the graphical construction of Fig. 11.14*b*, and by the fact that only seven increments are chosen for the corner expansion fan. For a more accurate calculation, finer increments should be used, resulting in a more closely spaced characteristic net throughout the nozzle.

Note that a small inconsistency is involved with the properties at point 1 in Fig. 11.14, as listed in the first line of Table 11.1. The entry in Table 11.1 for θ at point 1 is a nonzero (but small) number, namely 0.375°. This is inconsistent with the physical picture in Fig. 11.14, which shows point 1 on the nozzle centerline where $\theta = 0$. This inconsistency is due to the necessity of *starting* the calculations with the straight characteristic line, *a*–1, along which the value of θ is constant and equal to 0.375°. In reality, the characteristic *a*–1 is curved because of the nonuniform flow inside the region *a*–*b*–1 in Fig. 11.14, but we have no way of knowing what that nonuniform flow is for this problem. In Sec. 12.7, we will show that a finite-difference calculation in the throat region can provide such information. However, within the framework of the method of characteristics in the present section, we must live with this inconsistency. As long as the first characteristic line *a*–1 is taken as close as possible to the assumed straight sonic line, this inconsistency will be minimized.

11.8 | METHOD OF CHARACTERISTICS FOR AXISYMMETRIC IRROTATIONAL FLOW

For axisymmetric irrotational flow, the philosophy of the method of characteristics is the same as discussed earlier; however, some of the details are different, principally the compatibility equations. The purpose of this section is to illustrate those differences.

Consider a cylindrical coordinate system, as sketched in Fig. 11.15. The cylindrical coordinates are r, ϕ, and x, with corresponding velocity components v, w,

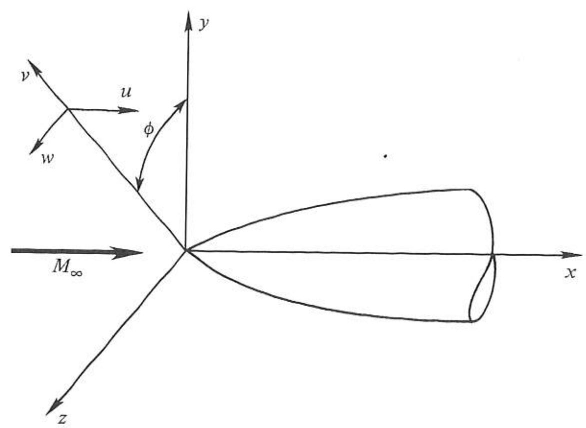

Figure 11.15 | Superposition of rectangular and cylindrical coordinate systems for axisymmetric flow.

and u, respectively. In these cylindrical coordinates, the continuity equation

$$\nabla \cdot (\rho \mathbf{V}) = 0$$

becomes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial r} + \frac{1}{r}\frac{\partial(\rho w)}{\partial \phi} + \frac{\rho v}{r} = 0 \tag{11.34}$$

Recalling from Sec. 10.1 that axisymmetric flow implies $\partial/\partial\phi = 0$, Eq. (11.34) becomes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial r} + \frac{\rho v}{r} = 0 \tag{11.35}$$

From Euler's equation for irrotational flow, Eq. (8.7),

$$dp = -\rho V\, dV = \frac{-\rho}{2} d(V^2) = -\frac{\rho}{2} d(u^2 + v^2 + w^2) \tag{11.36}$$

However, the speed of sound $a^2 = (\partial p/\partial \rho)_s = dp/d\rho$. Hence, along with $w = 0$ for axisymmetric flow, Eq. (11.36) becomes

$$d\rho = -\frac{\rho}{a^2}(u\, du + v\, dv) \tag{11.37}$$

from which follows

$$\frac{\partial \rho}{\partial x} = -\frac{\rho}{a^2}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right) \tag{11.38}$$

$$\frac{\partial \rho}{\partial r} = -\frac{\rho}{a^2}\left(u\frac{\partial u}{\partial r} + v\frac{\partial v}{\partial r}\right) \tag{11.39}$$

Substituting Eqs. (11.38) and (11.39) into Eq. (11.35), we obtain, after factoring,

$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial u}{\partial x} - \frac{uv}{a^2}\frac{\partial v}{\partial x} - \frac{uv}{a^2}\frac{\partial u}{\partial r} + \left(1 - \frac{v^2}{a^2}\right)\frac{\partial v}{\partial r} = -\frac{v}{r} \tag{11.40}$$

The condition of irrotationality is

$$\nabla \times \mathbf{V} = 0$$

which in cylindrical coordinates can be written as

$$\nabla \times \mathbf{V} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & \mathbf{e}_x \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial x} \\ v & rw & u \end{vmatrix} = 0 \tag{11.41}$$

For axisymmetric flow, Eq. (11.41) yields

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial x} \tag{11.42}$$

Substituting Eq. (11.42) into (11.40), we have

$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial u}{\partial x} - 2\frac{uv}{a^2}\frac{\partial v}{\partial x} + \left(1 - \frac{v^2}{a^2}\right)\frac{\partial v}{\partial r} = -\frac{v}{r} \tag{11.43}$$

Keeping in mind that $u = u(x, r)$ and $v = v(x, r)$, we can also write

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial r}dr = \frac{\partial u}{\partial x}dx + \frac{\partial v}{\partial x}dr \tag{11.44}$$

and

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial r}dr \tag{11.45}$$

Equations (11.43), (11.44), and (11.45) are three equations which can be solved for the three derivatives $\partial u/\partial x$, $\partial v/\partial x$, and $\partial v/\partial r$.

The reader should by now suspect that we are on the same track as in our previous development of the characteristic equations. Equations (11.43) through (11.45) for axisymmetric flow are analogous to Eqs. (11.5) through (11.7) for two-dimensional flow. To determine the characteristic lines and compatibility equations, solve Eqs. (11.43) through (11.45) for $\partial v/\partial x$ as follows:

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} 1 - \dfrac{u^2}{a^2} & -\dfrac{v}{r} & 1 - \dfrac{v^2}{a^2} \\ dx & du & 0 \\ 0 & dv & dr \end{vmatrix}}{\begin{vmatrix} 1 - \dfrac{u^2}{a^2} & -2\dfrac{uv}{a^2} & 1 - \dfrac{v^2}{a^2} \\ dx & dr & 0 \\ 0 & dx & dr \end{vmatrix}} = \frac{N}{D} \tag{11.46}$$

The characteristic directions are found by setting $D = 0$. This yields

$$\left(\frac{dr}{dx}\right)_{\text{char}} = \frac{-uv/a^2 \pm \sqrt{[(u^2 + v^2)/a^2] - 1}}{1 - (u^2/a^2)} \tag{11.47}$$

Equation (11.47) is identical to Eq. (11.10). The discussion following Eq. (11.10), leading to Eq. (11.14), also holds here. Consequently,

$$\boxed{\left(\frac{dr}{dx}\right)_{\text{char}} = \tan(\theta \mp \mu)} \tag{11.48}$$

and we see that *for axisymmetric irrotational flow, the characteristic lines are Mach lines*. The C_+ and C_- characteristics are the same as those sketched in Fig. 11.6.

The compatibility equations that hold along these characteristic lines are found by setting $N = 0$ in Eq. (11.46). The result is

$$\frac{dv}{du} = \frac{-\left(1 - \frac{u^2}{a^2}\right) - \frac{v}{r}\frac{dx}{du}}{\left(1 - \frac{v^2}{a^2}\right)\frac{dx}{dr}}$$

or

$$\frac{dv}{du} = -\frac{\left(1 - \frac{u^2}{a^2}\right)}{\left(1 - \frac{v^2}{a^2}\right)}\frac{dr}{dx} - \frac{\frac{v}{r}\frac{dr}{du}}{\left(1 - \frac{v^2}{a^2}\right)} \tag{11.49}$$

In Eq. (11.49), the term dr/dx is the characteristic direction given by Eq. (11.47). Hence, substituting Eq. (11.47) into (11.49), we have

$$\frac{dv}{du} = \frac{\frac{uv}{a^2} \mp \sqrt{\frac{u^2 + v^2}{a^2} - 1}}{\left(1 - \frac{v^2}{a^2}\right)} - \frac{\frac{v}{r}\frac{dr}{du}}{\left(1 - \frac{v^2}{a^2}\right)} \tag{11.50}$$

Note that Eq. (11.50) for axisymmetric flow differs from Eq. (11.16) for two-dimensional flow by the additional term involving dr/r. Referring again to Fig. 11.6, we make the substitution $u = V\cos\theta$ and $v = V\sin\theta$ into Eq. (11.50), which after algebraic manipulation becomes

$$d\theta = \mp\sqrt{M^2 - 1}\,\frac{dV}{V} \pm \frac{1}{\sqrt{M^2 - 1} \mp \cot\theta}\frac{dr}{r} \tag{11.51}$$

The first term on the right-hand side of Eq. (11.51) is the differential of the Prandtl–Meyer function, $d\nu$ (see Sec. 4.14). Hence, the final form of the compatibility equation is

$$\boxed{d(\theta + \nu) = \frac{1}{\sqrt{M^2 - 1} - \cot\theta}\frac{dr}{r} \quad \text{(along a } C_- \text{ characteristic)}} \tag{11.52}$$

$$\boxed{d(\theta - \nu) = -\frac{1}{\sqrt{M^2 - 1} + \cot\theta}\frac{dr}{r} \quad \text{(along a } C_+ \text{ characteristic)}} \tag{11.53}$$

Equations (11.52) and (11.53) are the compatibility equations for axisymmetric irrotational flow. Compare them with the analogous results for two-dimensional irrotational flow given by Eqs. (11.20) and (11.21). For axisymmetric flow, we note the following:

1. The compatibility equations are *differential* equations, not algebraic equations as before.
2. The quantity $\theta + \nu$ is no longer constant along a C_- characteristic. Instead, its value depends on the spatial location in the flowfield as dictated by the dr/r term in Eq. (11.52). The same qualification is made for $\theta - \nu$ along a C_+ characteristic.

For the actual numerical computation of an axisymmetric flowfield by the method of characteristics, the differentials in Eqs. (11.52) and (11.53) are replaced by finite differences (which are to be discussed later). The flow properties and their location are found by a step-by-step solution of Eqs. (11.52) and (11.53) coupled with the construction of the characteristics net using Eq. (11.48).

11.9 | METHOD OF CHARACTERISTICS FOR ROTATIONAL (NONISENTROPIC AND NONADIABATIC) FLOW

The assumption of irrotationality in the previous sections allows a great simplification. For example, Eq. (11.5) for two-dimensional irrotational flow contains only three velocity derivatives, namely, $\Phi_{xx} = \partial u/\partial x$, $\Phi_{yy} = \partial v/\partial y$, and $\Phi_{xy} = \partial u/\partial y = \partial v/\partial x$. The irrotationality condition allows the use of the velocity potential and, in particular, eliminates one of the possible velocity derivatives as an unknown via $\partial u/\partial y = \partial v/\partial x$. Along with Eqs. (11.6) and (11.7), we have a system of equations with three unknown velocity derivatives, which can be solved by means of three-by-three determinants, Eq. (11.8). Similarly, for axisymmetric irrotational flow, the irrotationality condition, Eq. (11.42), allows the derivation of a governing equation, Eq. (11.43), which contains only three unknown velocity derivatives. This again leads to a system of three-by-three determinants, namely, Eq. (11.46).

In contrast, rotational flow is more complex, although the philosophy of the method of characteristics remains the same. Only a brief outline of the rotational method of characteristics will be given here; the reader is referred to Shapiro (Ref. 16) for additional details.

Crocco's theorem, Eq. (6.60), repeated here,

$$T\nabla s = \nabla h_o - \mathbf{V} \times (\nabla \times \mathbf{V})$$

tells us that rotational flow occurs when nonisentropic and/or nonadiabatic conditions are present. An example of the former is the flow behind a curved shock wave (see Fig. 4.29), where the entropy increase across the shock is different for different streamlines. An example of the latter is a shock layer within which the static temperature is high enough for the gas to lose a substantial amount of energy due to thermal radiation.

Without the simplification afforded by the irrotationality condition, it is not possible to obtain a system of three independent equations with three unknown derivatives