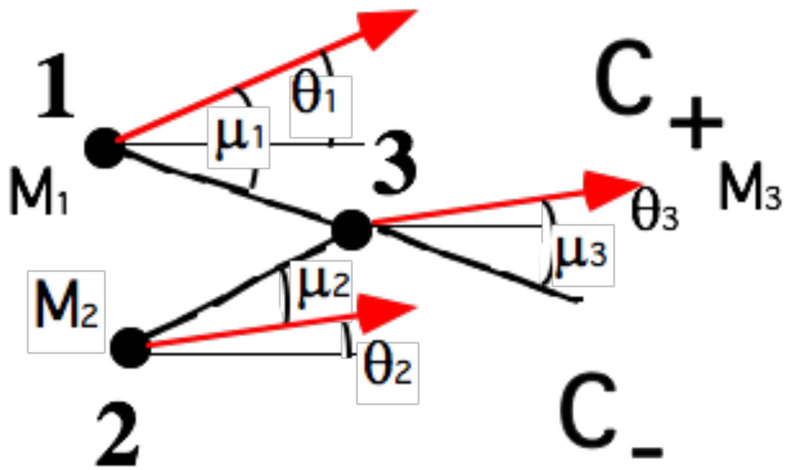
- Solve for $\{x_3,y_3\}$

$y_3=$

$$\frac{\tan\left(\frac{\pi}{180}(-19.79489)\right)\cdot\tan\left(\frac{\pi}{180}40.7731\right)\cdot(1.0-1.5)-2\tan\left(40.773123\frac{\pi}{180}\right)+1\tan\left(\frac{\pi}{180}(-19.79489)\right)}{\tan\left(\frac{\pi}{180}(-19.79489)\right)-\tan\left(\frac{\pi}{180}40.773123\right)}$$

$=1.57856$

Unit Process 1: Internal Flow Example (concluded)



$$\begin{bmatrix} M_1 \\ \theta_1 \\ x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 10^o \\ 1.0 \\ 2.0 \end{bmatrix} \rightarrow \begin{bmatrix} M_2 \\ \theta_2 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.75 \\ 5^o \\ 1.5 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} M_3 \\ \theta_3 \\ x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1.96198 \\ 11.0533 \\ 2.17091 \\ 1.57856 \end{bmatrix}$$

Using MOC for Supersonic Nozzle Design

In order to expand an internal steady flow through a duct from subsonic to supersonic speed, we established in Chap. 5 that the duct has to be convergent-divergent in shape,

Moreover, we developed relations

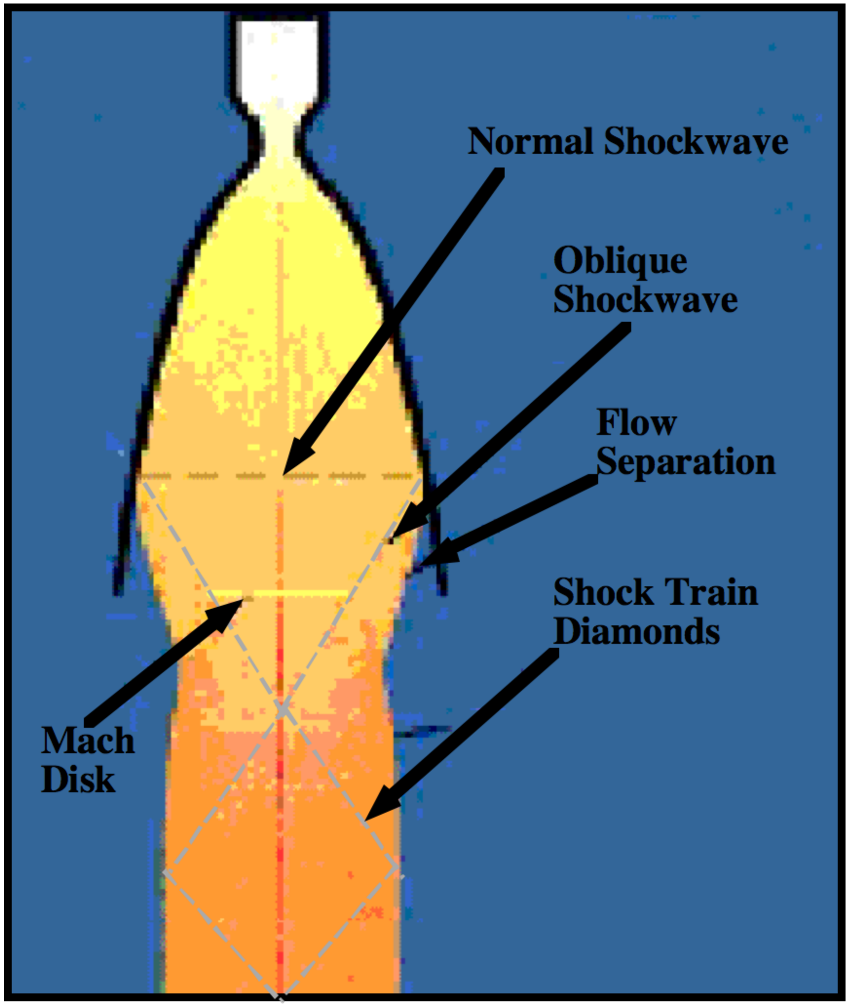
for the local Mach number, and hence the pressure, density, and temperature, as functions of local area ratio A/A^*.

However, these relations assumed quasi-one-

dimensional flow, whereas, strictly speaking, the flow two-dimensional. Moreover, the quasi-one-dimensional theory tells us nothing about the proper *contour* of the duct, i.e., what is the proper variation of area with respect to the flow direction $A = A(x)$. If the nozzle contour is not proper, shock waves may occur inside the duct.

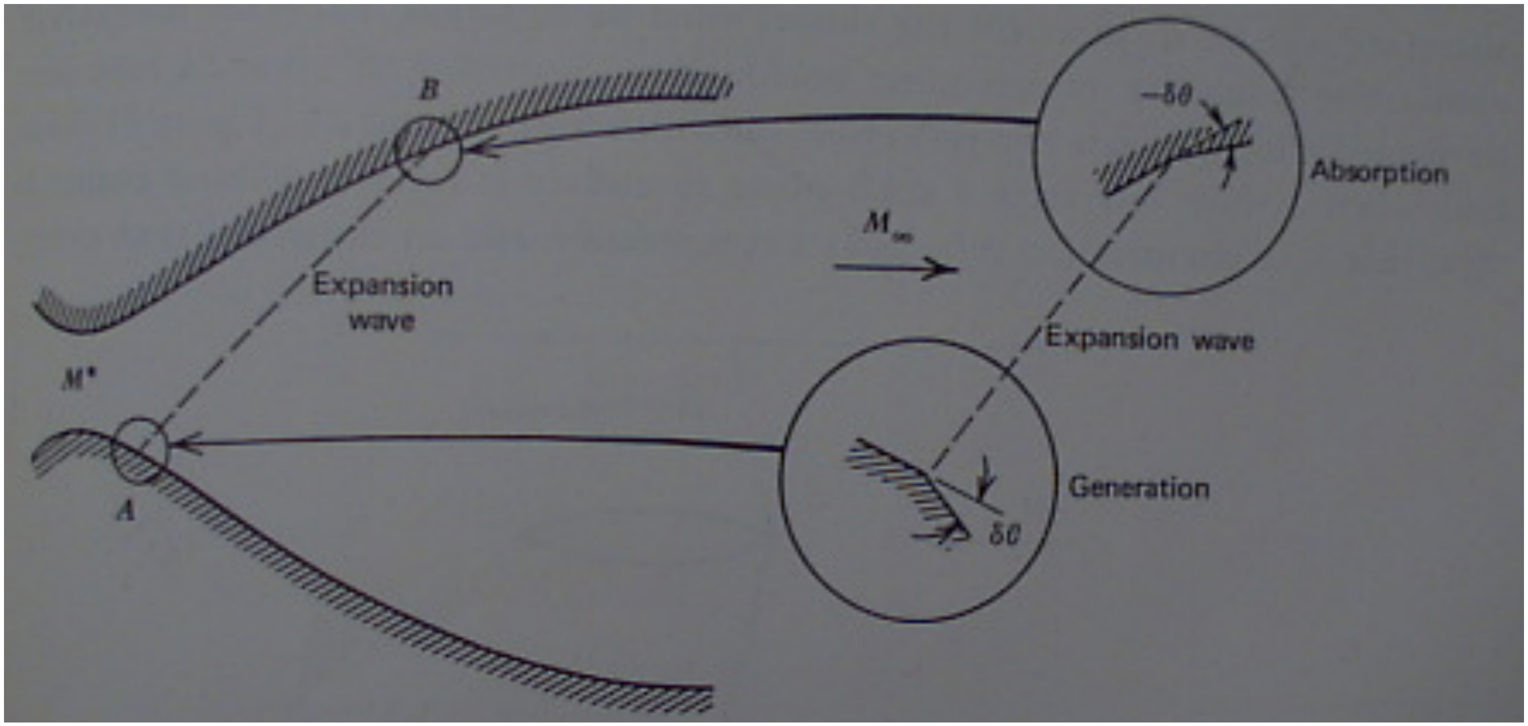
The method of characteristics provides a technique for properly designing the contour of a supersonic nozzle for shockfree, isentropic flow, taking into account the multidimensional flow inside the duct. The purpose of this section is to illustrate such an application.

What happens when a nozzle expands too quickly?



- *i.e. …. A mess …for a given Operating condition there is only so fast we can expand a Conventional Nozzle*

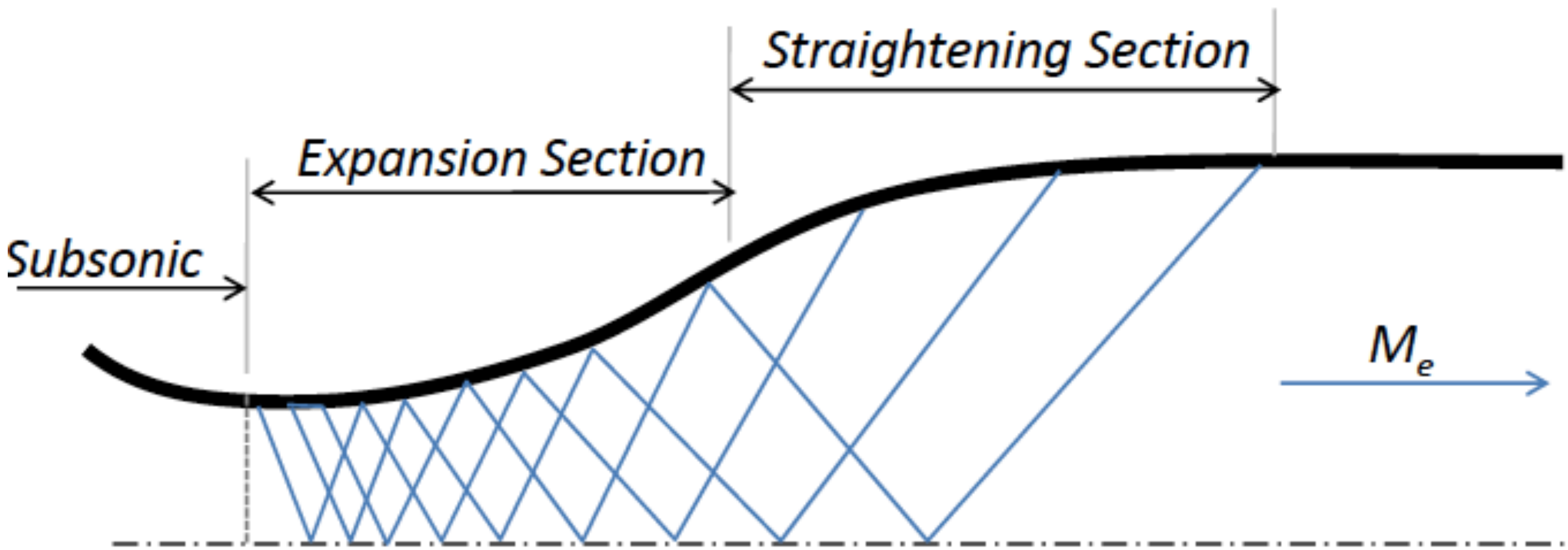
Supersonic Nozzle Design



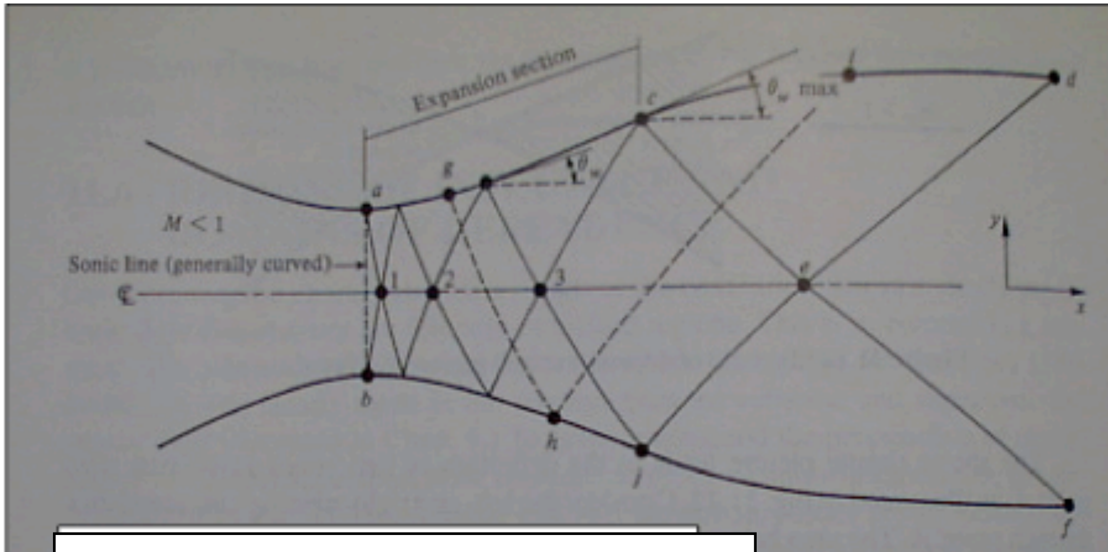
- Strategic contouring will “absorb” mach waves to give isentropic flow in divergent section

Using Method of Characteristics to Design a Bell Nozzle

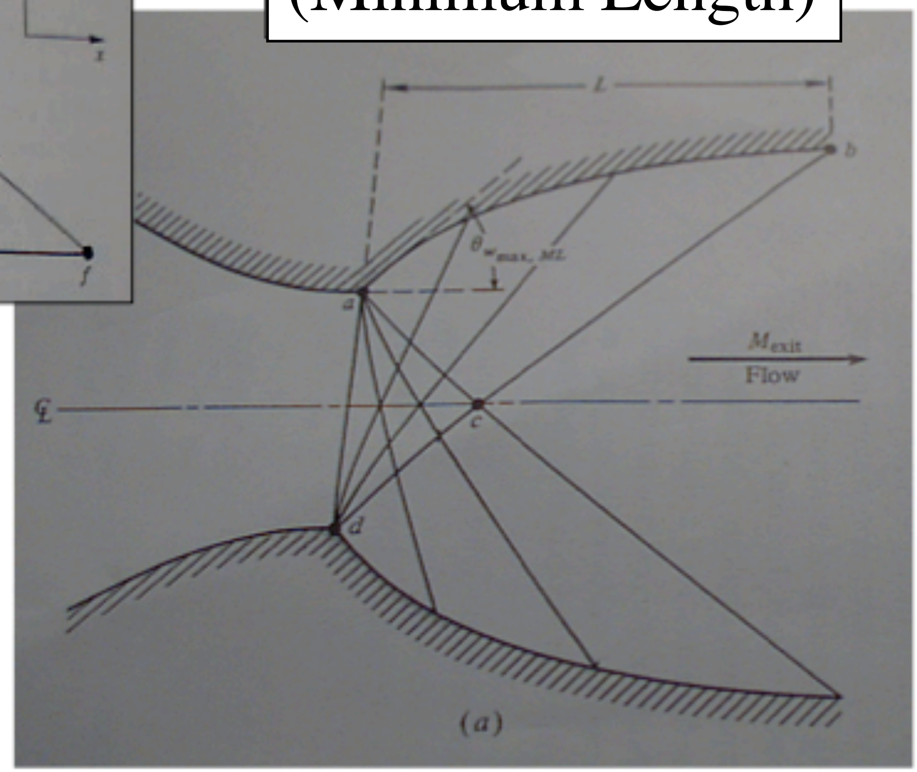
• This approach "prescribes" the expansion section of the nozzle, and then uses M.O.C to design turning section to achieve wave cancellation at wall …. And ensure isentropic flow



Supersonic Nozzle Design (cont'd)



• Bell Nozzle
(gradual expansion)



• Rocket Nozzle
(Minimum Length)

• Use compatibility eqs. to design boundary with shock free flow

Method of Characteristics

- Supersonic "compatibility" equations

$$\theta + \nu(M) = Const \equiv K_{-}$$

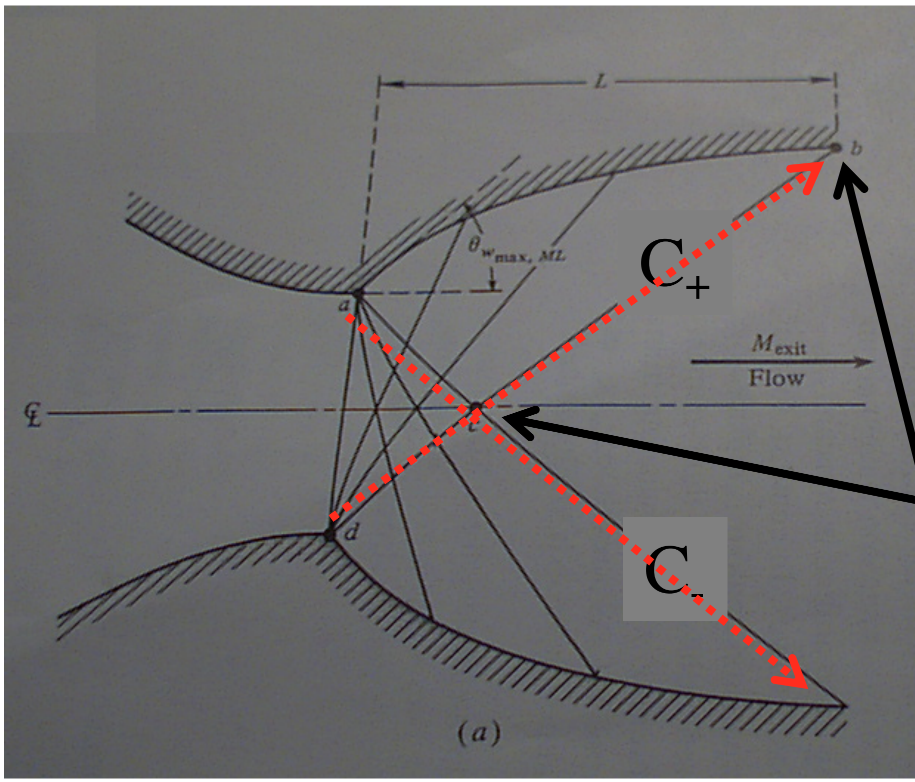
$$\theta - \nu(M) = Const \equiv K_{+}$$

- Apply along "characteristic lines" in flow field, and insure isentropic flow …

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right\} - \tan^{-1}\sqrt{M^2-1}$$

Minimum Length Nozzle Design (cont'd)

- **Find minimum length nozzle with shock-free flow → $\theta_{exit} = 0$**



- Along C_+ characteristic $\{d, c, exit\}$

$$\boxed{\begin{aligned}&\theta_c - \nu_c = \left(K_+\right)_c = \theta_{exit} - \nu_{exit}\\ &\rightarrow \theta_c = 0 \rightarrow \nu_c = \nu_{exit} - \theta_{exit}\end{aligned}}$$

- Along C_- characteristic $\{a, c, exit\}$

$$\boxed{\begin{aligned}&\theta_c + \nu_c = \left(K_-\right)_c = \theta_{exit} + \nu_{exit} \rightarrow\\ &\theta_c = 0 \rightarrow \nu_c = \left(K_-\right)_c = \theta_{exit} + \nu_{exit}\end{aligned}}$$

- Add

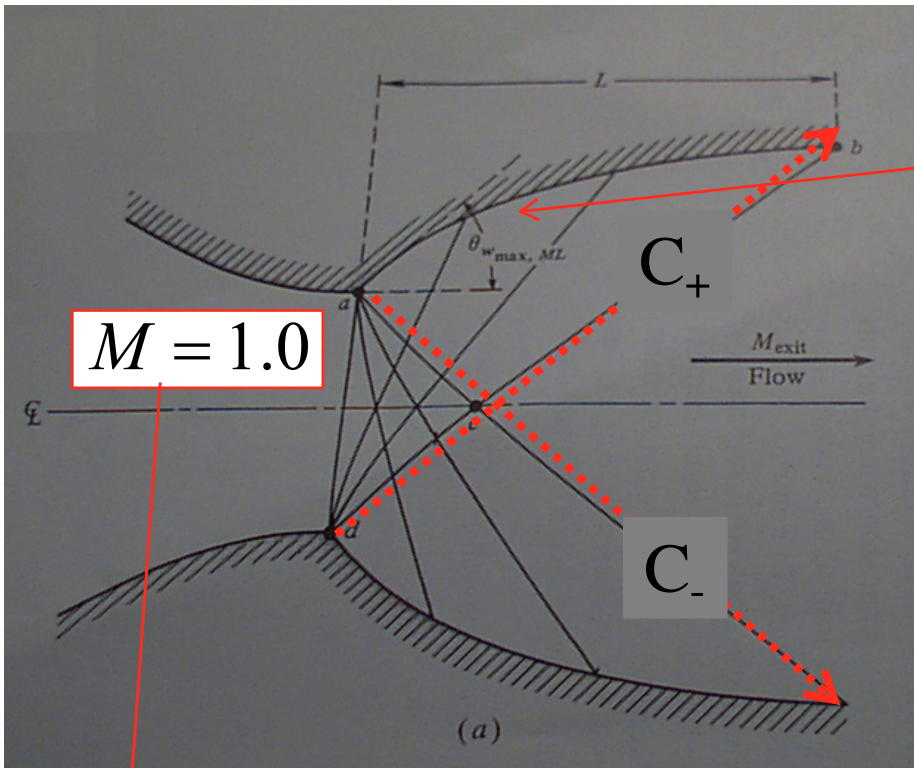
$$\nu_c = \nu_{exit} + \theta_{exit}$$

$$+ \quad \nu_c = \nu_{exit} - \theta_{exit} \quad \rightarrow \boxed{\nu_c = \nu_{exit}}$$

$$2\nu_c = 2\nu_{exit}$$

Minimum Length Nozzle Design (cont'd)

- **Find minimum length nozzle with shock-free flow**



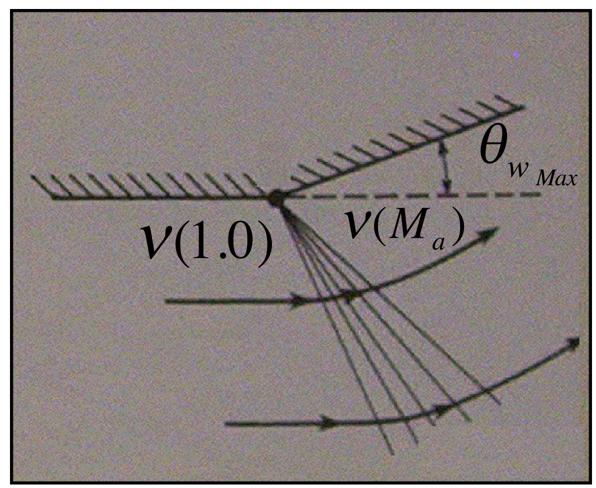
$$\nu(1.0) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(1.0^2-1\right)}\right\} - \tan^{-1}\sqrt{1.0^2-1} = 0$$

- Along C_- characteristic $\{a,c\}$ at point a $\quad \theta_c = 0$

$$\theta_{wall_{Max}} + \nu_a = (K_-)_c = \nu_c \rightarrow \boxed{\nu_c = \nu_{exit}}$$

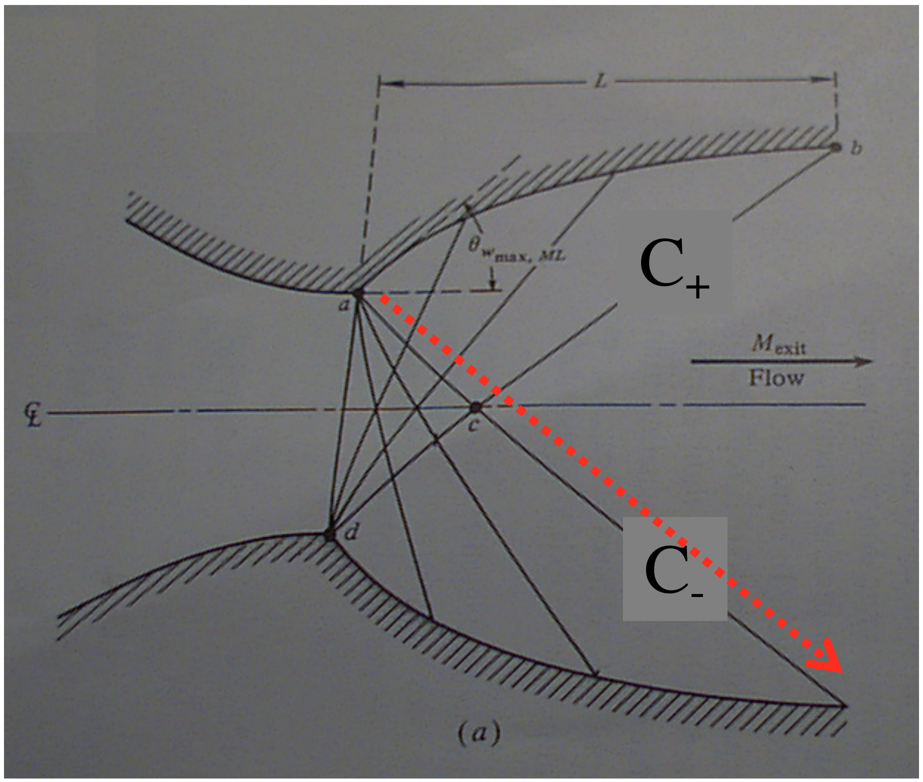
- But from Prandtl-Meyer expansion at point a

$$\theta_{wall_{Max}} = \nu_a - \cancelto{0}{\nu_{M=1.0}} = \nu_a$$



Minimum Length Nozzle Design (cont'd)

$$\nu(1.0) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}}\left(1.0^2-1\right)\right\} - \tan^{-1}\sqrt{1.0^2-1} = 0$$



$$\theta_{w_{Max}} = \nu(M_a) - 0 \rightarrow \theta_{w_{Max}} = \nu(M_a)$$

But as already shown

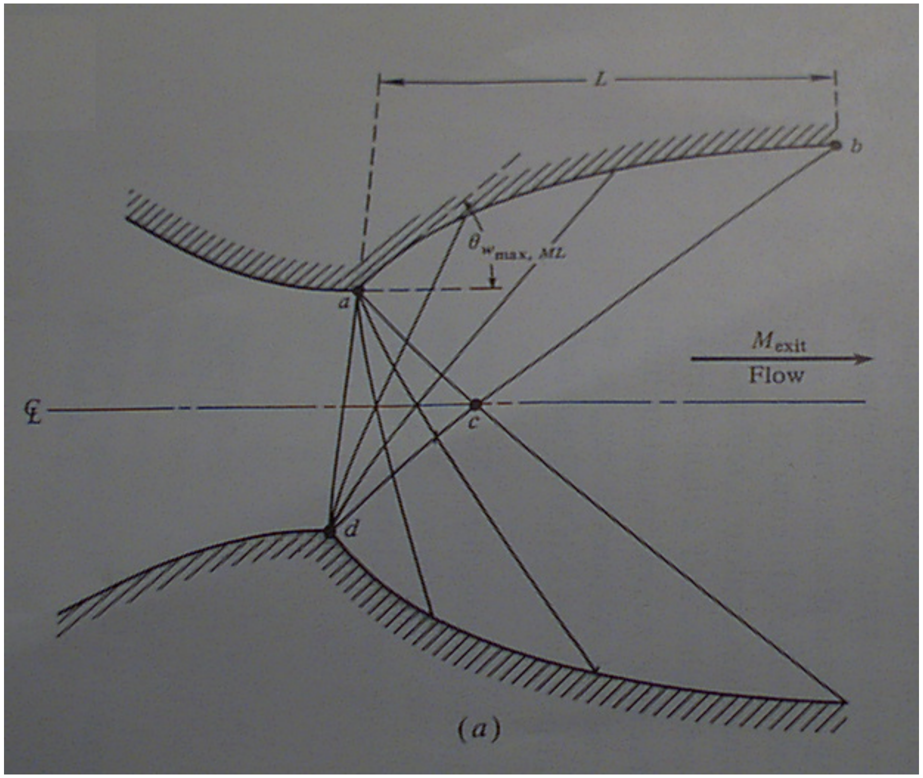
$$\theta_{w_{Max}} + \nu_a = \left(K_-\right)_c = \nu_{exit}$$

$$\theta_{w_{max}} - \nu_a = 0$$

$$2\theta_{w_{Max}} = \left(K_-\right)_c = \nu_{exit}$$

$$\boxed{\theta_{w_{Max}} = \frac{\nu_{exit}}{2}}$$

Minimum Length Nozzle Design (concluded)

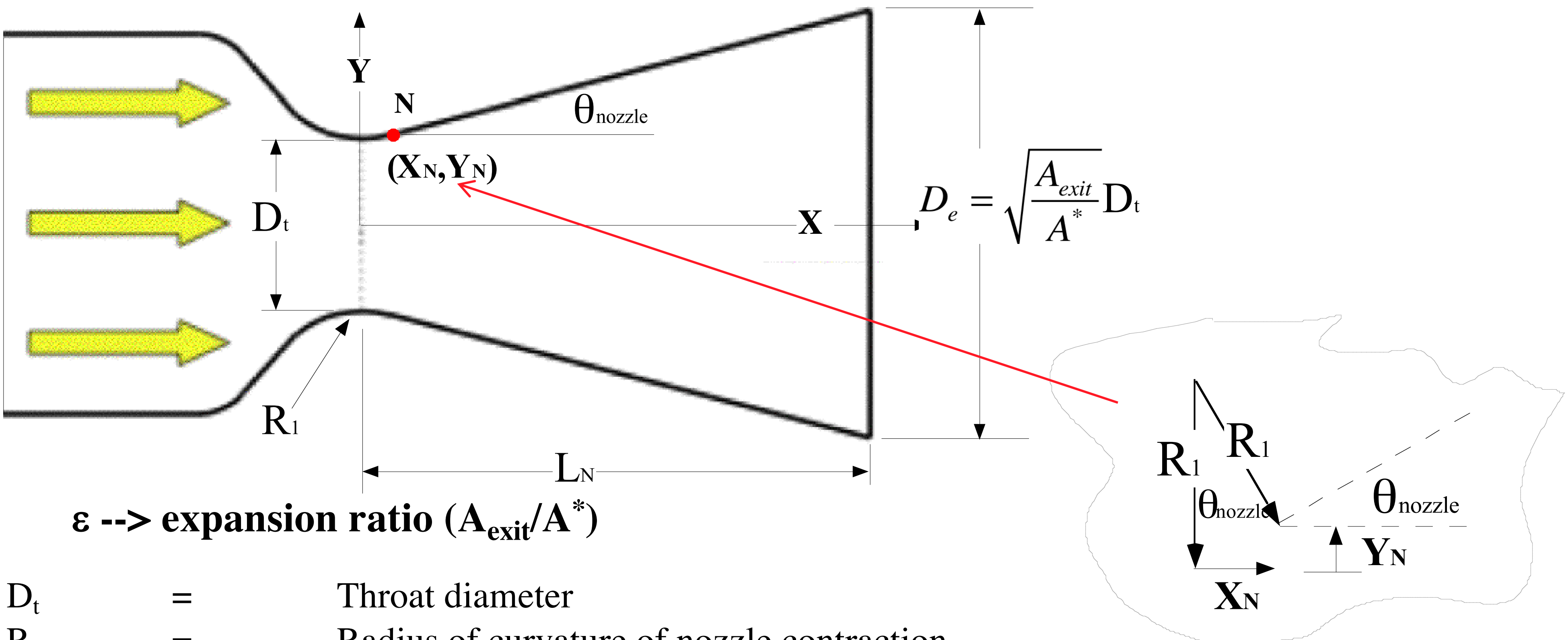


- **Criterion for Minimum Length Nozzle**

$$\theta_{w_{Max}} = \frac{\nu_{exit}}{2}$$

$$\nu(M_{exit}) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_{exit}^{\ 2}-1\right)}\right\} - \tan^{-1}\sqrt{M_{exit}^{\ 2}-1}$$

Typical Conical Nozzle Contour



$$D_e = \sqrt{\frac{A_{exit}}{A^*}} D_t$$

ε --> expansion ratio (A_{exit}/A^*)

D_t	=	Throat diameter
R_1	=	Radius of curvature of nozzle contraction
N	=	Transition point from circular contraction to conical nozzle
L_N	=	Nozzle Length
D_e	=	Exit diameter

$$X_N = R_1 \sin\left(\theta_{nozzle}\right)$$

$$Y_N = \frac{1}{2} D_{throat} + R_1 \left[\ 1 - \cos\left(\theta_{nozzle}\right)\ \right]$$

- $R_1 \sim 0.75 D_t$ is typical

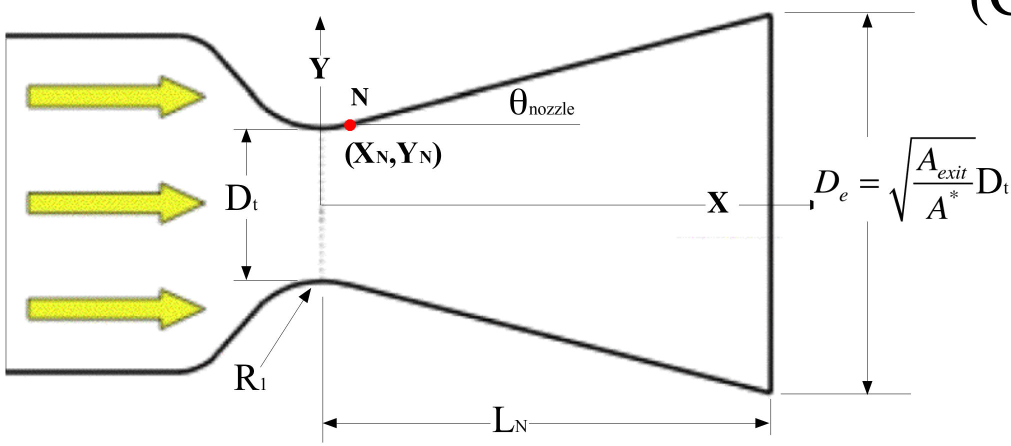
Credit:
Georgia Tech

Utah State University

Mechanical & Aerospace Engineering

Typical Conical Nozzle Contour

(Cont'd)



- Solve for Nozzle length in terms of other parameters

$$\tan(\theta_{nozzle}) = \frac{\frac{1}{2}D_e - \left\{\frac{1}{2}D_{throat} + R_1\left[\ 1-\cos\left(\theta_{nozzle}\right)\ \right]\right\}}{L_N - R_1\sin\left(\theta_{nozzle}\right)} =$$

$$\frac{\frac{1}{2}\left[D_e - D_{throat}\right] - R_1\left[\ 1-\cos\left(\theta_{nozzle}\right)\ \right]}{L_N - R_1\sin\left(\theta_{nozzle}\right)}$$

$$\rightarrow \left\{L_N - R_1\sin\left(\theta_{nozzle}\right)\right\}\tan(\theta_{nozzle}) = \frac{1}{2}\left[\sqrt{\frac{A_{exit}}{A^*}} - 1\right]D_{throat} - R_1\left[\ 1-\cos\left(\theta_{nozzle}\right)\ \right]$$

$$\rightarrow L_N\tan(\theta_{nozzle}) = \frac{1}{2}\left[\sqrt{\frac{A_{exit}}{A^*}} - 1\right]D_{throat} - R_1\left[\ 1-\cos\left(\theta_{nozzle}\right) - \tan(\theta_{nozzle})\sin\left(\theta_{nozzle}\right)\ \right]$$

UtahState UNIVERSITY

Mechanical & Aerospace Engineering

Typical Conical Nozzle Contour

- Using trig identities

(Cont'd)

$$1-\cos\left(\theta_{nozzle}\right)-\tan(\theta_{nozzle})\sin\left(\theta_{nozzle}\right)=1-\cos\left(\theta_{nozzle}\right)-\frac{\sin^2\left(\theta_{nozzle}\right)}{\cos\left(\theta_{nozzle}\right)}=$$

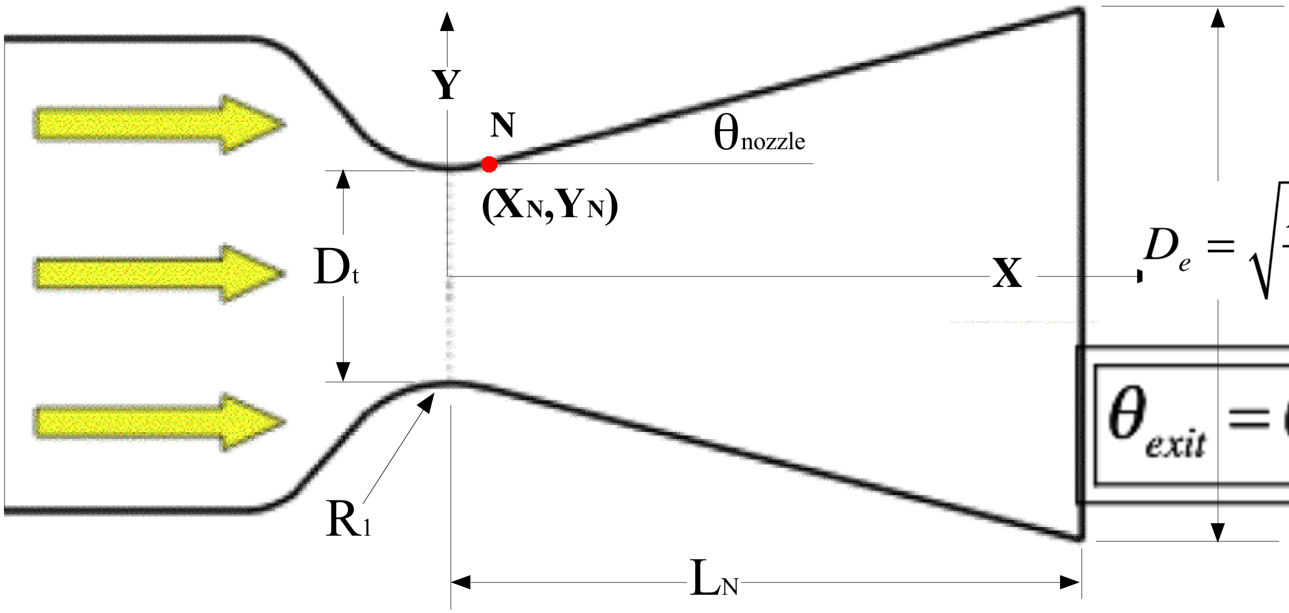
$$1-\frac{\cos^2\left(\theta_{nozzle}\right)+\sin^2\left(\theta_{nozzle}\right)}{\cos\left(\theta_{nozzle}\right)}=1-\frac{1}{\cos\left(\theta_{nozzle}\right)}$$

$$L_N\tan(\theta_{nozzle})=\frac{1}{2}\left[\sqrt{\frac{A_{exit}}{A^*}}-1\right]D_{throat}+R_1\left[\frac{1}{\cos\left(\theta_{nozzle}\right)}-1\right]$$

$$\boxed{L_N=\frac{\frac{1}{2}\left[\sqrt{\frac{A_{exit}}{A^*}}-1\right]D_{throat}+R_1\left[\frac{1}{\cos\left(\theta_{nozzle}\right)}-1\right]}{\tan(\theta_{nozzle})}}$$

- $R_1 \sim 0.75D_t$ is typical

Minimum Length Conical Nozzle (1)



$$D_e = \sqrt{\frac{A_{exit}}{A^*}} D_t$$

$$\boxed{\theta_{exit} = \theta_{nozzle} = \theta_{w_{\max}} \neq 0}$$

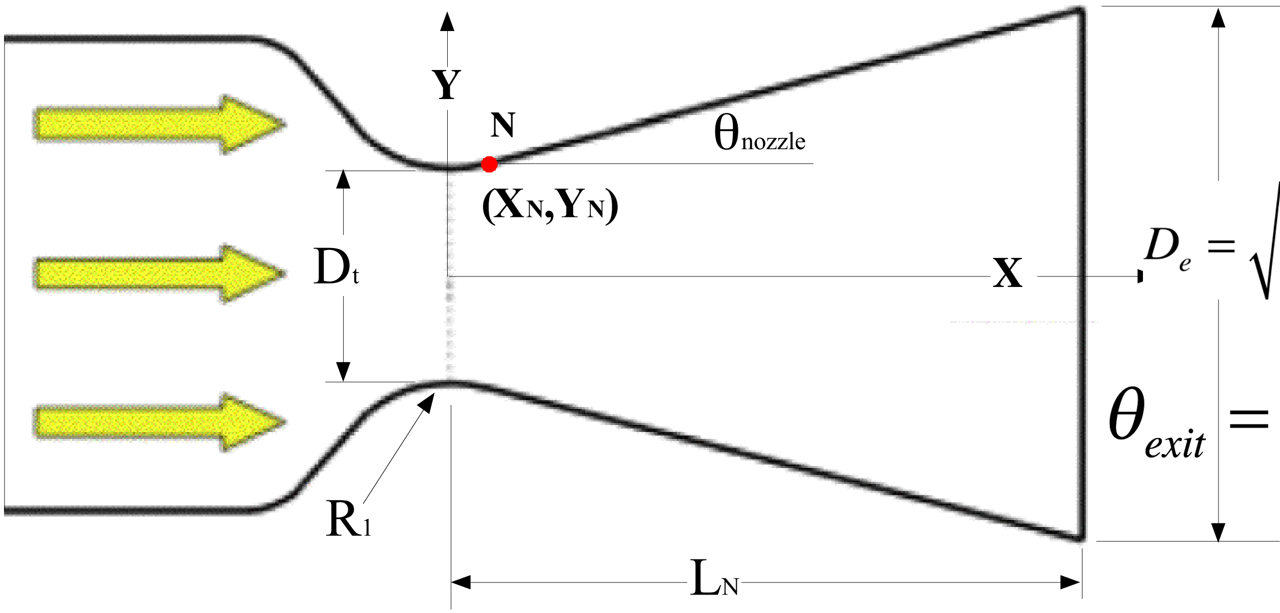
- Modify characteristic along C+ line from C_l to exit plane for non-zero Exit angle

$$\nu_{cl} = \nu_{exit} - \theta_{exit} = \nu_{exit} - \theta_{w_{\max}}$$

- From earlier Minimum Length Nozzle derivation ,,,

$$\theta_{w_{\max}} = \frac{\nu_{cl}}{2} = \frac{\nu_{exit} - \theta_{w_{\max}}}{2}$$

Minimum Length Conical Nozzle (2)



$$D_e = \sqrt{\frac{A_{exit}}{A^*}} D_t$$

$$\theta_{exit} = \theta_{nozzle} = \theta_{w_{max}} \neq 0$$

- Simplify

$$\rightarrow \frac{3}{2}\theta_{w_{max}} = \frac{\nu_{exit}}{2}$$

$$\boxed{\rightarrow \theta_{w_{max}} = \frac{2}{3}\frac{\nu_{exit}}{2} = \frac{\nu_{exit}}{3}}$$

"Two-thirds rule-of-thumb"
Applies strictly for conical nozzles
Generally applied as "safety factor"
for most nozzles

Minimum Length Conical Nozzle

- Example… given

D_{throat} = 1 cm
A_e/A^* = 8
γ = 1.2

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = 8.0 =$$

$$\frac{\left(\left(\frac{2}{1.2+1}\right)\left(1+\frac{1.2-1}{2}(3.122^2)\right)\right)^{\frac{1.2+1}{2(1.2-1)}}}{3.122}$$

$\boxed{M_{exit} = 3.122}$

Minimum Length Conical Nozzle

(cont'd)

$$\boxed{M_{exit} = 3.122}$$

$$\nu(M_{exit}) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_{exit}{}^{2}-1\right)}\right\} - \tan^{-1}\sqrt{M_{exit}{}^{2}-1}$$

$$= \frac{180}{\pi}\left(\left(\frac{1.2+1}{1.2-1}\right)^{0.5}\operatorname{atan}\left(\left(\frac{1.2-1}{1.2+1}\left(3.122^{2}-1\right)\right)^{0.5}\right) - \operatorname{atan}\left(\left(3.122^{2}-1\right)^{0.5}\right)\right)$$

$$= 67.06^{\circ} \longrightarrow \boxed{\theta_{w_{Max}} = \frac{\nu_{exit}}{2}} = 33.53^{\circ}$$

Apply 2/3'rds rule

$$\boxed{\theta_{w_{\max}} = \frac{2}{3}\frac{\nu_{exit}}{2} = 22.35^{o}}$$

Minimum Length Conical Nozzle

(cont'd)

- $R_1 \sim 0.75D_t$ is typical …. R_1=0.75 cm

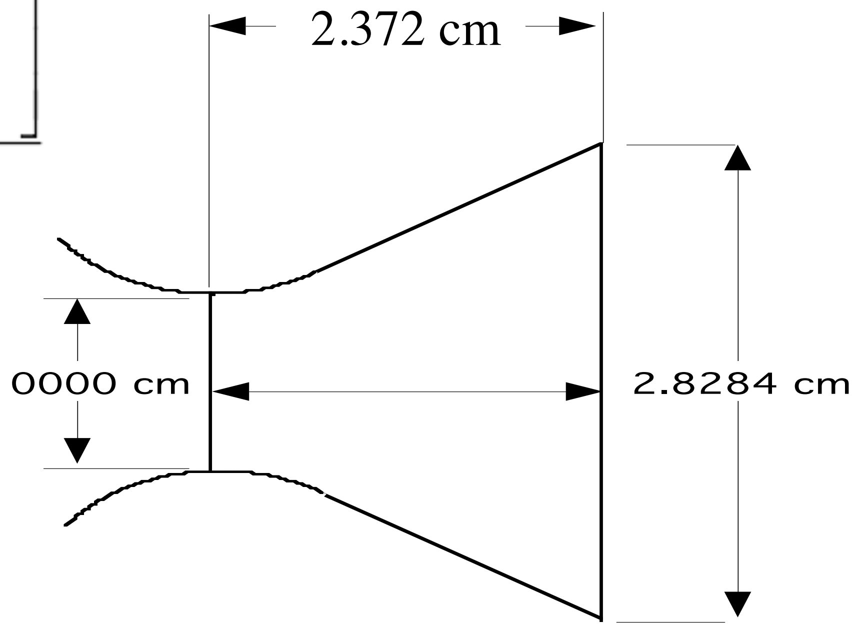
- Any shorter and you have "problems"

$$\boxed{\theta_{w_{max}} = \frac{2}{3}\frac{\nu_{exit}}{2} = 22.35^o}$$

$$[L_N]_{min} = \frac{\frac{1}{2}\left[\sqrt{\frac{A_{exit}}{A^*}} - 1\right]D_{throat} + R_1\left[\frac{1}{\cos\left(\theta_{nozzle_{max}}\right)} - 1\right]}{\tan(\theta_{nozzle_{max}})}$$

$$= \frac{\frac{1}{2}(8^{0.5} - 1)\,1.0 + 0.75\cdot 1\left(\frac{1}{\cos\left(\frac{\pi}{180}22.35\right)} - 1\right)}{\tan\left(\frac{\pi}{180}22.35\right)}$$

= 2.372 cm



Comparison of Cone and Bell Nozzles

For the same ε, we would expect $\lambda_{bell} > \lambda_{cone}$

A bell nozzle, while more complex to build, will generally yield a more efficient exhaust than a cone in a shorter nozzle length.

Same nozzle efficiency factor can be reached with about 70% of the length of a cone nozzle.

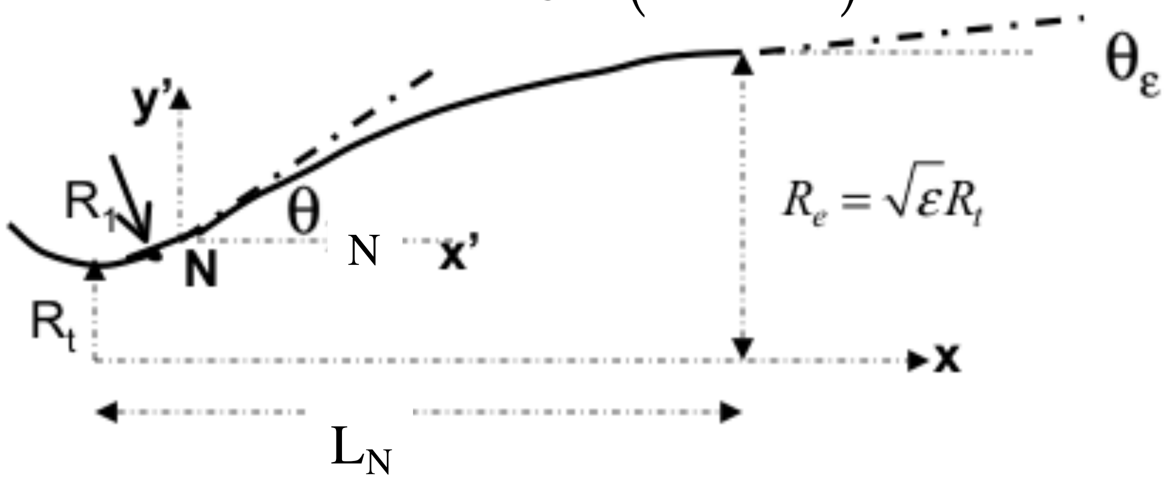
Alternatively, efficiency factor can be increased from about 98% for a cone to about 99.2% for a bell of the same length

Credit: Narayanan Komerath Georgia Tech

Bell Nozzle Contour Design

approximate shape can be formed from a parabola

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2}$$



$R_1 = 1.5R_t$ **Upstream of the throat**

ε --> expansion ratio (A_{exit}/A^*)

$R_1 = 0.382R_t$ **Downstream of the throat**

- $X = X' + X_N$
- $Y = Y' + Y_N$

$$X_N = R_1 \sin\left(\theta_{nozzle}\right)$$

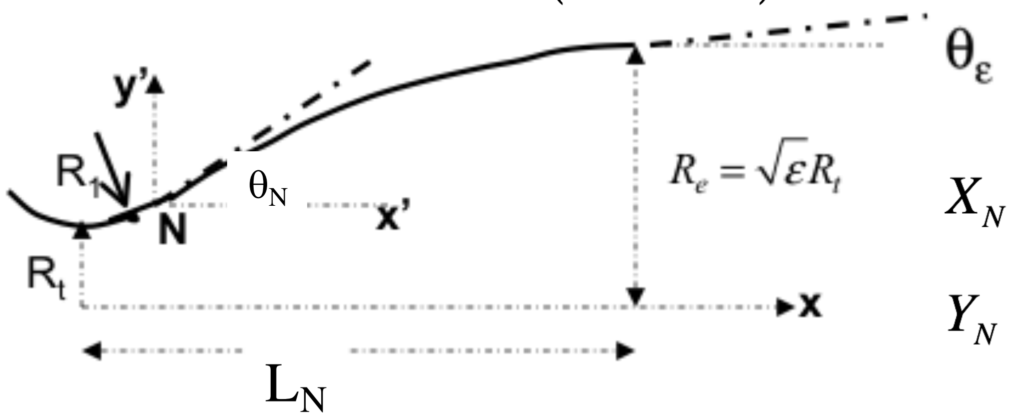
$$Y_N = \frac{1}{2} D_{throat} + R_1 \left[\ 1 - \cos\left(\theta_{nozzle}\right)\ \right]$$

Bell Nozzle Contour Design (cont'd)

- 4 unknowns in parabolic segment (P,Q,S,T)
- 4 boundary conditions

ε --> expansion ratio (A_{exit}/A^*)

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2}$$



$$X_N = R_1 \sin\left(\theta_{nozzle}\right)$$

$$Y_N = \frac{1}{2} D_{throat} + R_1 \left[\ 1 - \cos\left(\theta_{nozzle}\right)\ \right]$$

- Boundary Conditions

$$@N: \quad X'_N = 0 \quad Y'_N = 0$$

$$@exit: \quad \begin{bmatrix} X'_e = X_e - X_N \\ = L_N - X_N \end{bmatrix} \quad \begin{bmatrix} Y'_e = Y_e - Y_N \\ = \sqrt{\varepsilon} R_t - Y_N \end{bmatrix}$$

- θ_e
- θ_N

Given

Credit: Narayanan Komerath Georgia Tech

Bell Nozzle Contour Design (cont'd)

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2}$$

- Evaluate position boundary condition at *N*

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2} \rightarrow$$

$$@N : e = P \times \left[L_N - X_N\right] + Q + \left(S \times 0 + T\right)^{1/2} \rightarrow \boxed{Q^2 = T}$$

- Evaluate slope boundary condition at *N*

$$\tan\theta_N = \left(\frac{dY'}{dX'}\right)_N = P + \frac{1}{2}\frac{1}{\left(S \times X'_N + T\right)^{1/2}} \times S =$$

$$P + \frac{S}{2}\frac{1}{\left(S \times 0 + T\right)^{1/2}} \rightarrow Q = -T^{1/2} \rightarrow \boxed{\tan\theta_N = P - \frac{S}{2Q}}$$

Bell Nozzle Contour Design (cont'd)

$$Y^{'} = PX^{'} + Q + \left(SX^{'} + T\right)^{1/2}$$

- Rearranging slope boundary condition at *N*

$$Q = -\frac{S}{2\left(\tan\theta_N - P\right)}$$

- Evaluate Slope Boundary condition at *e*

$$\tan\theta_e = \left(\frac{dY^{'}}{dX^{'}}\right)_e = P + \frac{1}{2}\frac{S}{\left(S \times X^{'}_e + T\right)^{1/2}} \rightarrow$$

$$rearranging \rightarrow \left(S \times X^{'}_e + T\right)^{1/2} = \frac{S}{2\left(\tan\theta_e - P\right)}$$

Bell Nozzle Contour Design (cont'd)

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2}$$

- Evaluate Position Boundary Condition at e

$$Y'_e = PX'_e + Q + \left(SX'_e + T\right)^{1/2} \rightarrow$$

$$\left(SX'_e + T\right)^{1/2} = Y'_e - PX'_e$$

- And the Collection expressions are

$$\left(SX'_e + T\right)^{1/2} = Y'_e - PX'_e$$

$$Q = -T^{1/2}$$

$$Q = -\frac{S}{2\left(\tan\theta_N - P\right)}$$

$$\left(SX'_e + T\right)^{1/2} = \frac{S}{2\left(\tan\theta_e - P\right)}$$

Bell Nozzle Contour Design (cont'd)

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2}$$

1) $\left(SX'_e + T\right)^{1/2} = Y'_e - PX'_e$

2) $Q = -T^{1/2}$

3) $Q = -\dfrac{S}{2\left(\tan\theta_N - P\right)}$

4) $\left(SX'_e + T\right)^{1/2} = \dfrac{S}{2\left(\tan\theta_e - P\right)}$

- 4 equations in 4 unknowns
- Analytical Solution is a Mess getting there .. But result is OK

$$\boxed{\begin{aligned} X &= X' + X_N \\ Y &= Y' + Y_N \end{aligned}}$$

$$X_N = R_1 \sin\left(\theta_{nozzle}\right)$$

$$Y_N = \frac{1}{2} D_{throat} + R_1\left[\ 1 - \cos\left(\theta_{nozzle}\right)\ \right]$$

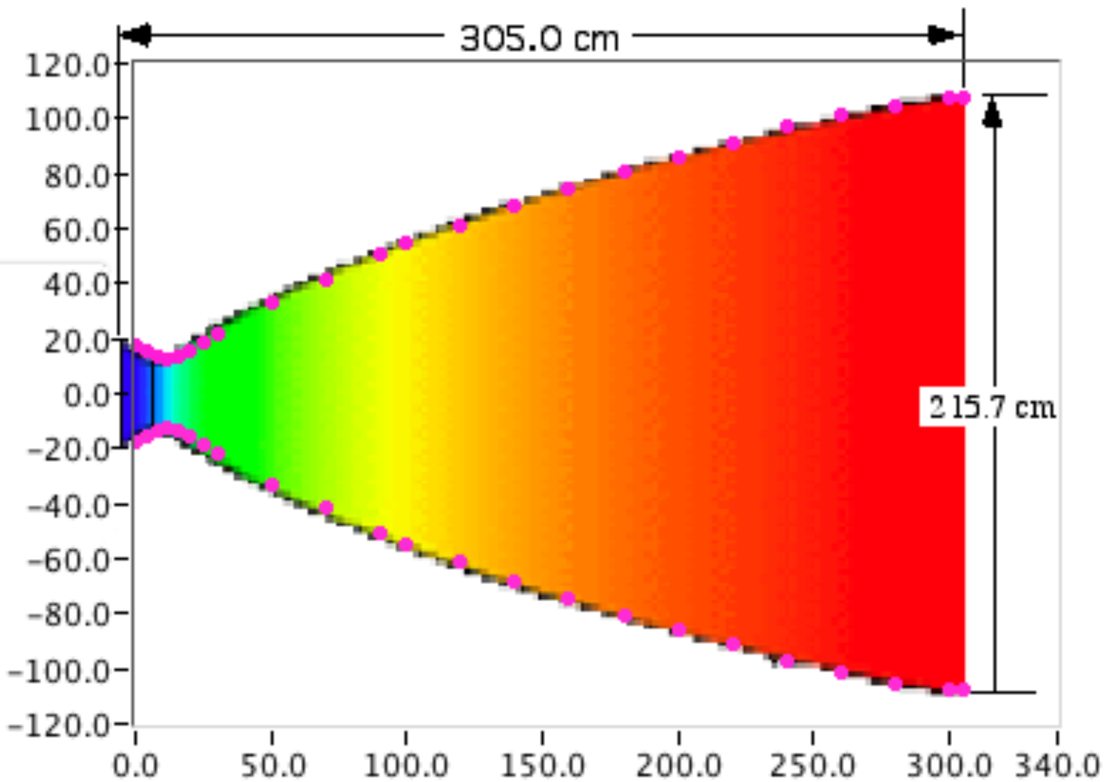
$$i) \rightarrow P = \frac{Y'_e\left(\tan\theta_N + \tan\theta_e\right) - 2X'_e \tan\theta_e \tan\theta_N}{2Y'_e - X'_e \tan\theta_N - X'_e \tan\theta_e}$$

$$ii) \rightarrow S = \frac{\left(Y'_e - PX'_e\right)^2\left(\tan\theta_N - P\right)}{X'_e \tan\theta_N - Y'_e}$$

$$iii) \rightarrow Q = -\frac{S}{2\left(\tan\theta_N - P\right)}$$

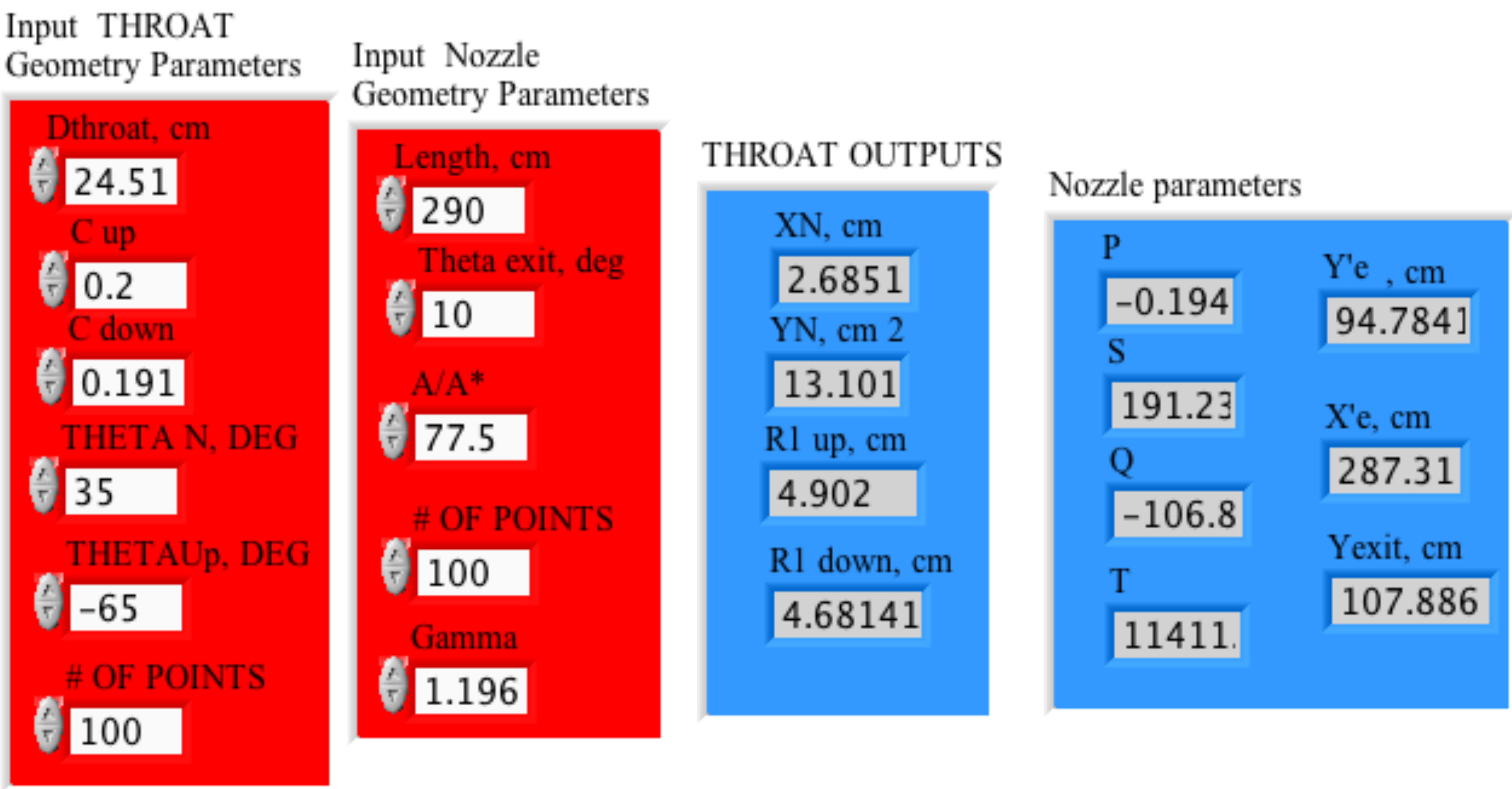
$$iv) \rightarrow T = Q^2$$

SSME Nozzle example



SSME Nozzle example (cont'd)

- Fit with Parabolic bell profile

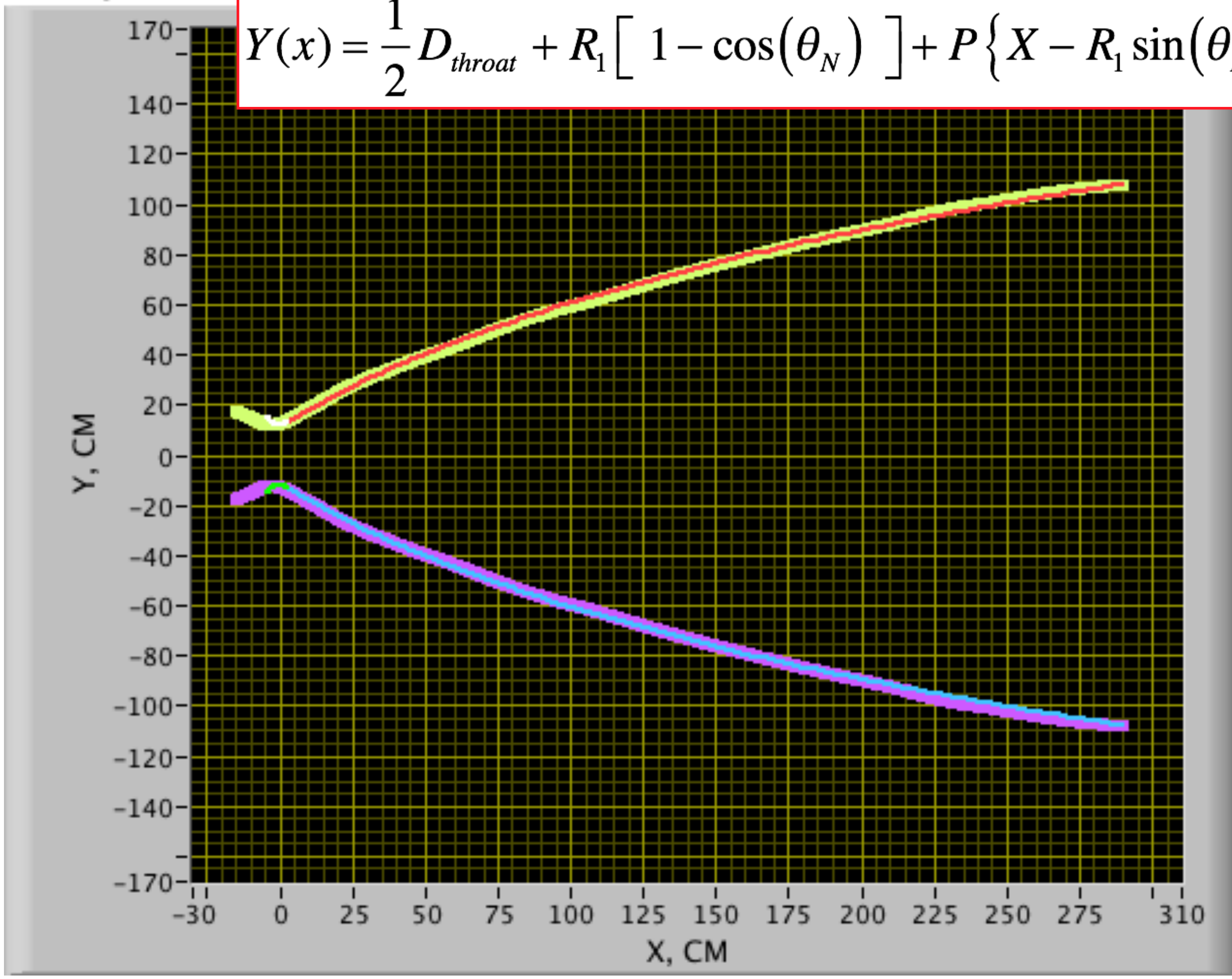


SSME Nozzle example (cont'd)

- Fit with Parabolic bell profile

XY Graph

$$Y(x) = \frac{1}{2} D_{throat} + R_1\left[\ 1 - \cos(\theta_N)\ \right] + P\left\{X - R_1 \sin(\theta_N)\right\} + Q + \left(S\left\{X - R_1 \sin(\theta_N)\right\} + T\right)^{1/2}$$



BOUNDARY CONDITIONS

θ_e	$=10°$
θ_N	$=35\ °$
D_{throat}	$=24.5$ cm
A_e/A^*	$=77.5$
R_1	$=4.681$cm

- Pretty good model

SSME Nozzle example (Cont'd)

- $M_{exit} = 4.677$

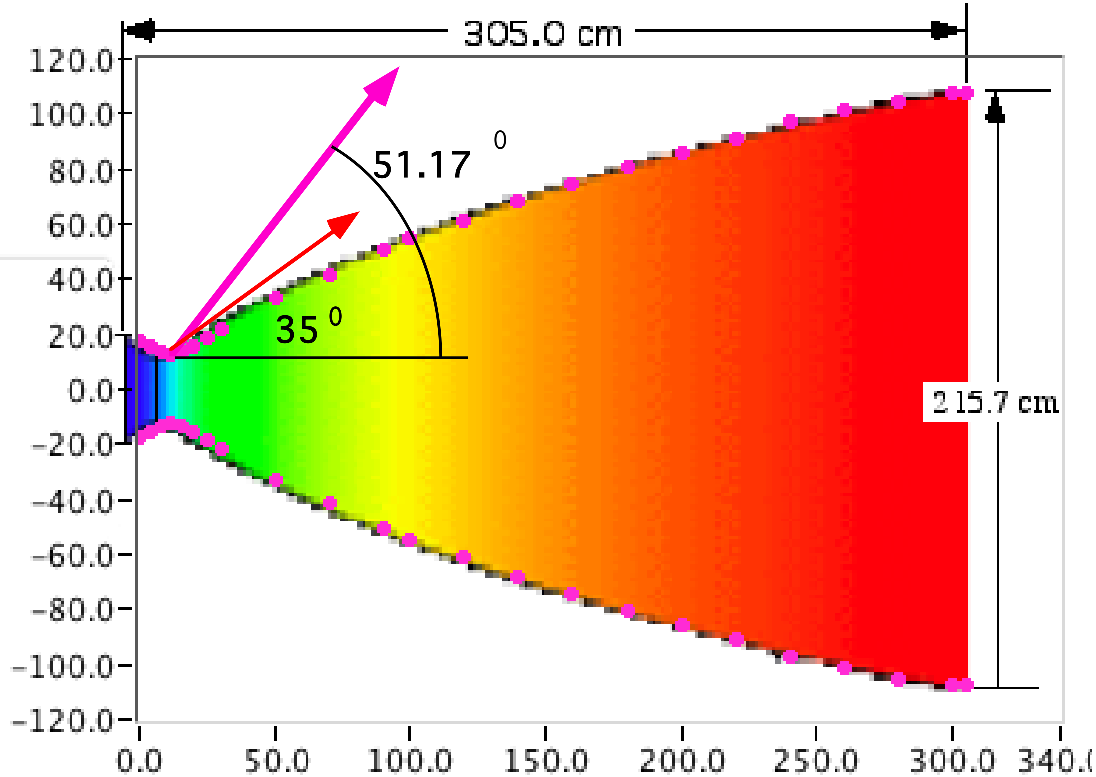
$$\nu(M_{exit}) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_{exit}{}^2-1\right)}\right\} - \tan^{-1}\sqrt{M_{exit}{}^2-1}$$

$$= \frac{180}{\pi}\left(\left(\frac{1.196+1}{1.196-1}\right)^{0.5} \text{atan}\left(\left(\frac{1.196-1}{1.196+1}(4.677^2-1)\right)^{0.5}\right) - \text{atan}\left(\left(4.677^2-1\right)^{0.5}\right)\right)$$

$$= 102.34° \longrightarrow \boxed{\theta_{w_{Max}} = \frac{\nu_{exit}}{2}} = 51.17°$$

SSME Nozzle example (cont'd)

$$\boxed{\theta_{w_{Max}} = \frac{\nu_{exit}}{2}} = 51.17°$$



- ***SSME is definitely not a minimum length nozzle***

$$35/51.7 = 0.677$$

"two thirds rule"

SSME Nozzle example (cont'd)

$$\boxed{\theta_{w_{Max}} = \frac{\nu_{exit}}{2}} = 51.17°$$

• ~ "minimum length SSME Nozzle

Rule of Thumb
Use $\theta_N < 2/3 \ \theta_{max}$

"two thirds rule"

XY Graph