# Rocket Science 102:
# How High will My Rocket Go?

## Newton's Laws as Applied to "Rocket Science"

**... its not just a job ... its an adventure**

**Sellers: Chapter 14**

# RS 101: Summary

***•Rocket Thrust Equation***

$$F_{thrust} = \dot{m}V_{exit} + A_{exit}\left(P_{exit} - P_{atmosphere}\right)$$

***•Specific Impulse .. Total …. Instantaneous)***

$$I_{sp} = \frac{I_{mpulse}}{g_0 M_{propellant}} = \frac{\int_0^t F_{thrust}\,dt}{g_0 \int_0^t \dot{m}_{propellant}\,dt} \qquad \frac{F_{thrust}}{g_0\,\dot{m}_{propellant}}$$

***•Rocket Equation***

$$\Delta V_{Burn} = g_0 I_{sp} \ln\left[1 + \frac{m_{propellant}}{M_{final}}\right]$$

***•Propellant Mass Budget Equation***

$$M_{\substack{fuel\\+\,oxidizer}} = \left[M_{dry} + M_{payload}\right]\left[e^{\left[\frac{\Delta V_{burn}}{g_0 I_{sp}}\right]} - 1\right]$$

# Summary (2)

- *Propellant Mass Budget Equation*

$$M_{\substack{\text{fuel}\\+\text{oxidizer}}} = \left[M_{dry} + M_{payload}\right]\left[e^{\left[\frac{\Delta V_{burn}}{g_0 I_{sp}}\right]} - 1\right]$$

- *Load Mass Fraction*

$$L_{mf} \equiv \frac{M_{propellant}}{M_{initial}} = \frac{M_{propellant}}{M_{final} + M_{initial}} 1 - e^{-\left(\frac{\Delta V}{g_0 \cdot I_{sp}}\right)}$$

# Summary (4)

Available $\Delta V$

$$\Delta V_{t_{burn}} = g_0 \cdot I_{sp}\left[\ln\left(1+P_{mf}\right)\right] - $$

$$\text{"}combustion\ \Delta V\text{"}$$

$$\int_0^{t_{burn}} g(t)\cdot\sin\theta\, dt - \sqrt{\int_0^{t_{burn}} \frac{\rho V^3}{\beta} dt}$$

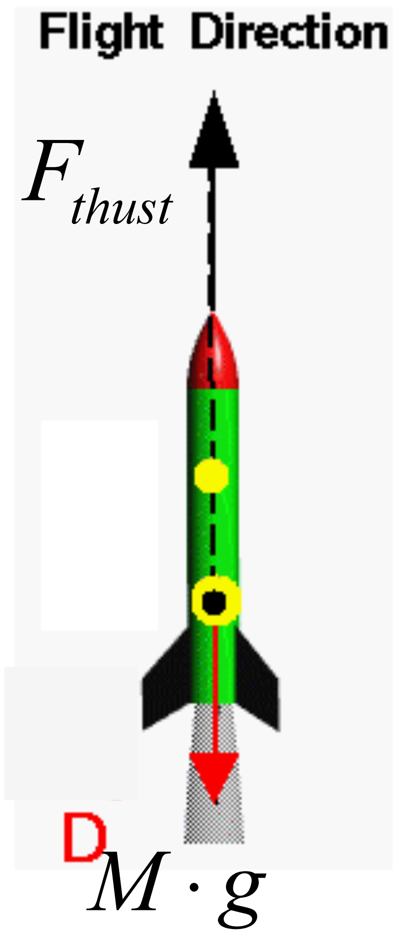
$$\text{"}gravity\ loss\text{"} \qquad \text{"}drag\ loss\text{"}$$

$$\beta = \frac{M}{C_D \cdot A_{ref}}$$

Path dependent velocity losses

# Summary (5)

## $\Delta V$ for a Vertically Accelerating Vehicle (7)




$$\boxed{\Delta V_{t_{burn}} = g_0 \cdot I_{sp}\left[\ln\left(1 + P_{mf}\right)\right] - g_0 \cdot t_{burn}}$$

$$\rightarrow \boxed{\begin{array}{l} P_{mf} = \dfrac{M_{propellant}}{M_{final}} \\ \\ t_{burn} = \dfrac{g_0 \cdot I_{sp} \cdot m_{propellamt}}{F_{thrust}} \end{array}}$$

# $\Delta V$ for a Vertically Accelerating Vehicle ...

•*Calculate burnout altitude*

$Instantaneously:$

$$\frac{dh}{dt} = \left( g_0 \cdot I_{sp} \ln\left( \frac{M_{initial}}{M_{initial} - \dot{m} \cdot t} \right) - g_0 \cdot t \right)$$

$at\ Burnout: (Above\ ground\ level - AGL)$

$$h_{t_{burn}} = \int_0^{t_{burn}} \left( g_0 \cdot I_{sp} \ln\left( \frac{M_{initial}}{M_{initial} - \dot{m} \cdot \tau} \right) - g_0 \cdot \tau \right) \cdot d\tau$$

•*After a lot of arithmetic!*

$$\boxed{h_{t_{burn}} = -\frac{g_0 \cdot t_{burn}{}^2}{2} + \left( g_0 \cdot I_{sp} \right) \cdot \left( \frac{M_{initial}}{\dot{m}} \cdot \ln\left( \frac{M_{final}}{M_{initial}} \right) + t_{burn} \cdot \left( 1 + \ln\left( \frac{M_{initial}}{M_{final}} \right) \right) \right)}$$

# ΔV for a Vertically Accelerating Vehicle (6)

**•*Collecting terms and simplifying***

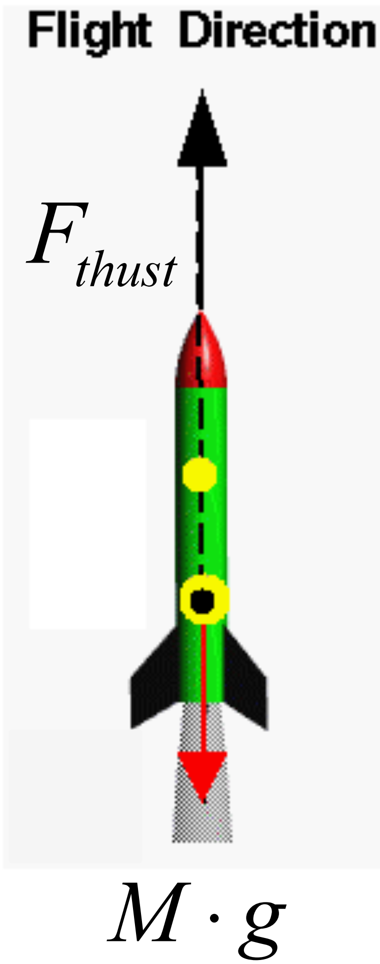
$$h_{t_{burn}} = -\frac{g_0 \cdot t_{burn}^{\ 2}}{2} + \left(g_0 \cdot I_{sp}\right) \cdot \left( \frac{M_{initial}}{\dot{m}} \cdot \ln\left(\frac{M_{final}}{M_{initial}}\right) + t_{burn} \cdot \left(1 + \ln\left(\frac{M_{initial}}{M_{final}}\right)\right)\right)$$

$$\frac{M_{initial}}{\dot{m}} = \frac{M_{final} + m_{propellant}}{m_{propellant} / t_{burn}}$$

$$\rightarrow \boxed{h_{t_{burn}} = g_0 \cdot t_{burn} \left\{ I_{sp} \cdot \left(1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}}\right) - \frac{t_{burn}}{2} \right\}}$$

# ΔV for a Vertically Accelerating Vehicle (7)

**•*Summary***




$$h_{t_{burn}} = g_0 \cdot t_{burn} \left\{ I_{sp} \cdot \left( 1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}} \right) - \frac{t_{burn}}{2} \right\}$$

$$V_{t_{burn}} = g_0 \cdot I_{sp} \left[ \ln\left(1 + P_{mf}\right) \right] - g_0 \cdot t_{burn}$$

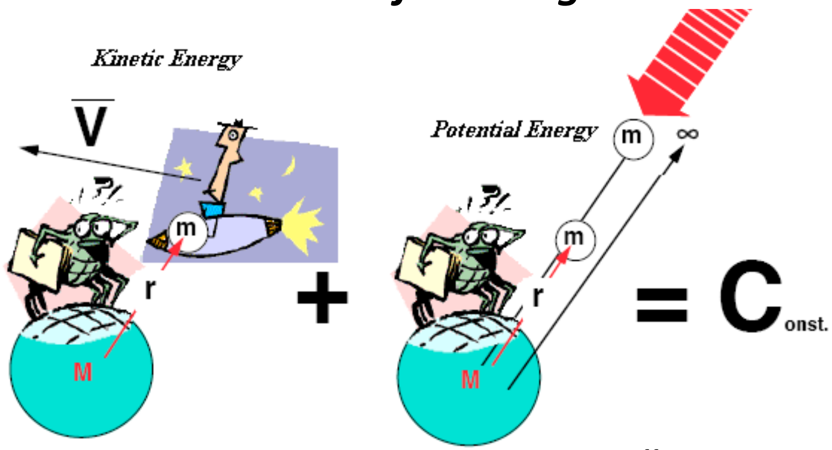
$\rightarrow$

$$P_{mf} = \frac{M_{propellant}}{M_{final}}$$

$$t_{burn} = \frac{g_0 \cdot I_{sp} \cdot m_{propellamt}}{F_{thrust}}$$

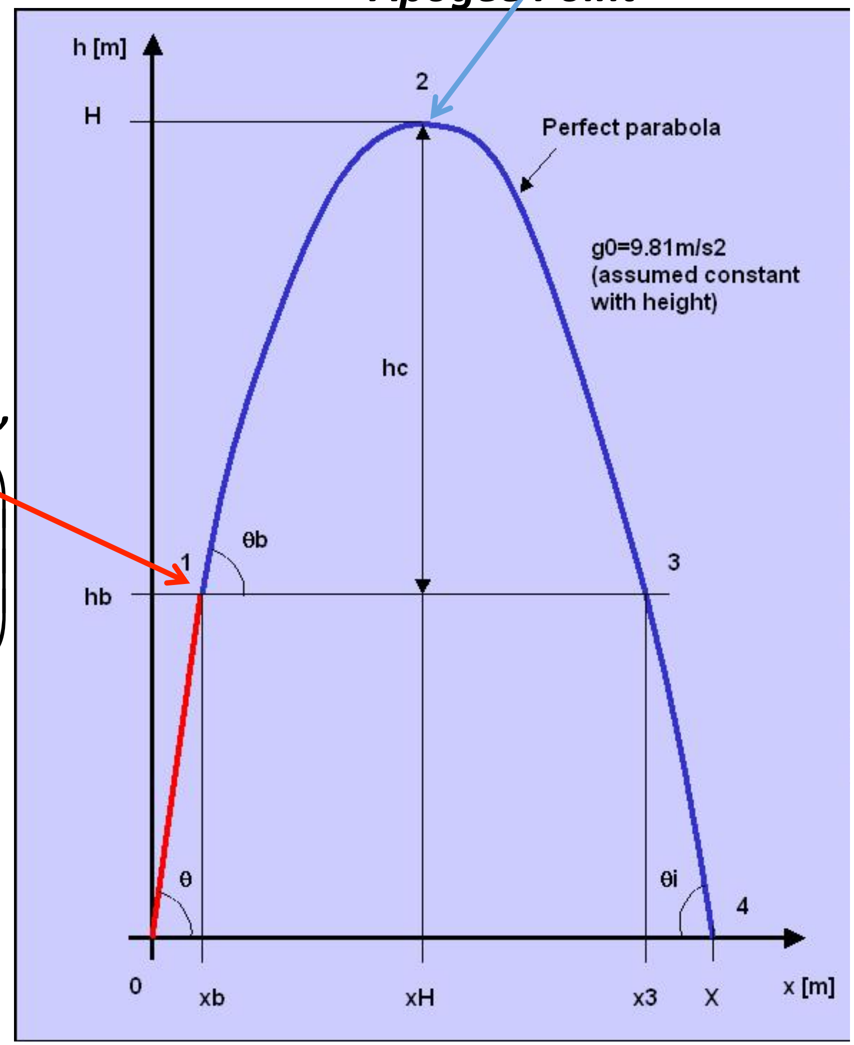
# How High will my Rocket go?

***In the absence of dissipative (friction, etc) forces … total mechanical energy of rocket remains constant following motor burnout***




**"Apogee Point"**

**"Motor Burnout"**




***At Motor Burnout…***

$$E_{mech} = M_{final}\left(\frac{\left(V_{burnout}\right)^2}{2} + g_0 \cdot h_{burnout}\right)$$

***At Apogee Point…***

$$E_{mech} = M_{final}\left(\frac{\left(V_{apogee}\right)^2}{2} + g_0 \cdot h_{apogee}\right)$$

$$\approx M_{final} \cdot g_0 \cdot h_{apogee}$$

# How High will my Rocket go? (2)

***Solving for*** **h**$_{\text{apogee}}$

$$E_{mech} = M_{final}\left(\frac{\left(V_{apogee}\right)^2}{2} + g_0 \cdot h_{apogee}\right) \approx M_{final} \cdot g_0 \cdot h_{apogee}$$

$$h_{apogee} \approx \frac{E_{mech}}{M_{final} \cdot g_0} = \frac{\left(V_{burnout}\right)^2}{2 \cdot g_0} + h_{burnout}$$

$$\boxed{h_{apogee} \approx g_0 \left[\frac{1}{2}\left(I_{sp}\left[\ln\left(1+P_{mf}\right)\right] - g_0 \cdot t_{burn}\right)^2 + t_{burn}\left\{I_{sp} \cdot \left(1 - \frac{\ln\left(1+P_{mf}\right)}{P_{mf}}\right) - \frac{t_{burn}}{2}\right\}\right]}$$

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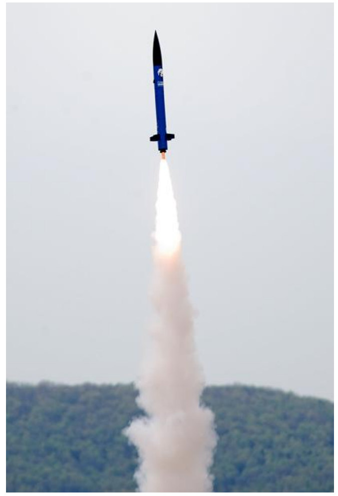
# Example Calculation



*2009 USLI Rocket*

*AMW L777 Motor*

**"Dry" vehicle mass : *11.2451 kg*, Propellant mass: 1.7623 kg**

**Propellant $I_{sp}$: 181.49sec, Mean Motor Thrust: 774.475 Newtons**

$$P_{mf} = \frac{m_{propellant}}{M_{final}} = \frac{1.7623}{11.2451} = 0.156717$$

$$t_{burn} = \frac{g_0 \cdot I_{sp} \cdot M_{propellamt}}{F_{thrust}} = \frac{9.8067 \cdot 181.49 \cdot 1.7623}{774.475} = 4.04993 \text{ sec}$$



CHIMAERA

# Example Calculation (2)

$$P_{mf} = \frac{m_{propellant}}{M_{final}} = 0.156717 \qquad t_{burn} = \frac{g_0 \cdot I_{sp} \cdot M_{propellamt}}{F_{thrust}} = 4.04993 \text{ sec}$$

$$h_{t_{burn}} = g_0 \cdot t_{burn} \left\{ I_{sp} \cdot \left( 1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}} \right) - \frac{t_{burn}}{2} \right\} =$$

$$9.8067 \cdot 4.04933 \left( 181.49 \left( 1 - \frac{\ln\left(1 + 0.156717\right)}{0.156717} \right) - \frac{4.04993}{2} \right)$$ **= 431.5 meters**

$$V_{t_{burn}} = g_0 \cdot I_{sp} \left[ \ln\left(1 + P_{mf}\right) \right] - g_0 \cdot t_{burn} =$$

$$9.8067 \cdot 181.49 \left( \ln\left(1 + 0.156717\right) \right) - 9.8067 \cdot 4.004993$$ **= 219. 5 m/sec**

# Example Calculation (3)

*Calculate Apogee Altitude (above ground level)*

$$h_{apogee} \approx \frac{E_{mech}}{M_{final} \cdot g_0} = \frac{\left(V_{burnout}\right)^2}{2 \cdot g_0} + h_{burnout} =$$

$$\frac{219.5^2}{2 \cdot 9.8067} + 431.5$$ = 2888 meters

# Compare to Simulation Results

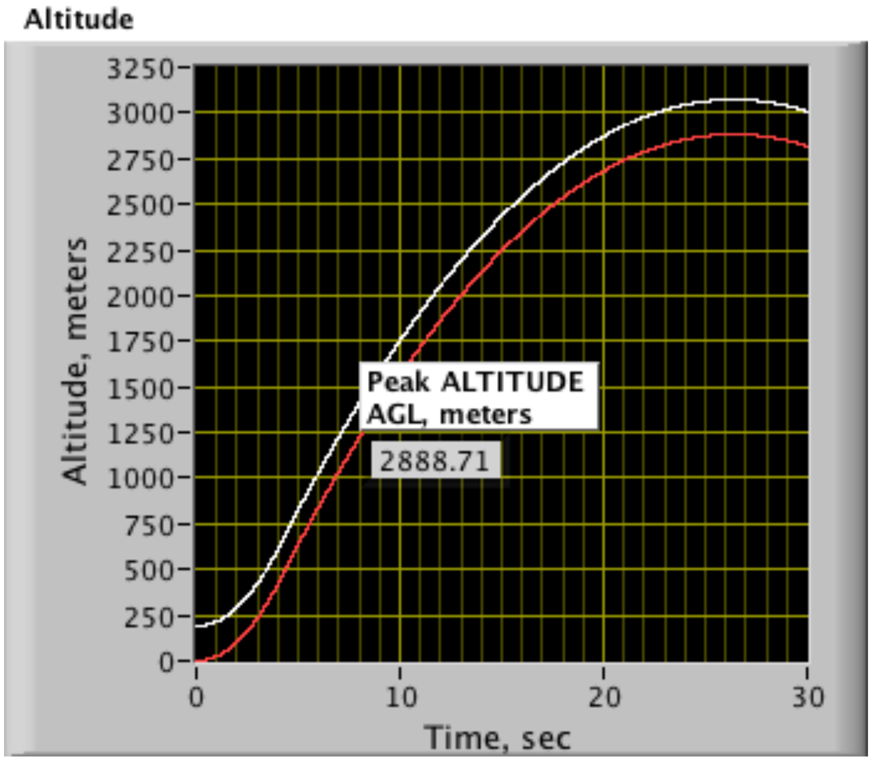
***We will build this simulation later***

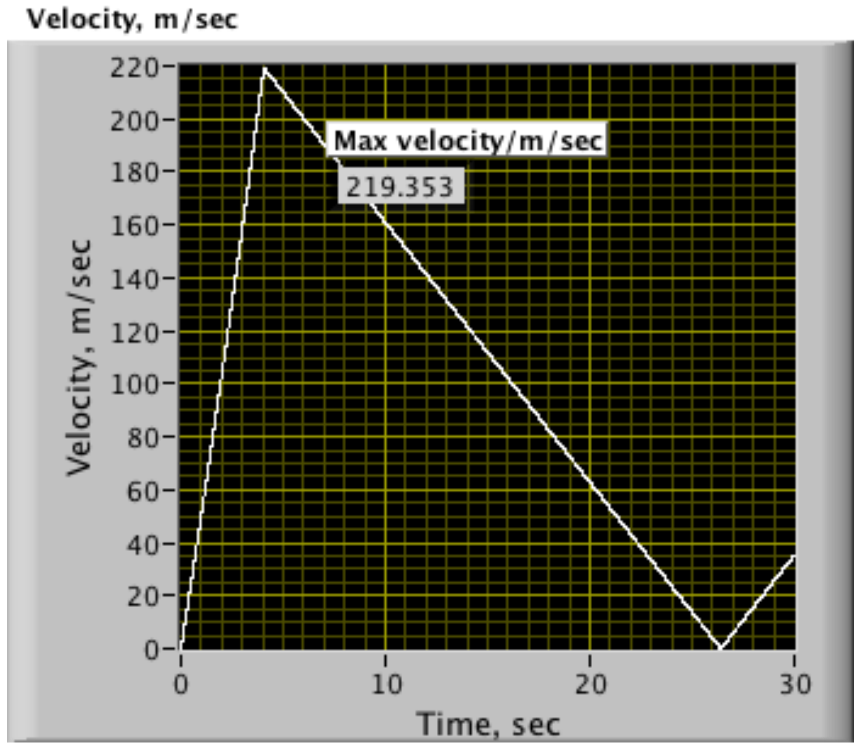
$$h_{apogee} = 2888.71 \ m$$
$$V_{burnout} = 219.34 \ m/\sec$$

Analytical Solution
***=> 2888 meters***

***= 219. 5 m/sec***

***Ignoring drag for now!***



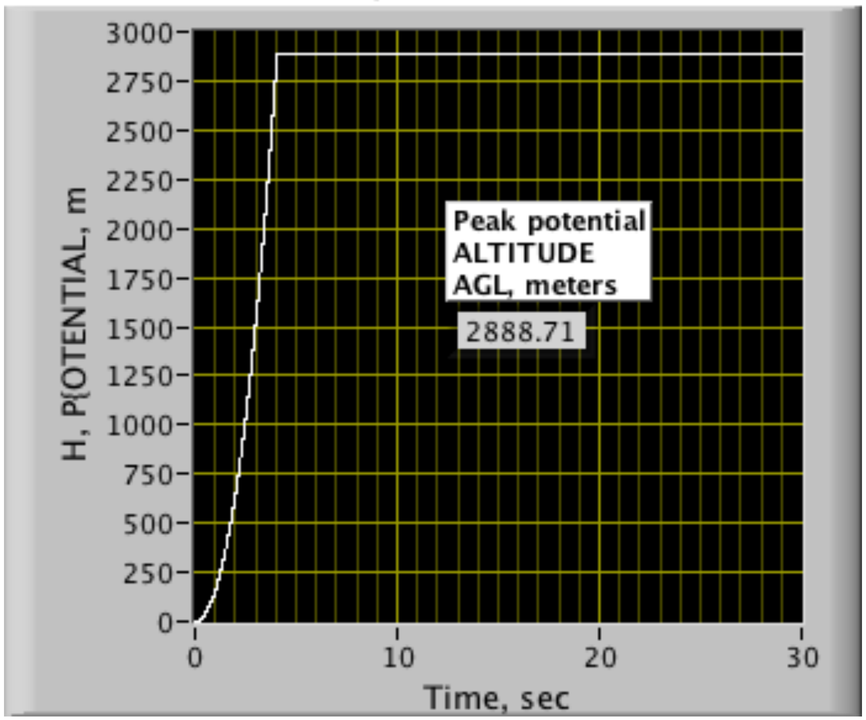

***Better than 0.056%***

# Compare to Simulation Results (2)

$$\boxed{\begin{aligned} h_{apogee} &= 2888.71\ m \\ V_{burnout} &= 219.34\ m/\sec \end{aligned}}$$

***Ignoring drag for now!***

POTENTIAL ALTITUDE, agl




$$h_{apogee} \approx h_{potential} =$$

$$\frac{E_{mech}}{M_{final} \cdot g_0} = \frac{\left(V_{burnout}\right)^2}{2 \cdot g_0} + h_{burnout}$$

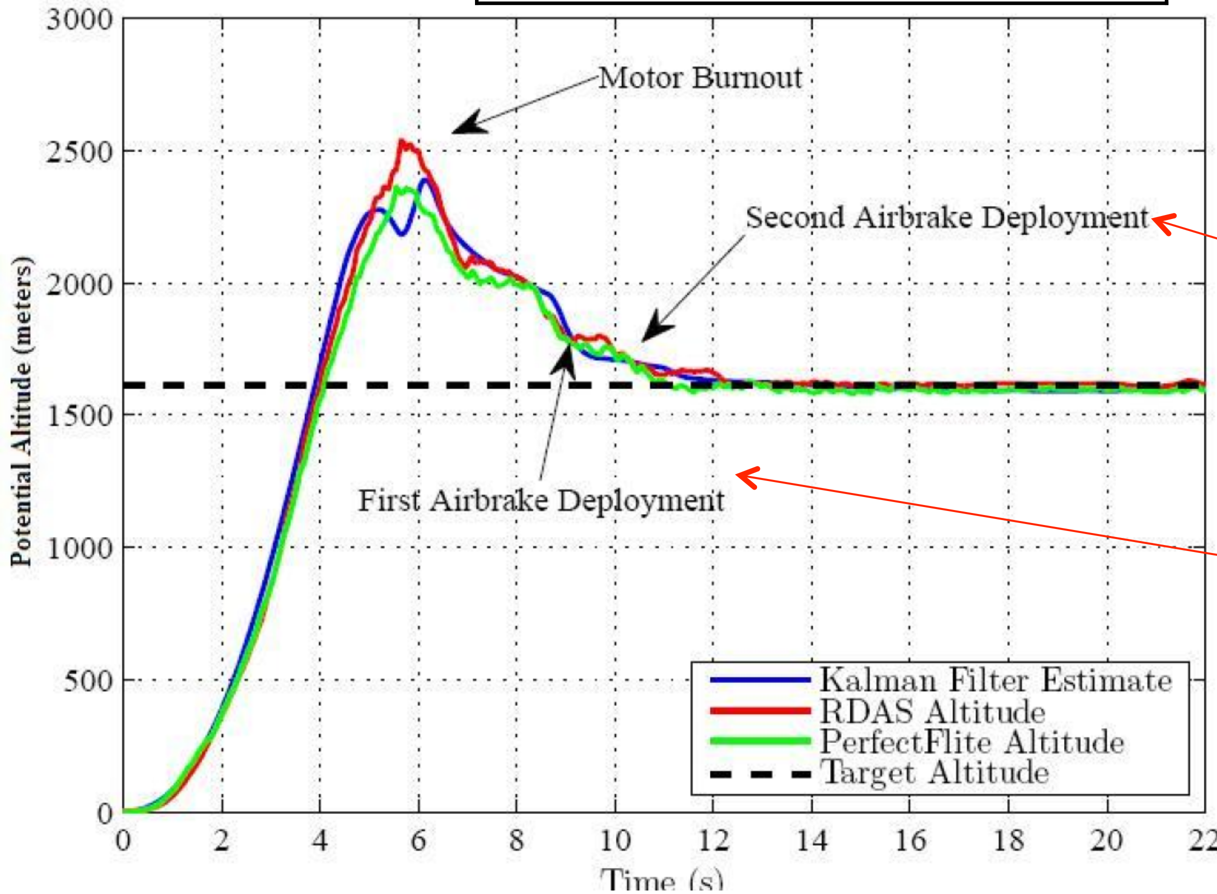
# Compare to Fight Data

$$h_{apogee} = 2888.71\ m$$
$$V_{burnout} = 219.34\ m / \sec$$

**Why the difference?**

**Primarily …. Drag!**

*~ 15% error in peak altitude*




$$h_{potential} = h + \frac{V^2}{2 \cdot g_0}$$

# Compare to Fight Data (2)

**Why the difference? Drag!**

$$\frac{(\Delta V_{apogee})^2}{2} = \frac{E_{mech}}{M_{final}} \rightarrow \Delta V_{apogee} = \sqrt{\frac{2 \cdot E_{mech}}{M_{final}}} = \sqrt{2 \cdot g_0 \cdot h_{potential}}$$

$$\frac{(\Delta V_{apogee})_{calc} - (\Delta V_{apogee})_{flight}}{\frac{(\Delta V_{apogee})_{calc} + (\Delta V_{apogee})_{flight}}{2}} \times 100_{\%} =$$

$$\frac{(2888.71 \cdot 2 \cdot 9.8067)^{0.5} - (2500 \cdot 2 \cdot 9.8067)^{0.5}}{\frac{(2888.71 \cdot 2 \cdot 9.8067)^{0.5} + (2500 \cdot 2 \cdot 9.8067)^{0.5}}{2}} = 7.22\%$$

***~ 7.2 % error in delivered apogee $\Delta V$***

# How Drag Losses Effect Peak Altitude

$$\Delta V_{drag} = \sqrt{2\int_0^t \frac{D_{rag}V}{M}dt} = \sqrt{2\int_0^t \frac{C_D A_{ref}\frac{1}{2}\rho V^2 \cdot V}{M}dt} = \sqrt{\int_0^t \frac{\rho \cdot V^3}{\beta}dt}$$

$$\Delta h_{drag} = \frac{\Delta V_{drag}{}^2}{2 \cdot g_0} = \frac{1}{2 \cdot g_0 \cdot \beta}\left[\int_0^t \rho \cdot V^3 dt\right]$$

**Check units!**

$$\sim \frac{1}{\frac{m}{\sec^2} \cdot \frac{kg}{m^2}} \cdot \frac{kg}{m^3} \cdot \left(\frac{m}{\sec}\right)^3 \cdot \sec = \frac{\sec^2 \cdot m^5}{m \cdot kg} \cdot \frac{kg}{m^3} \cdot \frac{1}{\sec^2} = m$$

# How Drag Losses Effect Peak Altitude (2)

$$\Delta h_{drag} = \frac{1}{2 \cdot g_0 \cdot \beta}\left[\int_0^t \rho \cdot V^3 dt\right]$$

***Correct peak altitude estimate***

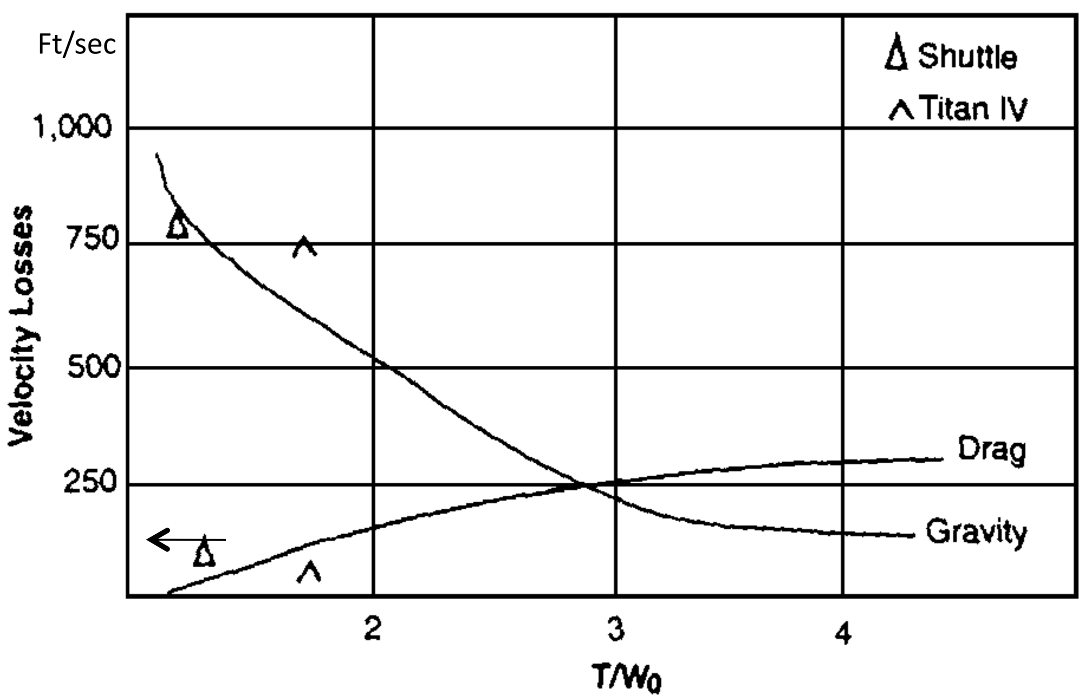
$$h_c = \frac{g_0}{4} \cdot \left(2 + P_{mf}\right) \cdot \left(I_{sp} \cdot \ln\left(1 + P_{mf}\right)\right)^2 - \Delta h_{drag} =$$

$$\underbrace{\frac{g_0}{4} \cdot \left(2 + P_{mf}\right) \cdot \left(I_{sp} \cdot \ln\left(1 + P_{mf}\right)\right)^2}_{\textit{Path Independent}} - \underbrace{\frac{1}{2 \cdot g_0 \cdot \beta}\left[\int_0^t \rho \cdot V^3 dt\right]}_{\textit{Path Dependent}}$$

"Rule of thumb" ~ drag loss is about 5-10% of delivered $\Delta V$ from motor

# Drag Losses (3)

$$D_{rag} = C_D A_{ref} \frac{1}{2} \rho V^2 \rightarrow \Delta V_{drag} = \sqrt{A_{ref} \int_0^t \frac{C_D \rho V^3}{m} dt} = \sqrt{\int_0^t \frac{\rho V^3}{\beta} dt}$$




Depending
On thrust to-weight
Off of the pad
drag losses
can be significant
During motor burn

As much as 12-15% of
Potential altitude

***… path dependent!***

***Must simulate trajectory***

# Drag Coefficient is Configuration Dependent

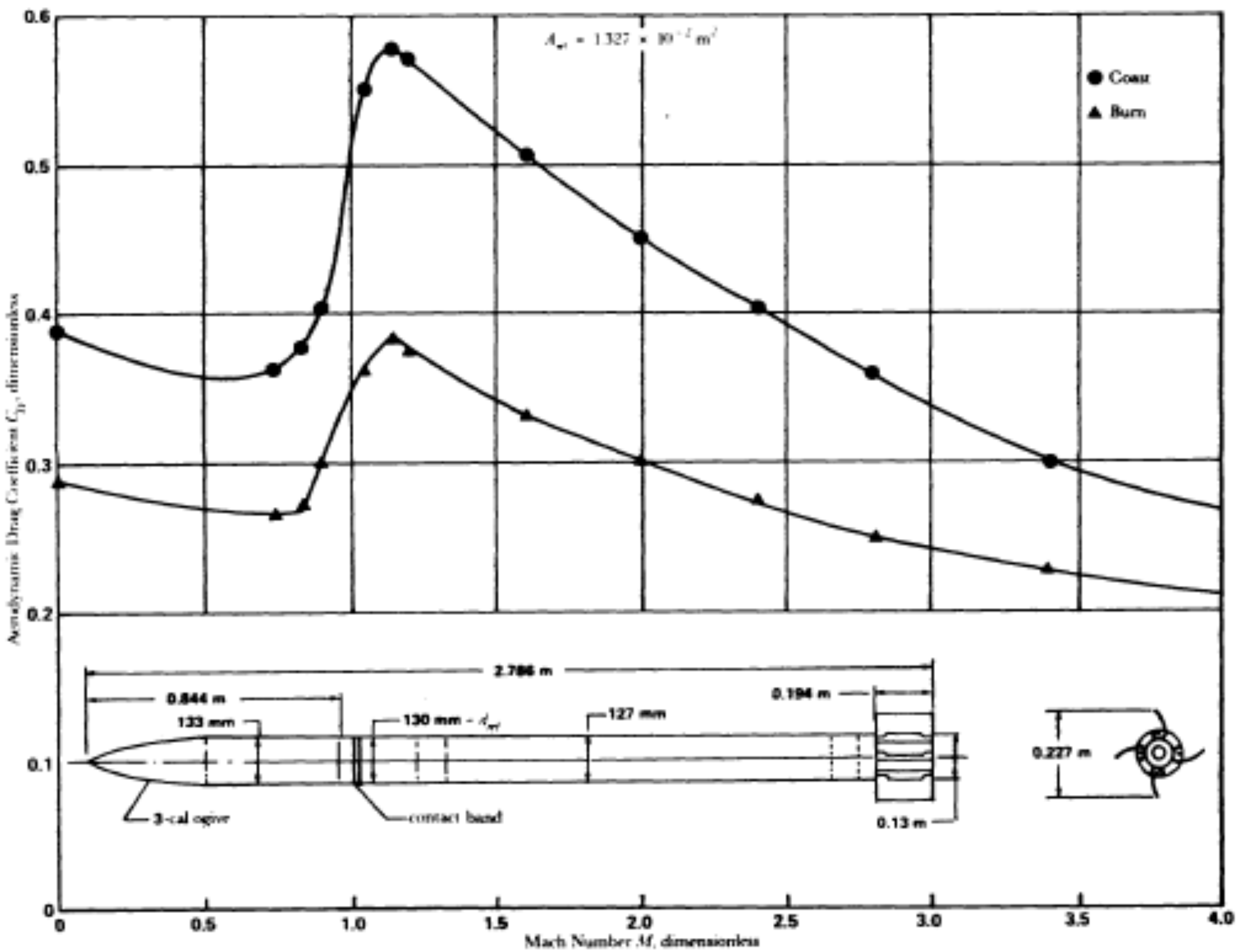



Figure 3-4. Drag Coefficient vs Mach Number-HO-mm Rocket

# Drag Coefficient is Configuration Dependent (2)

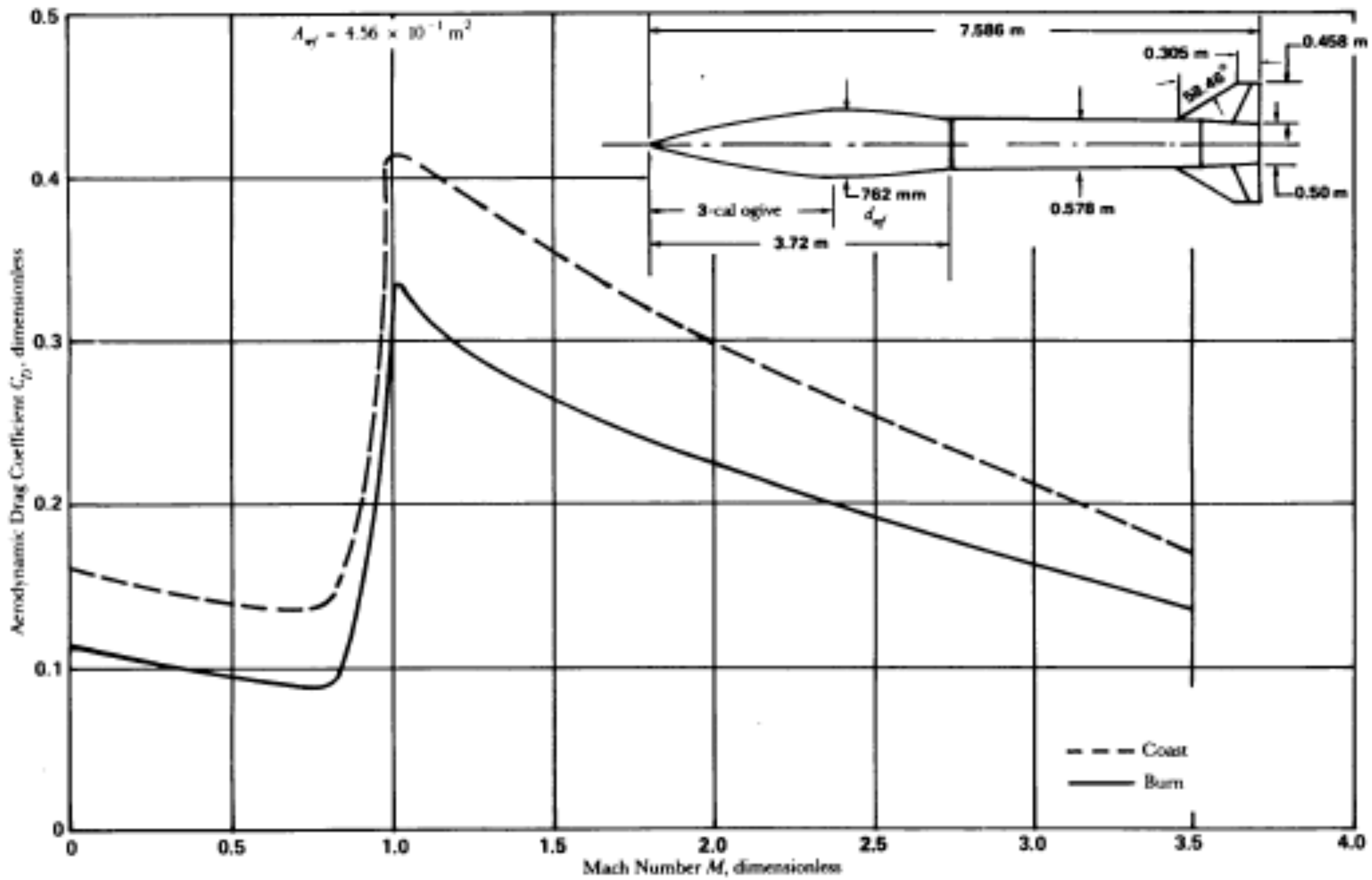



Figure 3-5. Drag Coefficient vs Mach Number—762-mm Rocket

# Drag Coefficient is Configuration Dependent (3)

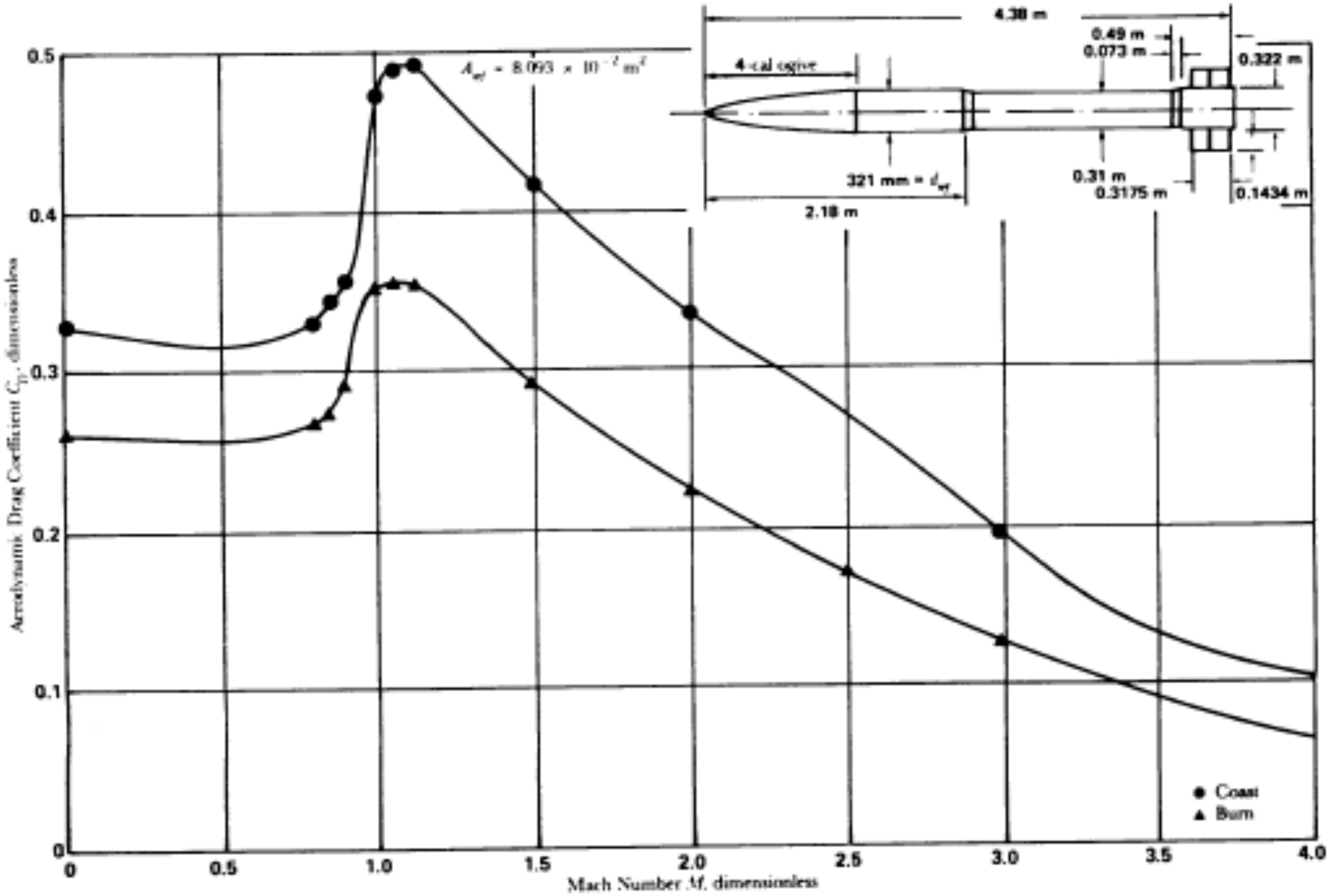



Figure 3-6. Drag Coefficient vs Mach Numbr-321-mm Rocket

# A Recipe for Energy Management

$$\textit{Conservation of Energy:}$$

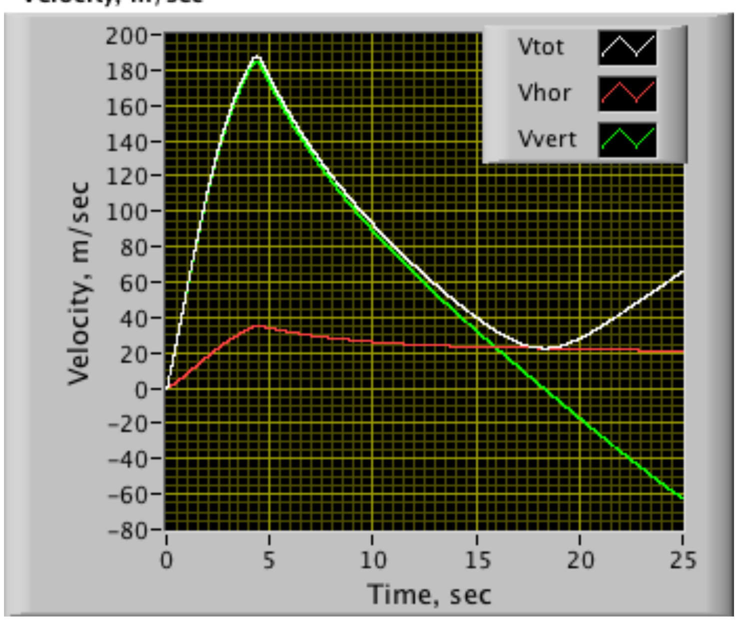
$$\textit{Potential + Kinetic Energy} = \textit{Constant} - \textit{Dissipated Energy}$$

$$g \cdot h_{apogee} + \frac{V^2_{apogee}}{2} = h_{burnout} + \frac{V^2_{burnout}}{2} - \int_{t_{burnout}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

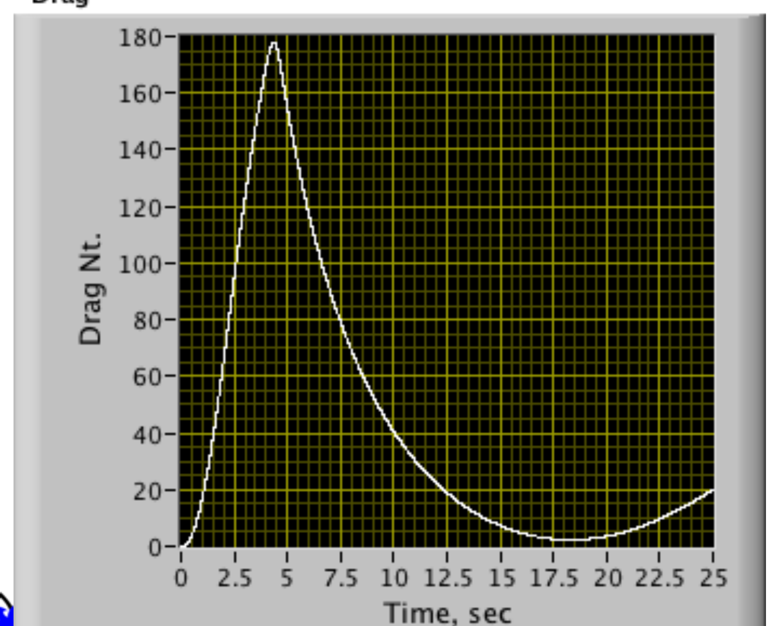
$$\rightarrow h_{apogee} = h_{burnout} + \left( \frac{V^2_{burnout}}{2 \cdot g} - \frac{V^2_{apogee}}{2 \cdot g} \right) - \frac{1}{g} \int_{t_{burnout}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

- **Velocity and drag are Very high just after motor burnout .. But diminish near apogee**
- **Specific Energy of Rocket becomes "more constant" with time**

Velocity, m/sec

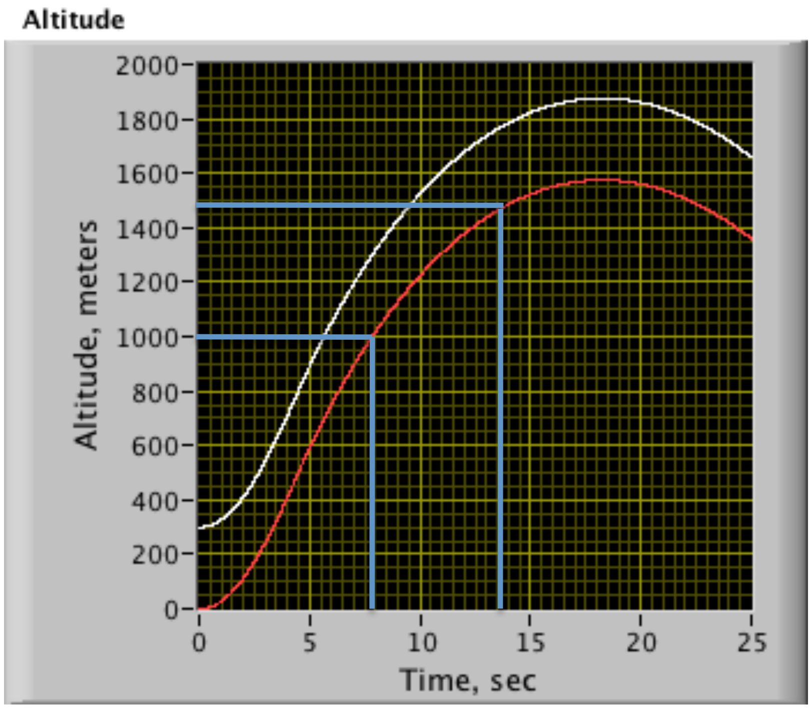


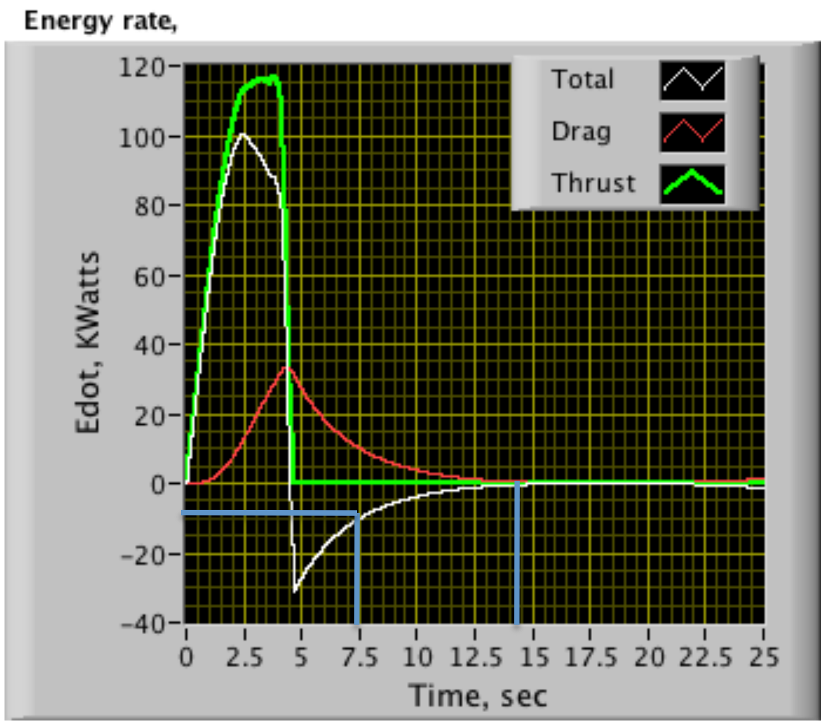

Drag



# A Recipe for Energy Management (2)

- **Specific Energy of Rocket becomes "more constant" with time (*and altitude*)**



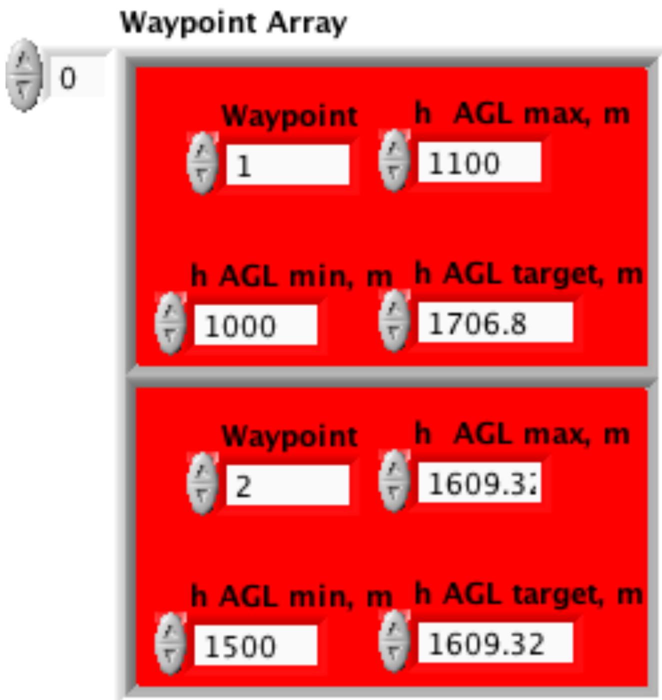

- At motor burn out Drag Energy Dissipation rate is ~3.5 times higher than at 1000 m AGL

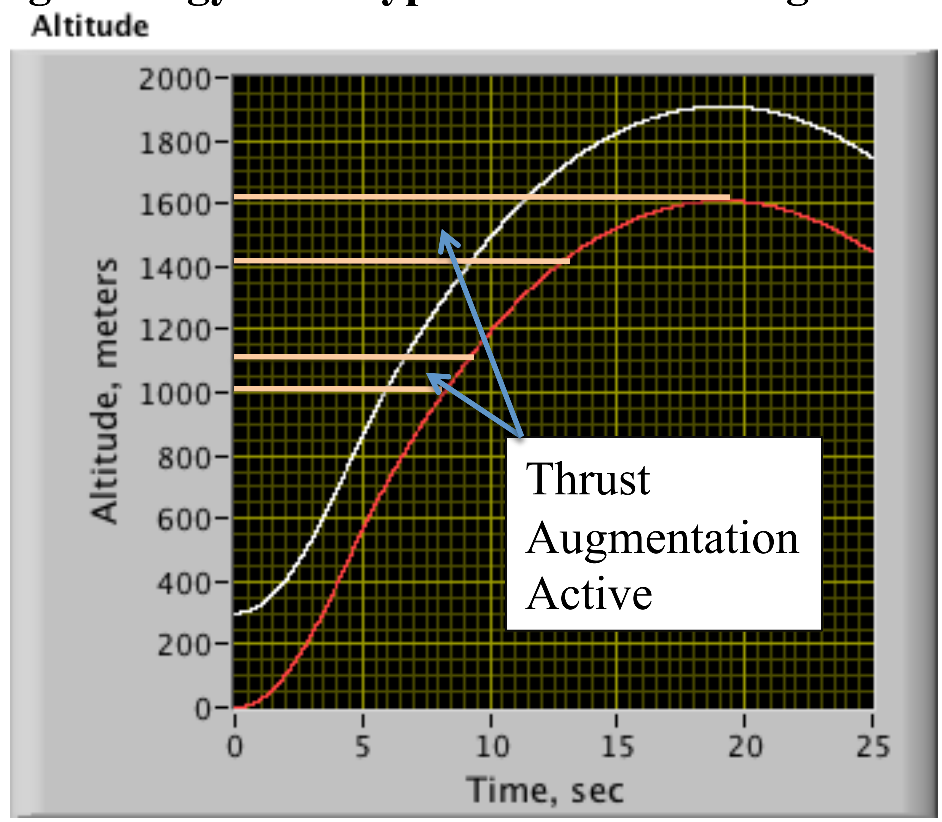
- At 1500 m AGL Drag Energy Dissipation is essentially zero ..
  Estimated energy level~ constant

# A Recipe for Energy Management (3)

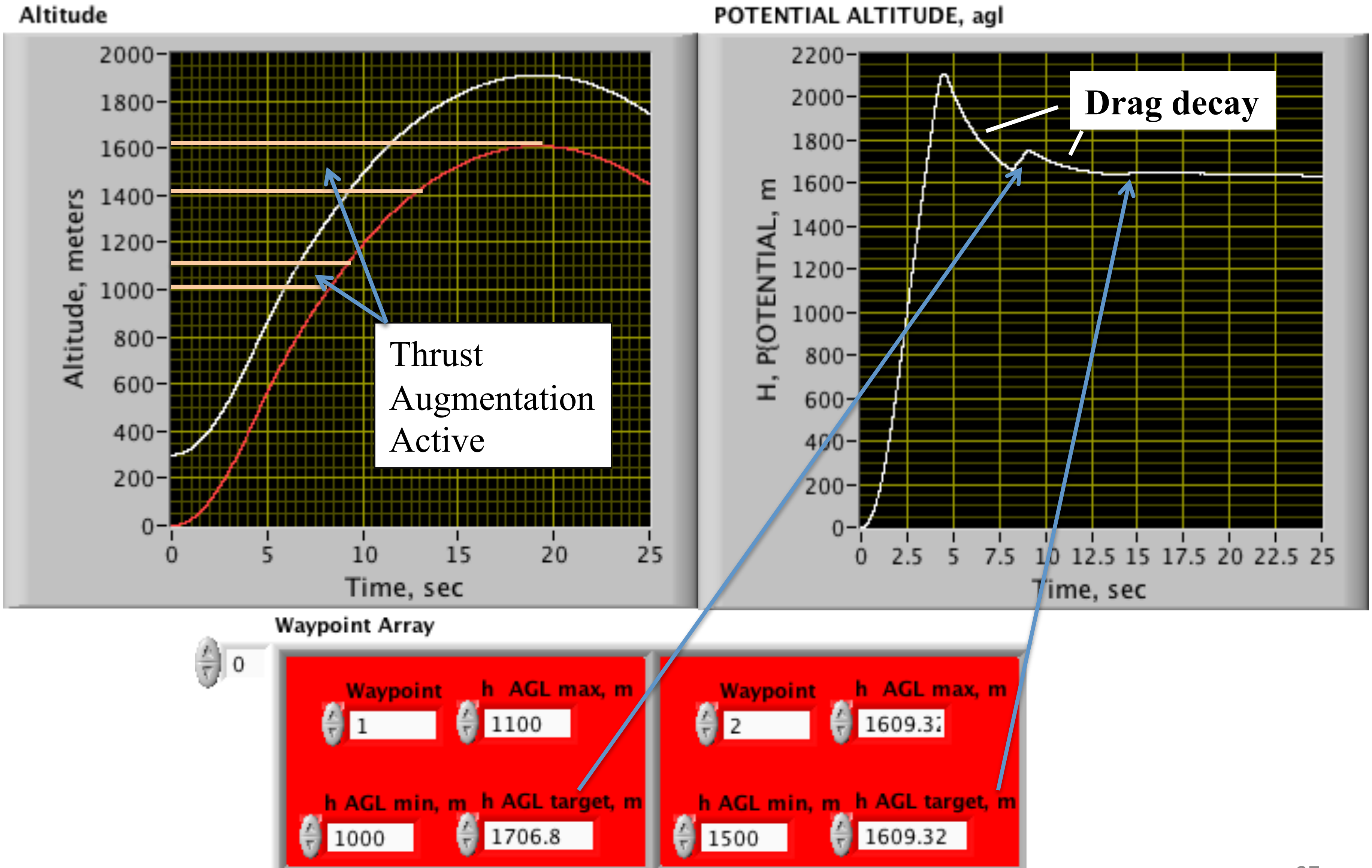
- **Potential Altitude as an Estimator of "Achievable Altitude" Becomes Increasingly More accurate as Apogee is Approached**

- **Use Augmentation Thrust to "Manage Energy" at waypoints of Increasing Altitude Along Probably trajectory**

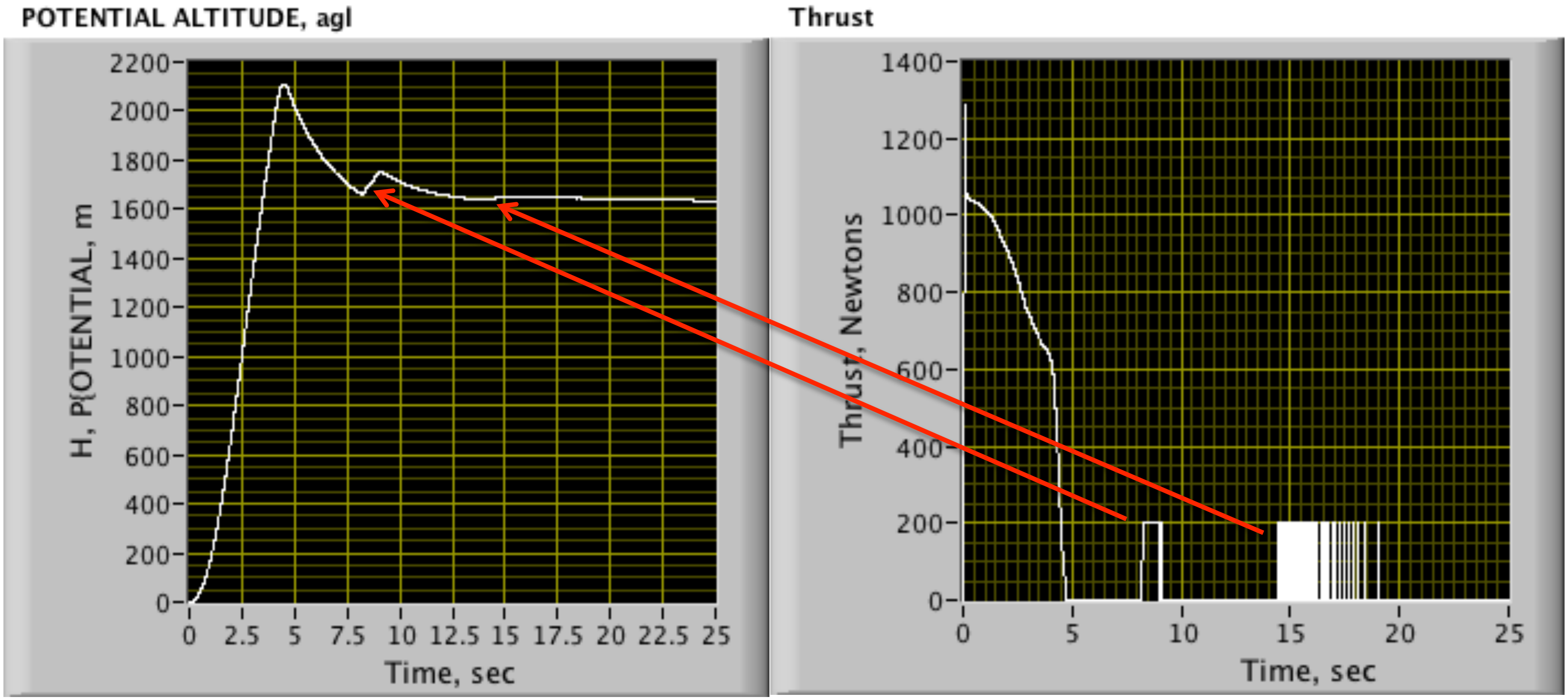


# A Recipe for Energy Management (4)



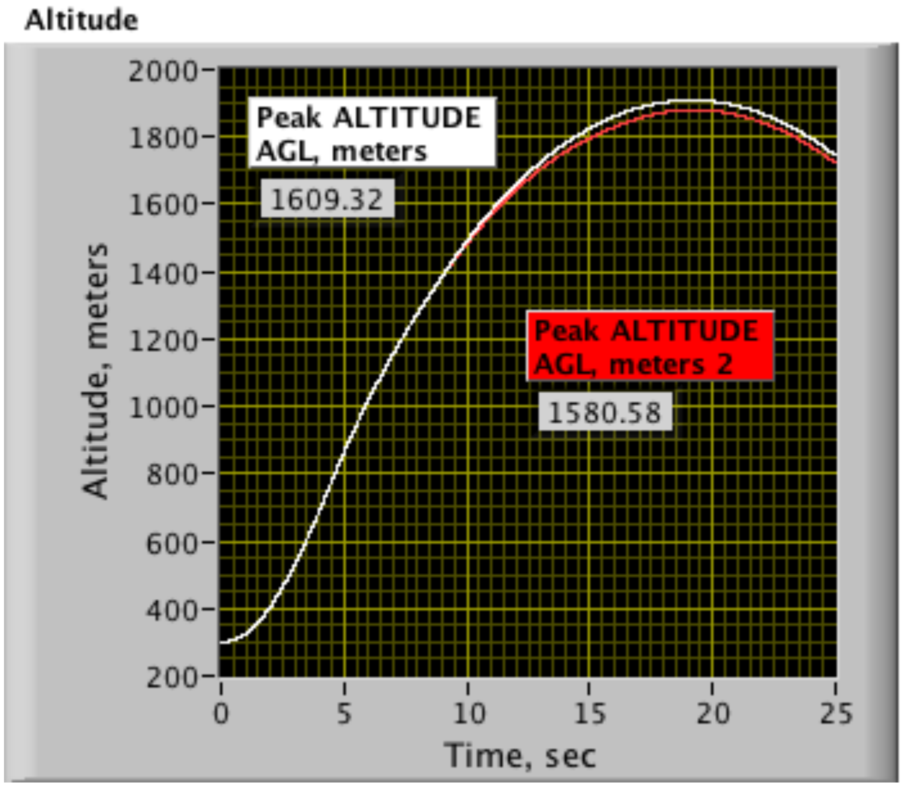
# A Recipe for Energy Management (4)

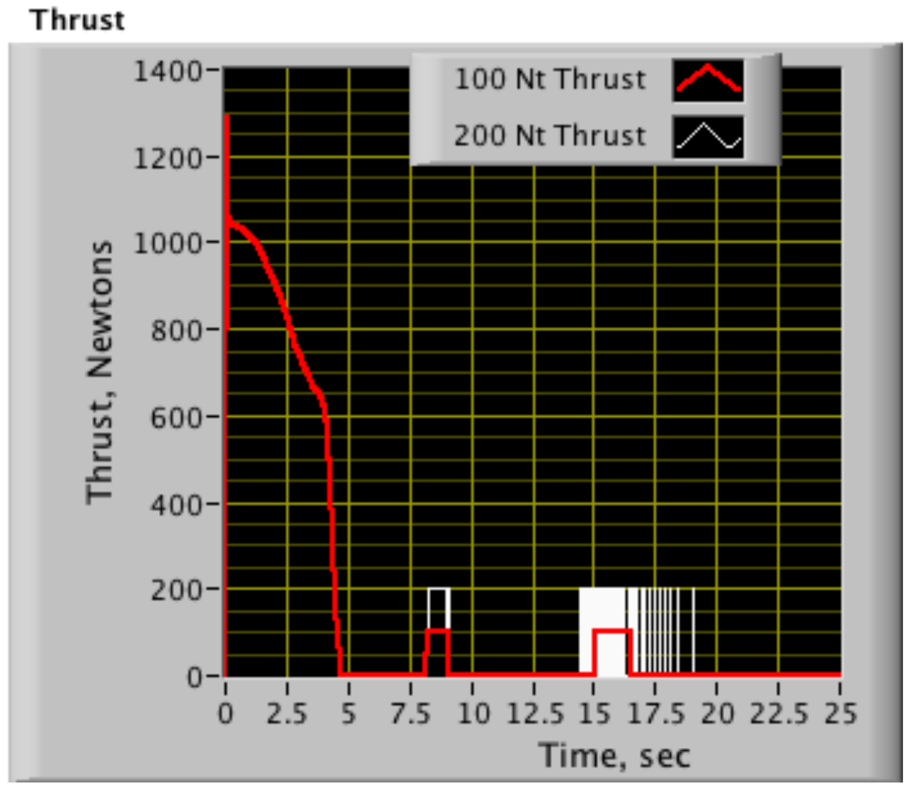



• First (mostly constant) Augmentation Impulse Boosts Energy to "achievable level"
Once we have calculated energy state (using IMU) … *1706.8 m = 5600 ft*

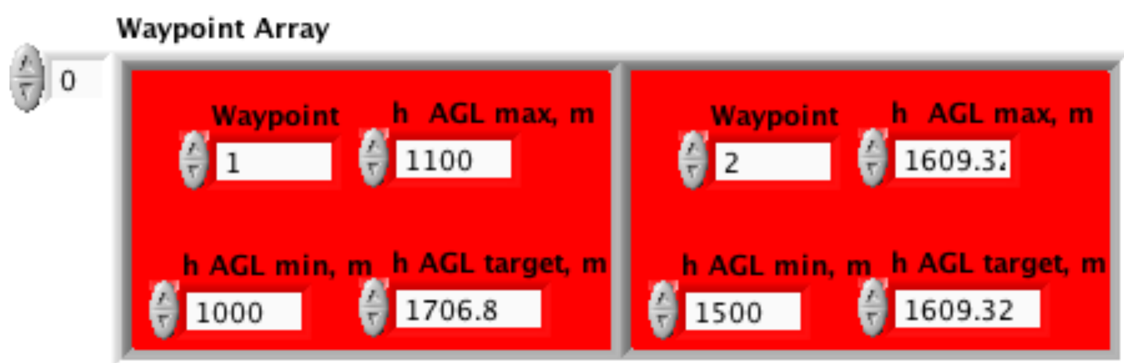
• Second Augmentation Impulse Boosts Maintains Energy level at Desired (Target Level)
Eneregy Level using Pulsed-modulation … *1609.32 m = 5280 ft*

# A Recipe for Energy Management (5)

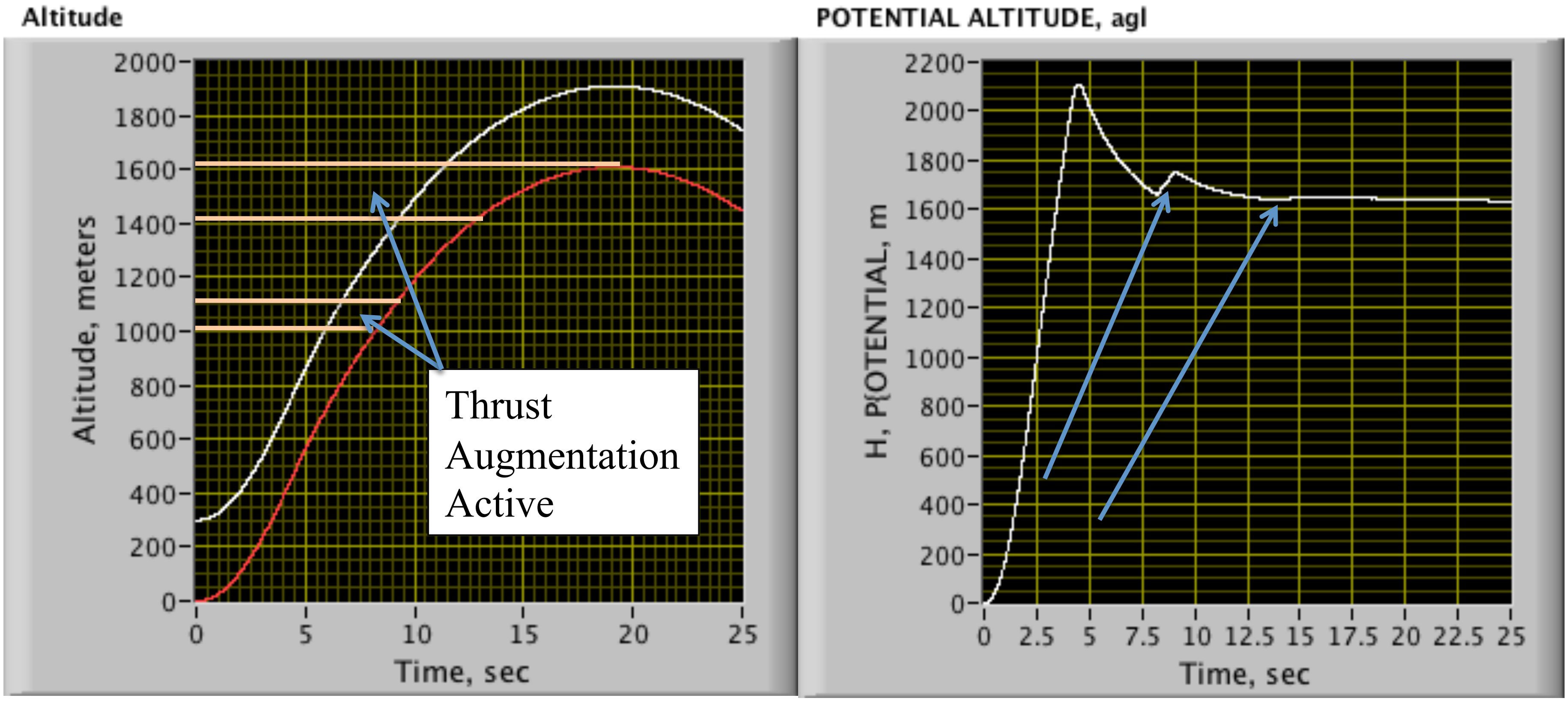



- Lower Augmentation Thrust levels allow for more precise modulation, but are less efficient and must "burn longer"

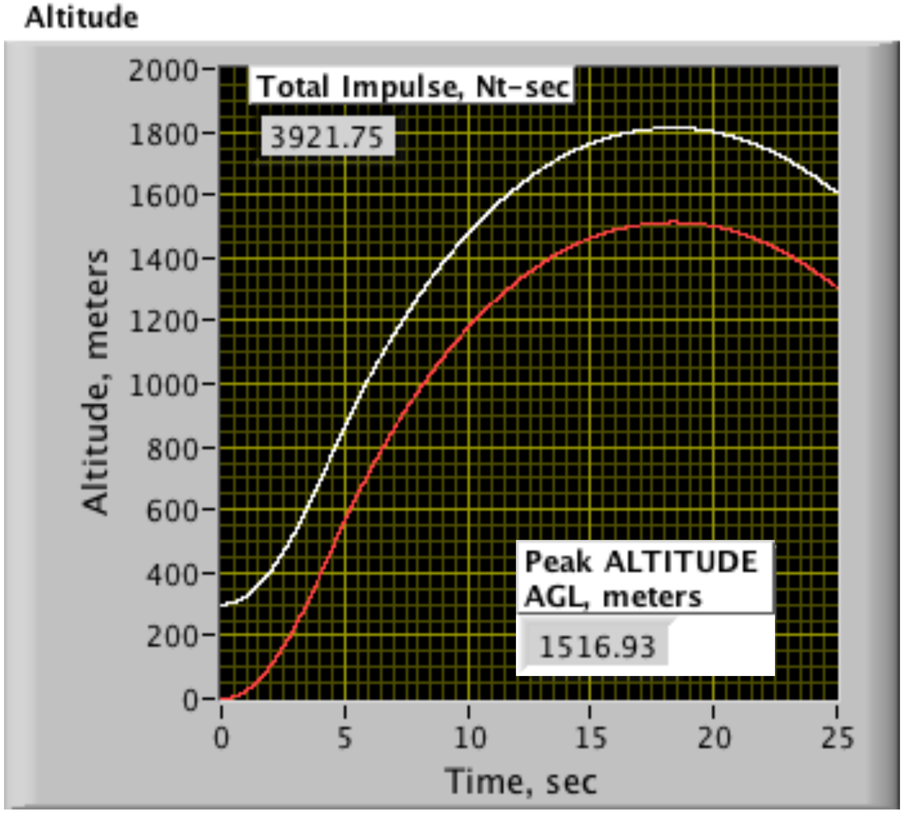


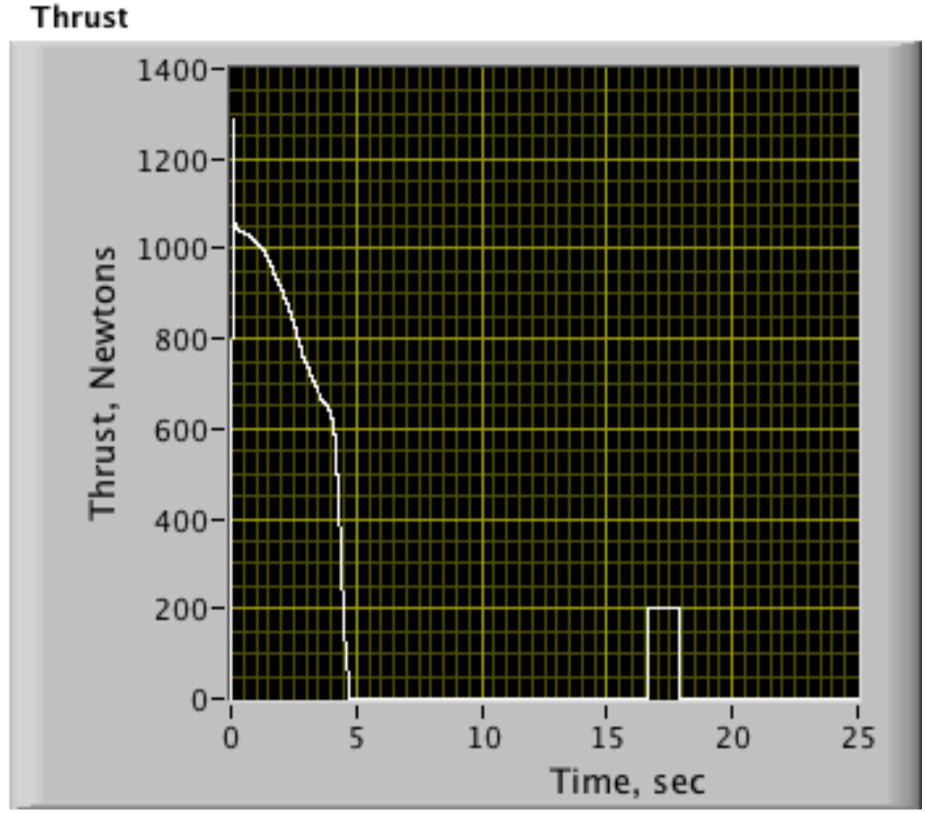
# A Recipe for Energy Management (6)



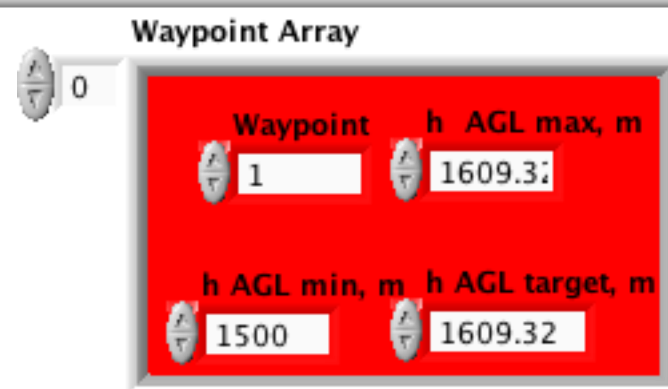

- Early Energy Management is More Effective, But less Precise

# A Recipe for Energy Management (7)



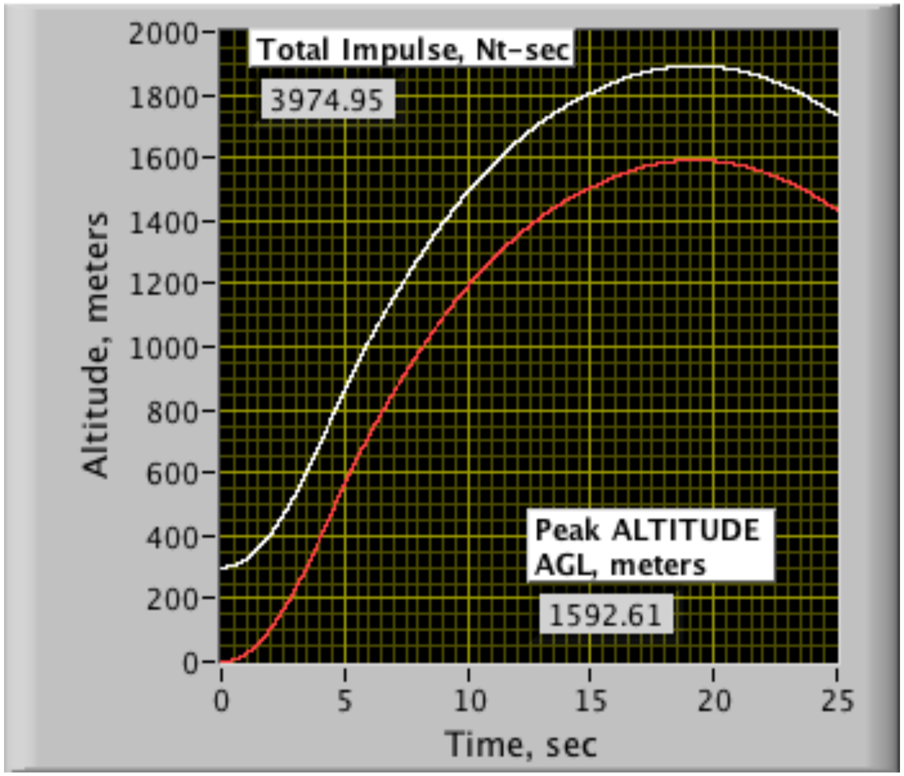




- Single Waypoint Energy Management Near Apogee
- *Insufficient Propellant to hit target*

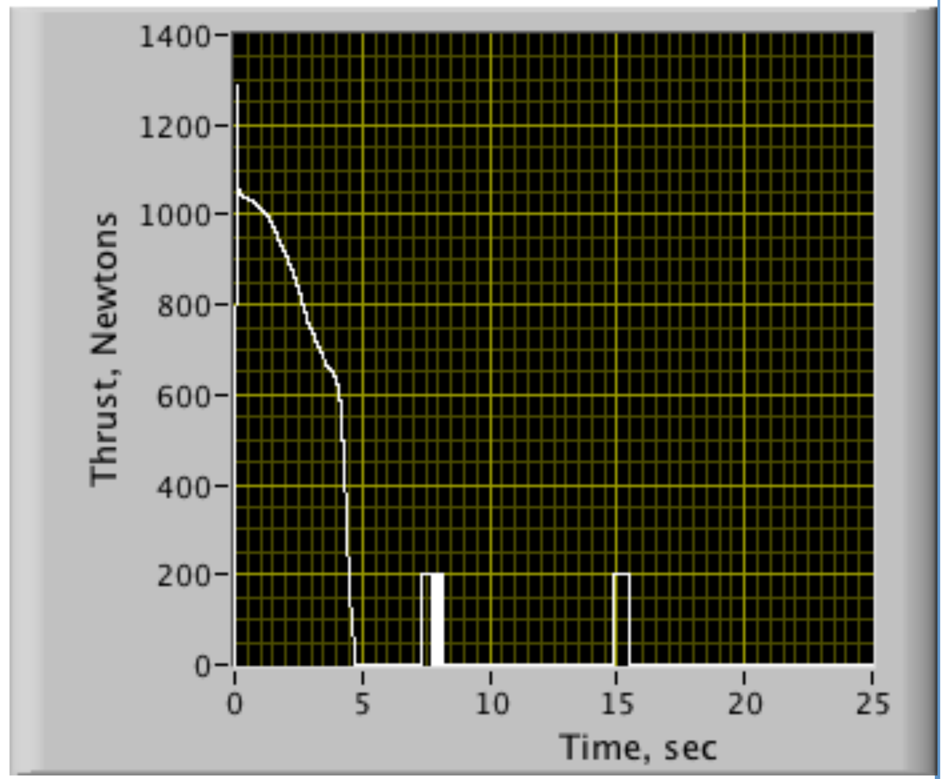


# A Recipe for Energy Management (8)

Altitude



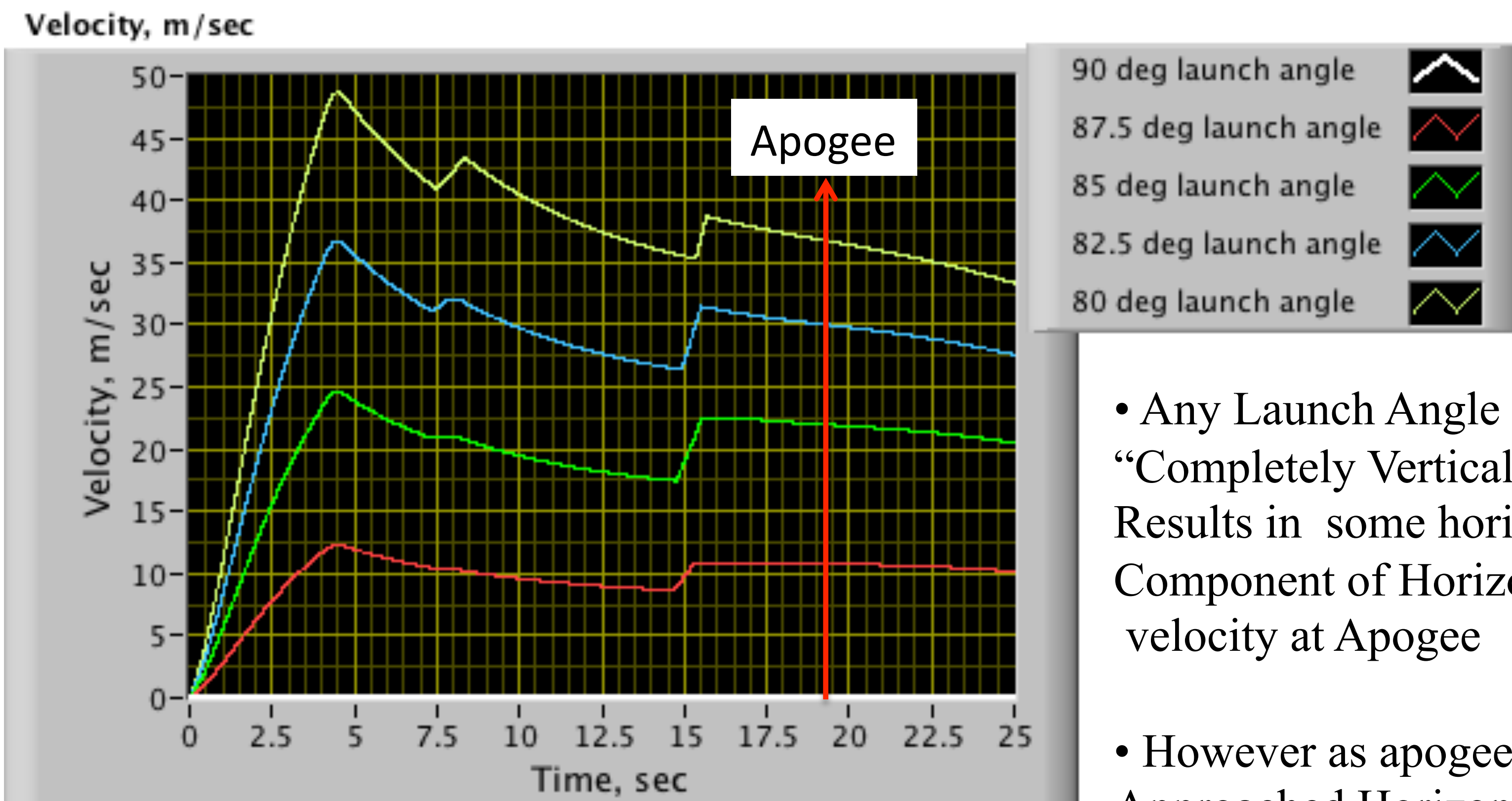

Thrust




- Earlier Implementation of First Waypoint
- Insufficient Accuracy to Hit target

- ***There will definitely be a Design "Sweet spot" .. here***

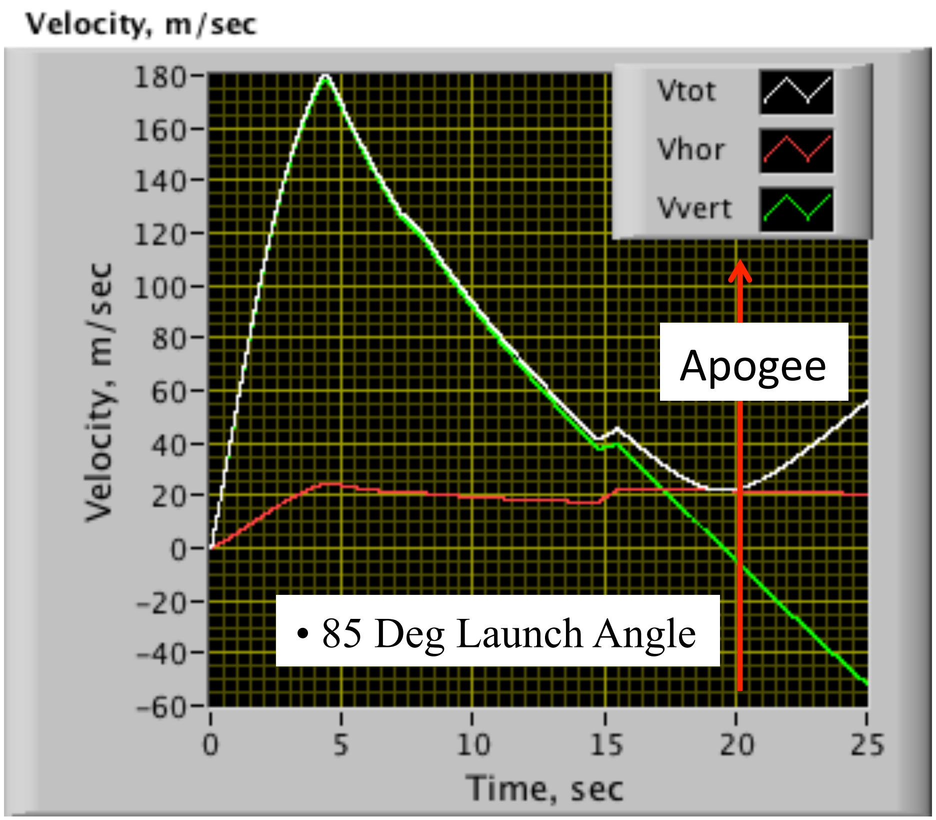
# Adjusting Potential Altitude Estimate for Effects of Horizontal Velocity




- Any Launch Angle not "Completely Vertical" Results in some horizontal Component of Horizontal velocity at Apogee

- However as apogee is Approached Horizontal Velocity Component becomes ~ constant

# Adjusting Potential Altitude Estimate for Effects of Horizontal Velocity (2)




• Compared to total velocity of Vehicle Horizontal component ~ constant Very soon after motor burn out

• $V_{hor_{waypoint}} \sim V_{apogee}$

$$V_{hor\ waypoint} = V_{waypoint} \cdot \cos(\gamma) \approx V_{apogee}$$

$$\rightarrow \left| \begin{array}{l} \gamma = flight\ path\ angle \\ = \tan^{-1} \dfrac{\dot{h}}{V_{hor}} \end{array} \right.$$

# Adjusting Potential Altitude Estimate for Effects of Horizontal Velocity (3)

$$h_{apogee} = h_{waypoint} + \left( \frac{V^2_{\ waypoint}}{2 \cdot g} - \frac{V^2_{\ apogee}}{2 \cdot g} \right) - \frac{1}{g} \int_{t_{waypoint}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt =$$

$$= h_{waypoint} + \left( \left( \frac{V^2_{\ waypoint}}{2 \cdot g} \right)_{vertical} + \left( \frac{V^2_{\ waypoint}}{2 \cdot g} \right)_{horizontal} - \frac{V^2_{\ apogee}}{2 \cdot g} \right) - \frac{1}{g} \int_{t_{waypoint}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

$$\boxed{\rightarrow \left( \frac{V^2_{\ waypoint}}{2 \cdot g} \right)_{horizontal} \approx \frac{V^2_{\ apogee}}{2 \cdot g}}$$

$$\rightarrow h_{apogee} = h_{waypoint} + \left( \frac{V^2_{\ waypoint}}{2 \cdot g} - \frac{V^2_{\ apogee}}{2 \cdot g} \right) - \frac{1}{g} \int_{t_{waypoint}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt =$$

$$\rightarrow h_{apogee} = h_{waypoint} + \left( \frac{V^2_{\ waypoint}}{2 \cdot g} \right)_{vertical} - \frac{1}{g} \int_{t_{waypoint}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

***"0" Near Apogee***

# Adjusting Potential Altitude Estimate for Effects of Horizontal Velocity (4)

$$\rightarrow h_{apogee} = h_{waypoint} + \left(\frac{V^2_{waypoint}}{2 \cdot g}\right)_{vertical} - \frac{1}{g}\int_{t_{waypoint}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

$$\rightarrow \boxed{\hat{h}_{potential} = h_{waypoint} + \frac{V^2_{waypoint} \cdot \sin^2(\gamma)}{2 \cdot g}}$$

- Non-optimal strategy
.. But it works pretty well

- Some potential that non-linear "bang-bang" or Dead-band controller may Be more propellant efficient

- But $\hat{h}_{potential}$ is a critical feedback parameter

*continuously estimate ...*

$$\left(\hat{h}_{potential}\right)_t = h_{(t)} + \frac{V^2_{(t)} \cdot \sin^2\left(\gamma_{(t)}\right)}{2 \cdot g_{(t)}}$$

*at waypoint ...we have a very simple control strategy....*

$$...if\left[(h_{min} \le h \le h_{max})\ \&\&\left(h < \hat{h}_{potential}\right)\right]$$

$$\text{"thrust on"}$$

$$...\ else$$

$$\text{"thrust off"}$$

# Accounting for Drag Losses In Potential Altitude

**Ignoring drag …. At any point along the trajectory …**

$$h_{potential} = h(t) + \frac{V(t)\cdot\sin(\gamma)}{2\cdot g}$$

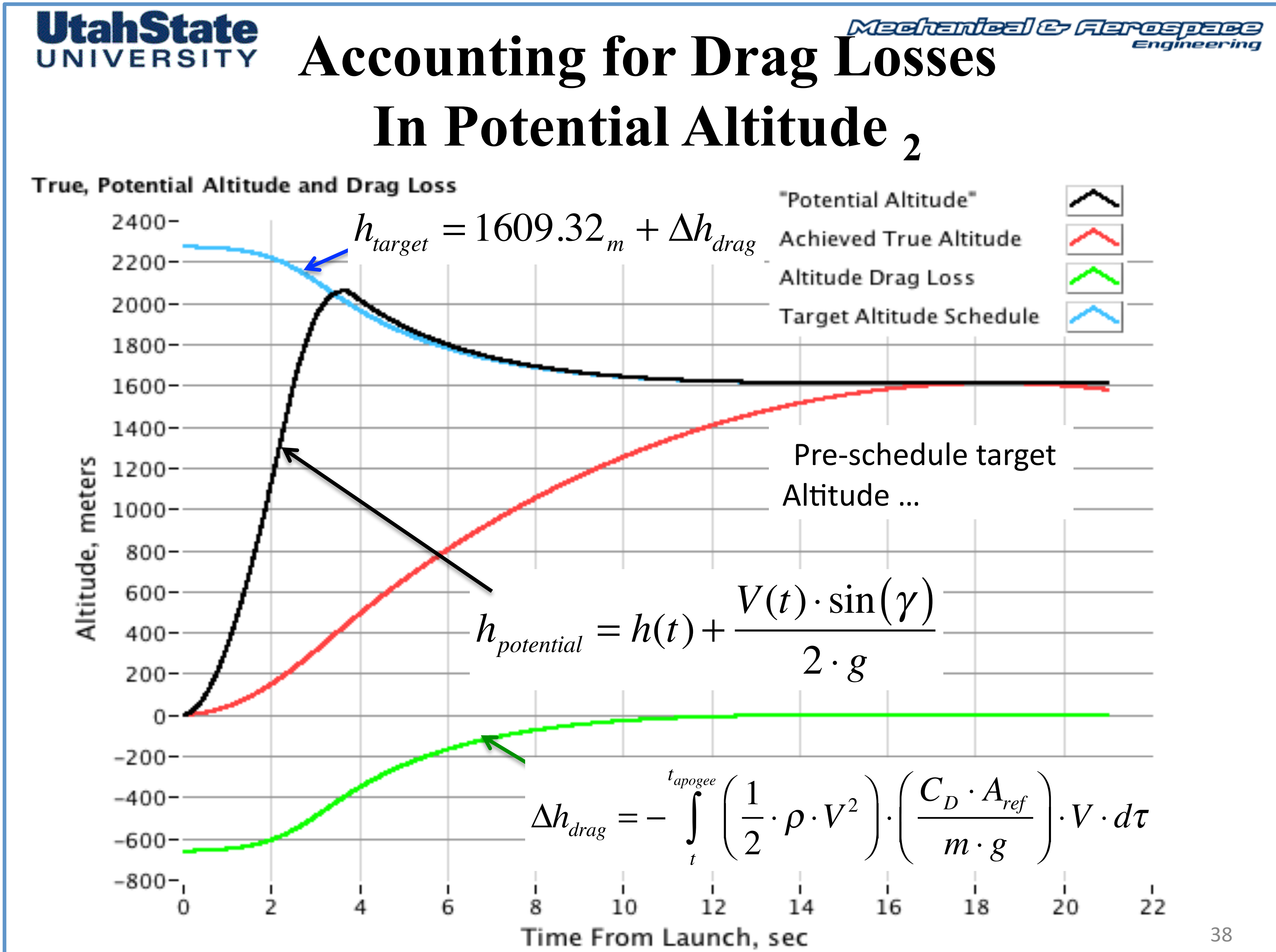
$$since \rightarrow V_{hor} = V(t)\cdot\cos(\gamma) \approx constant$$

**But because of drag …. The true apogee will be…**

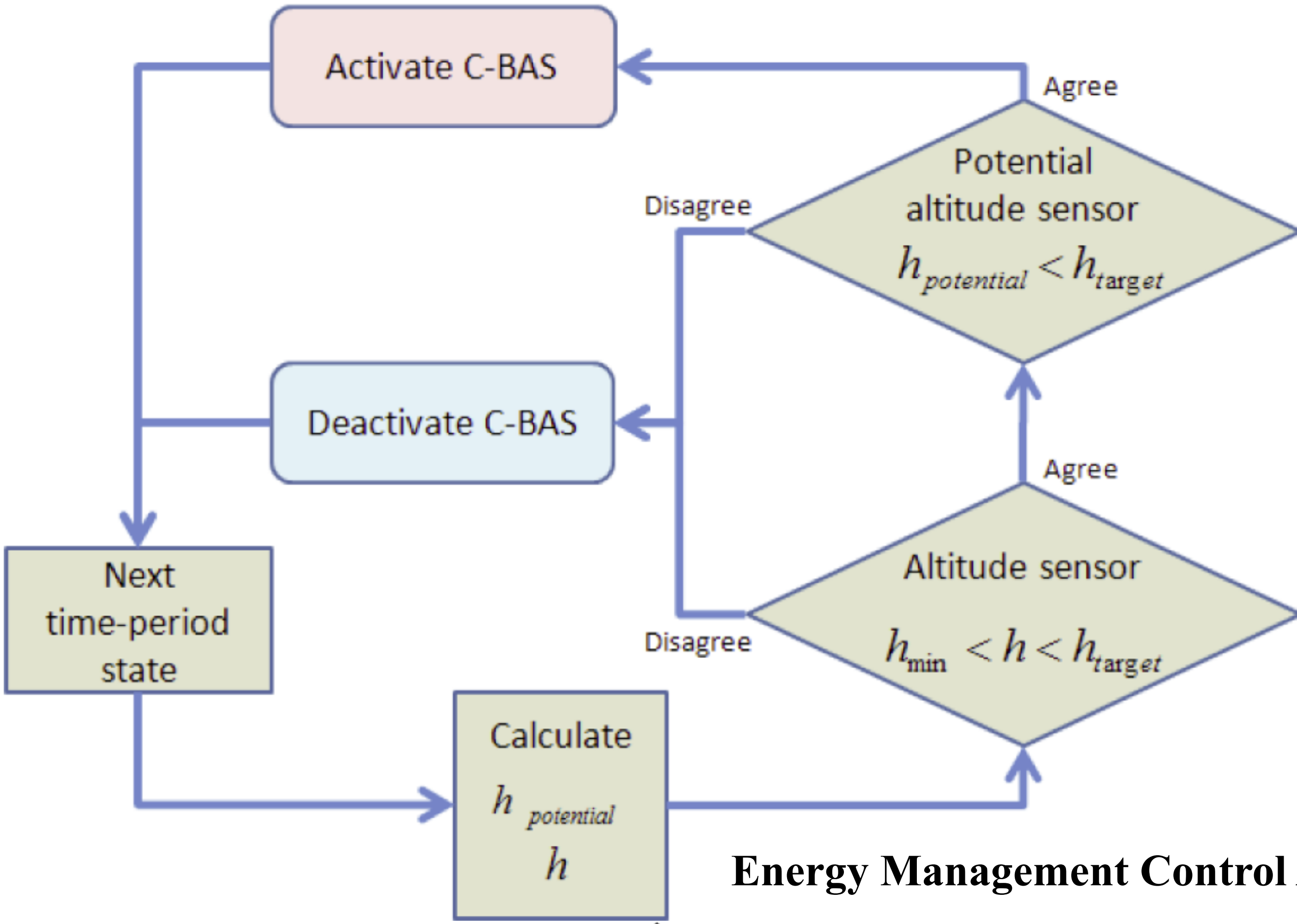
Drag Loss

$$h_{apogee} = h_{potential} - \boxed{\int_{t}^{t_{apogee}} \left(\frac{1}{2}\cdot\rho\cdot V^2\right)\cdot\left(\frac{C_D\cdot A_{ref}}{m\cdot g}\right)\cdot V\cdot d\tau}$$

# Accounting for Drag Losses In Potential Altitude 2



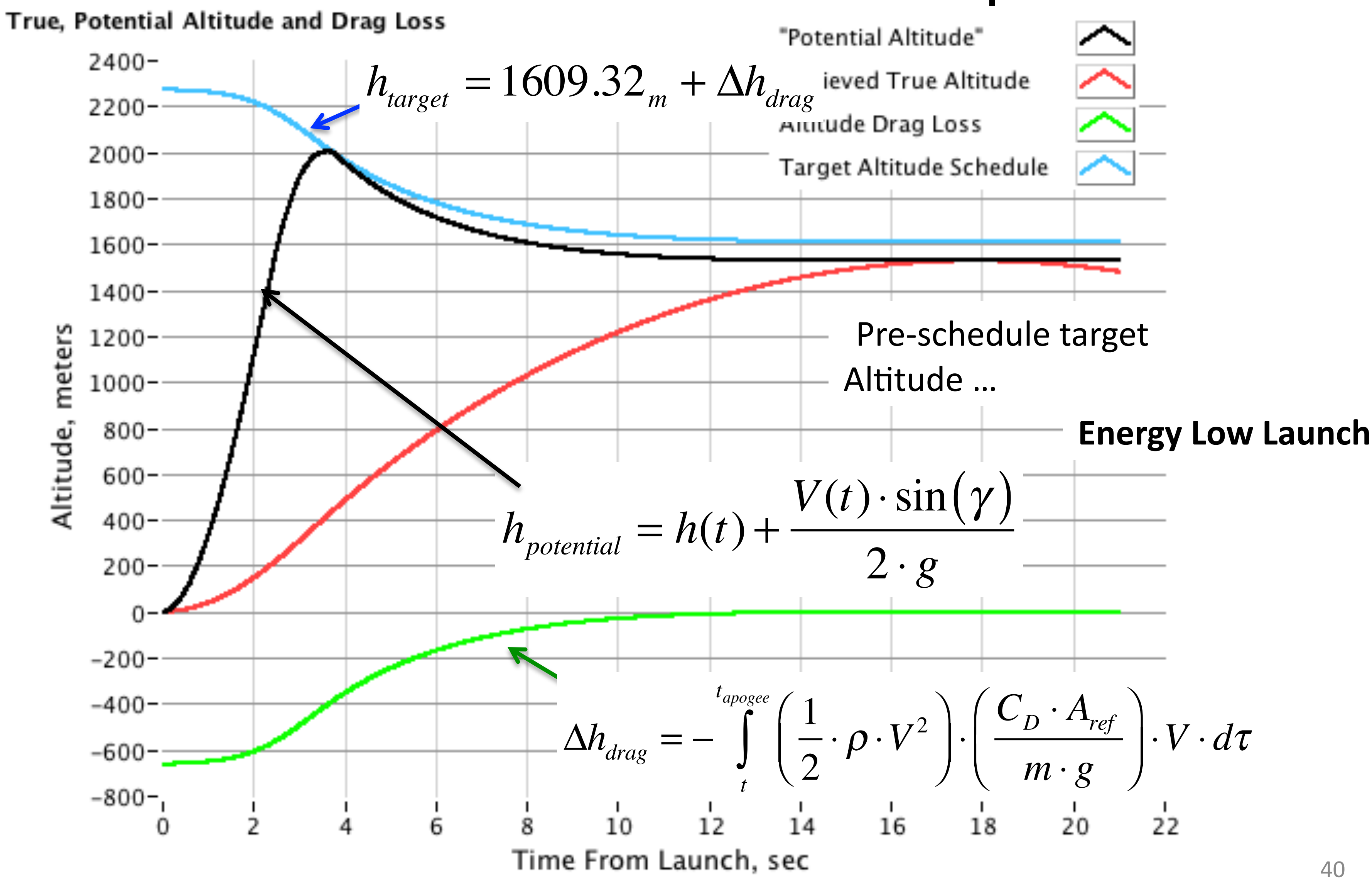
# Accounting for Drag Losses In Potential Altitude 3



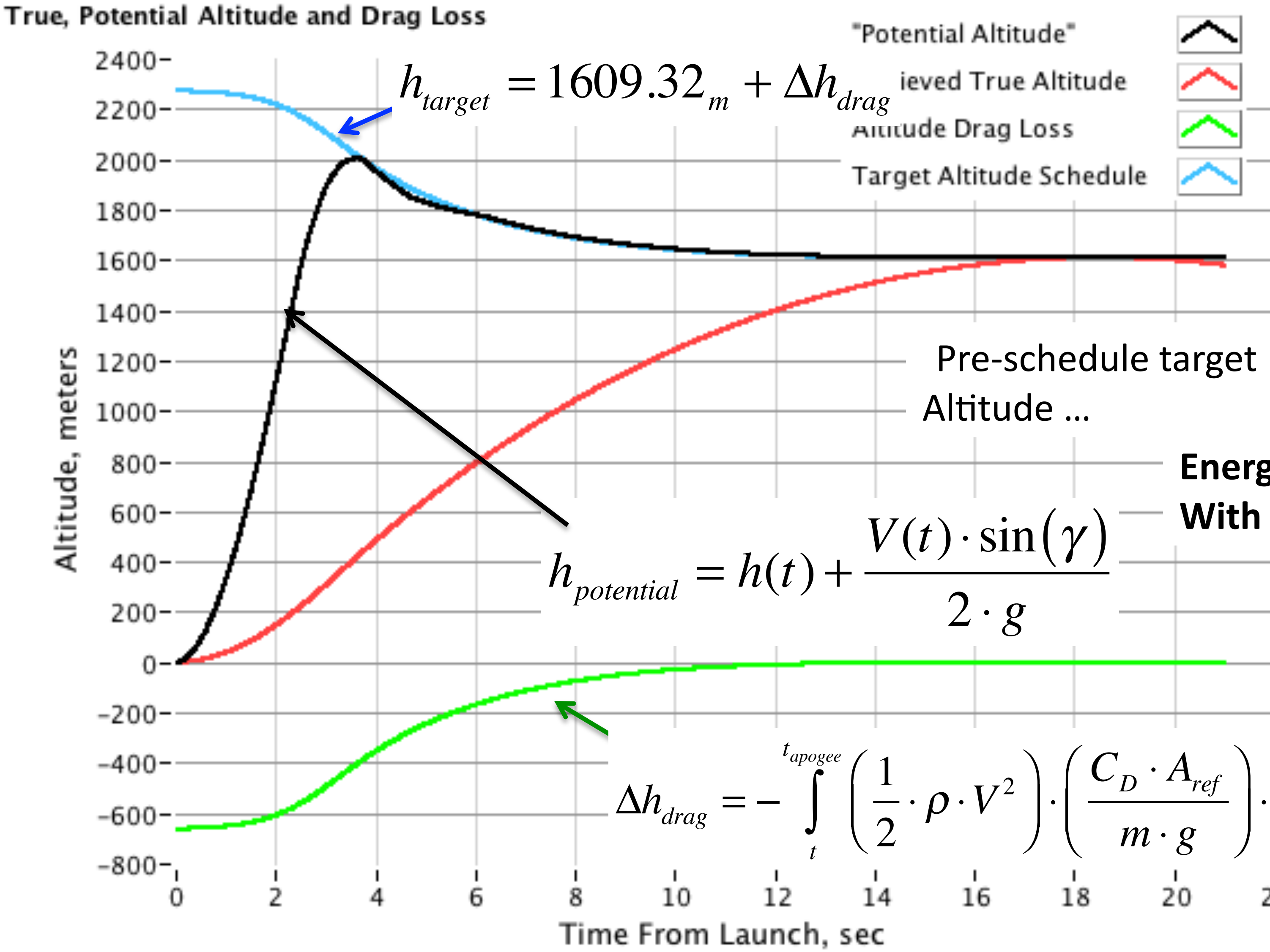

**Algorithm Active once Acceleration Drops below 1-g**

**Energy Management Control Algorithm.**

# Accounting for Drag Losses In Potential Altitude 4



# Accounting for Drag Losses In Potential Altitude 5




$$h_{target} = 1609.32_{m} + \Delta h_{drag}$$

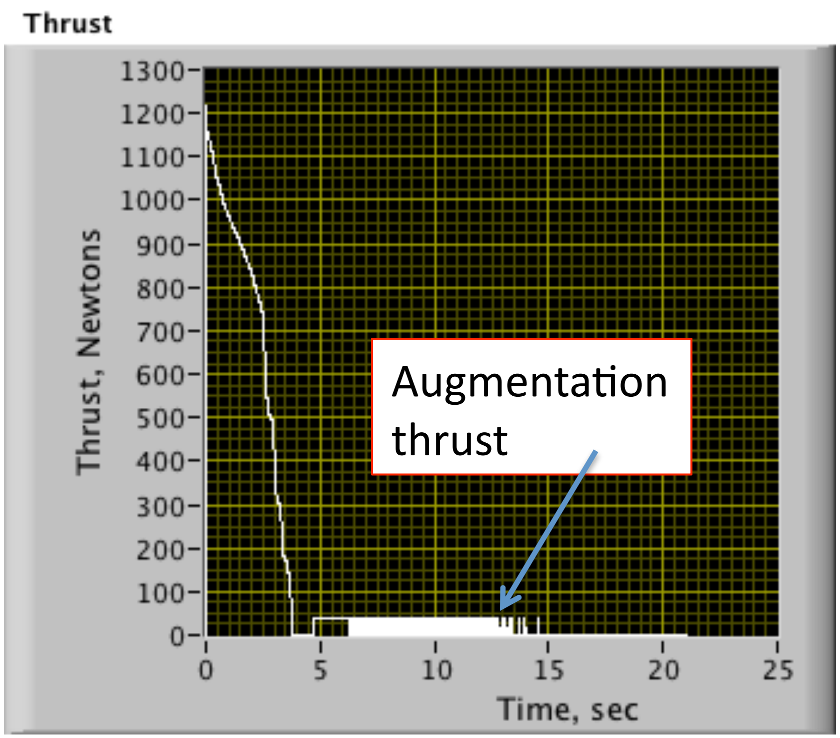
Pre-schedule target Altitude …

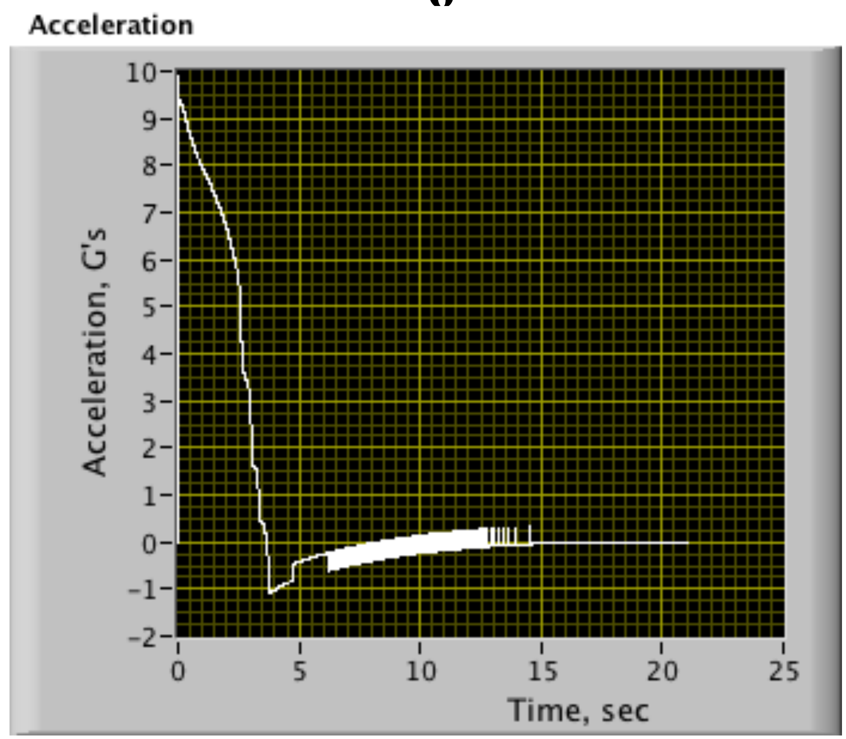
**Energy Low Launch With augmentation**

$$h_{potential} = h(t) + \frac{V(t)\cdot\sin(\gamma)}{2\cdot g}$$

$$\Delta h_{drag} = -\int_{t}^{t_{apogee}} \left(\frac{1}{2}\cdot\rho\cdot V^2\right)\cdot\left(\frac{C_D\cdot A_{ref}}{m\cdot g}\right)\cdot V\cdot d\tau$$

# Accounting for Drag Losses In Potential Altitude 6




Questions??