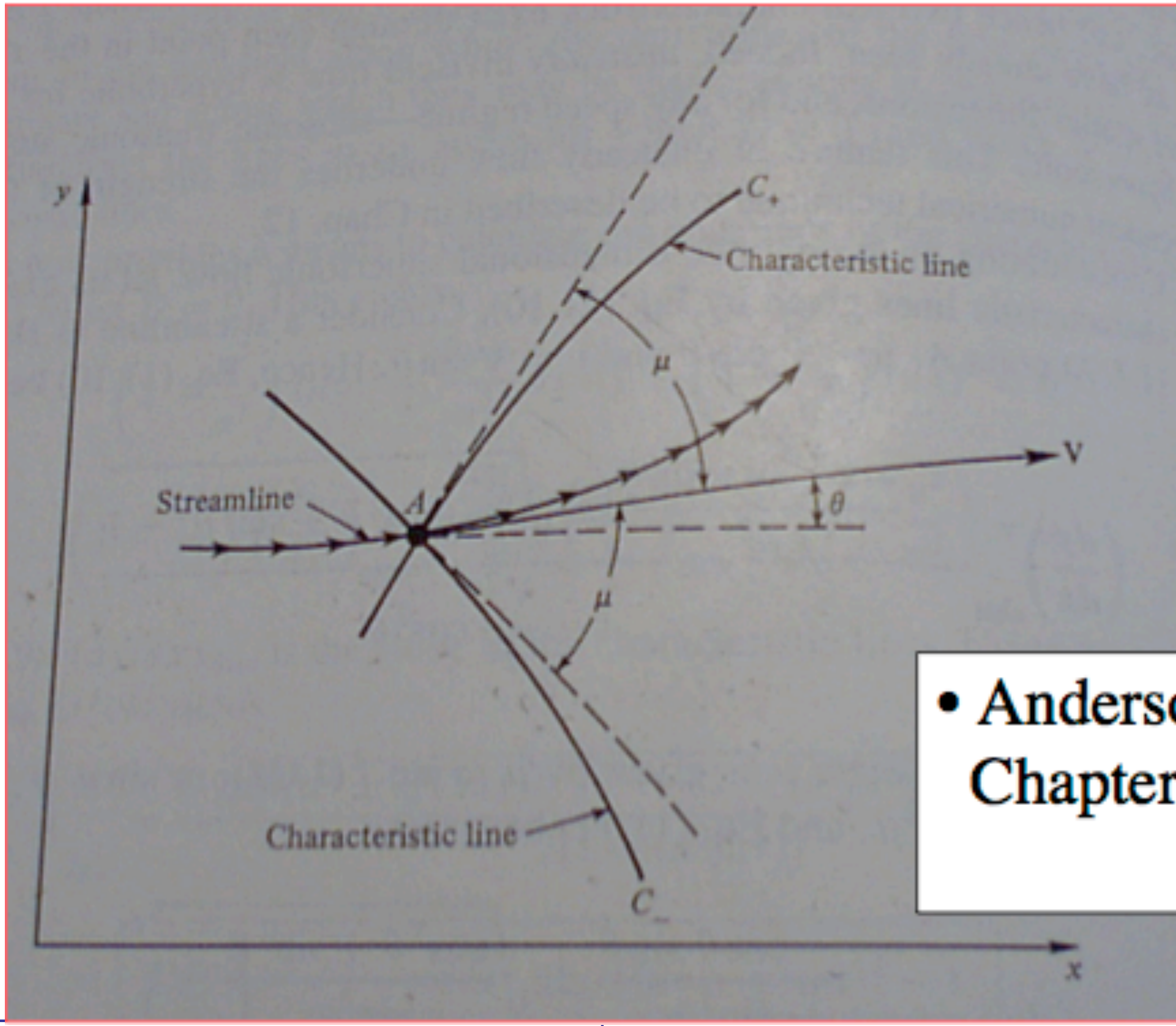
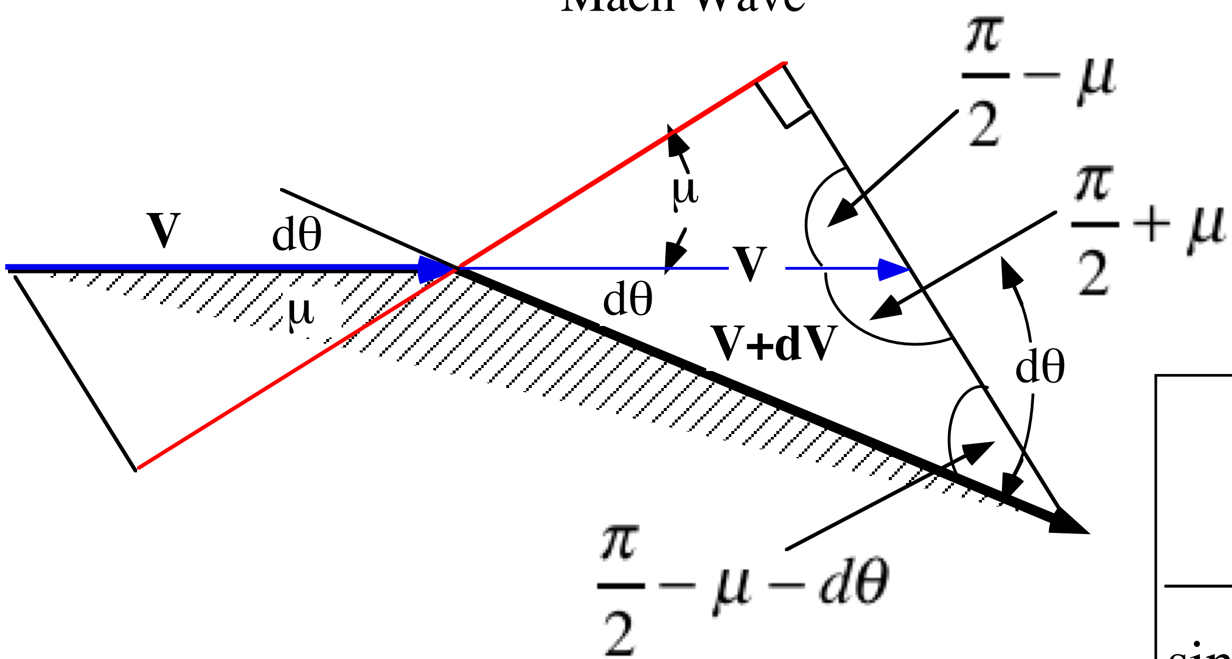
Section 1 Lecture 1: Review of the Two-Dimensional Method of Characteristics



- Anderson, Chapter 11 pp. 377-403

Prandtl-Meyer Expansion Fan: Mathematical Analysis Revisited

- Consider flow expansion around an infinitesimal corner



Infinitesimal Expansion Fan Flow Geometry

- From Law of Sines

$$\frac{V}{\sin\left(\frac{\pi}{2} - \mu - d\theta\right)} = \frac{V + dV}{\sin\left(\frac{\pi}{2} + \mu\right)}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Expanding and collecting terms

$$1+\frac{dV}{V}=\frac{\cos(\mu)}{\cos(\mu)\cos(d\theta)-\sin(\mu)\sin(d\theta)}$$

- Letting dθ is considered be infinitesimal

$$\cos(d\theta)=1\rightarrow\sin(d\theta)=d\theta$$

$$\boxed{1+\frac{dV}{V}=\frac{\cos(\mu)}{\cos(\mu)-\sin(\mu)(d\theta)}=\frac{1}{1-\tan(\mu)(d\theta)}}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Exploiting the form of the power series (expanded about dV/V=0)

$$\frac{1}{1-\frac{dV}{V}} = 1 + \frac{1}{\left(1-\frac{dV}{V}\right)^2}\Bigg|_{\frac{dV}{V}=0}\left(\frac{dV}{V}-0\right) + O\left(\frac{dV}{V}^2\right)$$

- … truncate after first order term

$$\frac{1}{1-\frac{dV}{V}} \approx 1 + \frac{dV}{V} \rightarrow \frac{1}{1-\frac{dV}{V}} \approx \frac{1}{1-\tan(\mu)(d\theta)}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- Solve for $d\theta$ in terms of dV/V

$$\frac{1}{1-\frac{dV}{V}} \approx 1+\frac{dV}{V} \rightarrow \frac{1}{1-\frac{dV}{V}} \approx \frac{1}{1-\tan(\mu)(d\theta)}$$

$$1-\tan(\mu)(d\theta) = 1-\frac{dV}{V} \rightarrow \boxed{d\theta = \frac{1}{\tan(\mu)}\frac{dV}{V}}$$

- Since disturbance is infinitesimal (mach wave)

$$\boxed{\sin(\mu) = \frac{1}{M}}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- " Differential form" of Prandtl-Meyer wave

$$d\theta = \frac{1}{\tan(\mu)} \frac{dV}{V}$$

- For an infinitesimal disturbance (mach wave)

$$\sin(\mu) = \frac{1}{M}$$

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

$$\sin(\mu) = \frac{1}{M} \rightarrow \sin^2(\mu) = \frac{1}{M^2} = \frac{\sin^2(\mu) + \cos^2(\mu)}{M^2} \rightarrow$$

$$M^2 = \frac{\sin^2(\mu) + \cos^2(\mu)}{\sin^2(\mu)} = 1 + \frac{1}{\tan^2(\mu)} \rightarrow \frac{1}{\tan^2(\mu)} = M^2 - 1$$

$$\frac{1}{\tan(\mu)} = \pm\sqrt{M^2 - 1}$$

- and ….

$$d\theta = \pm\sqrt{M^2 - 1}\,\frac{dV}{V}$$

keep ± sign

Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont'd)

- As demonstrated in section 6.2 of *MAE 5420*

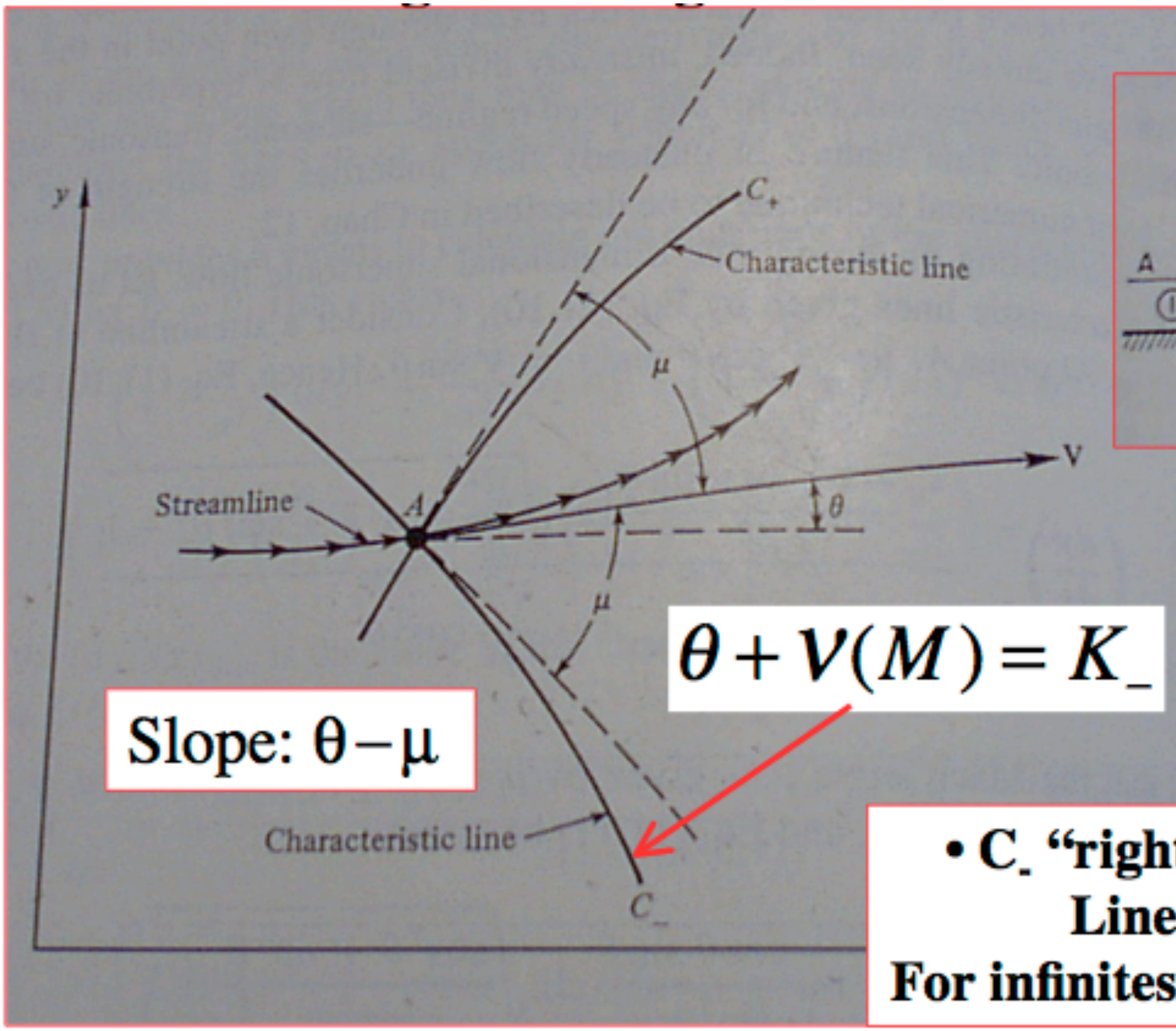
$$\theta = \int \pm\sqrt{M^2 - 1}\,\frac{dV}{V} = \pm\nu(M) + Const \rightarrow$$

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right\} - \tan^{-1}\sqrt{M^2-1}$$

$$\theta \pm \nu(M) = Const$$

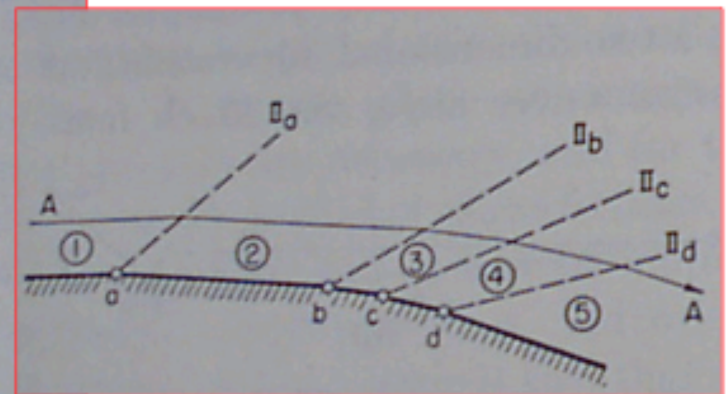
Characteristic Lines

- Right running characteristic lines



Slope: $\theta - \mu$

$$\theta + \nu(M) = K_-$$



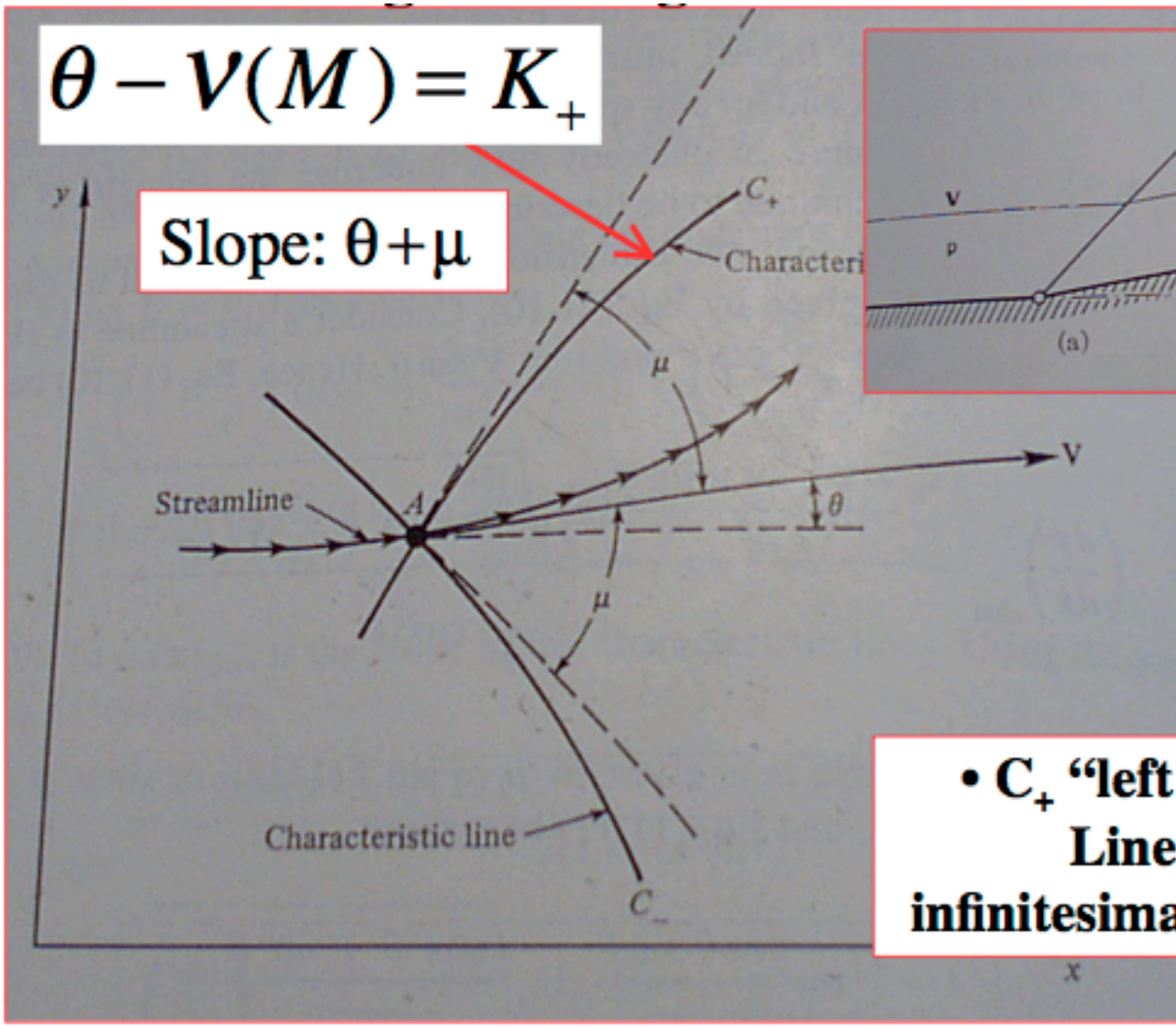
- **C_- "right running" characteristic Line is a Generalization For infinitesimal expansion corner flow**

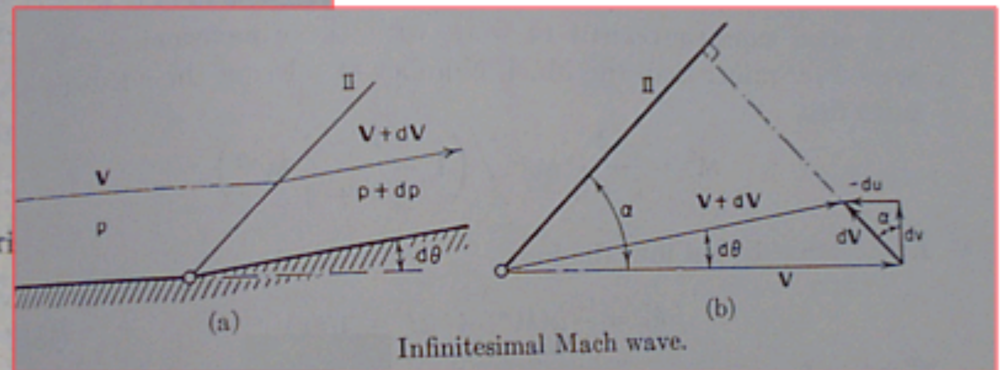
Characteristic Lines (cont'd)

- Left running characteristic lines

$$\theta - \nu(M) = K_+$$

Slope: $\theta + \mu$





- **C_+ "left running" characteristic Line is a Generalization infinitesimal compression corner flow**

Characteristic Lines

- Supersonic "compatibility" equations

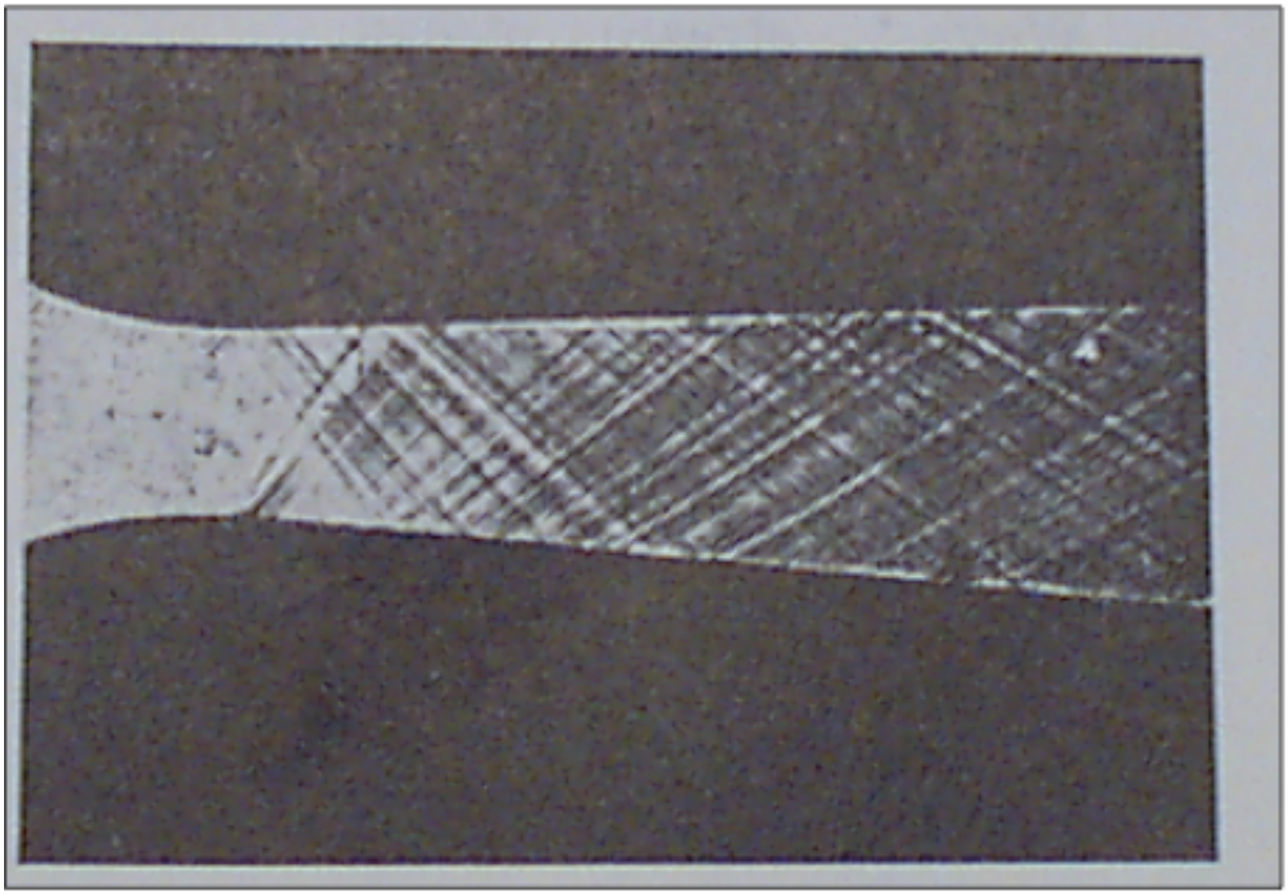
$$\theta + \nu(M) = Const \equiv K_{-}$$

$$\theta - \nu(M) = Const \equiv K_{+}$$

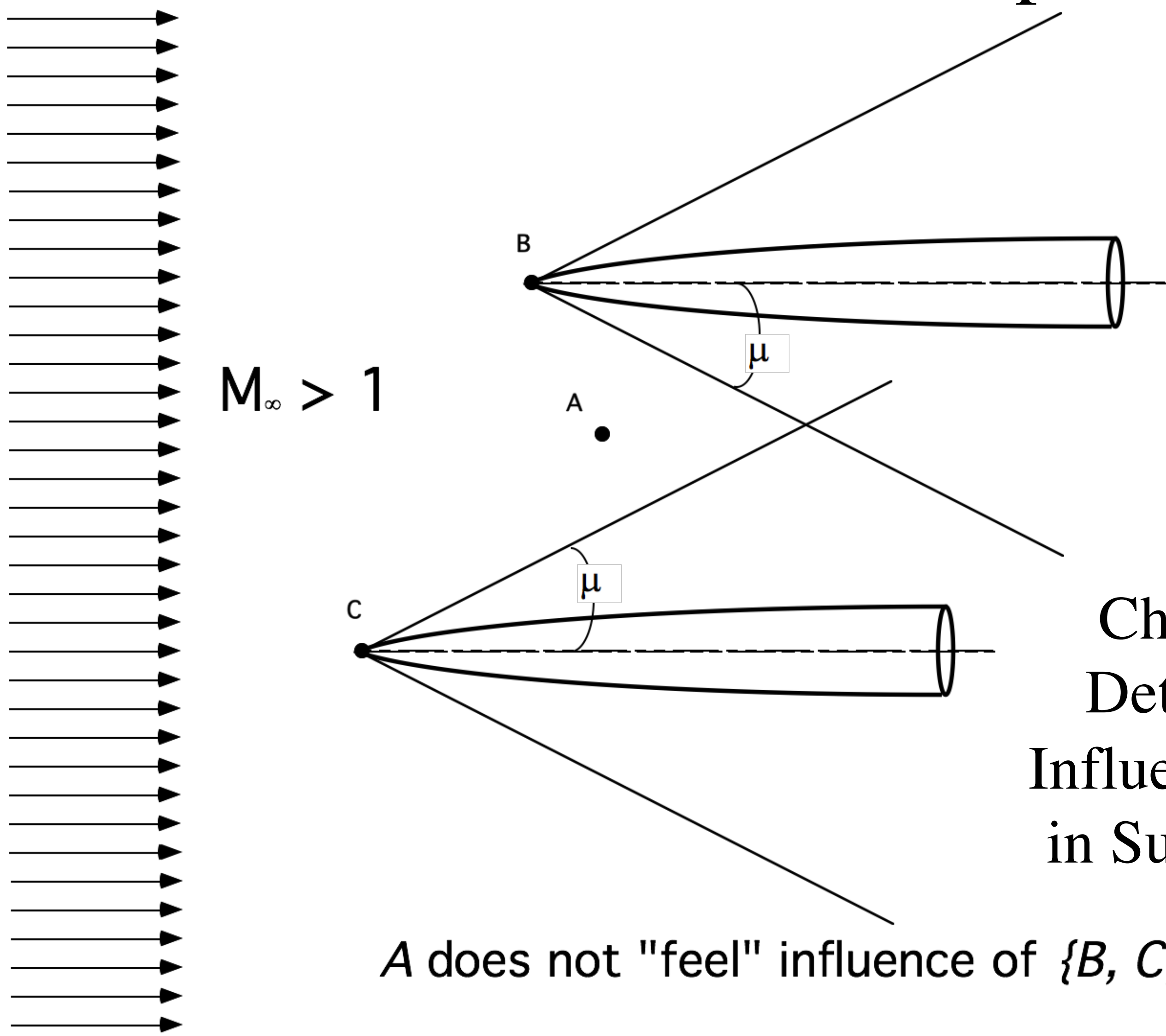
- Apply along "characteristic lines" in flow field

Physical Meaning of Characteristic Lines

- Schlieren Photo of Supersonic nozzle flow with roughened wall



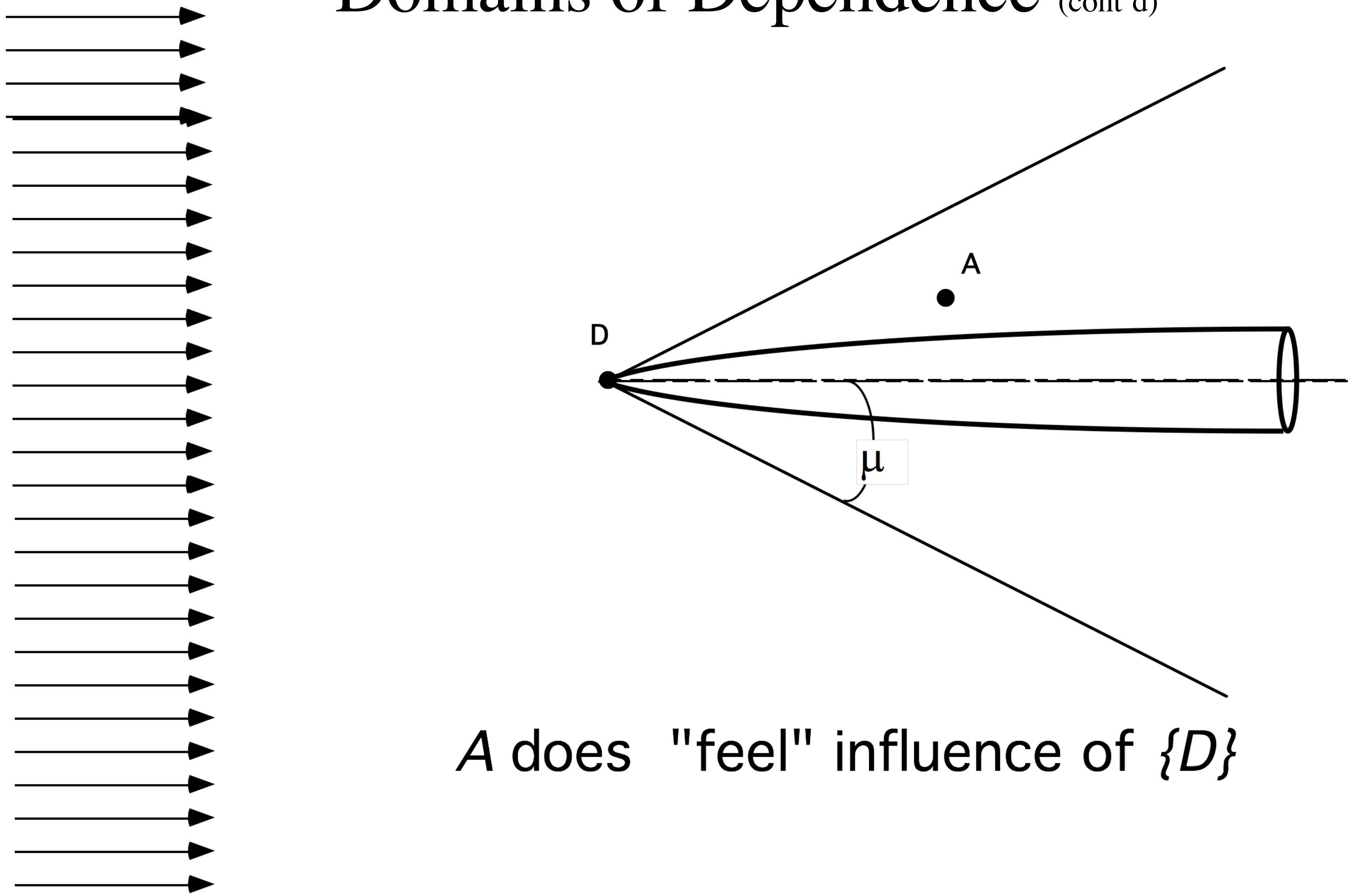
Regions of Influence and Domains of Dependence



Characteristics Lines Determine Regions of Influence and Dependence in Supersonic Flow Field

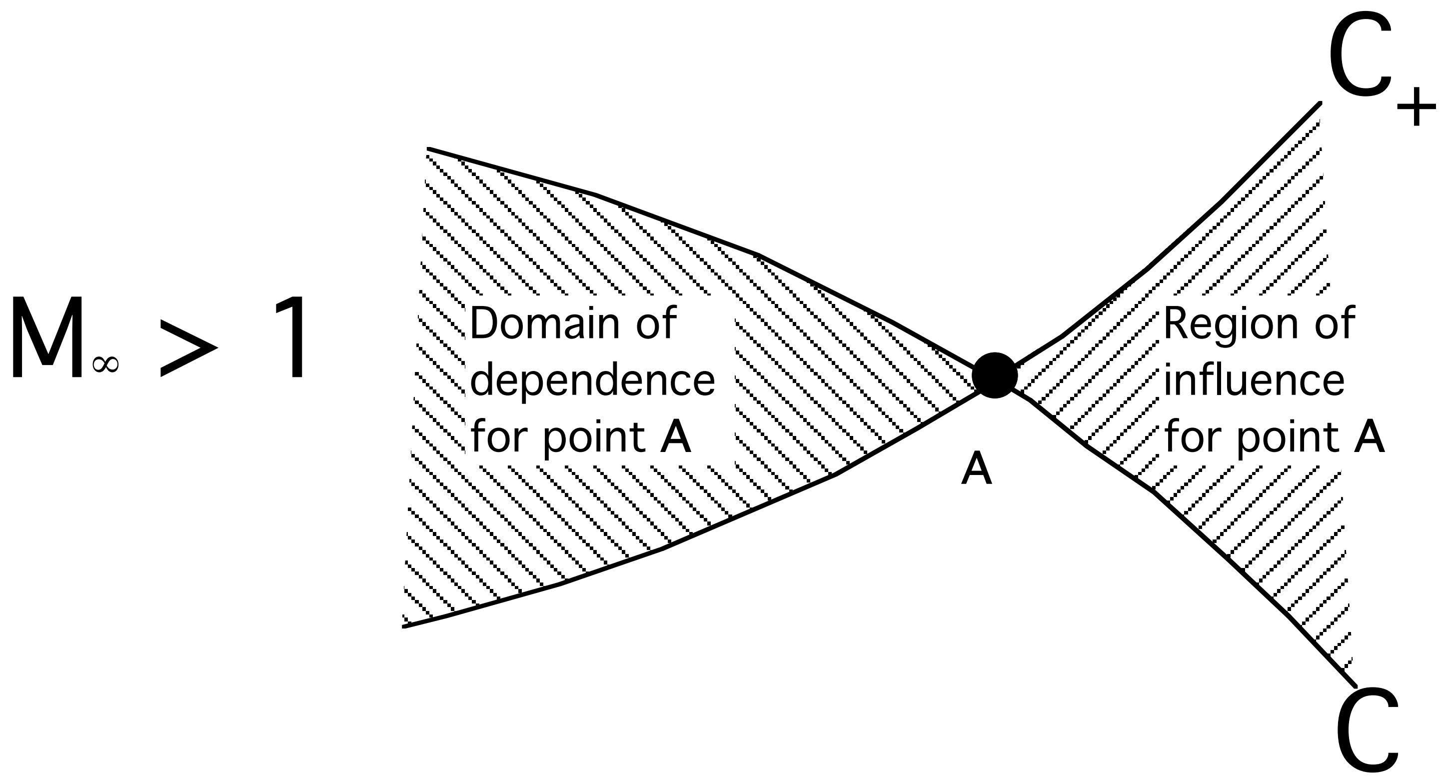
A does not "feel" influence of *{B, C}*

Regions of Influence and Domains of Dependence (cont'd)



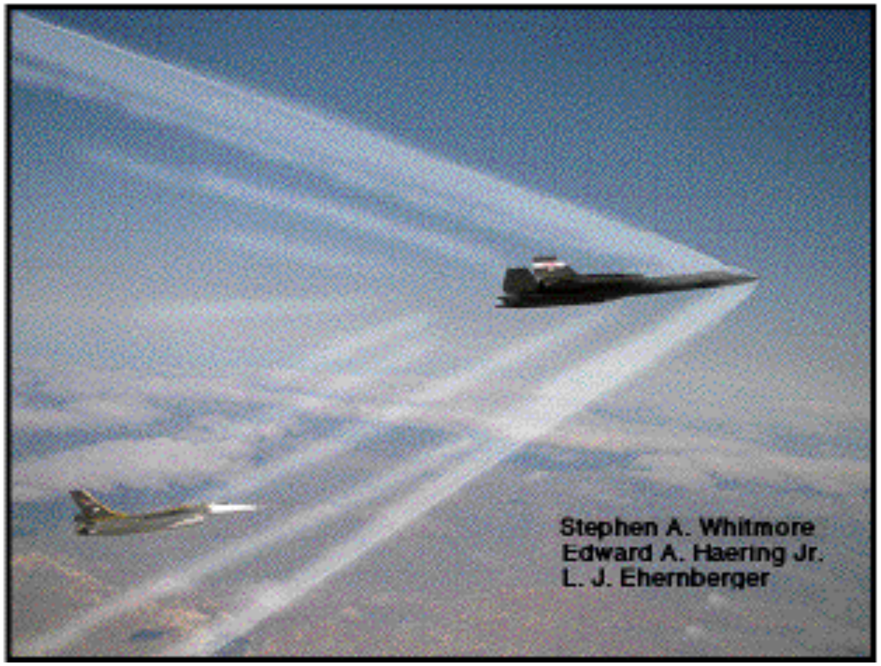
A does "feel" influence of *{D}*

Regions of Influence and Domains of Dependence (concluded)

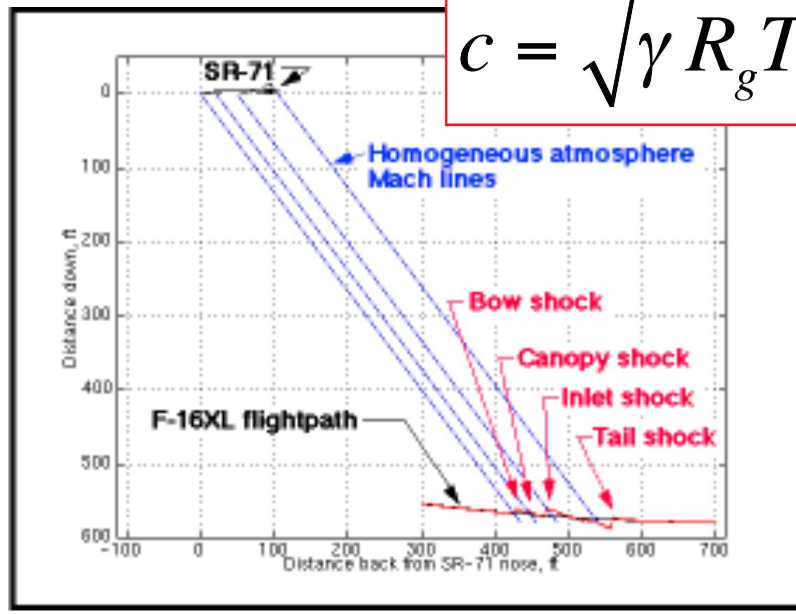


SR-71 Near Field Shock Wave Patterns

PRELIMINARY AIRBORNE MEASUREMENTS FOR THE SR-71 SONIC BOOM PROPAGATION EXPERIMENT



$$c = \sqrt{\gamma R_g T}$$



Speed of sound across each successive shock wave is Higher (temperature increases) … wave catch up and Reinforce each other

"Method of Characteristics"

- **Basic principle of Methods of Characteristics**

-- If supersonic flow properties
are known at two points in a flow field,

-- There is one and only one set of properties
compatible[*] with these at ***a*** third point,

-- Determined by the intersection
of *characteristics*, or *mach* ***waves*****,** from the two
original points.

[*]Root of term *"compatibility equations"*

"Method of Characteristics" (cont'd)

• Compatibility Equations relate the velocity magnitude and direction along the characteristic line.

• In 2-D and quasi 1-D flow, compatibility equations are Independent of spatial position, in 3-D methods, space Becomes a player and complexity goes up considerably

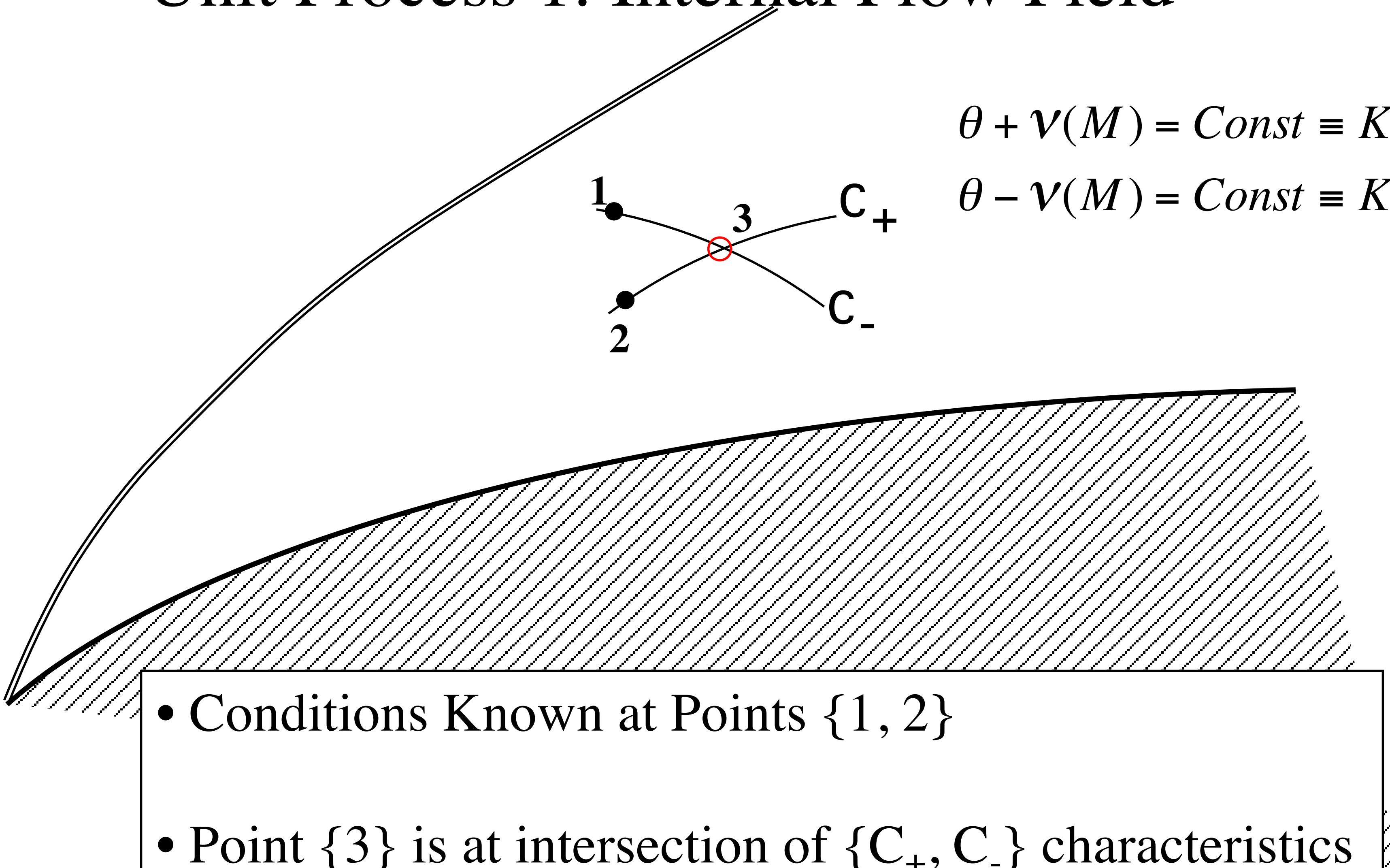
• Computational Machinery for applying the method of Characteristics are the so-called "unit processes"

• By repeated application of unit processes, flow field Can be solved in entirety

Unit Process 1: Internal Flow Field

$$\theta + \nu(M) = Const \equiv K_-$$

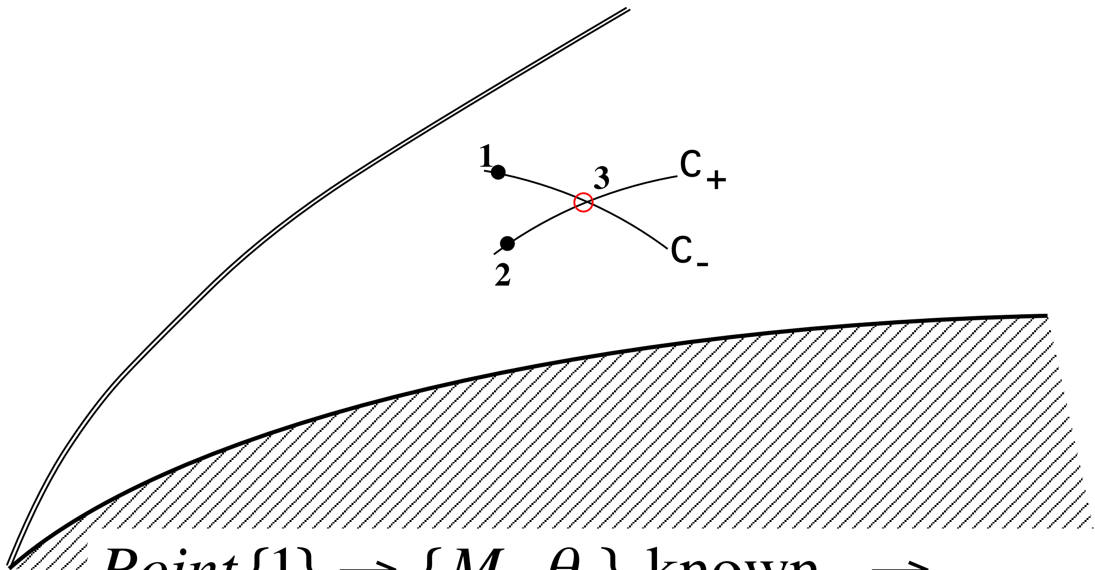
$$\theta - \nu(M) = Const \equiv K_+$$



- Conditions Known at Points {1, 2}

- Point {3} is at intersection of $\{C_+, C_-\}$ characteristics

Unit Process 1: Internal Flow Field (cont'd)

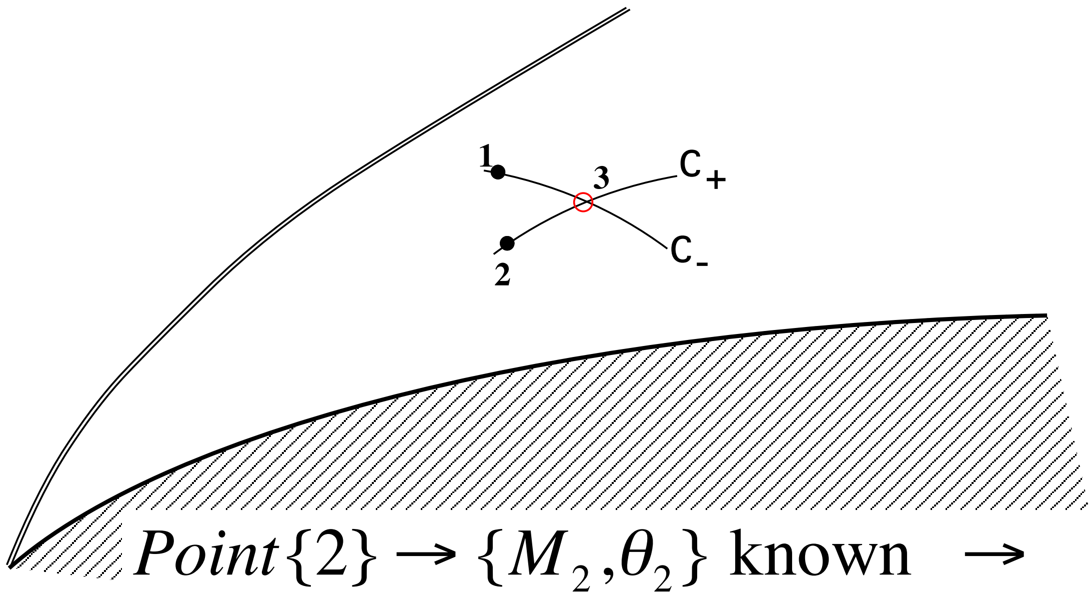


$Point\{1\} \rightarrow \{M_1, \theta_1\}$ known $\rightarrow$

$$\nu_1 = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_1^{\,2}-1\right)}\right\} - \tan^{-1}\sqrt{M_1^{\,2}-1}$$

$$Along\ \{C_-\} \rightarrow\ \theta_1 + \nu_1 = const = \left(K_-\right)_1$$

Unit Process 1: Internal Flow Field (cont'd)

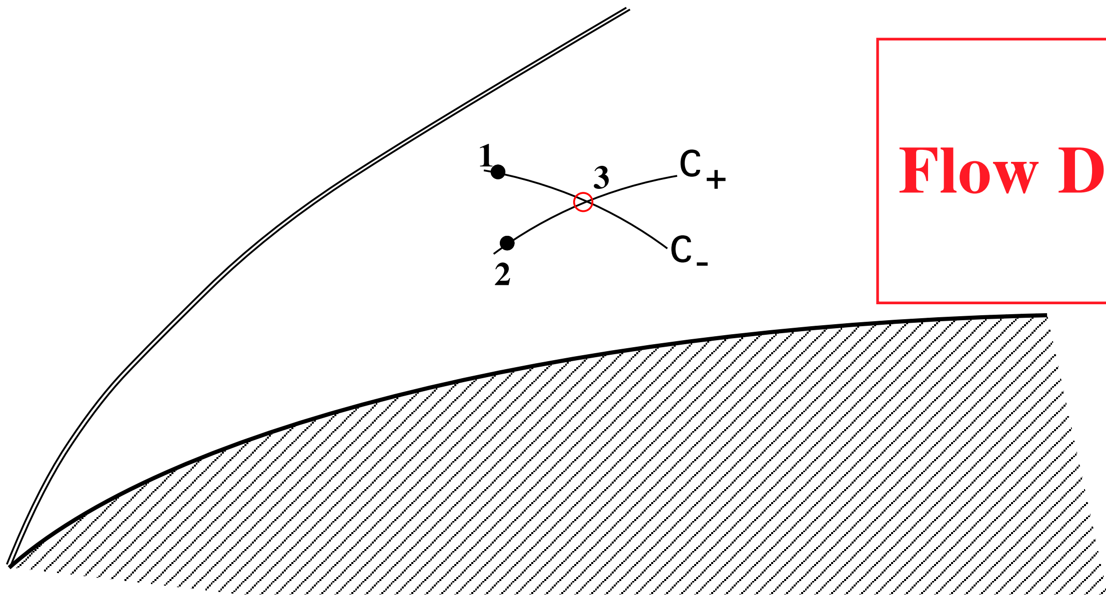


$$Point\{2\} \rightarrow \{M_2, \theta_2\} \text{ known } \rightarrow$$

$$\nu_2 = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_2{}^2-1\right)}\right\} - \tan^{-1}\sqrt{M_2{}^2-1}$$

$$Along\ \{C_+\} \rightarrow \theta_2 - \nu_2 = const = \left(K_+\right)_2$$

Unit Process 1: Internal Flow Field (cont'd)



Mach and Flow Direction solved for at Point 3

$$\theta + \nu(M) = Const \equiv K_-$$

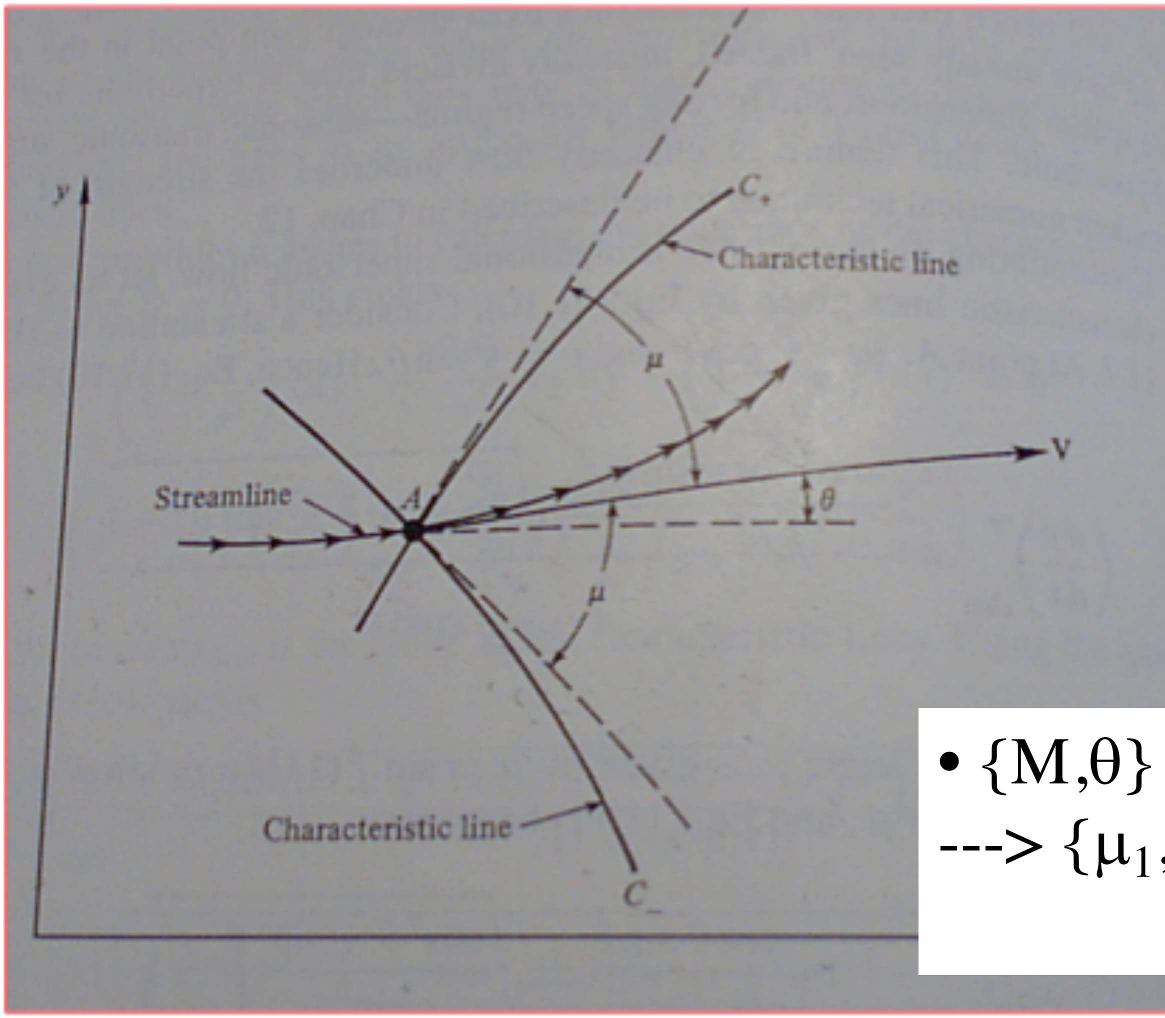
$$\theta - \nu(M) = Const \equiv K_+$$

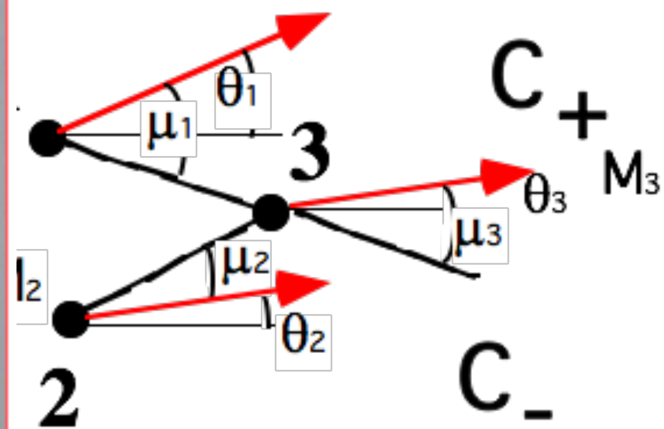
$$Point\{3\} \rightarrow \begin{bmatrix} \theta_1 + \nu_1 = \theta_3 + \nu_3 \\ \theta_2 - \nu_2 = \theta_3 - \nu_3 \end{bmatrix} \rightarrow \begin{bmatrix} \theta_3 = \dfrac{(\theta_1 + \nu_1) + (\theta_2 - \nu_2)}{2} = \dfrac{(K_-)_1 + (K_+)_2}{2} \\ \nu_3 = \dfrac{(\theta_1 + \nu_1) - (\theta_2 - \nu_2)}{2} = \dfrac{(K_-)_1 - (K_+)_2}{2} \end{bmatrix}$$

$$M_3 = Solve\left[\nu_3 = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_3^{\ 2} - 1\right)}\right\} - \tan^{-1}\sqrt{M_3^{\ 2} - 1}\right]$$

Unit Process 1: Internal Flow Field (cont'd)

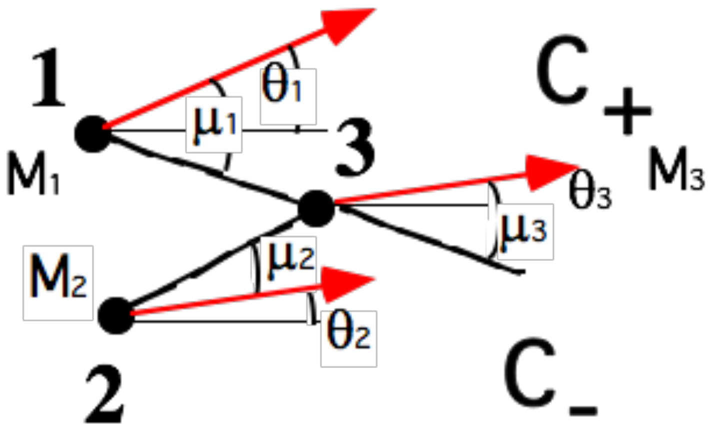
But where is Point {3} ?





- {M,θ} known at points {1,2,3} ---> $\{\mu_1,\mu_2,\mu_3\}$ known

Unit Process 1: Internal Flow Field (concluded)



But where is Point {3} ?

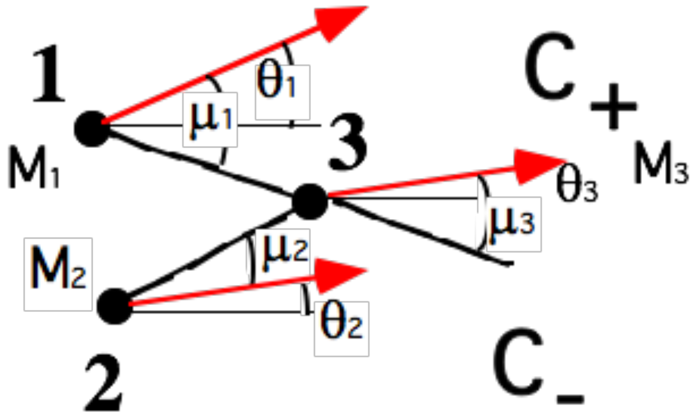
- Slope of characteristics lines approximated by:

$$slope\{C_-\} = \frac{(\theta_1 - \mu_1) + (\theta_3 - \mu_3)}{2}$$

Intersection locates point 3

$$slope\{C_+\} = \frac{(\theta_2 + \mu_2) + (\theta_3 + \mu_3)}{2}$$

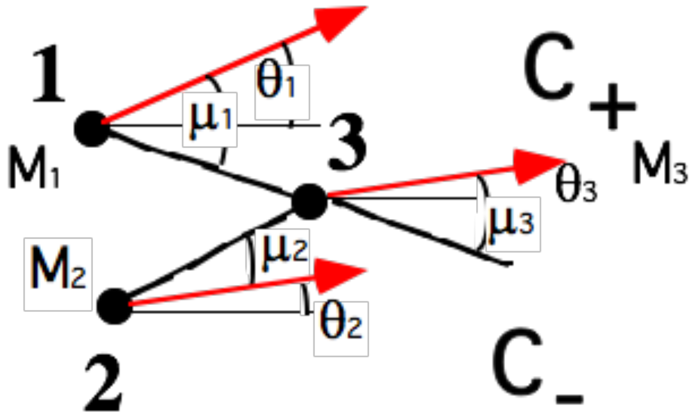
Unit Process 1: Internal Flow Example (cont'd)



- Solve for $\{x_3,y_3\}$

$$\begin{bmatrix} \dfrac{y_3 - y_1}{x_3 - x_1} = \tan\left[slope\left\{C_-\right\}\right] \\ \dfrac{y_3 - y_2}{x_3 - x_2} = \tan\left[slope\left\{C_+\right\}\right] \end{bmatrix} \rightarrow \begin{bmatrix} y_3 = \left(x_3 - x_1\right)\tan\left[slope\left\{C_-\right\}\right] + y_1 \\ y_3 = \left(x_3 - x_2\right)\tan\left[slope\left\{C_+\right\}\right] + y_2 \end{bmatrix}$$

Unit Process 1: Internal Flow Example (cont'd)



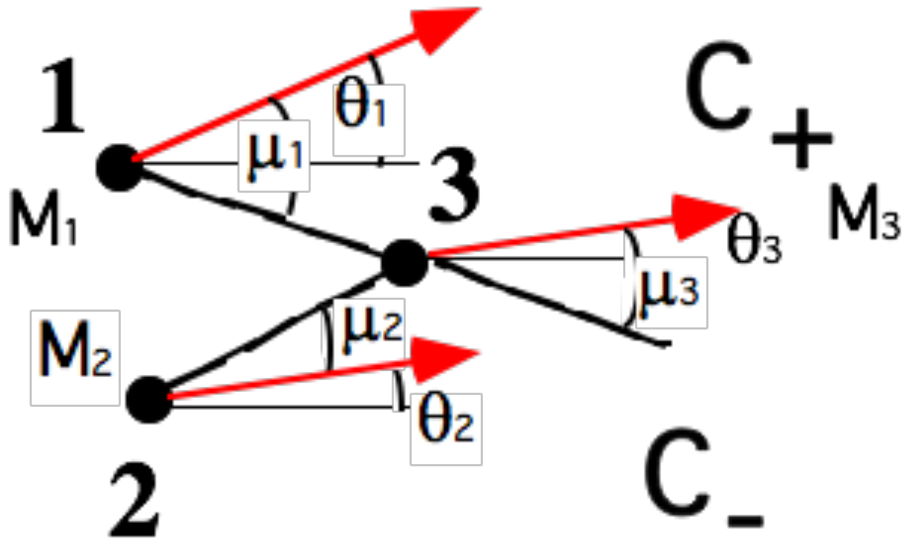
- Solve for $\{x_3, y_3\}$

Summary

$$x_3 = \frac{x_1 \cdot \tan\left[slope\{C_-\}\right] - x_2 \cdot \tan\left[slope\{C_+\}\right] + (y_2 - y_1)}{\left\{\tan\left[slope\{C_-\}\right] - \tan\left[slope\{C_+\}\right]\right\}}$$

$$y_3 = \frac{\tan\left[slope\{C_-\}\right] \cdot \tan\left[slope\{C_+\}\right] \cdot (x_1 - x_2) + \tan\left[slope\{C_-\}\right] \cdot y_2 - \tan\left[slope\{C_+\}\right] \cdot y_1}{\tan\left[slope\{C_-\}\right] - \tan\left[slope\{C_+\}\right]}$$

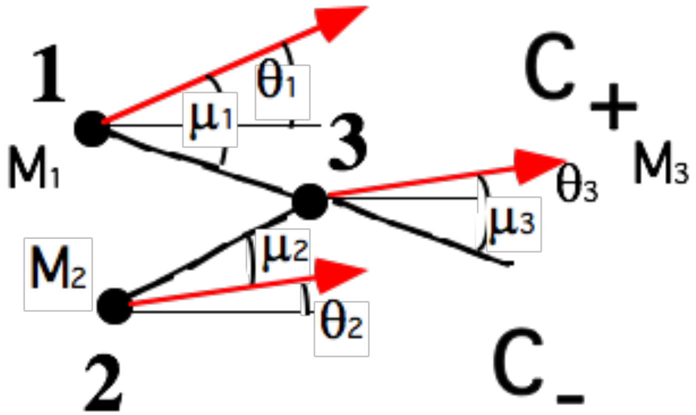
Unit Process 1: Internal Flow Example



$$M_1 = 2.0,\ \theta_1 = 10^o,\ \{x_1, y_1\} = \{1.0, 2.0\}$$

$$M_2 = 1.75,\ \theta_2 = 5^o,\ \{x_2, y_2\} = \{1.5, 1.0\}$$

Unit Process 1: Internal Flow Example (cont'd)



- Point 1, compute

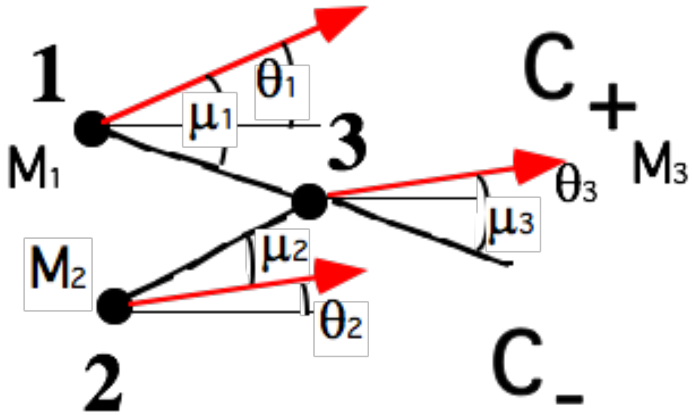
$$\left\{ \nu_1, \mu_1, (K_-)_1 \right\}$$

$$\nu_1 = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{ \sqrt{\frac{\gamma-1}{\gamma+1}\left(2.0^2-1\right)} \right\} - \tan^{-1}\sqrt{2.0^2-1} = 26.37976^{\circ}$$

$$\mu_1 = \frac{180}{\pi}\sin^{-1}\left[\frac{1}{2.0}\right] = 30^{o}$$

$$(K_-)_1 = \theta_1 + \nu_1 = 10^{o} + 26.37976^{\circ} = 36.37976^{\circ}$$

Unit Process 1: Internal Flow Example (cont'd)



- Point 2, compute

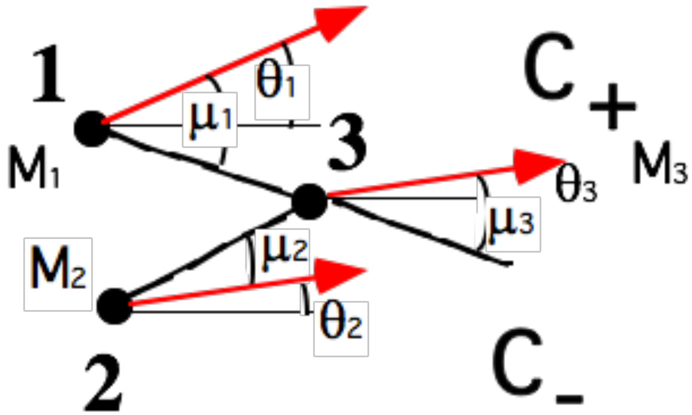
$$\left\{\nu_2, \mu_2, \left(K_+\right)_2\right\}$$

$$\nu_2 = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(1.75^2-1\right)}\right\} - \tan^{-1}\sqrt{1.75^2-1} = 19.27319^{\circ}$$

$$\mu_2 = \frac{180}{\pi}\sin^{-1}\left[\frac{1}{1.75}\right] = 34.84990^{o}$$

$$\left(K_+\right)_2 = \theta_2 - \nu_2 = 5^{o} - 19.27319^{\circ} = -14.27319^{\circ}$$

Unit Process 1: Internal Flow Example (cont'd)



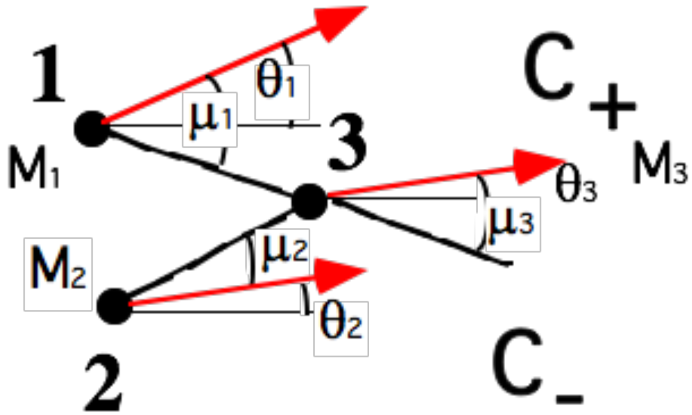
• Point 3 Solve for

$$\left\{\theta_3, \nu_3\right\}$$

$$\theta_3 = \frac{\left(K_-\right)_1 + \left(K_+\right)_2}{2} = \frac{36.37976 + (-14.27319)}{2} = 11.0533 \text{ deg.}$$

$$\nu_3 = \frac{\left(K_-\right)_1 - \left(K_+\right)_2}{2} = \frac{36.37976 - (-14.27319)}{2} = 25.3265 \text{ deg.}$$

Unit Process 1: Internal Flow Example (cont'd)



- Point 3 Solve for

$$\left\{M_3, \mu_3\right\}$$

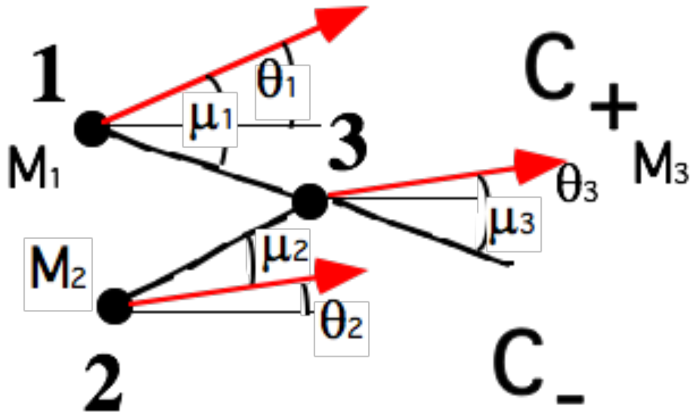
$$M_3 = Solve\left[25.3265\ \frac{\pi}{180} = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_3^{\ 2}-1\right)}\right\} - \tan^{-1}\sqrt{M_3^{\ 2}-1}\right]$$

$$\sin(\mu) = \frac{1}{M}$$

$M_3 = 1.96198$

---> $\mu_3 =$ 30.6431°

Unit Process 1: Internal Flow Example (cont'd)



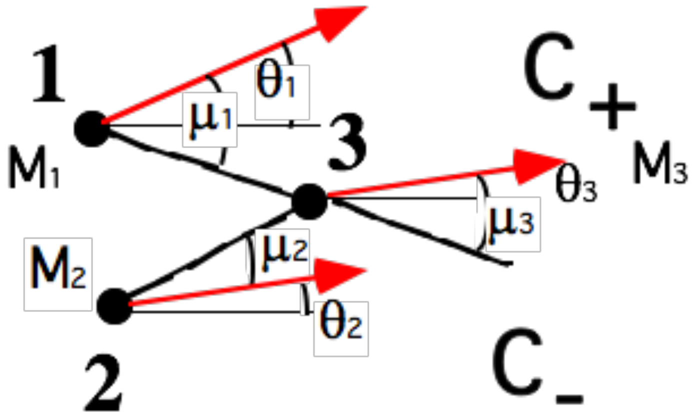
- Locate Point 3

- Line Slope Angles

$$slope\{C_-\} = \frac{(\theta_1 - \mu_1) + (\theta_3 - \mu_3)}{2} = \frac{(10 - 30) + (11.053 - 30.6431)}{2} = -19.795 \text{ deg}$$

$$slope\{C_+\} = \frac{(\theta_2 + \mu_2) + (\theta_3 + \mu_3)}{2} = \frac{(5 + 34.8499) + (11.053 + 30.6431)}{2} = 40.773 \text{ deg}$$

Unit Process 1: Internal Flow Example (cont'd)

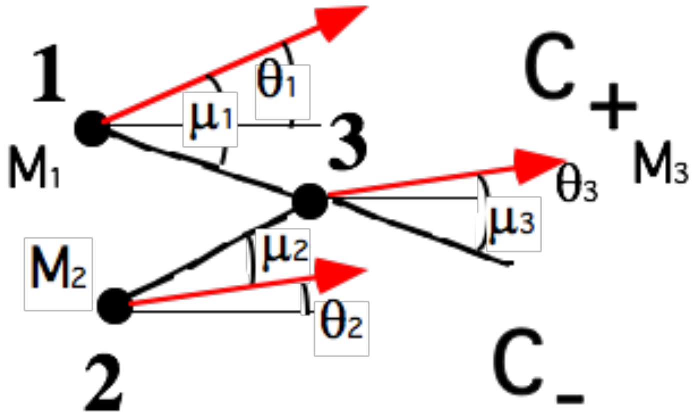


- Solve for $\{x_3, y_3\}$

$$x_3 = \frac{1\tan\left(\frac{\pi}{180}(-19.794887)\right) - 1.5\tan\left(\frac{\pi}{180}40.773123\right) + 1 - 2}{\tan\left(\frac{\pi}{180}(-19.7949)\right) - \tan\left(\frac{\pi}{180}40.7731\right)}$$

$$= 2.17091$$

Unit Process 1: Internal Flow Example (cont'd)



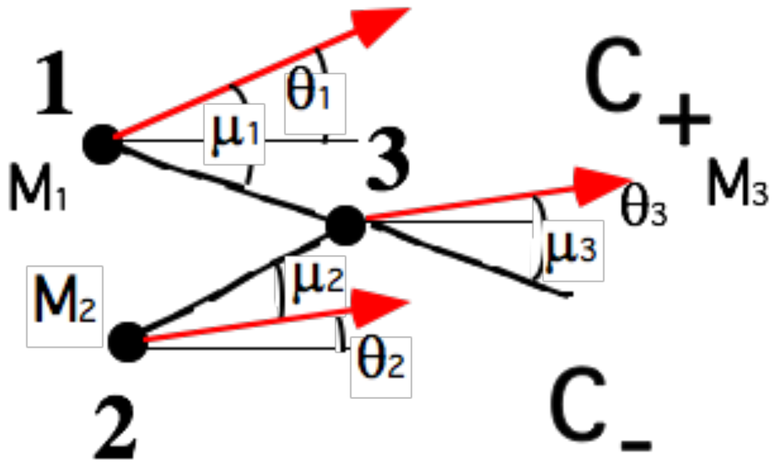
- Solve for $\{x_3,y_3\}$

$y_3=$

$$\frac{\tan\left(\frac{\pi}{180}(-19.79489)\right)\cdot\tan\left(\frac{\pi}{180}40.7731\right)\cdot(1.0-1.5)-2\tan\left(40.773123\frac{\pi}{180}\right)+1\tan\left(\frac{\pi}{180}(-19.79489)\right)}{\tan\left(\frac{\pi}{180}(-19.79489)\right)-\tan\left(\frac{\pi}{180}40.773123\right)}$$

$=1.57856$

Unit Process 1: Internal Flow Example (concluded)

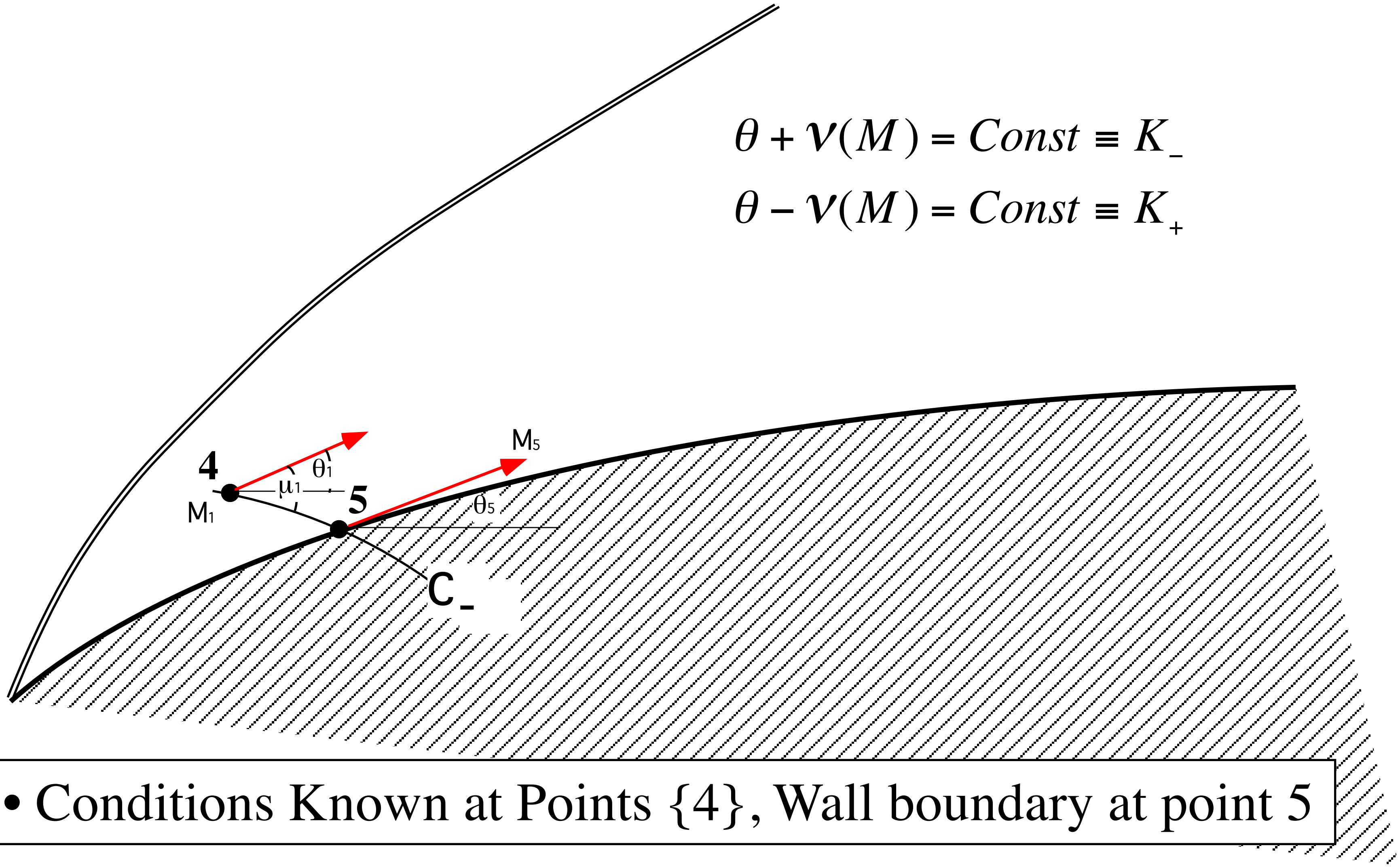


$$\begin{bmatrix} M_1 \\ \theta_1 \\ x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 10^o \\ 1.0 \\ 2.0 \end{bmatrix} \rightarrow \begin{bmatrix} M_2 \\ \theta_2 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.75 \\ 5^o \\ 1.5 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} M_3 \\ \theta_3 \\ x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1.96198 \\ 11.0533 \\ 2.17091 \\ 1.57856 \end{bmatrix}$$

Unit Process 2: Wall Point

$$\theta + \nu(M) = Const \equiv K_-$$

$$\theta - \nu(M) = Const \equiv K_+$$

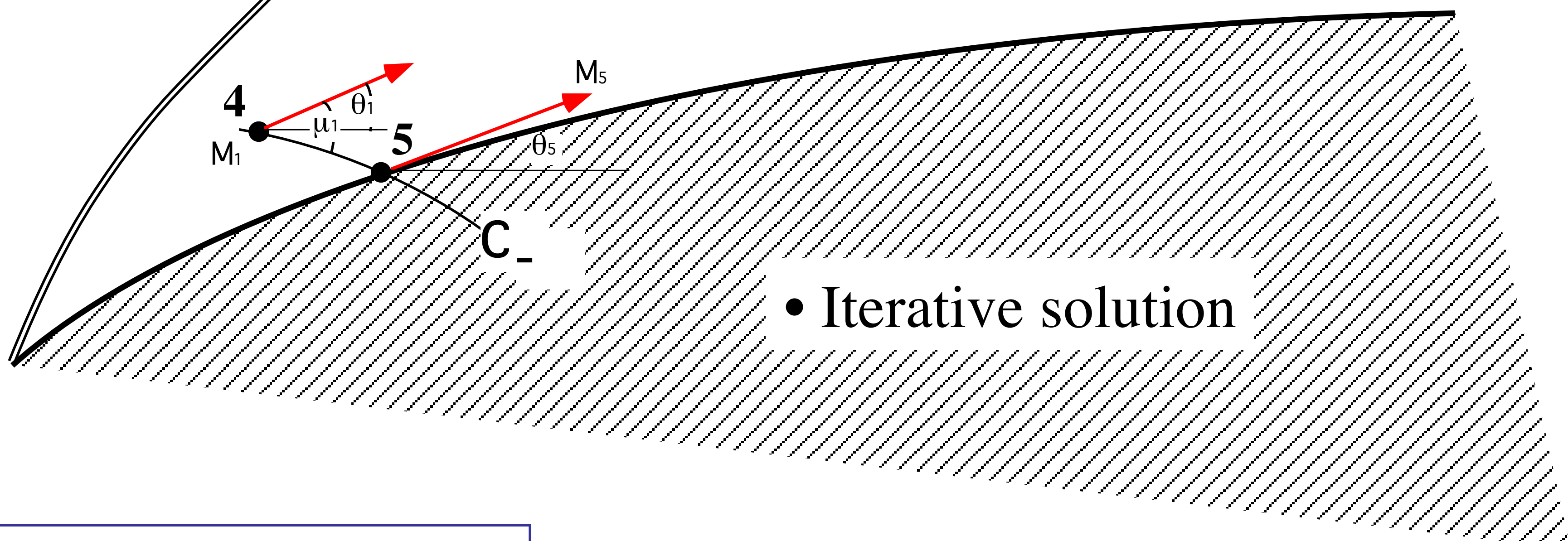


• Conditions Known at Points {4}, Wall boundary at point 5

Unit Process 2: Wall Point (cont'd)

$$\theta_5 + \nu_5 = \theta_4 + \nu_4 \rightarrow \boxed{\nu_5 = \theta_4 + \nu_4 - \theta_5}$$

$$slope\{C_-\} = \frac{(\theta_4 - \mu_4) + (\theta_5 - \mu_5)}{2}$$



- Iterative solution

Unit Process 2: Wall Point (cont'd)

- Iterative solution process

$$\nu_4 = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_4{}^2-1\right)}\right\} - \tan^{-1}\sqrt{M_4{}^2-1}$$

$$Along\ \{C_-\} \rightarrow\ \theta_4 + \nu_4 = const = \left(K_-\right)_4$$

- Pick θ_5 $\quad \nu_5 = \theta_4 + \nu_4 - \theta_5$

$$M_5 = Solve\left[\nu_5\frac{\pi}{180} = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_5{}^2-1\right)}\right\} - \tan^{-1}\sqrt{M_5{}^2-1}\right]$$

Unit Process 2: Wall Point (concluded)

- Solve for Mach angle, C_- slope

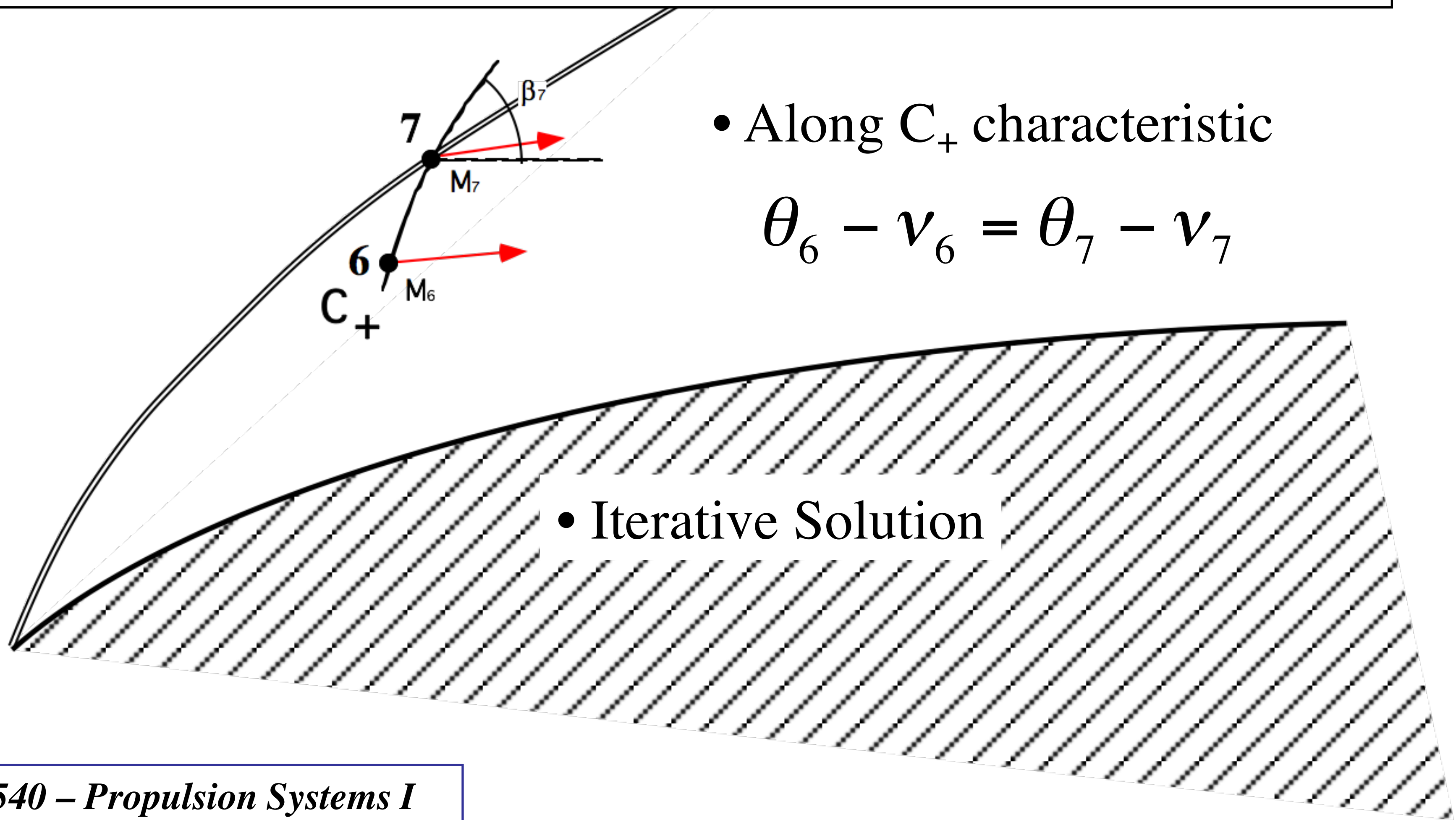
$$slope\left\{C_-\right\} = \frac{\left(\theta_4 - \mu_4\right) + \left(\theta_5 - \mu_5\right)}{2}$$

$$\sin\left(\mu_5\right) = \frac{1}{M_5}$$

- In Similar manner as before find intersection of C_- and surface mold line .. Get new θ_5, repeat iteration

Unit Process 3: Shock Point

- Conditions Known at Points {6}, Shock boundary at point 7
  Freestream Mach Number Known



- Along C_+ characteristic

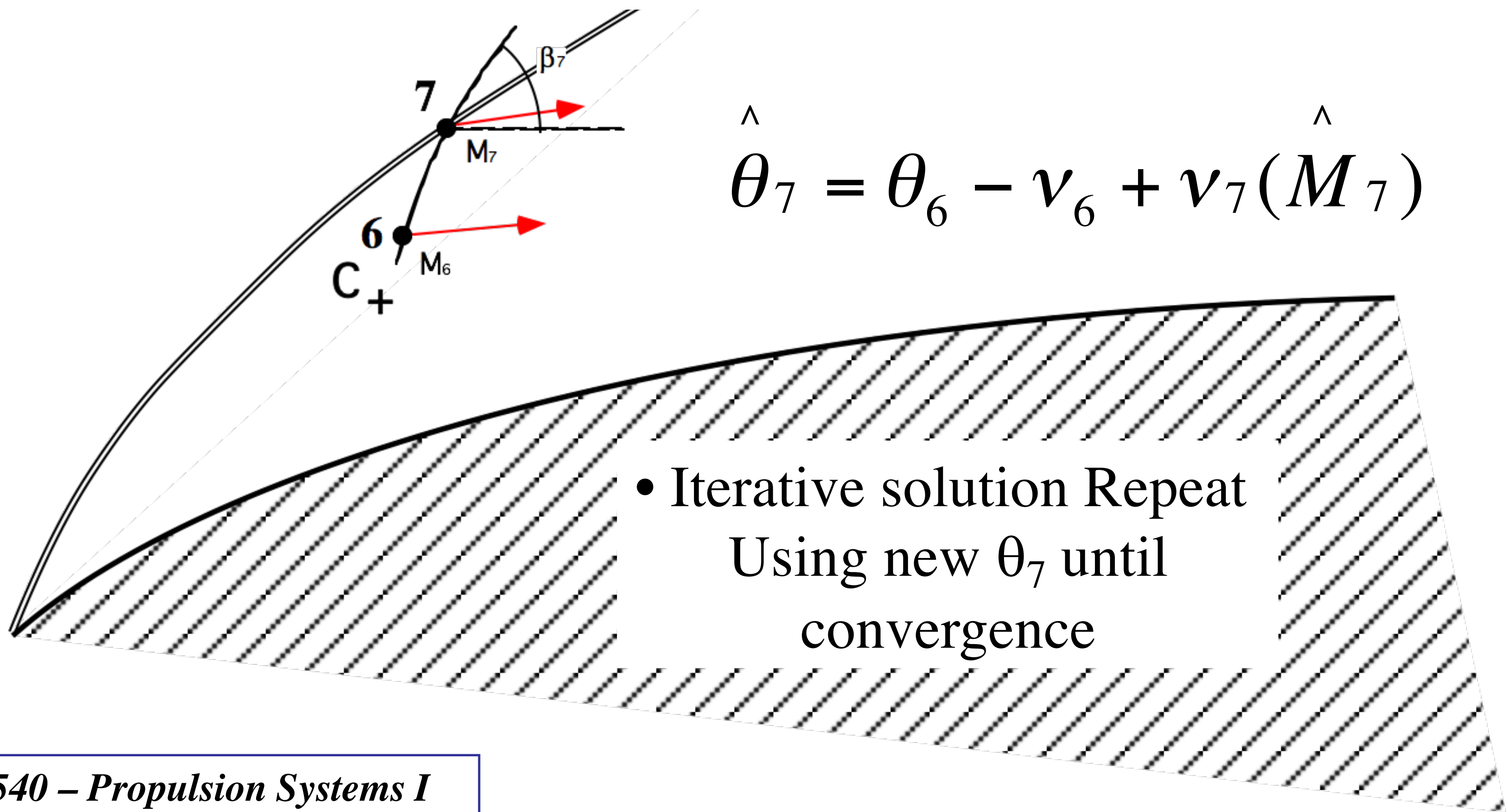
$$\theta_6 - \nu_6 = \theta_7 - \nu_7$$

- Iterative Solution

Unit Process 3: Shock Point

- Pick θ_7---> Oblique Shock wave solver

M_∞, θ_7 ---> M_7 (behind shock)



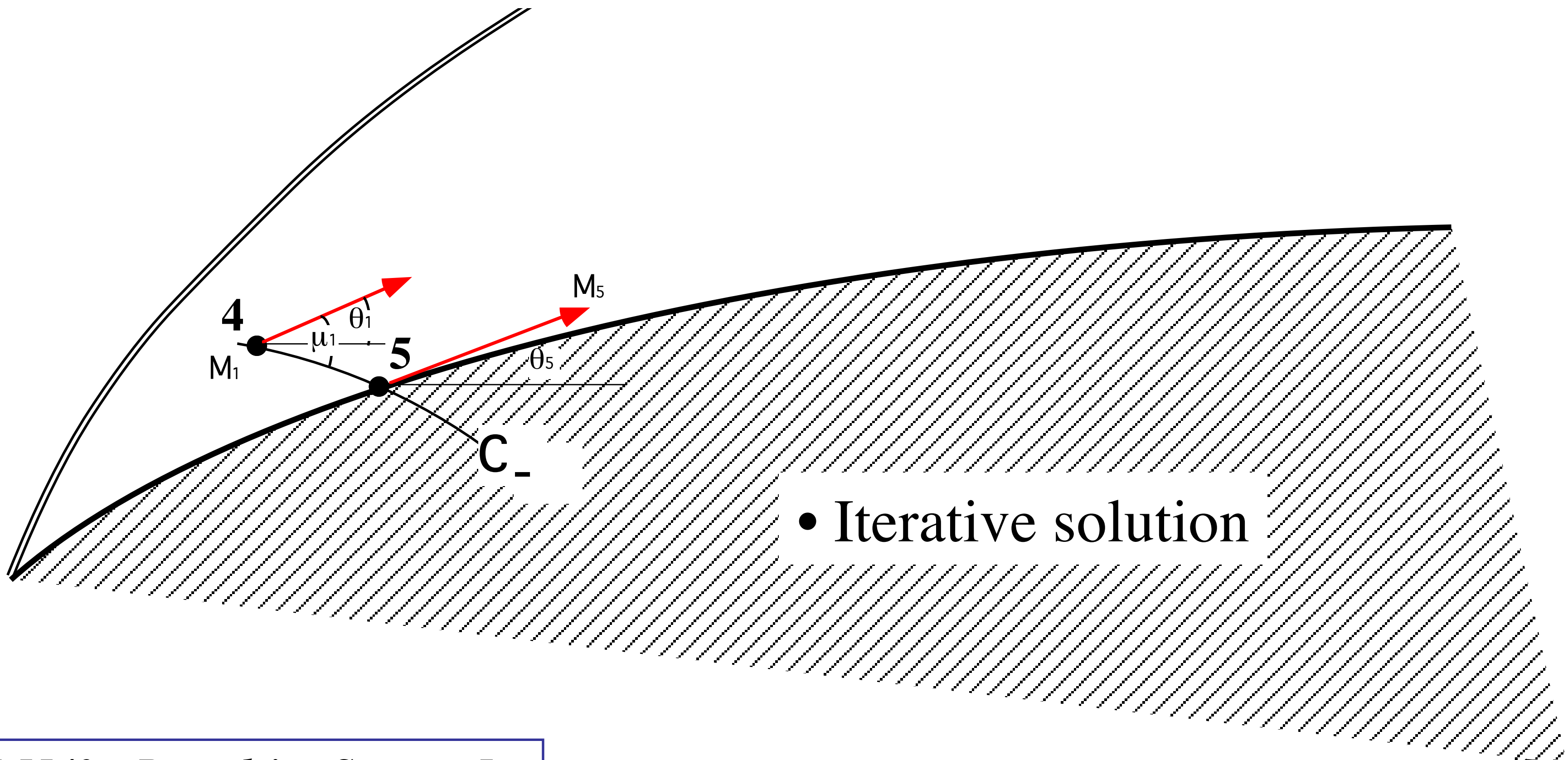
$$\hat{\theta}_7 = \theta_6 - \nu_6 + \nu_7(\hat{M}_7)$$

- Iterative solution Repeat Using new θ_7 until convergence

Unit Process 2: Shock Point (cont'd)

- Pick θ_7---> Oblique Shock wave solver

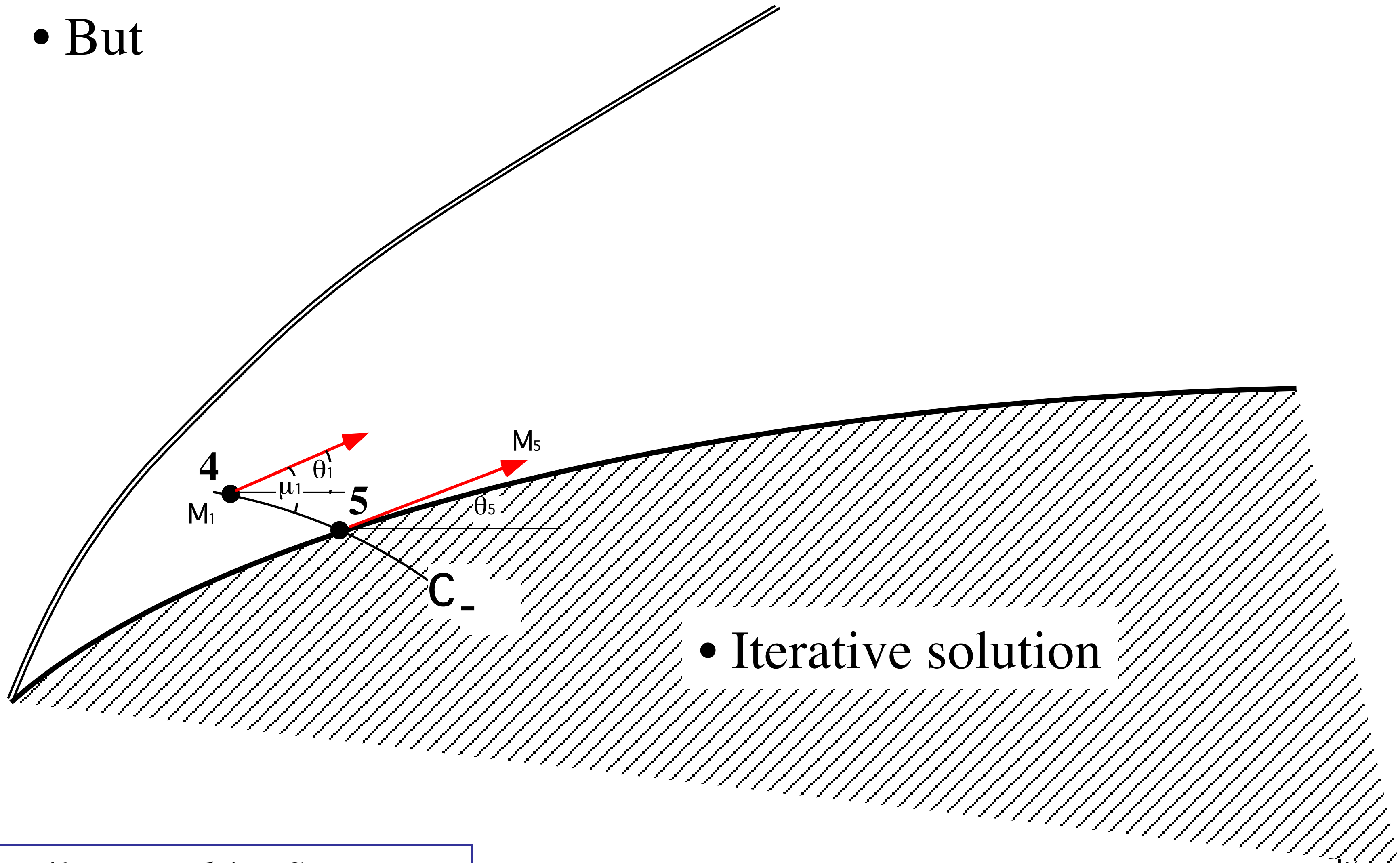
 ---> M7



- Iterative solution

Unit Process 2: Shock Point (cont'd)

- But



- Iterative solution

Using MOC for Supersonic Nozzle Design

In order to expand an internal steady flow through a duct from subsonic to supersonic speed, we established in Chap. 5 that the duct has to be convergent-divergent in shape,

Moreover, we developed relations

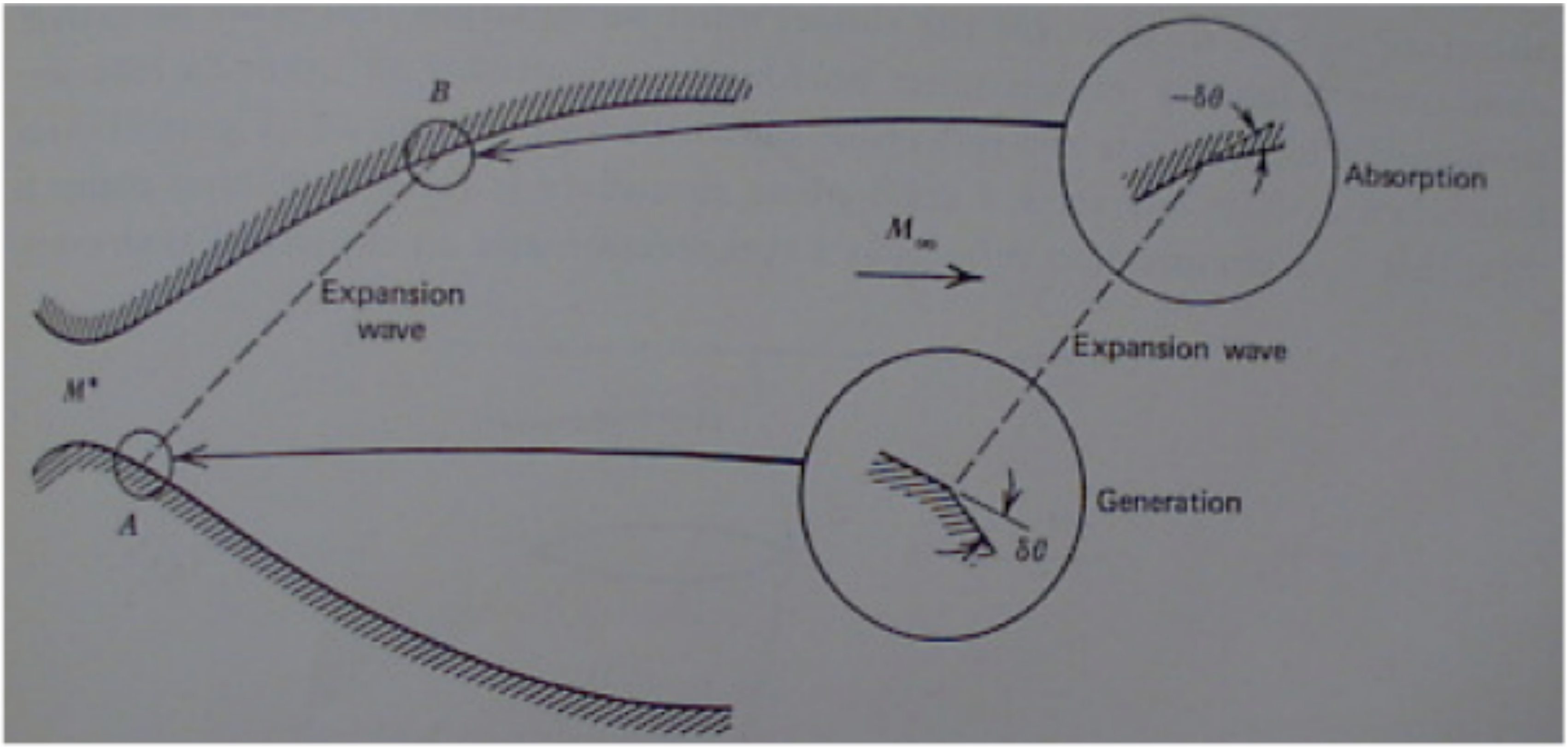
for the local Mach number, and hence the pressure, density, and temperature, as functions of local area ratio A/A^*.

However, these relations assumed quasi-one-

dimensional flow, whereas, strictly speaking, the flow two-dimensional. Moreover, the quasi-one-dimensional theory tells us nothing about the proper *contour* of the duct, i.e., what is the proper variation of area with respect to the flow direction $A = A(x)$. If the nozzle contour is not proper, shock waves may occur inside the duct.

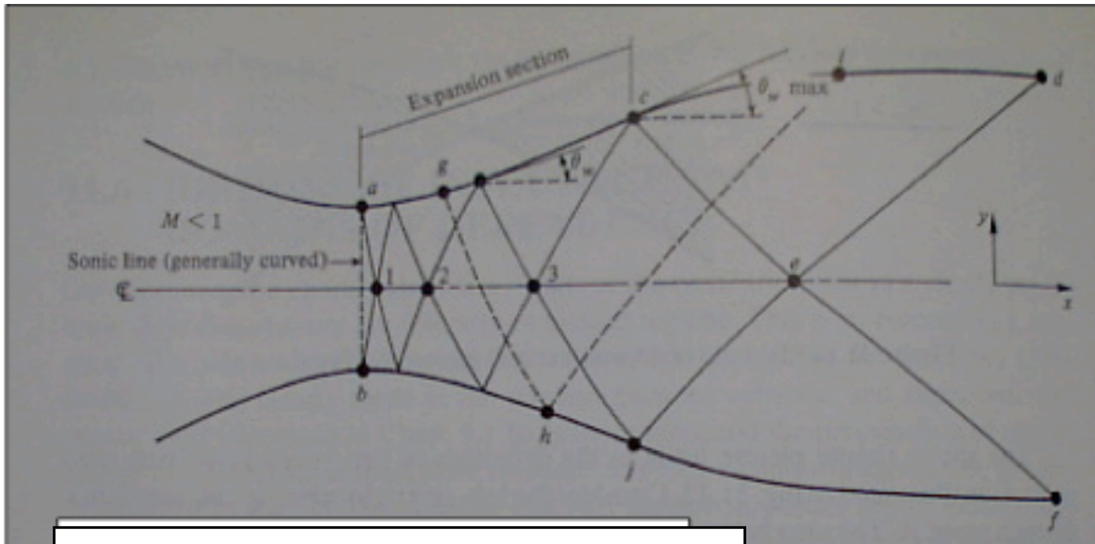
The method of characteristics provides a technique for properly designing the contour of a supersonic nozzle for shockfree, isentropic flow, taking into account the multidimensional flow inside the duct. The purpose of this section is to illustrate such an application.

Supersonic Nozzle Design

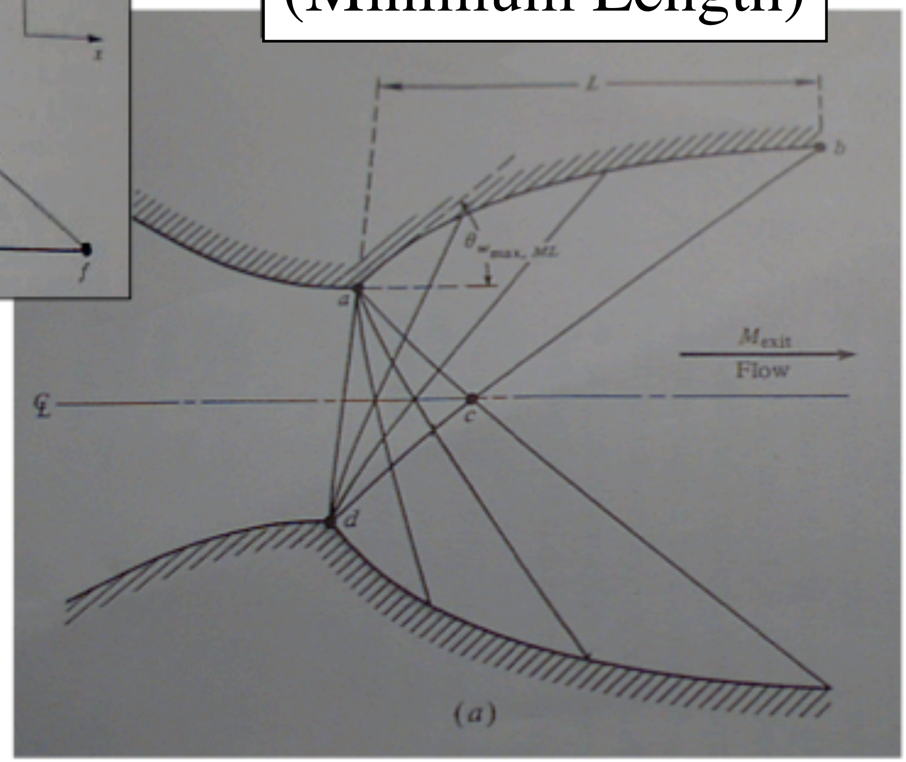


- Strategic contouring will "absorb" mach waves to give isentropic flow in divergent section

Supersonic Nozzle Design (cont'd)



• Bell Nozzle
(gradual expansion)

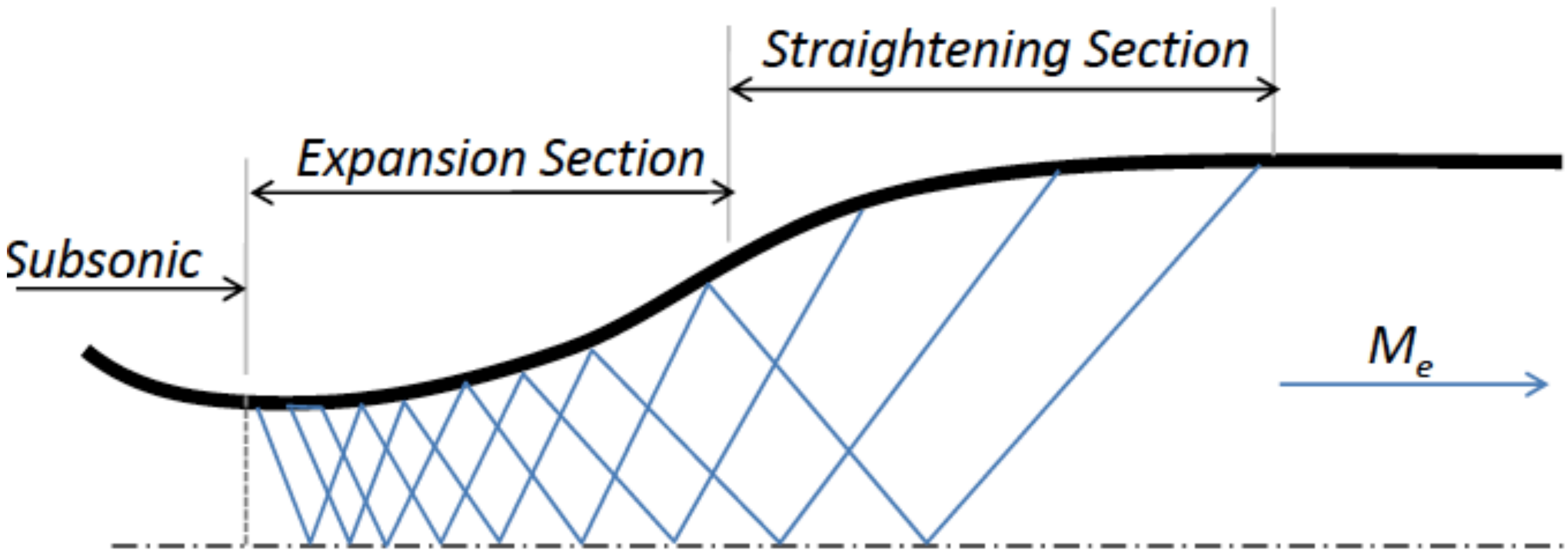


• Rocket Nozzle
(Minimum Length)

• **Use compatibility eqs. to design boundary with shock free flow**

Using Method of Characteristics to Design a Bell Nozzle

- This approach "prescribes" the expansion section of the nozzle, and then uses M.O.C to design turning section to achieve wave cancellation at wall …. And ensure isentropic flow

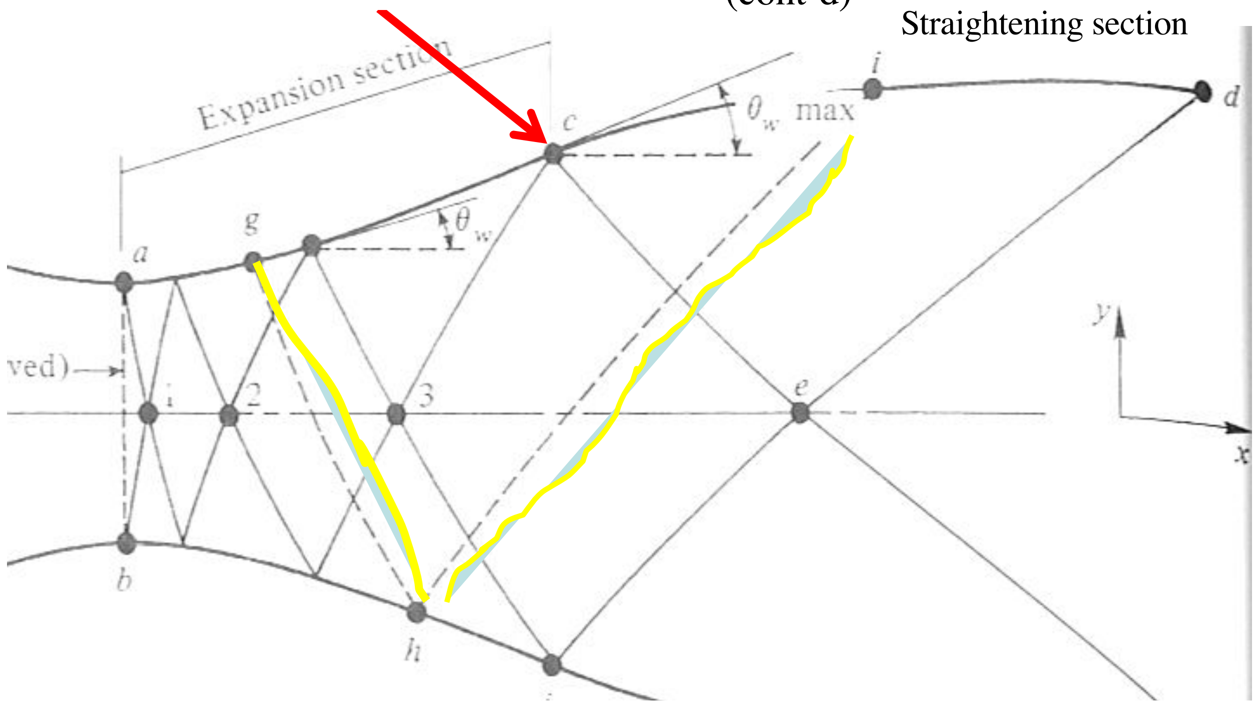


Method of Characteristics for Bell Nozzle Design

(cont'd)

Inflection Point

Straightening section



- Straightening section is designed to cancel all of the expansion waves generated by expansion section

- Wave generated at ***g***, reflects at ***h*** and is cancelled by wall at ***i***

Using Method of Characteristics to Design a Bell Nozzle (2)

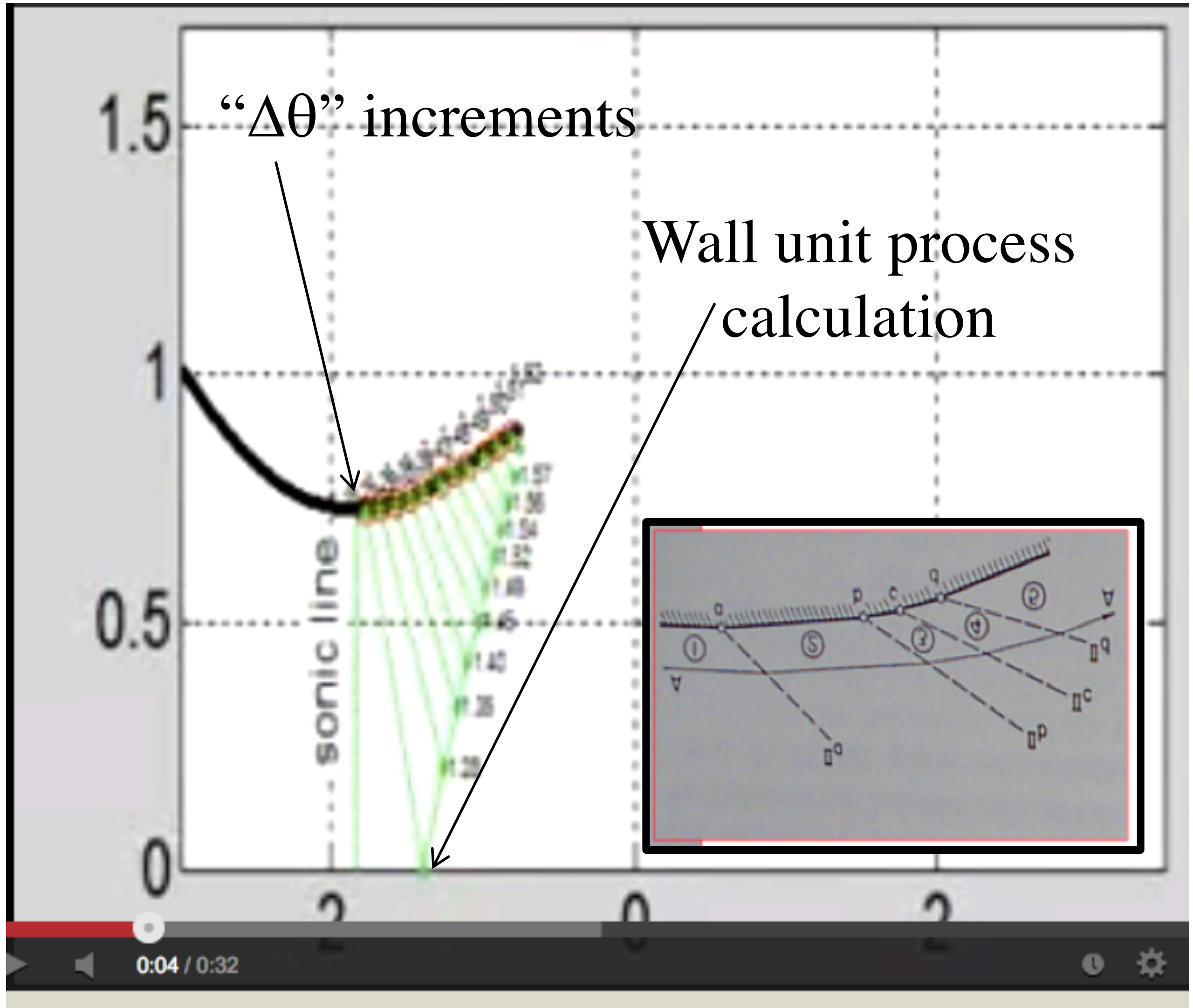
"Running Method"

- Pick "θ_{max}" and length of expansion section

- Use Prandtl-Meyer Expansion equations to calculate M, ν, μ at Each point along boundary

$$X_i = R_c \cdot \sin\theta_i$$

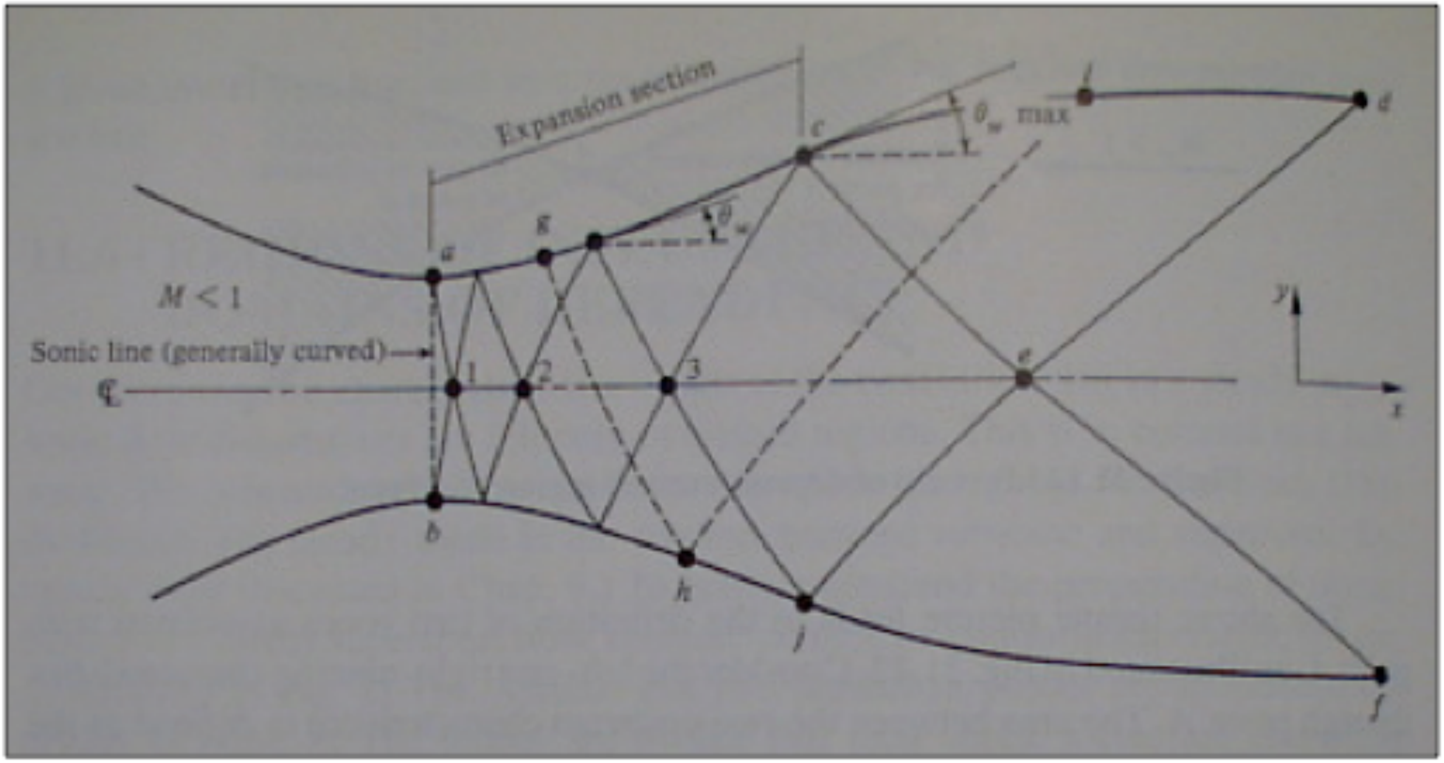
$$Y_i = \frac{1}{2} D_{throat} + R_c \cdot \left(1 - \cos\theta_i\right)$$



Initial Data Line

- Unit Processes must start somewhere .. Need a datum from which too start process

- Example nozzle flow … Throat



Initial Data Line

Using Method of Characteristics to Design a Bell Nozzle (2)

$$\Delta\theta = \frac{\theta_{MAX}}{N} \rightarrow \begin{array}{l} \theta_0 = 0 \\ M_0 = 1 \rightarrow \ \nu(1) = 0 \\ \theta_i = i \cdot \Delta\theta \end{array}$$

$$X_i = R_c \cdot \sin\theta_i$$

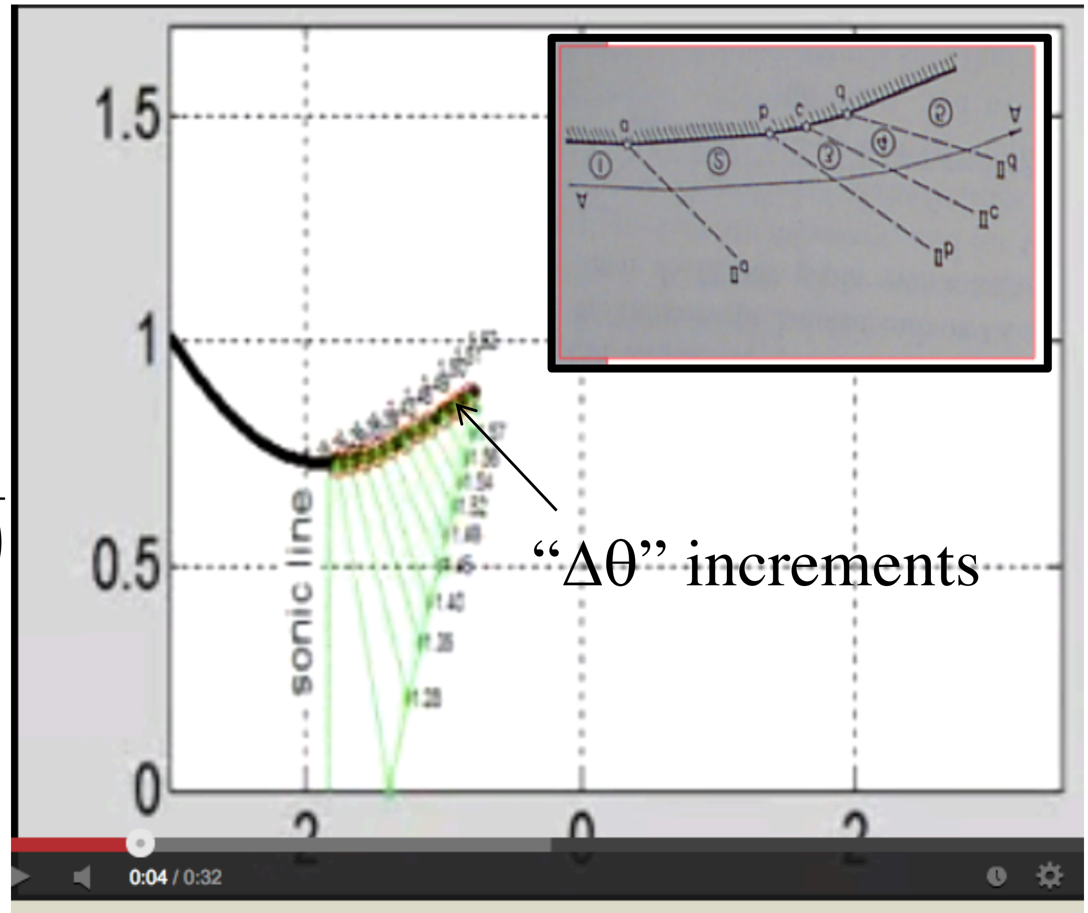
$$Y_i = \frac{1}{2} D_{throat} + R_c \cdot (1 - \cos\theta_i)$$

$$\rightarrow \begin{array}{l} \nu(M_i) - \theta_i = \nu(M_0) - \theta_0 \\ \nu(M_i) = \theta_i = i \cdot \Delta\theta \end{array}$$

$$Extract\ M_i \rightarrow \nu(M_i) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M_i^2 - 1)} - \tan^{-1}\sqrt{(M_i^2 - 1)}$$

$$i = \{1, \ldots N\}$$

$$\mu_i = \sin^{-1}\left(\frac{1}{M_i}\right)$$



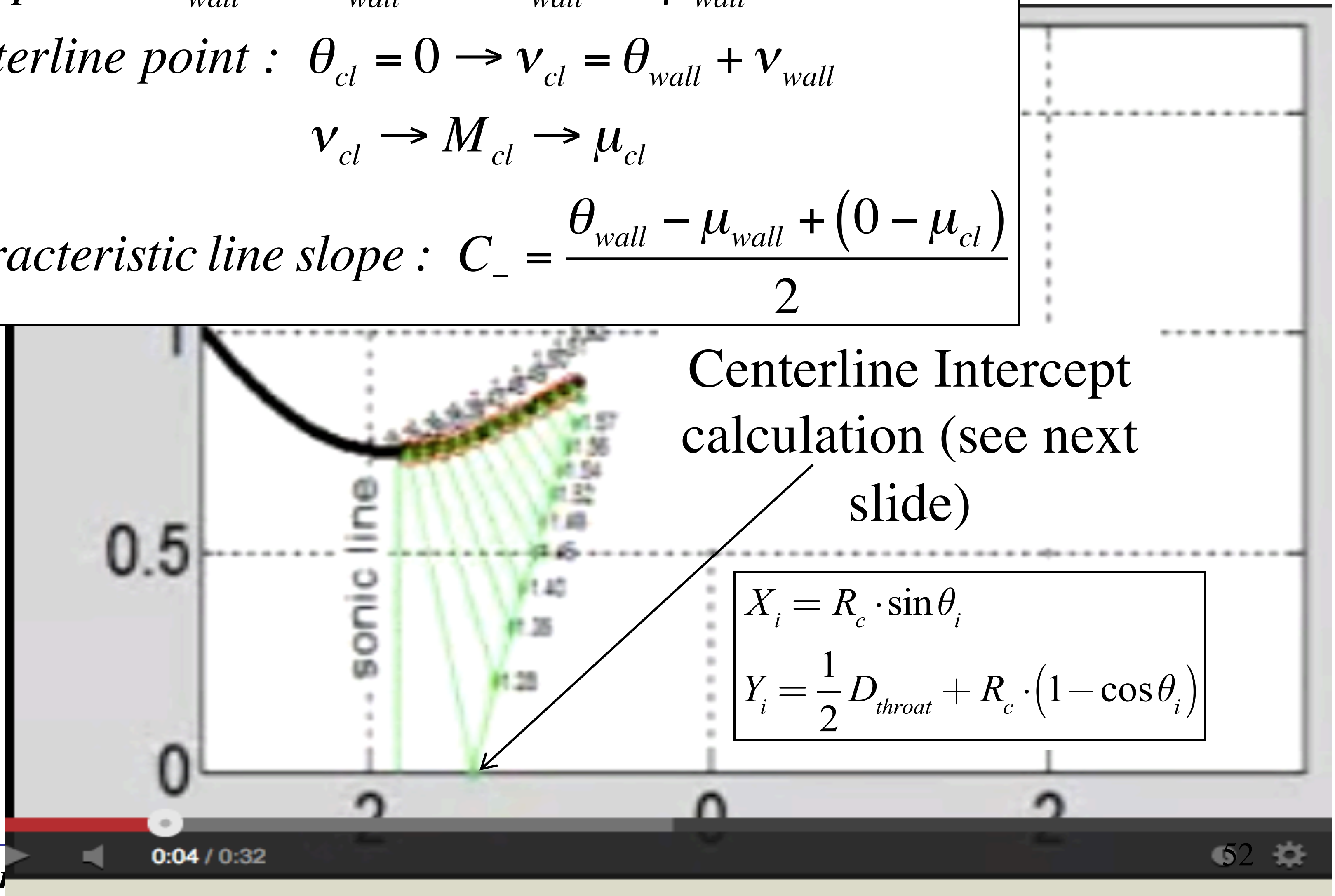
Using Method of Characteristics to Design a Bell Nozzle (3)

1) *Wall point* : $\theta_{wall} \rightarrow \nu_{wall} \rightarrow M_{wall} \rightarrow \mu_{wall}$

2) *Centerline point* : $\theta_{cl} = 0 \rightarrow \nu_{cl} = \theta_{wall} + \nu_{wall}$

$$\nu_{cl} \rightarrow M_{cl} \rightarrow \mu_{cl}$$

3) *Characteristic line slope* : $C_{-} = \dfrac{\theta_{wall} - \mu_{wall} + (0 - \mu_{cl})}{2}$



Centerline Intercept calculation (see next slide)

$$X_i = R_c \cdot \sin\theta_i$$

$$Y_i = \frac{1}{2} D_{throat} + R_c \cdot (1 - \cos\theta_i)$$

Centerline Intercept Solution

$$1)\ Initial\ Point:\ \{x_1, y_1, \theta_1, M_1\} \rightarrow \begin{bmatrix} \nu_1 \\ \mu_1 \end{bmatrix}$$

$$2)\ Centerline\ Intercept:\ \theta_{cl} = 0$$

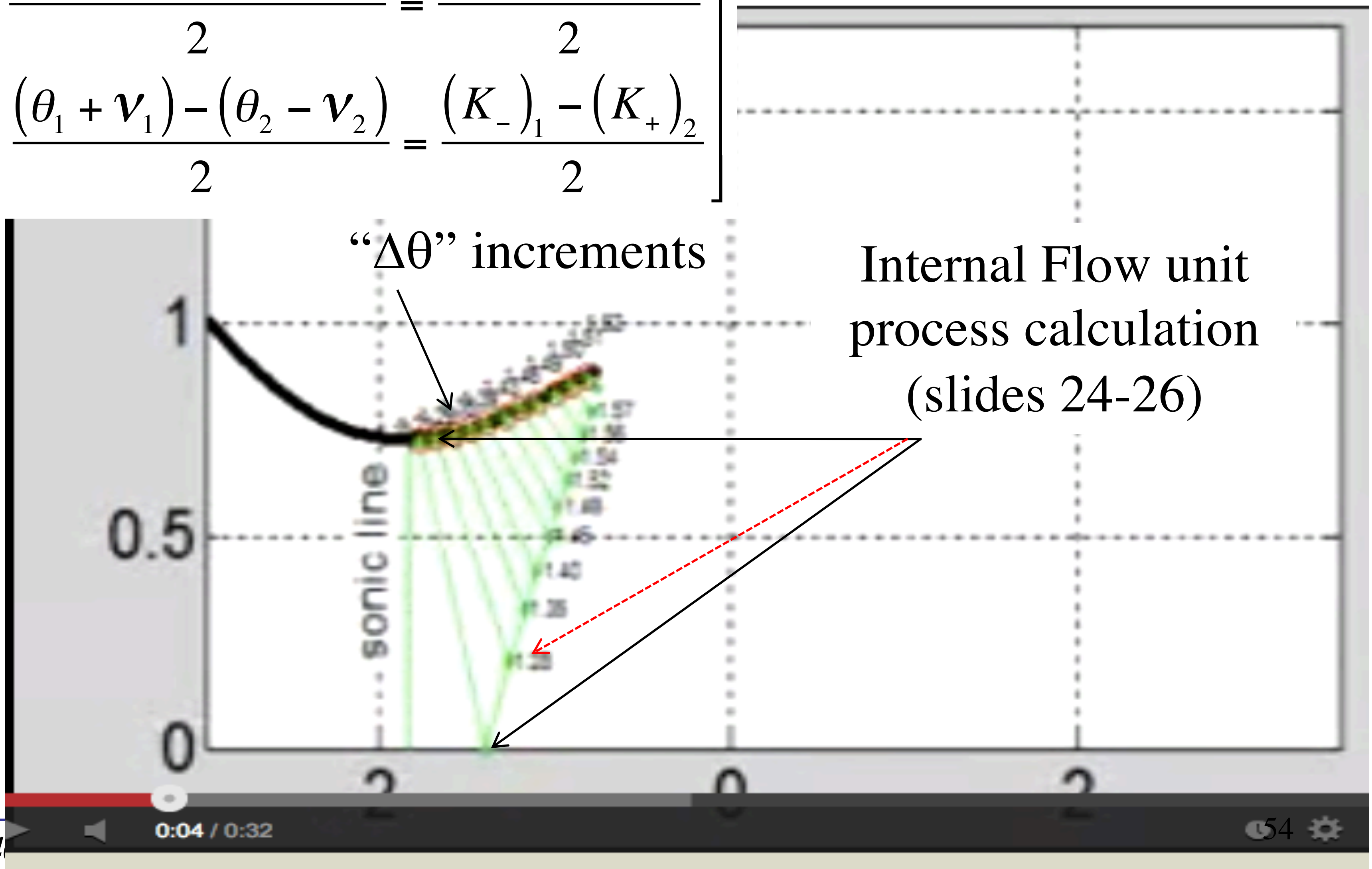
$$right\ running\ characteristic\ line \rightarrow \theta_{cl} + \nu_{cl} = \theta_1 + \nu_1$$

$$\rightarrow \nu_{cl} = \theta_1 + \nu_1 \rightarrow\rightarrow \begin{bmatrix} M_{cl} \\ \mu_{cl} \end{bmatrix} \rightarrow Slope(C_-) = \frac{\theta_1 - \mu_1 - \mu_{cl}}{2}$$

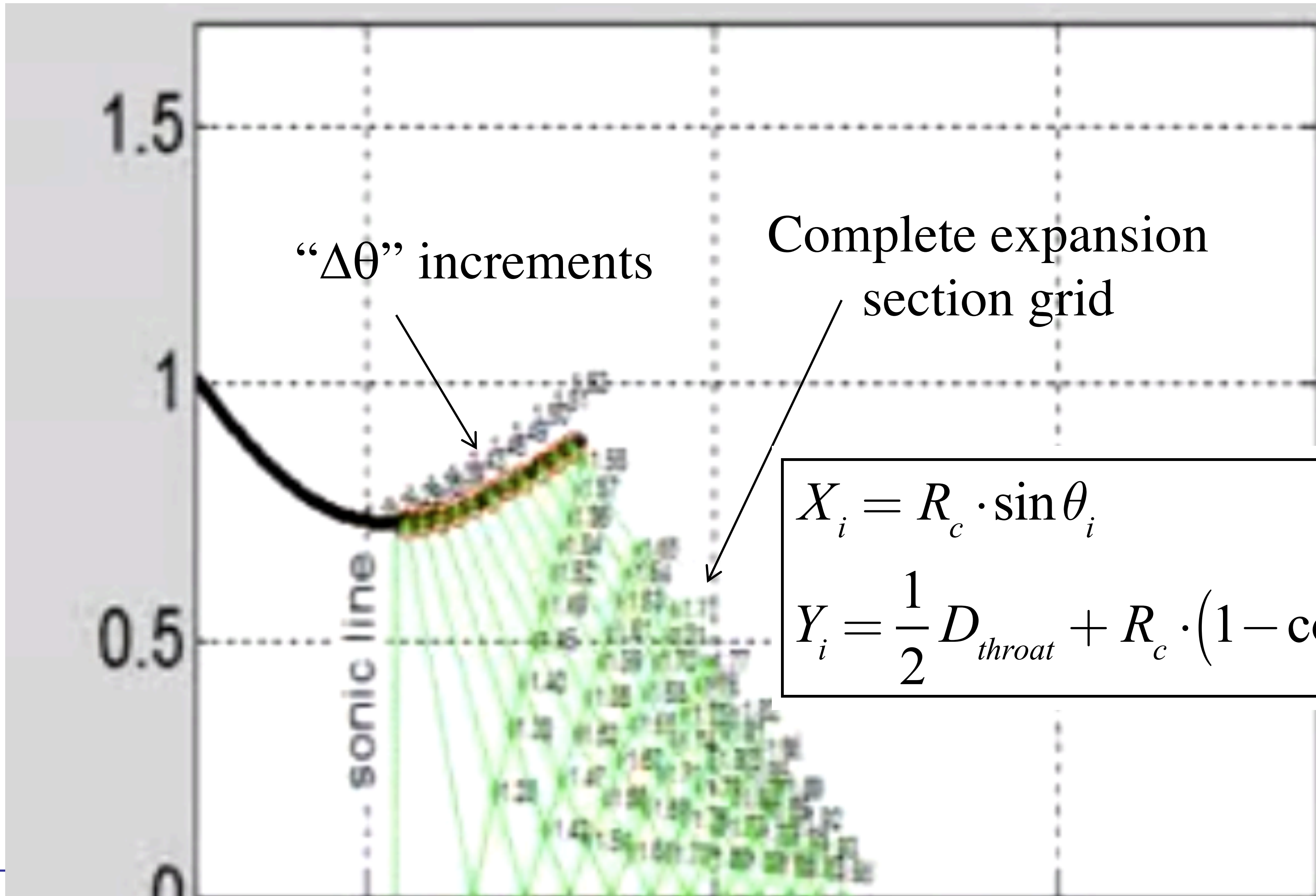
$$y_{cl} = 0 \rightarrow \frac{0 - y_1}{x_{cl} - x_1} = \tan\left(Slope(C_-)\right) \rightarrow \boxed{x_{cl} = -\frac{y_1}{\tan\left(Slope(C_-)\right)} + x_1}$$

Using Method of Characteristics to Design a Bell Nozzle *(4)*

$$\left[\begin{array}{l} \theta_3 = \dfrac{(\theta_1 + \nu_1) + (\theta_2 - \nu_2)}{2} = \dfrac{(K_-)_1 + (K_+)_2}{2} \\ \nu_3 = \dfrac{(\theta_1 + \nu_1) - (\theta_2 - \nu_2)}{2} = \dfrac{(K_-)_1 - (K_+)_2}{2} \end{array}\right]$$



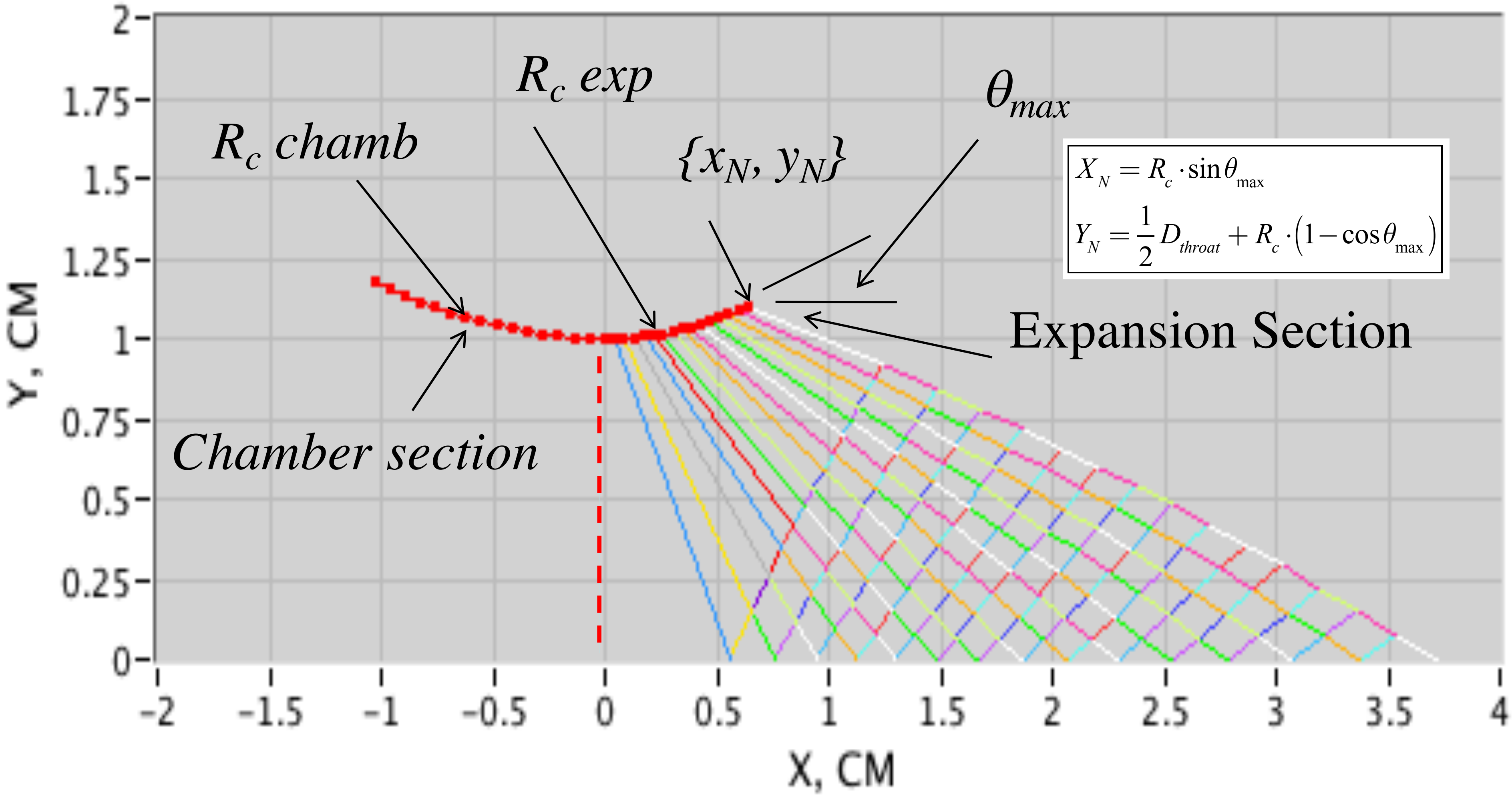
Using Method of Characteristics to Design a Bell Nozzle (5)



$$X_i = R_c \cdot \sin\theta_i$$

$$Y_i = \frac{1}{2} D_{throat} + R_c \cdot \left(1 - \cos\theta_i\right)$$

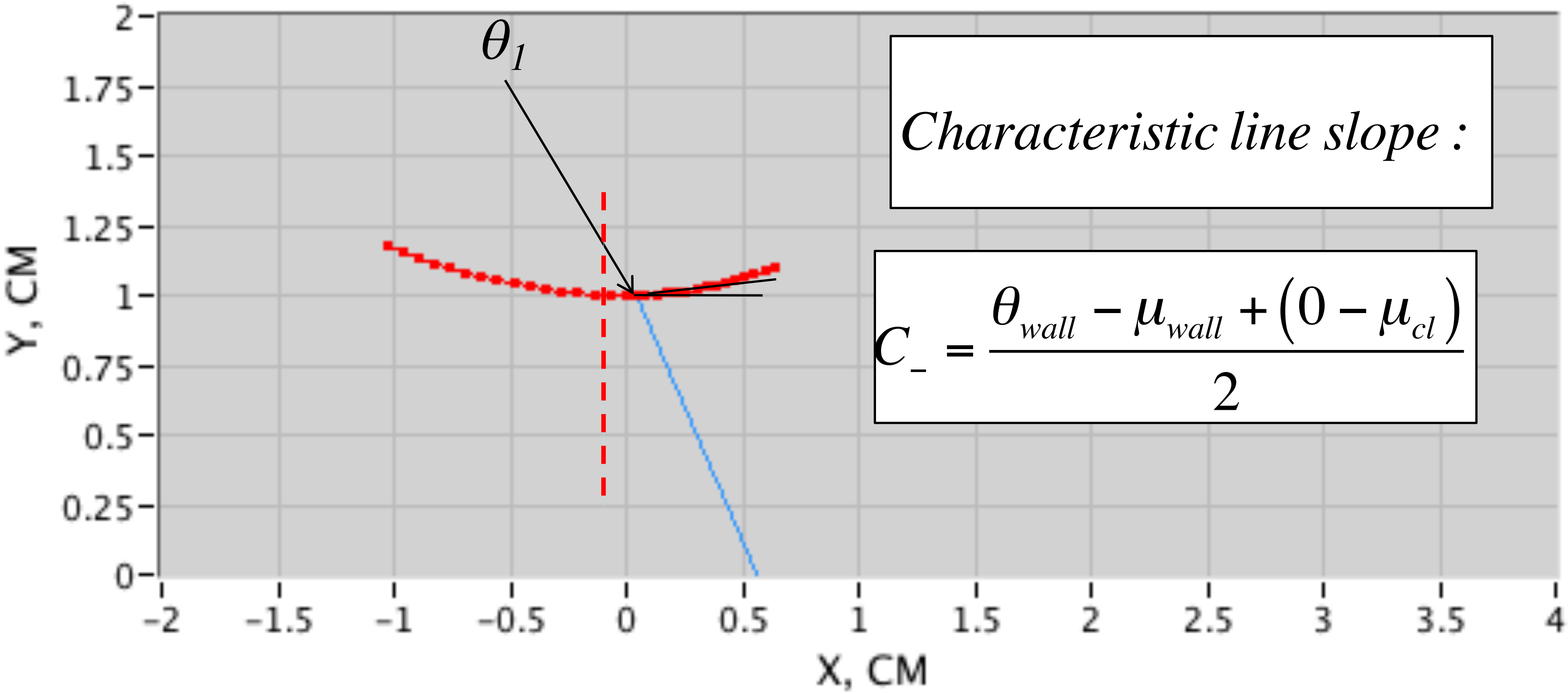
Nozzle Construction Example: Chamber and Expansion Section



Nozzle Construction Example: Initial characteristic line

$$Prantdl\text{ - }Meyer\ Expansion: \nu_1 = \nu_{throat} + \theta_1 \rightarrow M_1$$

$$C\_\ Characteristic\ Line: \theta_{cl} = 0 \rightarrow \nu_{cl} = \theta_1 + \nu_1$$



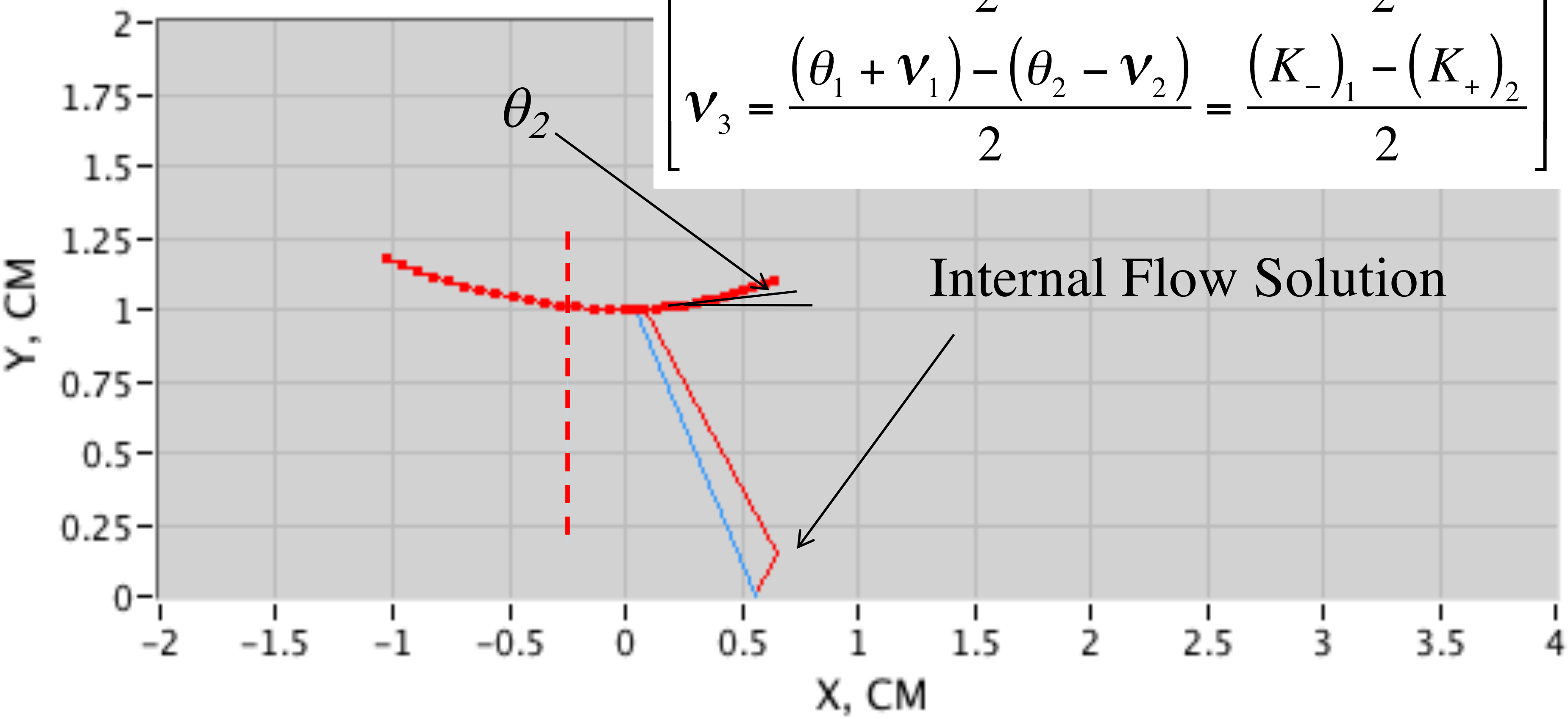
Characteristic line slope :

$$C\_ = \frac{\theta_{wall} - \mu_{wall} + (0 - \mu_{cl})}{2}$$

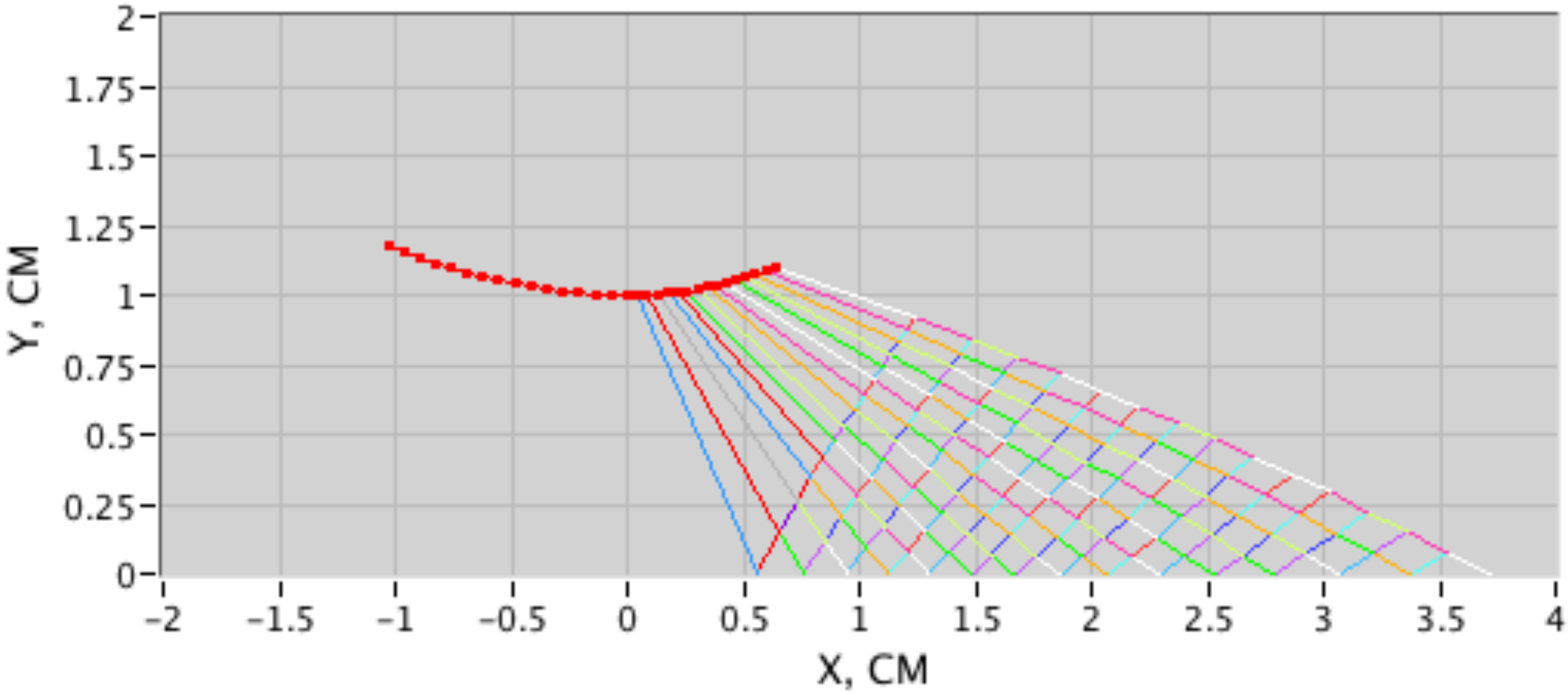
Nozzle Construction Example: Second characteristic line

$$Prantdl\text{ - }Meyer\ Expansion: \nu_2 = \nu_1 + (\theta_2 - \theta_1) \rightarrow M_2$$

$$\left[\begin{array}{l} \theta_3 = \dfrac{(\theta_1 + \nu_1) + (\theta_2 - \nu_2)}{2} = \dfrac{(K_-)_1 + (K_+)_2}{2} \\ \nu_3 = \dfrac{(\theta_1 + \nu_1) - (\theta_2 - \nu_2)}{2} = \dfrac{(K_-)_1 - (K_+)_2}{2} \end{array}\right]$$



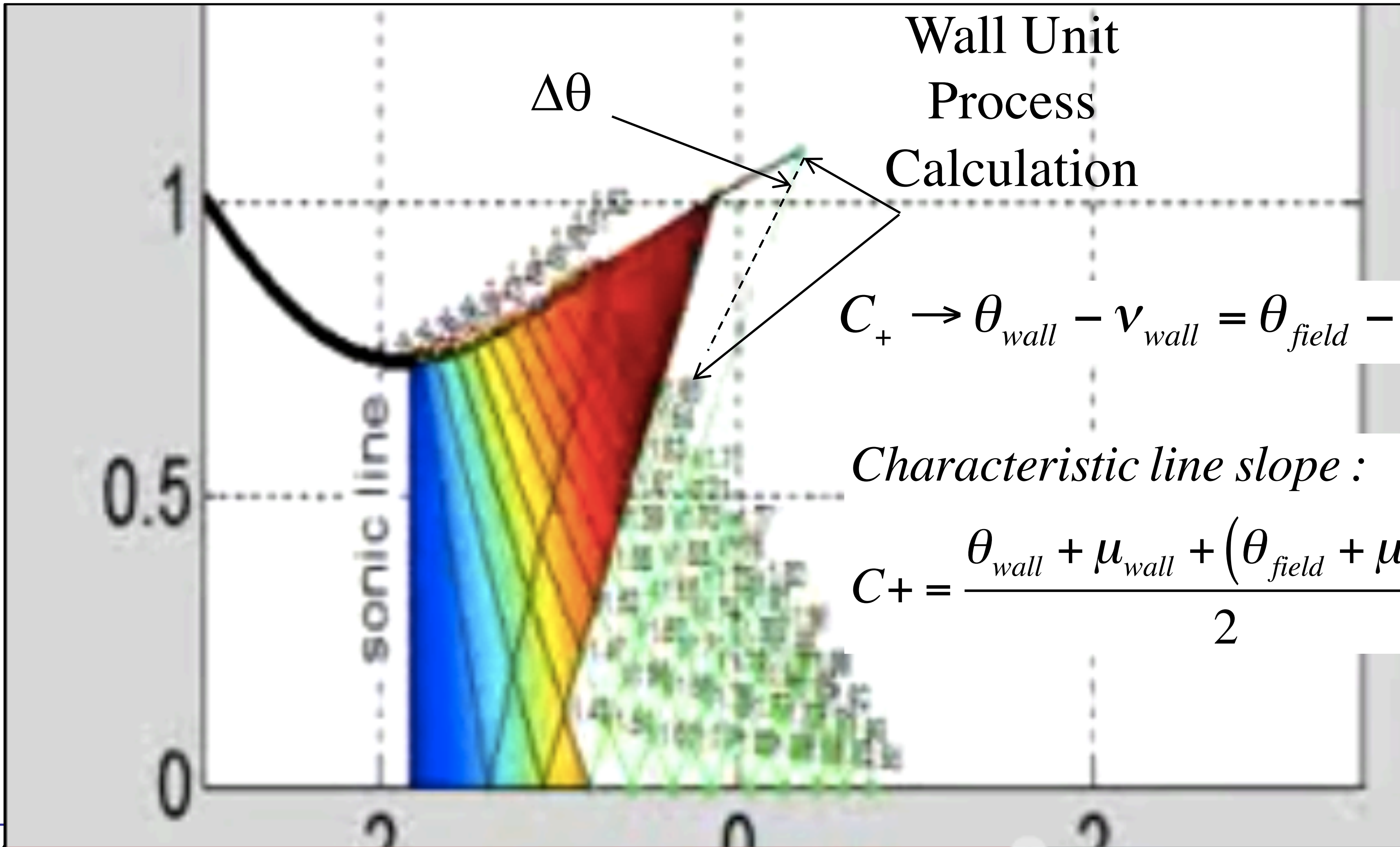
Nozzle Construction Example: Repeat Unit Processes to Complete Grid for Expansion Section



1. Prandtl-Meyer Expansion at Wall
2. Internal Flow
3. Centerline Intercept

Using Method of Characteristics to Design a Bell Nozzle (6)

Now begin straightening section



$$C_+ \rightarrow \theta_{wall} - \nu_{wall} = \theta_{field} - \nu_{field}$$

Characteristic line slope :

$$C+ = \frac{\theta_{wall} + \mu_{wall} + \left(\theta_{field} + \mu_{field}\right)}{2}$$

Wall Point Solution for Turning Section

Final Point on Expansion Grid : $\{x_{field}, y_{field}, \theta_{field}, M_{field}\} \rightarrow \begin{bmatrix} \nu_{field} \\ \mu_{field} \end{bmatrix}$

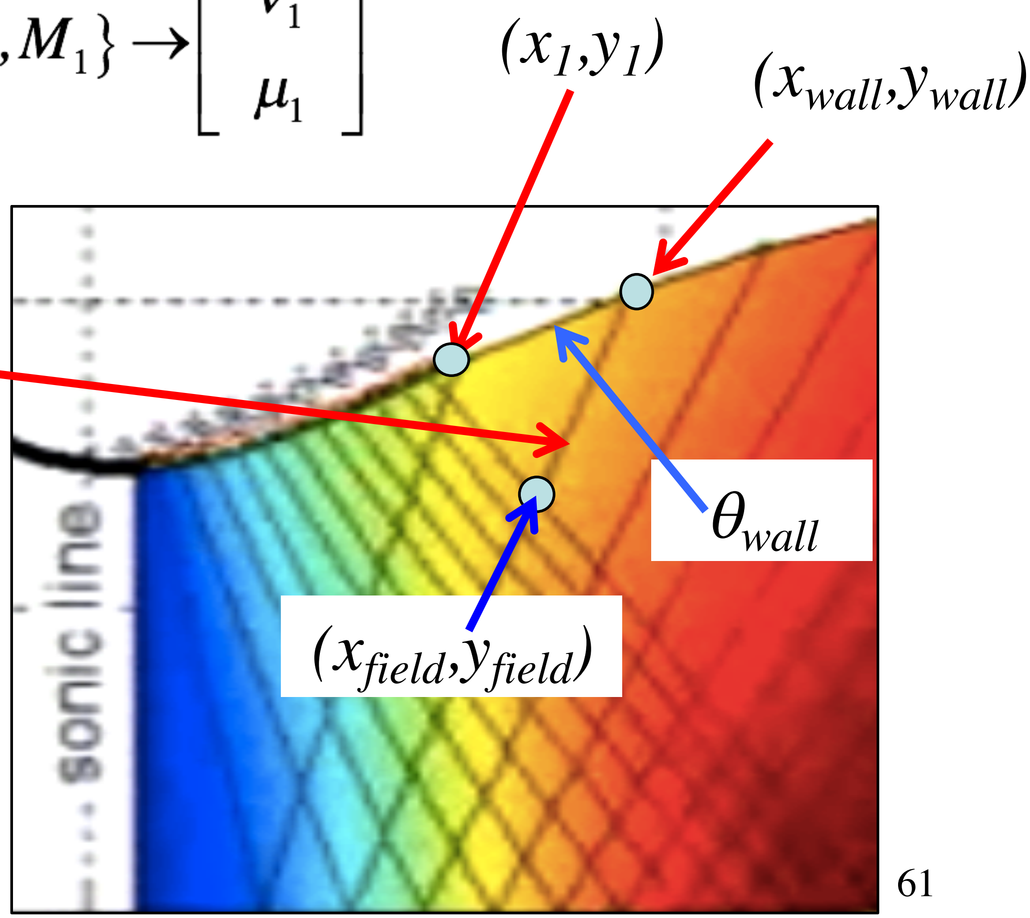
Previous Point on Wall : $\{x_1, y_1, \theta_1, M_1\} \rightarrow \begin{bmatrix} \nu_1 \\ \mu_1 \end{bmatrix}$

Current Wall Slope : θ_{wall}

Left running (C+) line

$$\nu_{wall} = \left(\theta_{wall} - \theta_{field}\right) + \nu_{field}$$

$$\rightarrow \begin{bmatrix} M_{wall} \\ \mu_{wall} \end{bmatrix}$$



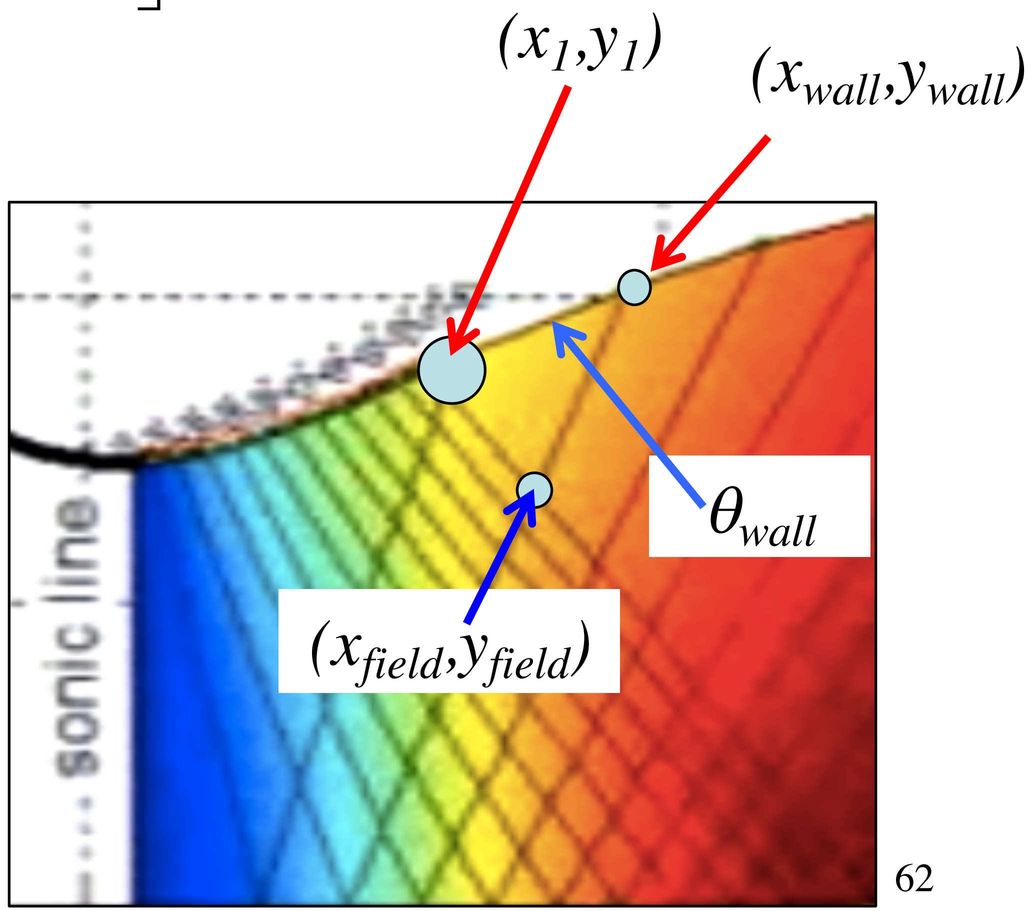
Wall Point Solution for Turning Section (2)

Left running (C+) line

$$\nu_{wall} = \left(\theta_{wall} - \theta_{field}\right) + \nu_{field} \rightarrow \begin{bmatrix} M_{wall} \\ \mu_{wall} \end{bmatrix} \quad slope\left\{C^+\right\} = \left(\frac{\theta_{field} + \mu_{field} + \theta_{wall} + \mu_{wall}}{2}\right)$$

$$\frac{y_{wall} - y_1}{x_w - x_1} = \tan\left(\theta_{wall}\right)$$

$$\frac{y_{wall} - y_{filed}}{x_{wall} - x_{field}} = \tan\left(slope\left\{C^+\right\}\right)$$



Wall Point Solution for Turning Section (3)

$$\frac{y_{wall} - y_1}{x_{wall} - x_1} = \tan\left(\theta_{wall}\right)$$

$$\frac{y_{wall} - y_{filed}}{x_{wall} - x_{field}} = \tan\left(slope\left\{C^+\right\}\right)$$

Regroup in Matrix Form
Solve via Cramer's Rule

$$\begin{bmatrix} y_1 - x_1 \cdot \tan\left(\theta_{wall}\right) \\ y_{field} - x_{field} \cdot \tan\left(slope\left\{C^+\right\}\right) \end{bmatrix} = \begin{bmatrix} -\tan\left(\theta_{wall}\right) & 1 \\ -\tan\left(slope\left\{C^+\right\}\right) & 1 \end{bmatrix} \begin{bmatrix} x_{wall} \\ y_{wall} \end{bmatrix}$$

$$\begin{bmatrix} x_{wall} \\ y_{wall} \end{bmatrix} = \frac{\begin{bmatrix} 1 & -1 \\ \tan\left(slope\left\{C^+\right\}\right) & -\tan\left(\theta_w\right) \end{bmatrix} \cdot \begin{bmatrix} y_1 - x_1 \cdot \tan\left(\theta_w\right) \\ y_{field} - x_{filed} \cdot \tan\left(slope\left\{C^+\right\}\right) \end{bmatrix}}{\tan\left(slope\left\{C^+\right\}\right) - \tan\left(\theta_w\right)}$$

Wall Point Solution for Turning Section (4)

$$\frac{y_{wall}-y_1}{x_{wall}-x_1}=\tan\left(\theta_{wall}\right)$$

$$\frac{y_{wall}-y_{filed}}{x_{wall}-x_{field}}=\tan\left(slope\left\{C^+\right\}\right)$$

Simplify Solution

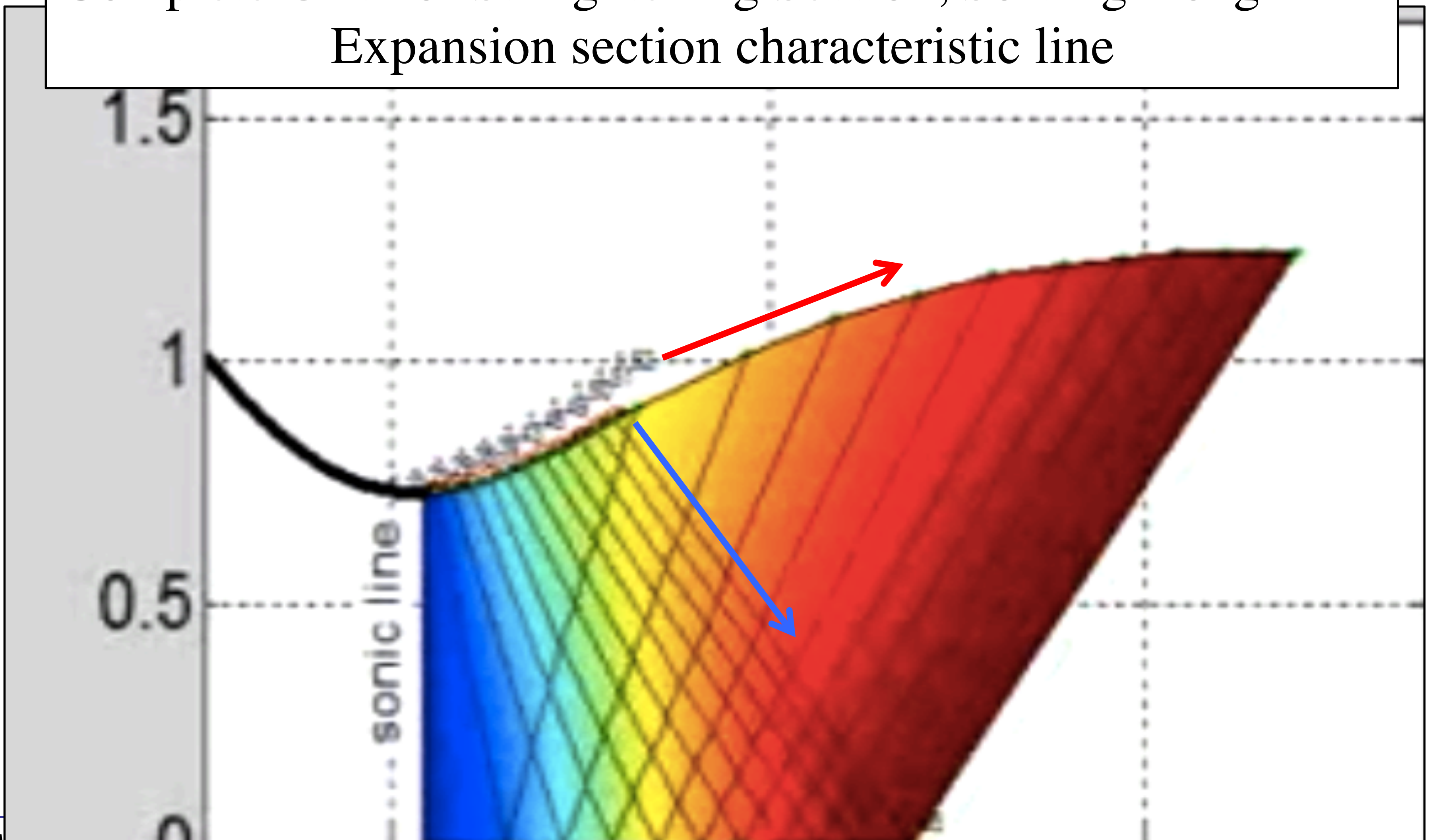
$$x_{wall}=\frac{x_1\cdot\tan\left(\theta_{wall}\right)-x_{field}\cdot\tan\left(slope\left\{C^+\right\}\right)+\left(y_{field}-y_1\right)}{\tan\left(\theta_{wall}\right)-\tan\left(slope\left\{C^+\right\}\right)}$$

$$y_{wall}=\frac{\tan\left(\theta_{wall}\right)\cdot\tan\left(slope\left\{C^+\right\}\right)\cdot\left(x_1-x_{field}\right)+\tan\left(\theta_{wall}\right)\cdot y_{field}-\tan\left(slope\left\{C^+\right\}\right)\cdot y_1}{\tan\left(\theta_{wall}\right)-\tan\left(slope\left\{C^+\right\}\right)}$$

$$\rightarrow slope\left\{C^+\right\}=\frac{\theta_{wall}+\mu_{wall}+\theta_{field}+\mu_{field}}{2}$$

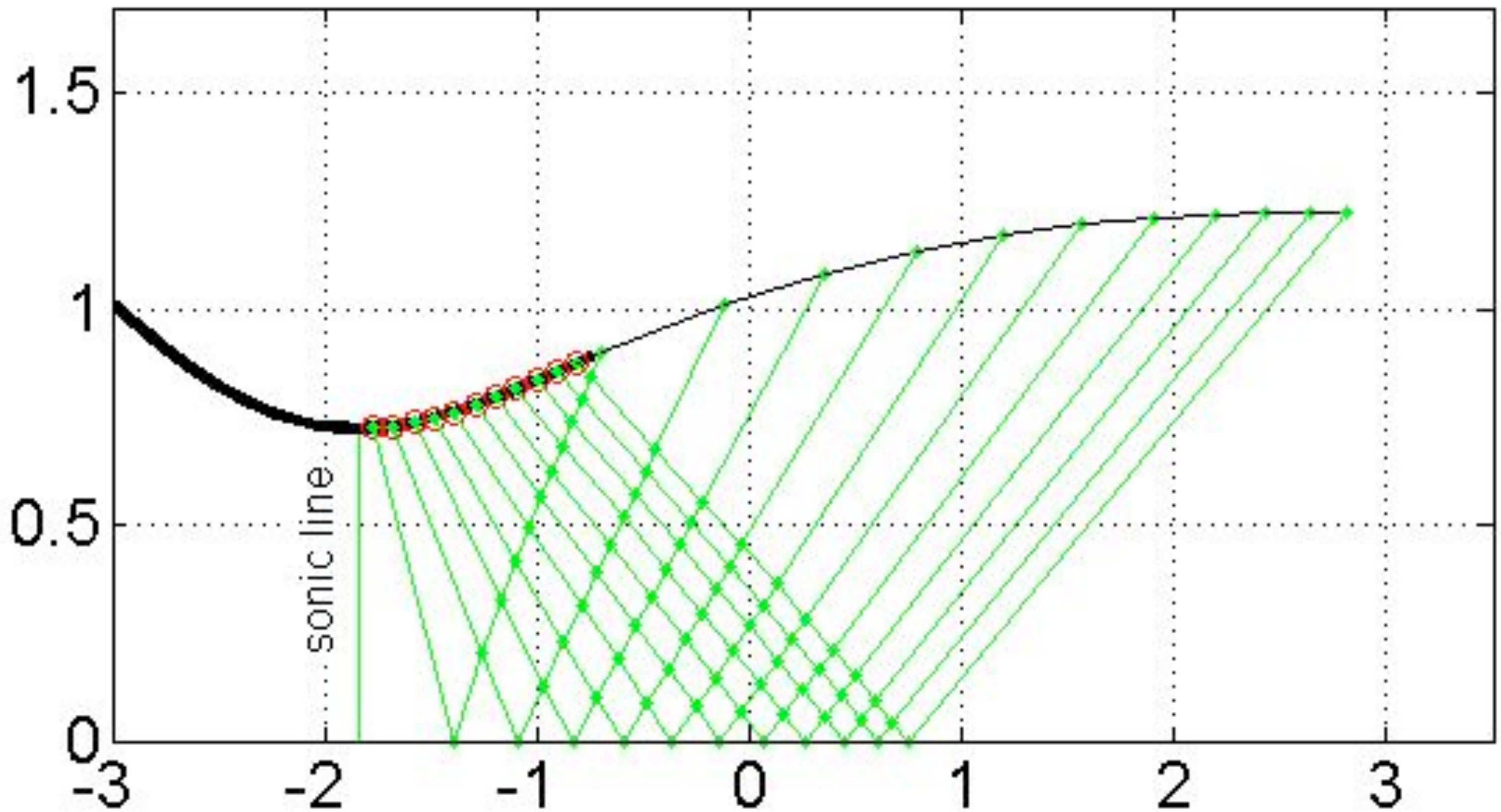
Using Method of Characteristics to Design a Bell Nozzle (7)

Complete Grid for Straightening Section, Solving along Final Expansion section characteristic line

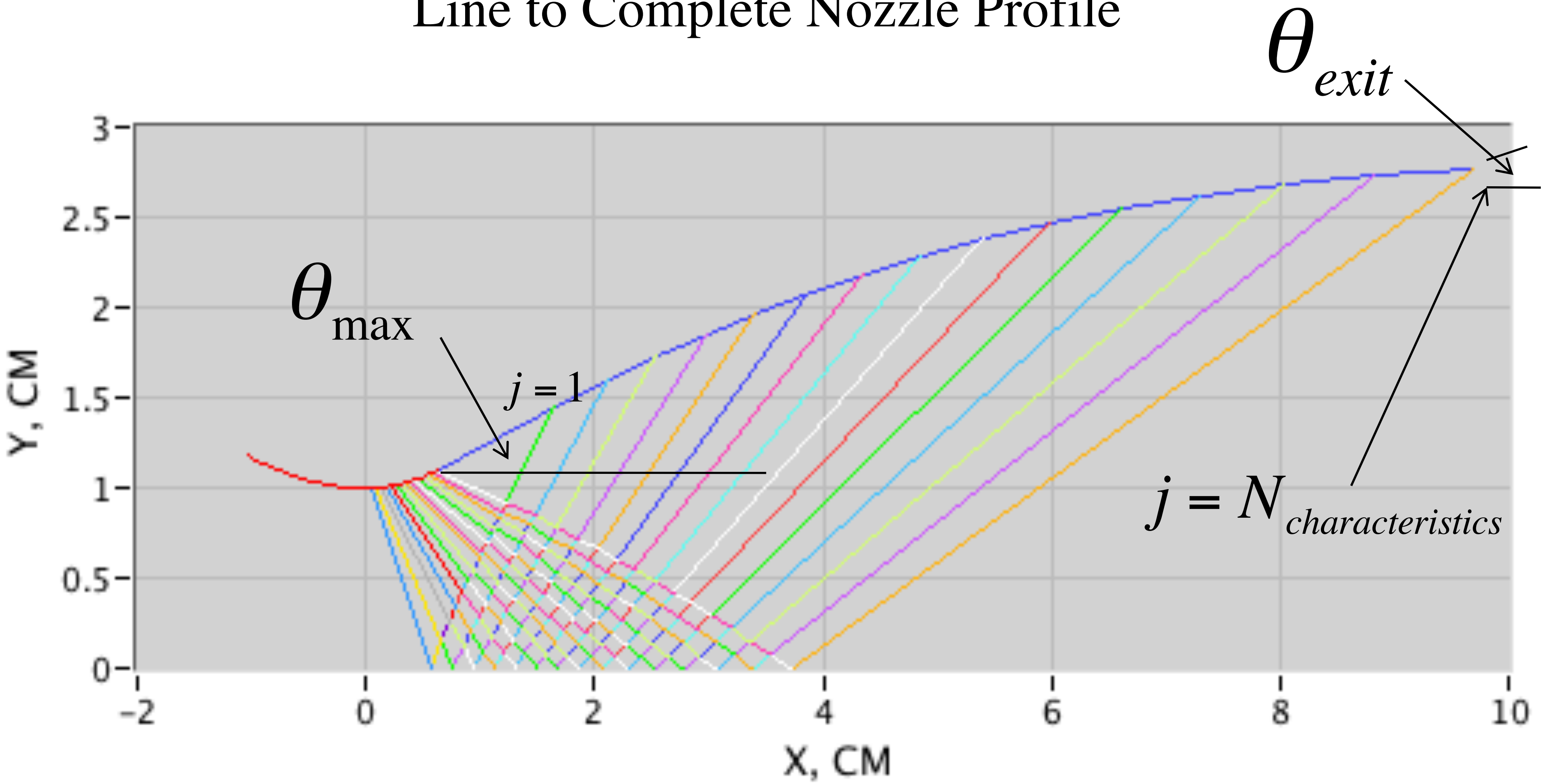


Using Method of Characteristics to Design a Bell Nozzle (8)

Completed Grid for Straightening Section

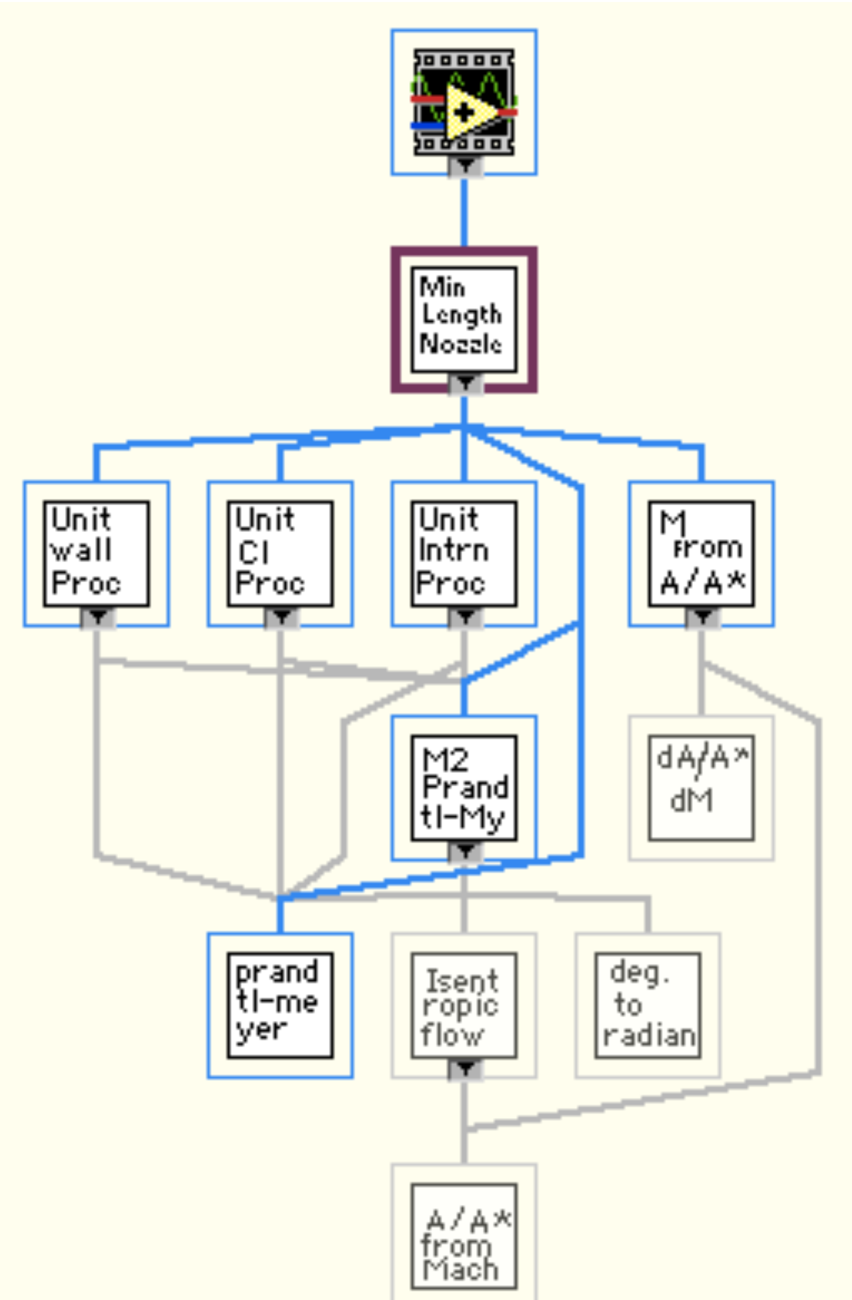


Nozzle Construction Example: Work from Final Characteristic Line to Complete Nozzle Profile



Assign "thetas" ….

$$\Delta\theta = \frac{\theta_{max} - \theta_{exit}}{N_{characteristics}} \rightarrow \theta_j = \theta_{max} - j \cdot \Delta\theta$$



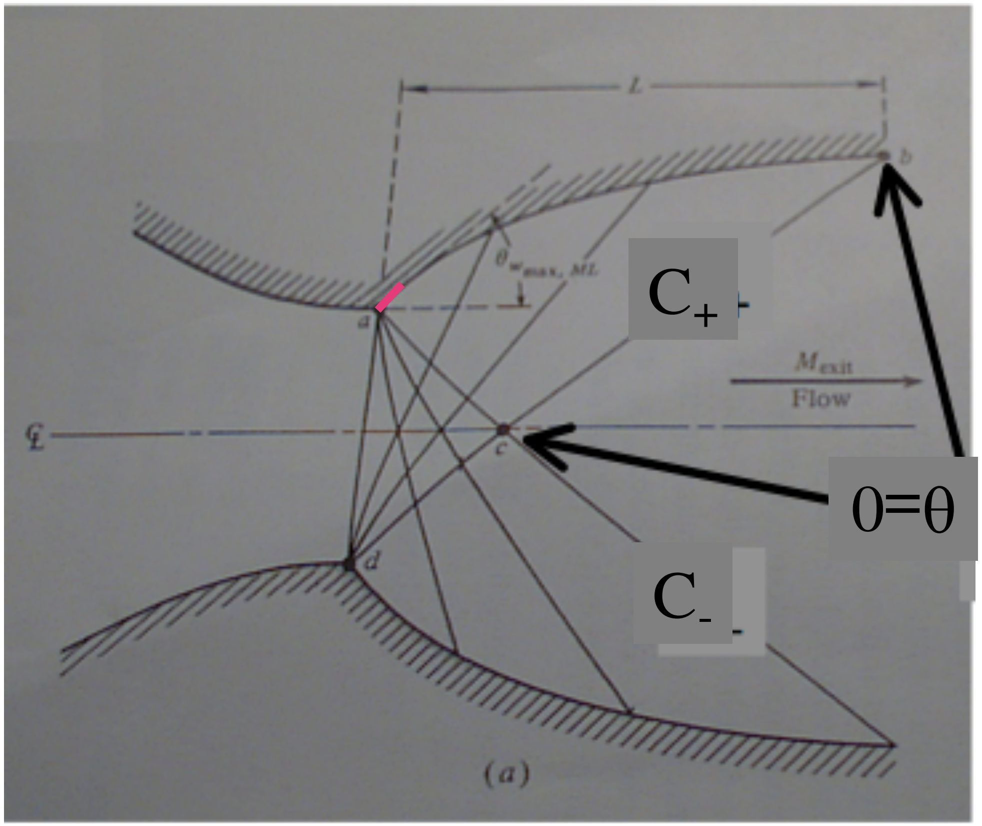
Nozzle Construction Example: Code Layout

1. Prandtl-Meyer Expansion at Wall
2. Internal Flow
3. Centerline Intercept
4. Wall Intercept

Minimum Length Nozzle Design (cont'd)

- **Find minimum length nozzle with shock-free flow → $\theta_{exit} = 0$**

$$\nu_{exit} \to M_{exit}$$



- Along C_+ characteristic *{b,c}*

$$\theta_{cl} - \nu_{cl} = \theta_{exit} - \nu_{exit}$$

$$\theta_{cl}, \theta_{exit} = 0 \to \boxed{\nu_{cl} = \nu_{exit}}$$

- *At expansion corners, {a,d}*

From Prandtl-Meyer Theory

$$\nu_{\{a,d\}} = \nu_{throat} + \theta_{w_{max}}$$

$$\nu_{throat} = \nu_{m=1} = 0$$

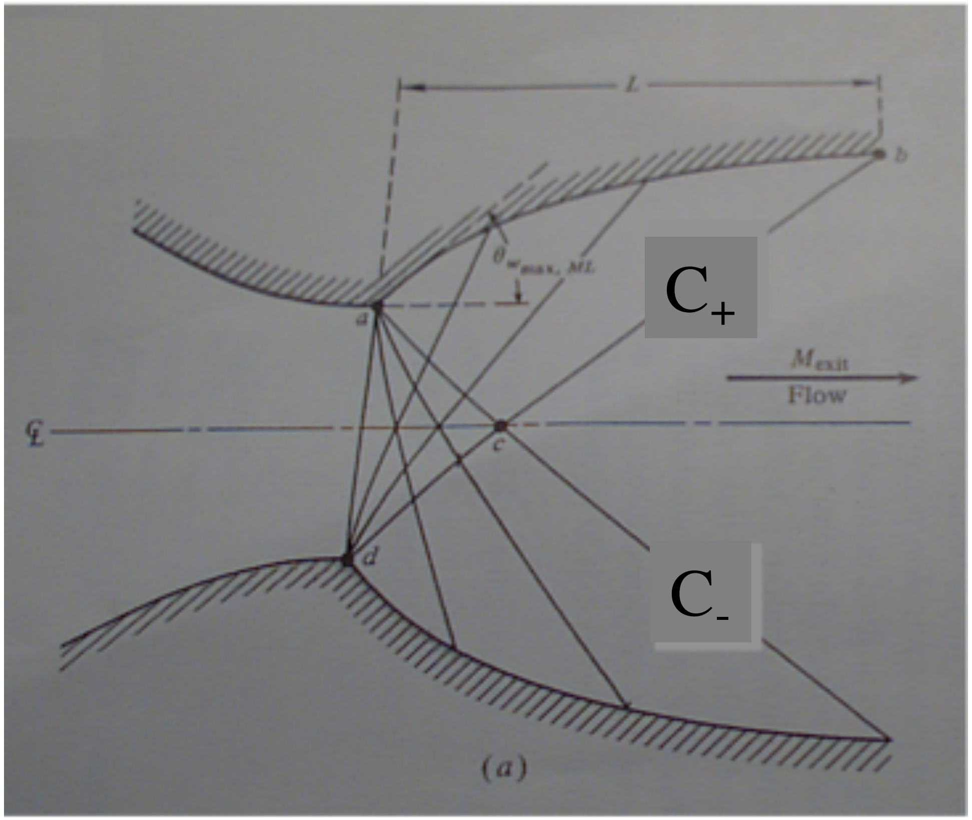
$$\boxed{\nu_{\{a,d\}} = \theta_{w_{max}}}$$

$$\nu(1.0) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(1.0^2-1\right)}\right\} - \tan^{-1}\sqrt{1.0^2-1} = 0$$

Minimum Length Nozzle Design (cont'd)

- **Find minimum length nozzle with shock-free flow → $\theta_{exit} = 0$**

$$\nu_{exit} \rightarrow M_{exit}$$



- Along C- characteristic {*a*,*c*} at point *a*

$$\theta_{w_{max}} + \nu_a = \theta_{cl} + \nu_{cl} \rightarrow \theta_{cl} = 0$$

$$\boxed{\theta_{w_{max}} + \nu_a = \nu_{cl}}$$

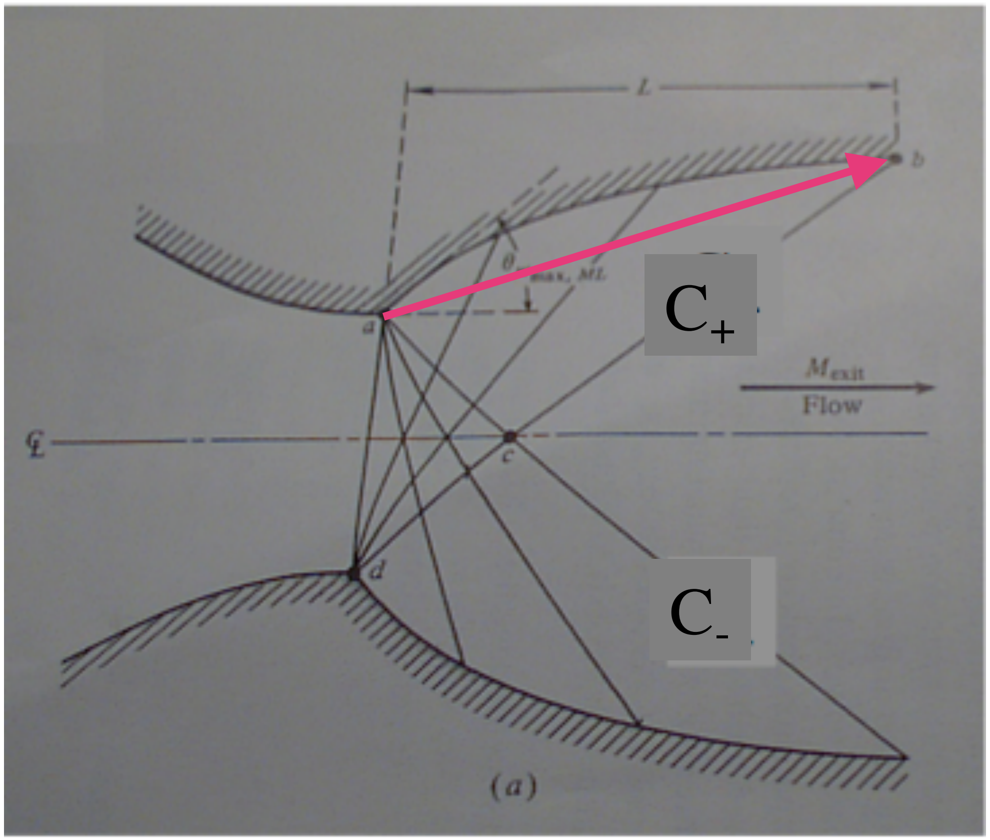
- Along C+ characteristic {*d*,*c*} at point *a*

$$\left(-\theta_{w_{max}}\right) - \nu_d = \theta_{cl} - \nu_{cl} \rightarrow \theta_{cl} = 0$$

$$\boxed{\theta_{w_{max}} + \nu_d = \nu_{cl}}$$

$$\boxed{\rightarrow \nu_a = \nu_d}$$

Minimum Length Nozzle Design (cont'd)



- Length for a given expansion angle is more important than the precise shape of nozzle …

But from Previous discussion

$$\boxed{\nu_{\{a,d\}} = \theta_{w_{\max}}} \ldots \rightarrow \ldots \boxed{\theta_{w_{\max}} + \nu_d = \nu_{cl}} \ldots$$

$$\rightarrow \ldots \boxed{\nu_a = \nu_d} \rightarrow \boxed{\nu_{cl} = \nu_{exit}}$$

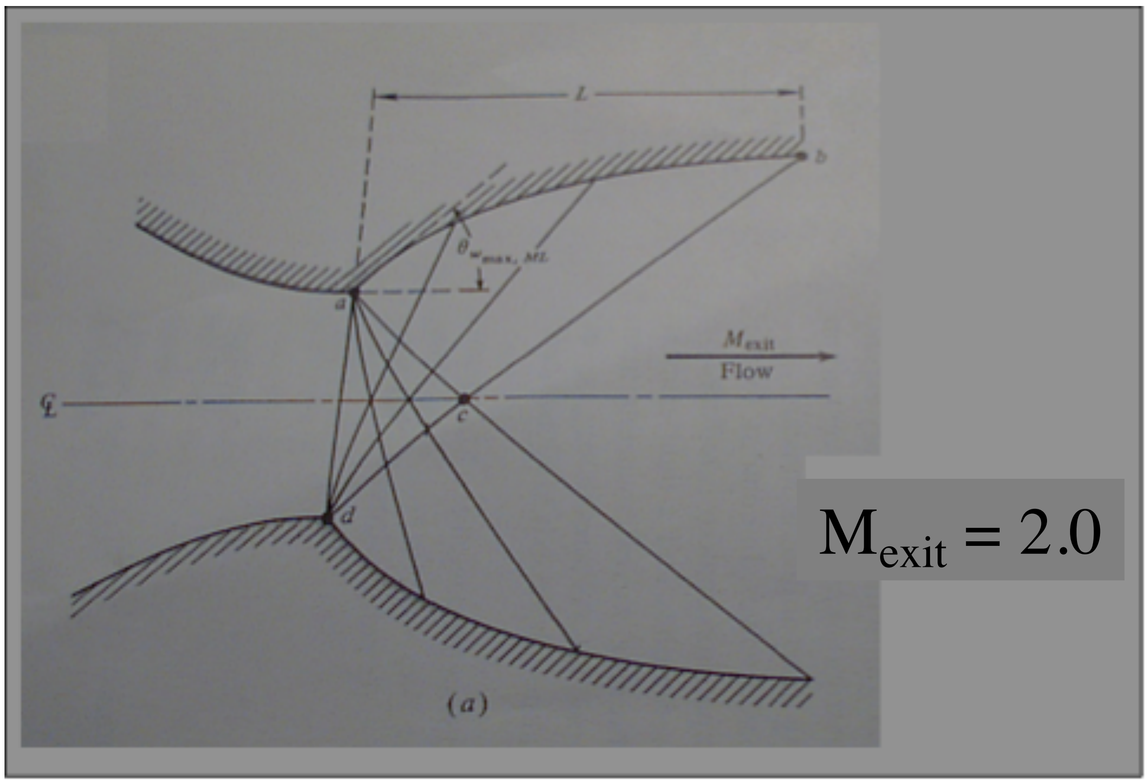
$$\rightarrow \theta_{w_{\max}} + \theta_{w_{\max}} = \nu_{cl} \rightarrow \boxed{\theta_{w_{\max}} = \frac{\nu_{cl}}{2} = \frac{\nu_{exit}}{2}}$$

- **Criterion for Minimum Length "Bell" Nozzle**

$$\theta_{w\ Max} = \frac{\nu_{exit}}{2}$$

Minimum Length Nozzle: Construction Example (from Anderson)

- Use Method of characteristics, compute and graph contour for 2-D minimum length nozzle for a design exit mach number of 2.0



Minimum Length Nozzle: Construction Example (cont'd)

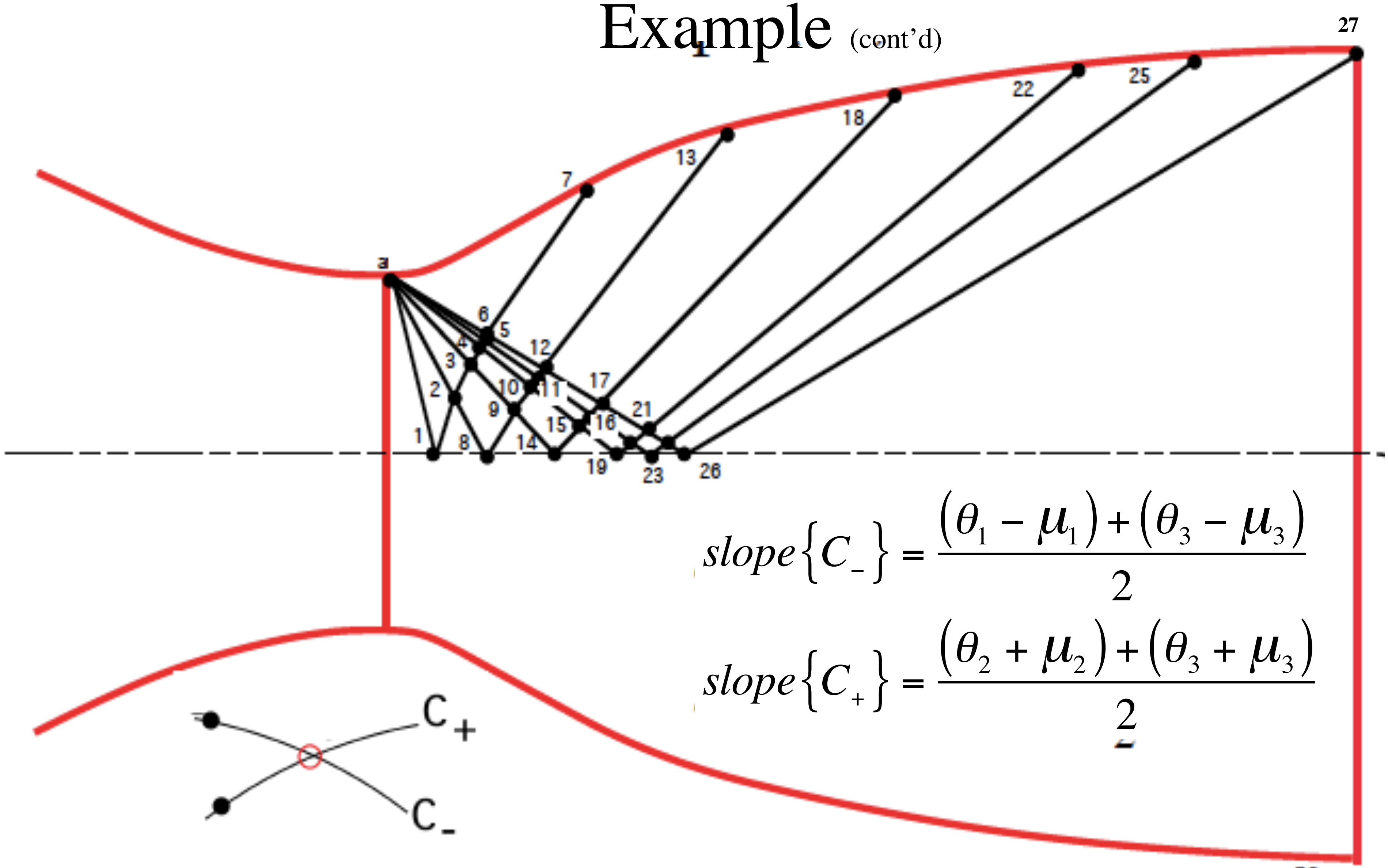
$$M_{exit} = 2.0 \rightarrow \nu(M_{exit}) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(2.0^2-1\right)}\right\} - \tan^{-1}\sqrt{2.0^2-1} =$$

$$\frac{180}{\pi}\left(\left(\frac{1.4+1}{1.4-1}\right)^{0.5}\operatorname{atan}\left(\left(\left(\frac{1.4-1}{1.4+1}\right)\left(2.0^2-1\right)\right)^{0.5}\right) - \operatorname{atan}\left(\left(\left(2.0^2-1\right)\right)^{0.5}\right)\right)$$

$$= 26.3798^{o}$$

$$\theta_{w_{Max}} = \frac{\nu_{exit}}{2} = 13.1899^{o}$$

Minimum Length Nozzle: Construction Example (cont'd)

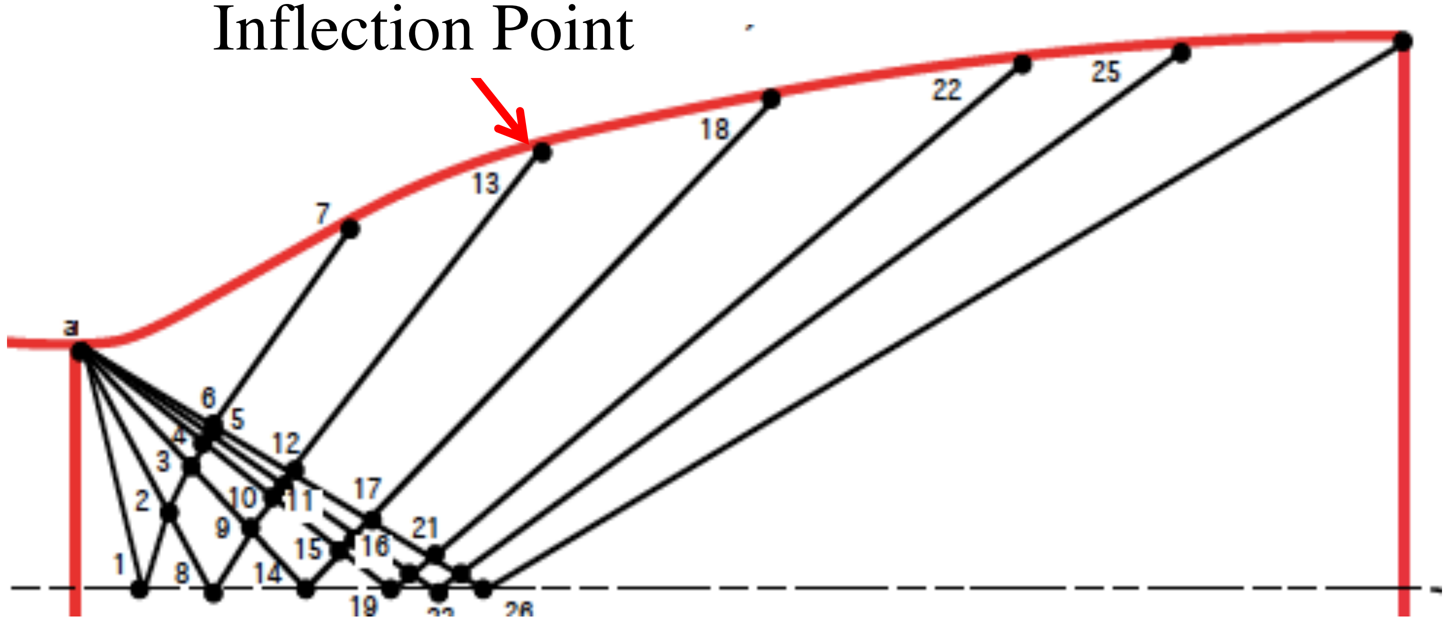


$$slope\{C_-\} = \frac{(\theta_1 - \mu_1) + (\theta_3 - \mu_3)}{2}$$

$$slope\{C_+\} = \frac{(\theta_2 + \mu_2) + (\theta_3 + \mu_3)}{2}$$

Minimum Length Nozzle: Construction Example (cont'd)

Inflection Point



- Section of Nozzle with θ_{wall} increasing called "expansion section"

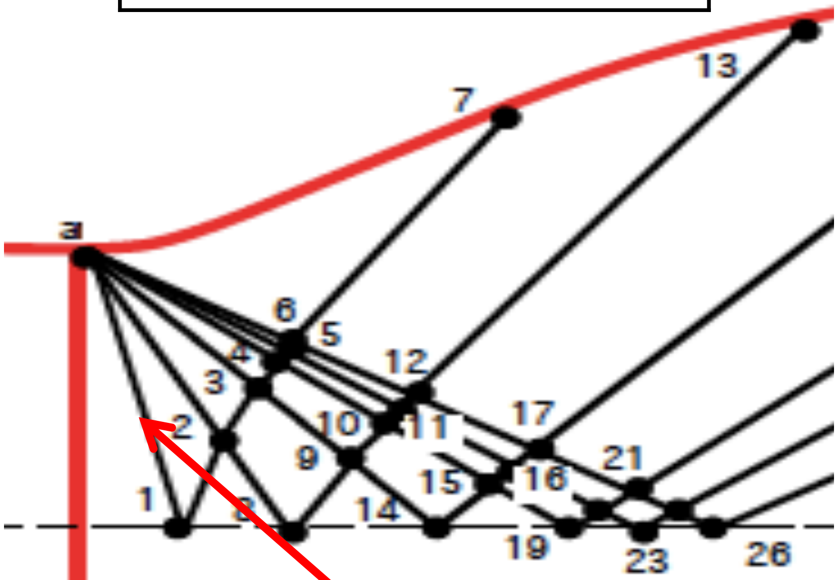
- Inflection point occurs at $\left(\theta_{wall}\right)_{\max}$

- Downstream of inflection point called "straightening" section

Minimum Length Nozzle: Construction Example (cont'd)

$$\nu(1) = 0 \rightarrow C_{-_a} = 0$$

- **Starting up the calculation …**

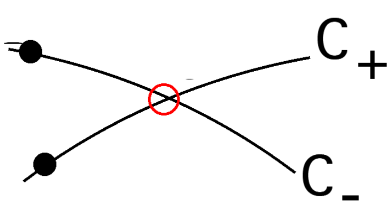


Point a … Need to Kick "a" just a bit Downstream of throat

$$\begin{Bmatrix} M = 1.026 \\ \theta_{wall} = 0.19 \end{Bmatrix}$$

$$\nu(1.026) = 0.19 \rightarrow \begin{pmatrix} C_{-_a} = \theta + \nu = 0.38 \\ \mu_a = 77.1^o \end{pmatrix}$$

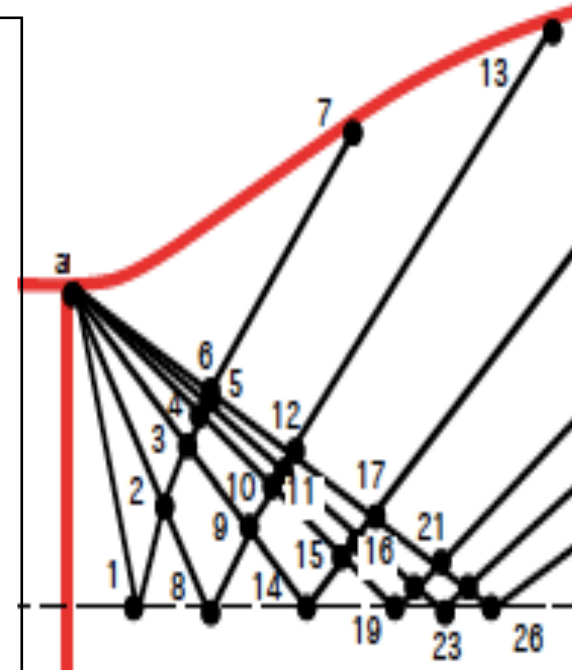
$$\theta_1 + \nu_1 = C_- = 0.38 \rightarrow \theta_1 = 0 \rightarrow \nu_1 = 0.38 \rightarrow \begin{pmatrix} M_1 = 1.042 \\ \mu_a = 73.58^o \end{pmatrix}$$

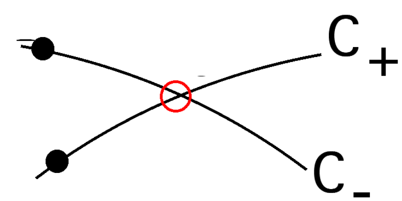


$$slope\{C_-\} \approx \frac{(\theta_a - \mu_a) + \theta_1 - \mu_1}{2} = -75.43^o$$

Minimum Length Nozzle: Construction Example (cont'd)

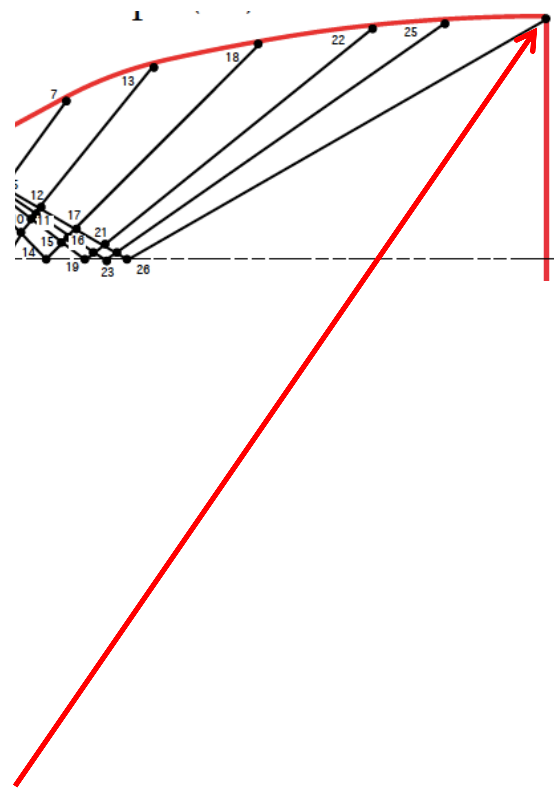
Point #	$C_- = \theta + \nu$ (deg),	$C_+ = \theta - \nu$ (deg),	$\theta = \frac{C_- + C_+}{2}$ (deg),	$\nu = \frac{C_- - C_+}{2}$ (deg).	M	μ (deg),
a	0.38	0.0	0.19	0.0	1.026	77.10
1	0.38	-0.38	0.0	0.38	1.042	77.14
2	2.38	0	1.19	1.19	1.092	66.29
3	8.38	0	4.19	4.19	1.225	54.88
4	14.38	0	7.19	7.19	1.337	48.42
5	20.38	0	10.19	10.19	1.442	43.92
6	26.38	0	**13.19**	13.19	1.544	40.40
7	**26.38**	**0**	**13.19**	**13.19**	**1.544**	**40.40**
8	2.38	-2.38	0	2.38	1.150	60.41
9	8.38	-2.38	3	5.38	1.271	51.90
10	14.38	-2.38	6	8.38	1.379	46.69
11	20.38	-2.38	9	11.38	1.482	42.44
12	26.38	-2.38	**12**	14.38	1.584	39.16
13	**26.38**	**-2.38**	**12**	**14.38**	**1.584**	**39.17**



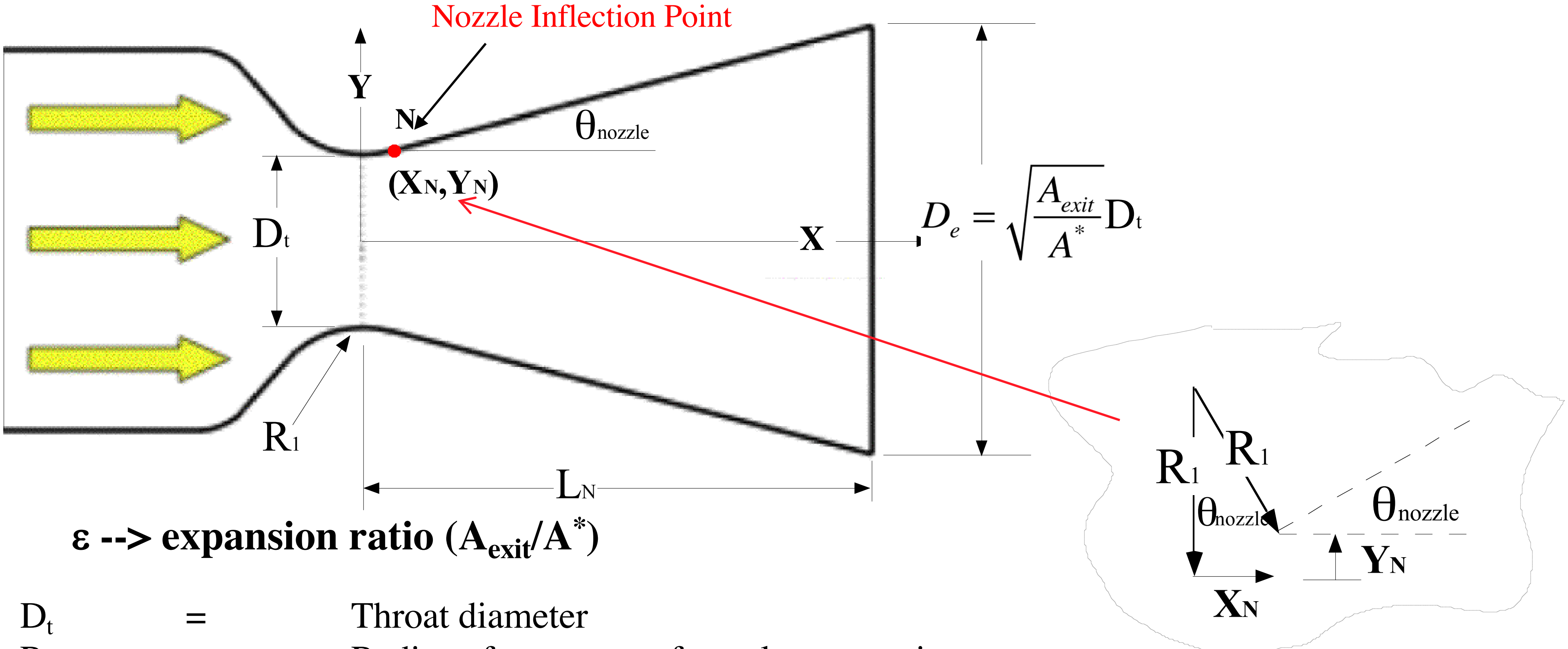


Minimum Length Nozzle: Construction Example (cont'd)

Point #	$C_- = \theta + \nu$ (deg),	$C_+ = \theta - \nu$ (deg),	$\theta = \frac{C_- + C_+}{2}$ (deg),	$\nu = \frac{C_- - C_+}{2}$ (deg).	M	μ (deg),
14	8.38	-8.38	0	8.38	1.379	46.49
15	14.38	-8.38	3	11.38	1.482	42.44
16	20.38	-8.38	6	14.38	1.584	39.16
17	26.38	-8.38	**9**	17.38	1.685	36.40
18	**26.38**	**-8.38**	**9**	**17.38**	**1.685**	**36.40**
19	14.38	-14.38	0	14.38	1.584	39.16
20	20.38	-14.38	3	17.38	1.684	36.40
21	26.38	-14.38	**6**	20.38	1.788	34.00
22	**26.38**	**-14.38**	**6**	**20.38**	**1.788**	**34.00**
23	20.38	-20.38	0	20.38	1.788	34.00
24	26.38	-20.38	**3**	23.38	1.893	31.89
25	**26.38**	**-20.38**	**3**	**23.38**	**1.893**	**31.89**
26	26.38	-26.38	0	26.38	2.000	30.00
27	**26.38**	**-26.38**	**0**	**26.38**	**2.000**	**30.00**



Typical Conical Nozzle Contour



ε --> expansion ratio (A_{exit}/A^*)

D_t	=	Throat diameter
R_1	=	Radius of curvature of nozzle contraction
N	=	Transition point from circular contraction to conical nozzle
L_N	=	Nozzle Length
D_e	=	Exit diameter

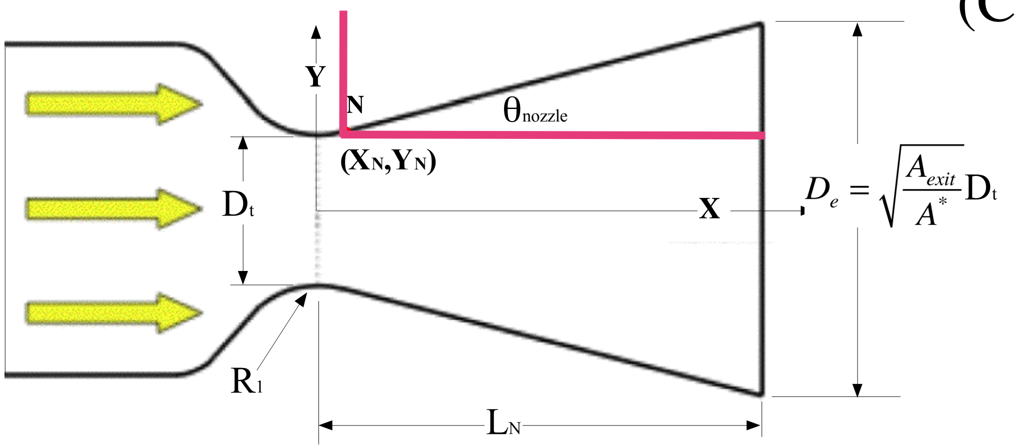
$$X_N = R_1 \sin(\theta_{nozzle})$$

$$Y_N = \frac{1}{2} D_{throat} + R_1 \left[1 - \cos(\theta_{nozzle}) \right]$$

- $R_1 \sim 0.75 D_t$ is typical

Typical Conical Nozzle Contour

(Cont'd)



- Solve for Nozzle length in terms of other parameters

$$\tan(\theta_{nozzle}) = \frac{\frac{1}{2}D_e - \left\{\frac{1}{2}D_{throat} + R_1\left[\,1-\cos\left(\theta_{nozzle}\right)\,\right]\right\}}{L_N - R_1\sin\left(\theta_{nozzle}\right)} =$$

$$\frac{\frac{1}{2}\left[D_e - D_{throat}\right] - R_1\left[\,1-\cos\left(\theta_{nozzle}\right)\,\right]}{L_N - R_1\sin\left(\theta_{nozzle}\right)}$$

$$\rightarrow \left\{L_N - R_1\sin\left(\theta_{nozzle}\right)\right\}\tan(\theta_{nozzle}) = \frac{1}{2}\left[\sqrt{\frac{A_{exit}}{A^*}} - 1\right]D_{throat} - R_1\left[\,1-\cos\left(\theta_{nozzle}\right)\right]$$

$$\rightarrow L_N\tan(\theta_{nozzle}) = \frac{1}{2}\left[\sqrt{\frac{A_{exit}}{A^*}} - 1\right]D_{throat} - R_1\left[\,1-\cos\left(\theta_{nozzle}\right) - \tan(\theta_{nozzle})\sin\left(\theta_{nozzle}\right)\,\right]$$

Typical Conical Nozzle Contour

- Using trig identities

(Cont'd)

$$1-\cos\left(\theta_{nozzle}\right)-\tan(\theta_{nozzle})\sin\left(\theta_{nozzle}\right)=1-\cos\left(\theta_{nozzle}\right)-\frac{\sin^2\left(\theta_{nozzle}\right)}{\cos\left(\theta_{nozzle}\right)}=$$

$$1-\frac{\cos^2\left(\theta_{nozzle}\right)+\sin^2\left(\theta_{nozzle}\right)}{\cos\left(\theta_{nozzle}\right)}=1-\frac{1}{\cos\left(\theta_{nozzle}\right)}$$

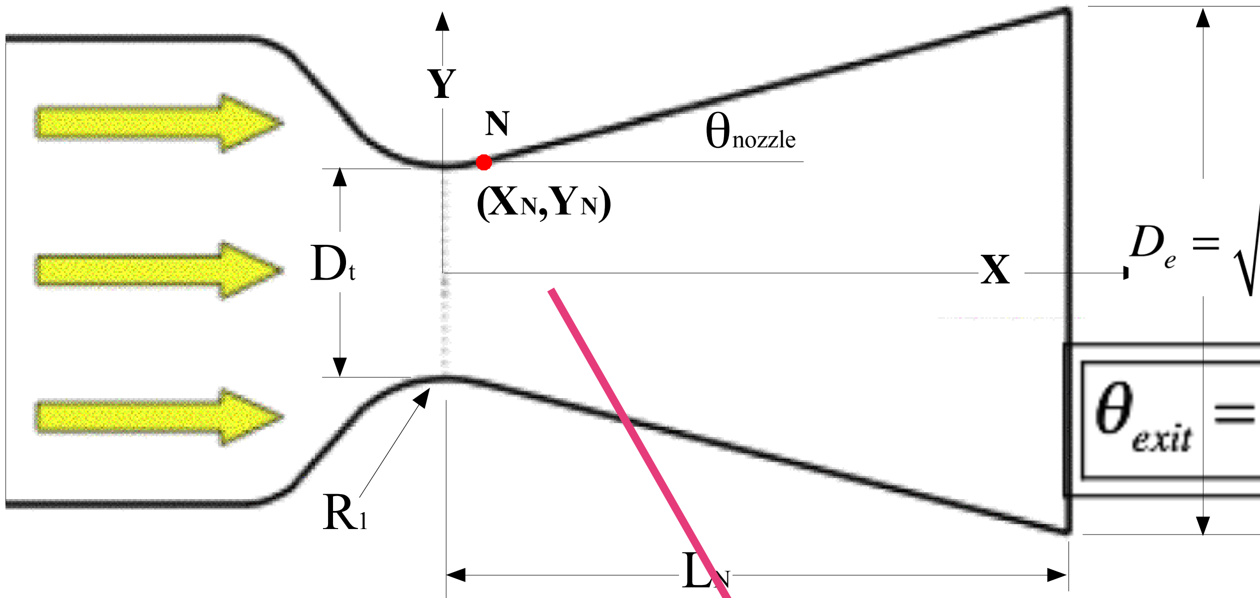
$$L_N\tan(\theta_{nozzle})=\frac{1}{2}\left[\sqrt{\frac{A_{exit}}{A^*}}-1\right]D_{throat}+R_1\left[\frac{1}{\cos\left(\theta_{nozzle}\right)}-1\right]$$

$$L_N=\frac{\frac{1}{2}\left[\sqrt{\frac{A_{exit}}{A^*}}-1\right]D_{throat}+R_1\left[\frac{1}{\cos\left(\theta_{nozzle}\right)}-1\right]}{\tan(\theta_{nozzle})}$$

- $R_1 \sim 0.75 D_t$ is typical

Nozzle Length Written In Terms of θ_N

Minimum Length Conical Nozzle (1)



$$D_e = \sqrt{\frac{A_{exit}}{A^*}}D_t$$

$$\theta_{exit} = \theta_{nozzle} = \theta_{w_{max}} \neq 0$$

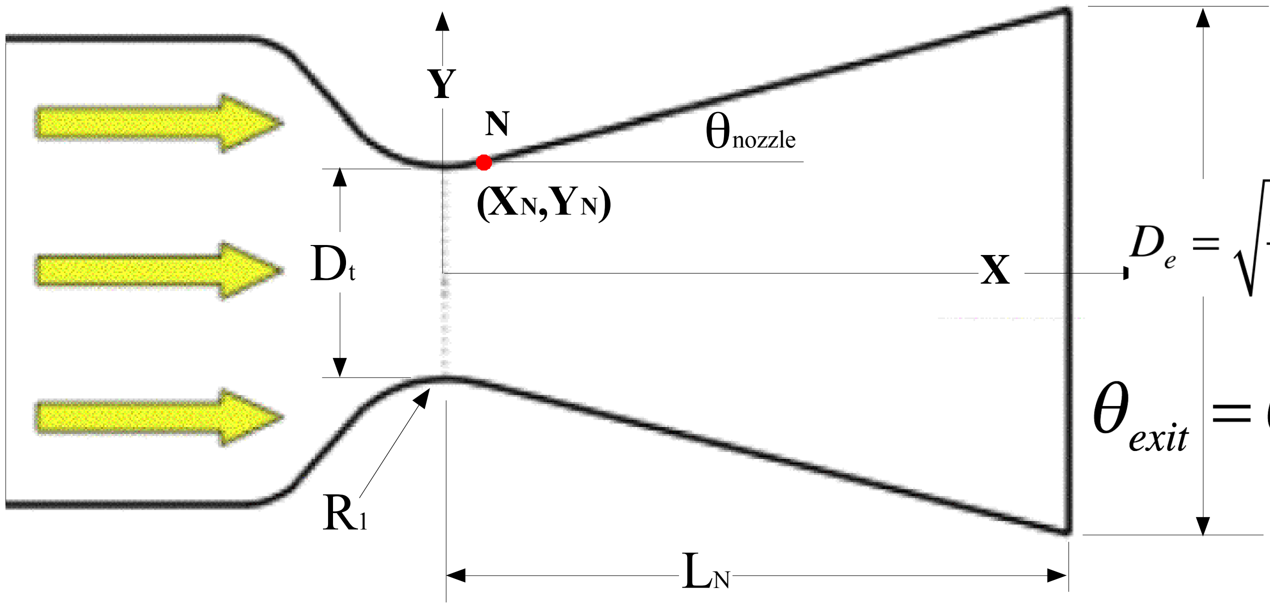
- Modify characteristic along C+ line from C_l to exit plane for non-zero Exit angle

$$\nu_{cl} = \nu_{exit} - \theta_{exit} = \nu_{exit} - \theta_{w_{max}}$$

- From earlier Minimum Length Nozzle derivation ,,,

$$\theta_{w_{max}} = \frac{\nu_{cl}}{2} = \frac{\nu_{exit} - \theta_{w_{max}}}{2}$$

Minimum Length Conical Nozzle (2)



$$D_e = \sqrt{\frac{A_{exit}}{A^*}} D_t$$

$$\theta_{exit} = \theta_{nozzle} = \theta_{w_{max}} \neq 0$$

- Simplify

$$\rightarrow \frac{3}{2}\theta_{w_{max}} = \frac{\nu_{exit}}{2}$$

$$\boxed{\rightarrow \theta_{w_{max}} = \frac{2}{3}\frac{\nu_{exit}}{2} = \frac{\nu_{exit}}{3}}$$

"Two-thirds rule-of-thumb"
Applies strictly for conical nozzles
Generally applied as "safety factor" for most nozzles

Minimum Length Conical Nozzle

- Example… given

D_{throat} = 1 cm

A_e/A^* = 8

γ = 1.2

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = 8.0 =$$

$$\frac{\left(\left(\frac{2}{1.2+1}\right)\left(1+\frac{1.2-1}{2}(3.122^2)\right)\right)^{\frac{1.2+1}{2(1.2-1)}}}{3.122}$$

$M_{exit} = 3.122$

Minimum Length Conical Nozzle

(cont'd)

$M_{exit} = 3.122$

$$\nu(M_{exit}) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}}\left(M_{exit}{}^2-1\right)\right\} - \tan^{-1}\sqrt{M_{exit}{}^2-1}$$

$$= \frac{180}{\pi}\left(\left(\frac{1.2+1}{1.2-1}\right)^{0.5}\operatorname{atan}\left(\left(\frac{1.2-1}{1.2+1}\,(3.122^2-1)\right)^{0.5}\right) - \operatorname{atan}\left(\left(3.122^2-1\right)^{0.5}\right)\right)$$

$$= 67.06^\circ \longrightarrow \theta_{w_{Max}} = \frac{\nu_{exit}}{2} = 33.53^\circ$$

Apply 2/3'rds rule

$$\theta_{w_{max}} = \frac{2}{3}\frac{\nu_{exit}}{2} = 22.35^o$$

Minimum Length Conical Nozzle

(cont'd)

- $R_1 \sim 0.75 D_t$ is typical …. $R_1 = 0.75$ cm

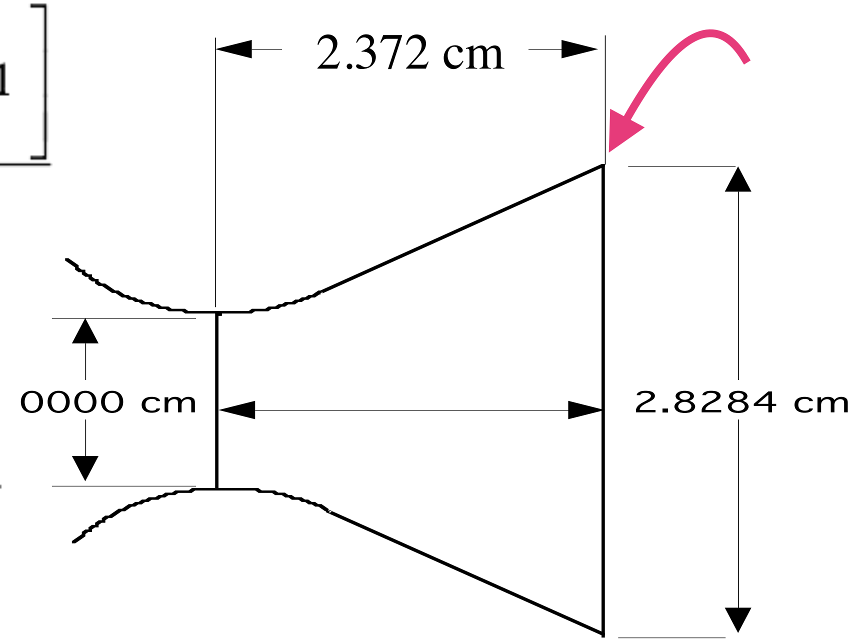
- Any shorter and you have "problems"

$$\theta_{w_{max}} = \frac{2}{3}\frac{v_{exit}}{2} = 22.35^o$$

$$[L_N]_{min} = \frac{\frac{1}{2}\left[\sqrt{\frac{A_{exit}}{A^*}} - 1\right] D_{throat} + R_1\left[\frac{1}{\cos\left(\theta_{nozzle_{max}}\right)} - 1\right]}{\tan(\theta_{nozzle_{max}})}$$

$$= \frac{\frac{1}{2}\,(8^{0.5} - 1)\,1.0 + 0.75 \cdot 1\left(\frac{1}{\cos\left(\frac{\pi}{180}22.35\right)} - 1\right)}{\tan\left(\frac{\pi}{180}22.35\right)}$$

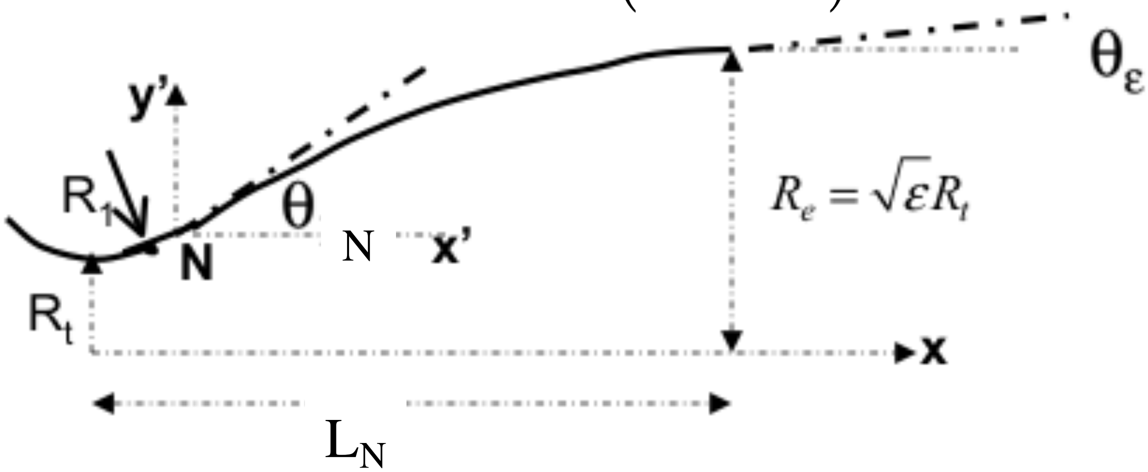
$= 2.372$ cm



3-D (Axi-Symmetric) Bell Nozzle Contour Design

approximate shape can be formed from a parabola

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2}$$



$R_1 = 1.5R_t$ **Upstream of the throat**

ε --> expansion ratio (A_{exit}/A^*)

$R_1 = 0.382R_t$ **Downstream of the throat**

- $X = X' + X_N$
- $Y = Y' + Y_N$

$$X_N = R_1 \sin\left(\theta_{nozzle}\right)$$

$$Y_N = \frac{1}{2} D_{throat} + R_1 \left[\, 1 - \cos\left(\theta_{nozzle}\right) \,\right]$$

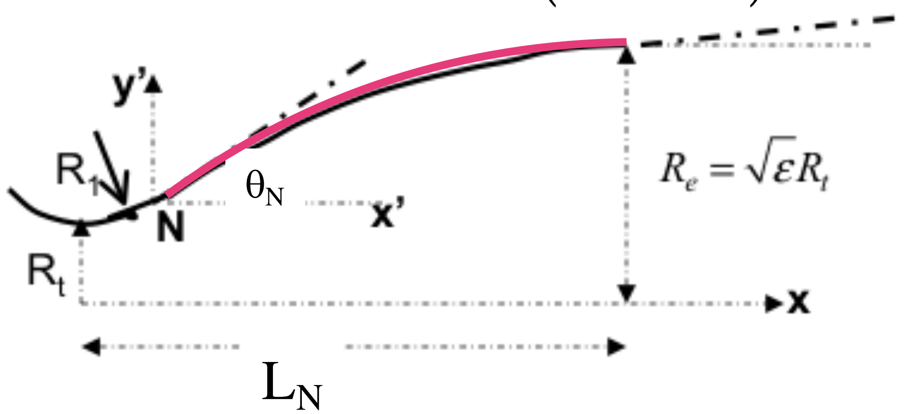
Credit: Narayanan Komerath Georgia Tech

Bell Nozzle Contour Design (cont'd)

- 4 unknowns in parabolic segment (P,Q,S,T)
- 4 boundary conditions

ε --> expansion ratio (A_{exit}/A^*)

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2}$$



$$X_N = R_1 \sin\left(\theta_{nozzle}\right)$$

$$Y_N = \frac{1}{2} D_{throat} + R_1 \left[\ 1 - \cos\left(\theta_{nozzle}\right)\ \right]$$

- Boundary Conditions

$$@N: \quad X'_N = 0 \quad Y'_N = 0$$

$$@exit: \quad \begin{bmatrix} X'_e = X_e - X_N \\ = L_N - X_N \end{bmatrix} \quad \begin{bmatrix} Y'_e = Y_e - Y_N \\ = \sqrt{\varepsilon} R_t - Y_N \end{bmatrix}$$

- θ_e
- θ_N

Given

Credit: Narayanan Komerath Georgia Tech

Bell Nozzle Contour Design (cont'd)

- Evaluate ***position*** boundary condition at N $\{X', Y'\} = \{0, 0\}$

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2} \ @ \ \{X', Y'\} = \{0, 0\}$$

$$0 = P\cdot 0 + Q + \left(S\cdot 0 + T\right)^{1/2} \rightarrow Q = -\left(T\right)^{1/2} \rightarrow \boxed{Q^2 = T}$$

- Evaluate ***slope*** boundary condition at N

$$\tan\theta_N = \left(\frac{dY'}{dX'}\right)_N = P + \frac{1}{2}\frac{1}{\left(S \times X'_N + T\right)^{1/2}} \times S =$$

$$P + \frac{S}{2}\frac{1}{\left(S \times 0 + T\right)^{1/2}} \rightarrow Q = -T^{1/2} \rightarrow \boxed{\tan\theta_N = P - \frac{S}{2Q}}$$

Bell Nozzle Contour Design (cont'd)

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2}$$

- Rearranging slope boundary condition at *N*

$$Q = -\frac{S}{2\left(\tan\theta_N - P\right)}$$

- Evaluate Slope Boundary condition at *e*

$$\tan\theta_e = \left(\frac{dY'}{dX'}\right)_e = P + \frac{1}{2}\frac{S}{\left(S \times X'_e + T\right)^{1/2}} \rightarrow$$

$$rearranging \rightarrow \left(S \times X'_e + T\right)^{1/2} = \frac{S}{2\left(\tan\theta_e - P\right)}$$

Bell Nozzle Contour Design (cont'd)

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2}$$

• Evaluate Position Boundary Condition at e

$$Y'_e = PX'_e + Q + \left(SX'_e + T\right)^{1/2} \rightarrow$$

$$\left(SX'_e + T\right)^{1/2} = Y'_e - PX'_e$$

• And the Collection expressions are

$$\left(SX'_e + T\right)^{1/2} = Y'_e - PX'_e$$

$$Q = -T^{1/2}$$

$$Q = -\frac{S}{2\left(\tan\theta_N - P\right)}$$

$$\left(SX'_e + T\right)^{1/2} = \frac{S}{2\left(\tan\theta_e - P\right)}$$

Bell Nozzle Contour Design (cont'd)

1) $\left(SX'_e + T\right)^{1/2} = Y'_e - PX'_e$

2) $Q = -T^{1/2}$

3) $Q = -\dfrac{S}{2\left(\tan\theta_N - P\right)}$

4) $\left(SX'_e + T\right)^{1/2} = \dfrac{S}{2\left(\tan\theta_e - P\right)}$

$$X = X' + X_N$$
$$Y = Y' + Y_N$$

$$X_N = R_1 \sin\left(\theta_{nozzle}\right)$$
$$Y_N = \frac{1}{2} D_{throat} + R_1\left[\ 1 - \cos\left(\theta_{nozzle}\right)\ \right]$$

$$Y' = PX' + Q + \left(SX' + T\right)^{1/2}$$

- 4 equations in 4 unknowns
- Analytical Solution is a Mess getting there .. But result is OK

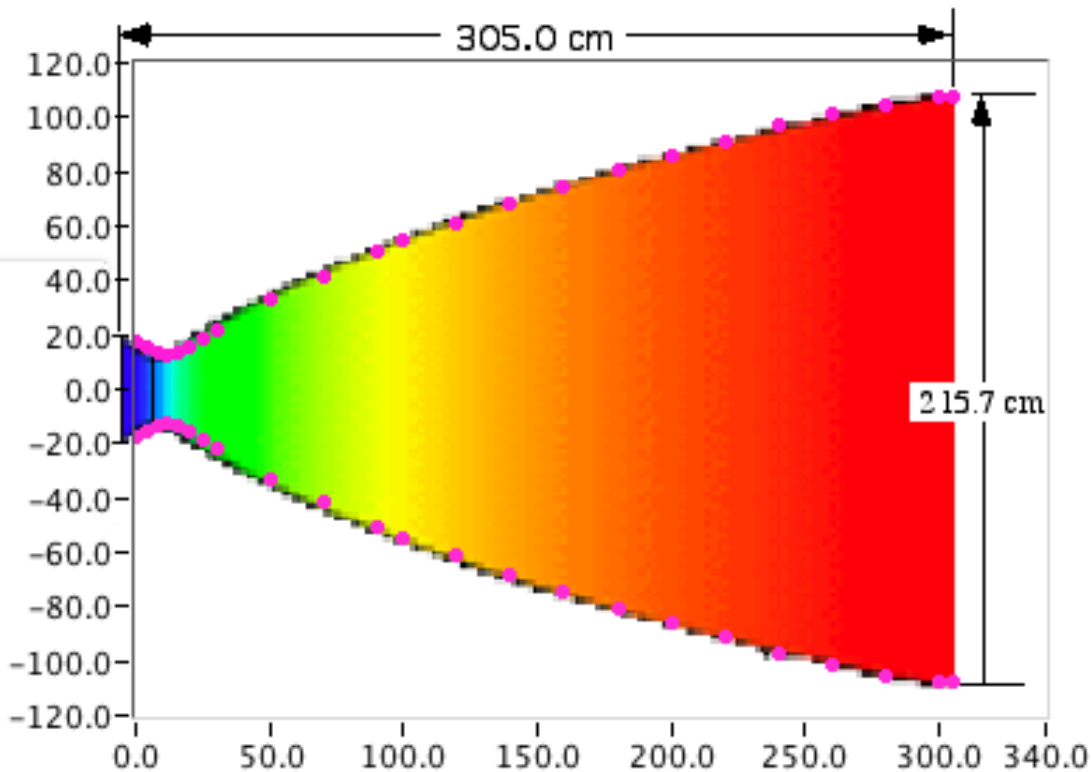
$$i) \rightarrow P = \frac{Y'_e\left(\tan\theta_N + \tan\theta_e\right) - 2X'_e \tan\theta_e \tan\theta_N}{2Y'_e - X'_e \tan\theta_N - X'_e \tan\theta_e}$$

$$ii) \rightarrow S = \frac{\left(Y'_e - PX'_e\right)^2 \left(\tan\theta_N - P\right)}{X'_e \tan\theta_N - Y'_e}$$

$$iii) \rightarrow Q = -\frac{S}{2\left(\tan\theta_N - P\right)}$$

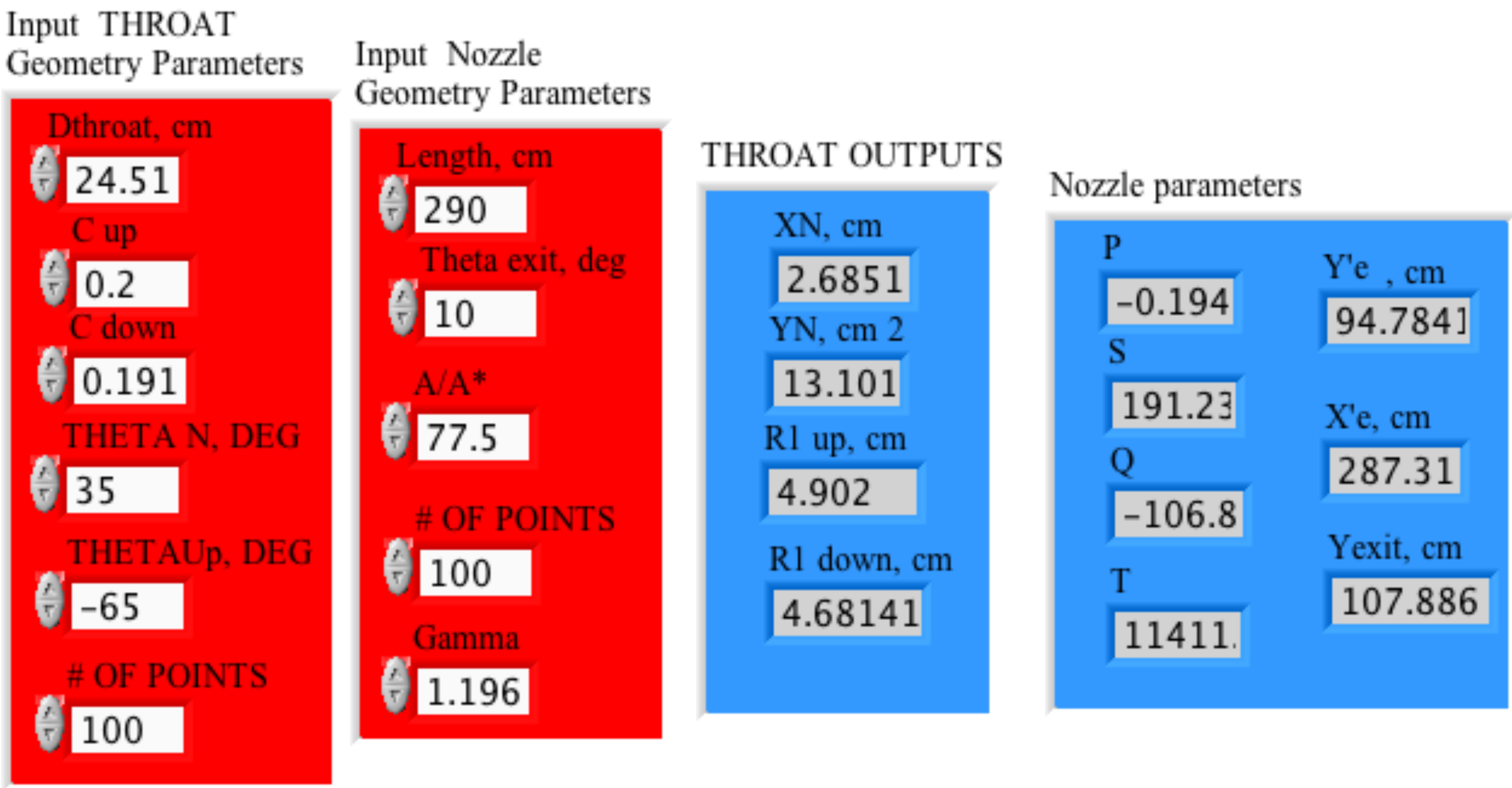
$$iv) \rightarrow T = Q^2$$

SSME Nozzle example



SSME Nozzle example (cont'd)

- Fit with Parabolic bell profile

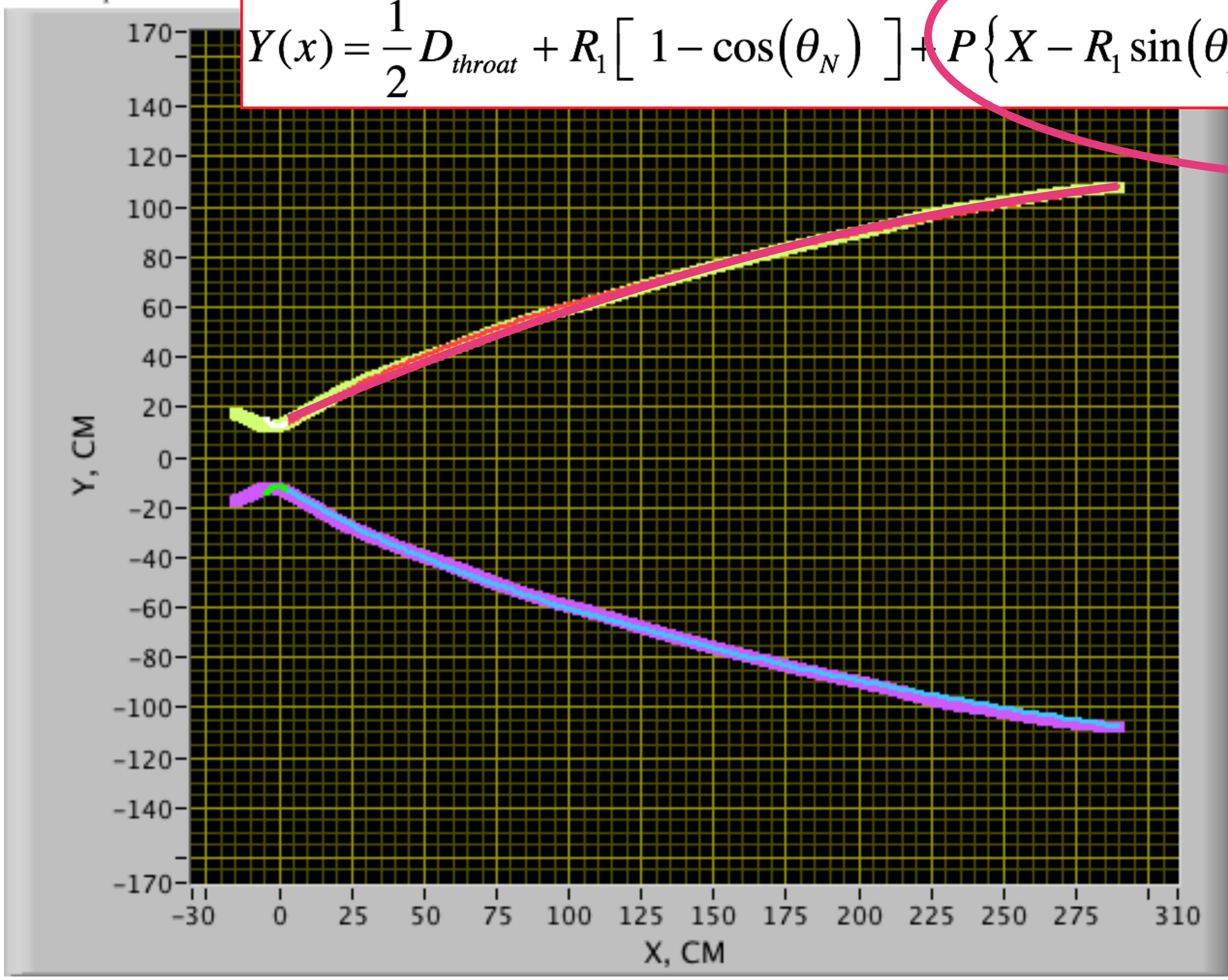


SSME Nozzle example (cont'd)

- Fit with Parabolic bell profile

$$Y(x)=\frac{1}{2}D_{throat}+R_1\left[\,1-\cos\left(\theta_N\right)\,\right]+P\left\{X-R_1\sin\left(\theta_N\right)\right\}+Q+\left(S\left\{X-R_1\sin\left(\theta_N\right)\right\}+T\right)^{1/2}$$

XY Graph



BOUNDARY CONDITIONS

- $\theta_e = 10°$
- $\theta_N = 35°$
- $D_{throat} = 24.5$ cm
- $A_e/A^* = 77.5$
- $R_1 = 4.681$ cm

- Pretty good model

SSME Nozzle example (Cont'd)

- $M_{exit} = 4.677$

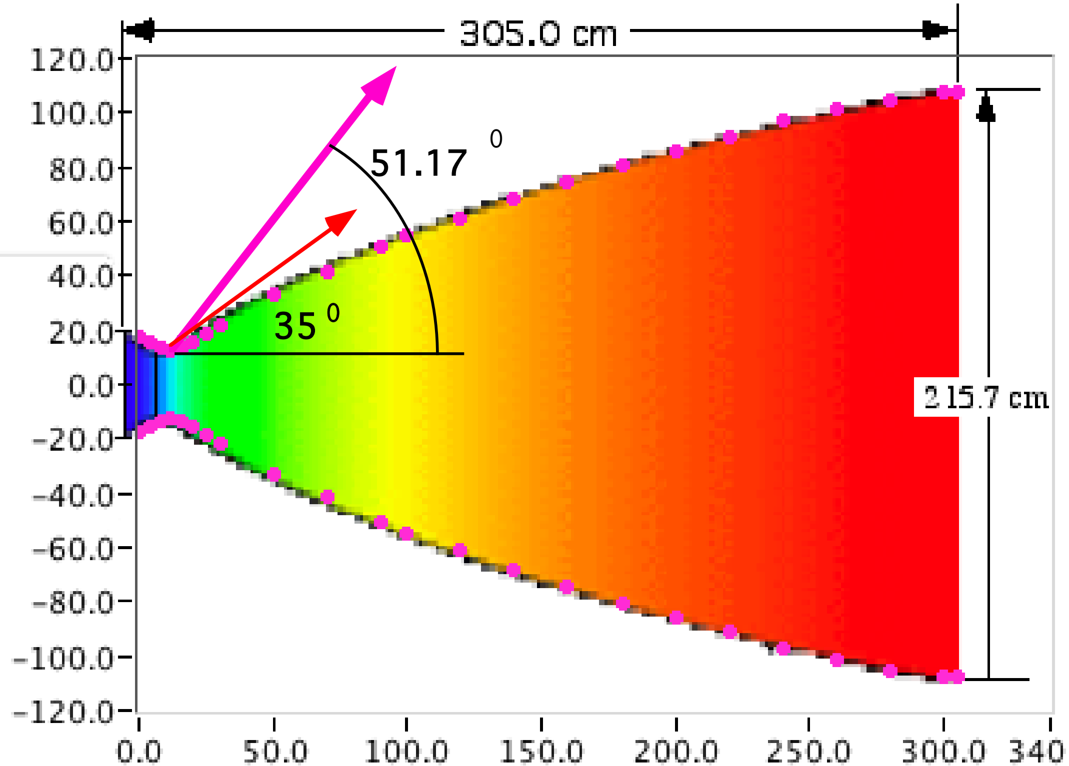
$$\nu(M_{exit}) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_{exit}^{\ 2}-1\right)}\right\} - \tan^{-1}\sqrt{M_{exit}^{\ 2}-1}$$

$$= \frac{180}{\pi}\left(\left(\frac{1.196+1}{1.196-1}\right)^{0.5} \text{atan}\left(\left(\frac{1.196-1}{1.196+1}\left(4.677^2-1\right)\right)^{0.5}\right) - \text{atan}\left(\left(4.677^2-1\right)^{0.5}\right)\right)$$

$$= 102.34° \longrightarrow \boxed{\theta_{w_{Max}} = \frac{\nu_{exit}}{2}} = 51.17°$$

SSME Nozzle example (cont'd)

$$\theta_{w_{Max}} = \frac{\nu_{exit}}{2} = 51.17°$$



- ***SSME is definitely not a minimum length nozzle***

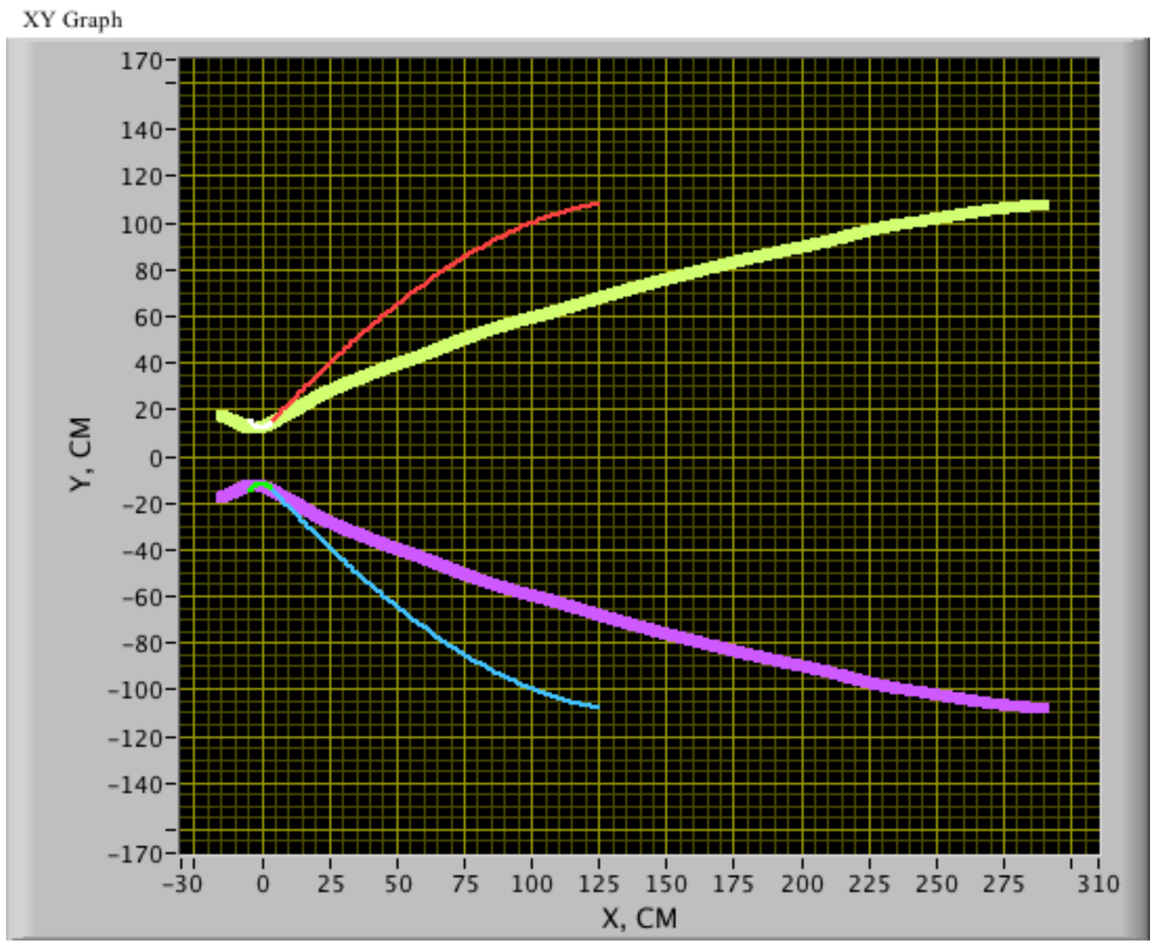
35/51.7 = 0.677

"two thirds rule"

SSME Nozzle example (cont'd)

$$\theta_{w_{Max}} = \frac{\nu_{exit}}{2} = 51.17°$$

• ~ "minimum length SSME Nozzle



Rule of Thumb
Use $\theta_N < 2/3 \;\; \theta_{max}$

"two thirds rule"

Comparison of Cone and Bell Nozzles

For the same ε, we would expect $\lambda_{bell} > \lambda_{cone}$

A bell nozzle, while more complex to build, will generally yield a more efficient exhaust than a cone in a shorter nozzle length.

Same nozzle efficiency factor can be reached with about 70% of the length of a cone nozzle.

Alternatively, efficiency factor can be increased from about 98% for a cone to about 99.2% for a bell of the same length

Credit: Narayanan Komerath Georgia Tech

Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow

• For 3-D flow using method of characteristics, Prandtl Meyer flow introduces an additional term to account for the ability of the flow to expand into the third (circumferential) dimension

$$d\theta = \mp\sqrt{M^2 - 1}\cdot\frac{dV}{V} \pm \frac{1}{\sqrt{M^2 - 1} \mp \frac{1}{\tan\theta}}\cdot\frac{dr}{r}$$

• Since the first term on right hand side is just the differential form of the Prandtl-Meyer function,

$$\theta = \int \pm\sqrt{M^2 - 1}\frac{dV}{V} = \pm\nu(M) + Const \rightarrow$$

$$d\nu = \sqrt{M^2 - 1}\cdot\frac{dV}{V}$$

Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (2)

- Thus the revised axi-compatibility equations .. $\left(x,y\right)_{3D} \rightarrow \left(x,r\right)_{axi}$ are

$$C_+ \; characteristic\, line \rightarrow \partial\left(\theta - \nu\right) = -\frac{1}{\sqrt{M^2-1}+1/\tan\theta}\frac{\partial r}{r}$$

$$C_- \; characteristic\, line \rightarrow \partial\left(\theta + \nu\right) = \frac{1}{\sqrt{M^2-1}-1/\tan\theta}\frac{\partial r}{r}$$

Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (3)

- Along the C^+ line

$$C_+ \ \textit{characteristic line} \rightarrow \partial(\theta - \nu) = -\frac{1}{\sqrt{M^2 - 1} + 1/\tan\theta}\frac{\partial r}{r}$$

$$\rightarrow \sqrt{M^2-1} = 1/\tan\mu \rightarrow \frac{\partial}{\partial K_+}(\theta - \nu) = -\frac{1}{1/\tan\mu + 1/\tan\theta}\frac{dr}{r} = -\frac{\tan\mu \cdot \tan\theta}{\tan\theta + \tan\mu}\frac{\partial r}{r} =$$

$$-\frac{\partial r}{\cos\mu \cdot \cos\theta \cdot \left(\frac{\sin\theta}{\cos\theta} + \frac{\sin\mu}{\cos\mu}\right)}\left(\frac{\sin\mu \cdot \sin\theta}{r}\right) = -\frac{dr}{(\sin\theta \cdot \cos\mu + \sin\mu \cdot \cos\theta)}\left(\frac{\sin\mu \cdot \sin\theta}{r}\right) =$$

$$-\frac{\partial r}{\left[\sin(\theta+\mu)\right]}\left(\frac{\sin\mu \cdot \sin\theta}{r}\right) \rightarrow \partial K_+ = \frac{\partial r}{\left[\sin(\theta+\mu)\right]} \rightarrow$$

$$\partial(\theta - \nu) = -\partial K_+\left(\frac{\sin\mu \cdot \sin\theta}{r}\right) \rightarrow \boxed{\frac{\partial(\theta-\nu)}{\partial K_+} = -\left(\frac{\sin\mu \cdot \sin\theta}{r}\right)}$$

$$\partial K_+ = \frac{\partial r}{\left[\sin(\theta+\mu)\right]} \rightarrow dr = \left[\sin(\theta+\mu)\right]\cdot \partial K_+$$

$$\frac{\partial r}{\partial x} = \tan(\theta+\mu) \rightarrow \left[\sin(\theta+\mu)\right]\cdot\partial K_+ = \tan(\theta+\mu)\cdot dx \rightarrow \boxed{\partial K_+ = \frac{\partial x}{\cos(\theta+\mu)}}$$

Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (4)

- Along the C^- line

$$C_- \ \textit{characteristic line} \rightarrow \partial(\theta+\nu) = \frac{1}{\sqrt{M^2-1}-1/\tan\theta}\frac{\partial r}{r}$$

$$\rightarrow \sqrt{M^2-1} = 1/\tan\mu \rightarrow \frac{\partial}{\partial K_-}(\theta+\nu) = \frac{1}{1/\tan\mu - 1/\tan\theta}\frac{dr}{r} = \frac{\tan\mu \cdot \tan\theta}{\tan\theta - \tan\mu}\frac{\partial r}{r} =$$

$$\frac{\partial r}{\cos\mu \cdot \cos\theta \cdot \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\mu}{\cos\mu}\right)}\left(\frac{\sin\mu \cdot \sin\theta}{r}\right) = \frac{dr}{(\sin\theta\cdot\cos\mu - \sin\mu\cdot\cos\theta)}\left(\frac{\sin\mu \cdot \sin\theta}{r}\right) =$$

$$\frac{\partial r}{\left[\sin(\theta-\mu)\right]}\left(\frac{\sin\mu \cdot \sin\theta}{r}\right) \rightarrow \partial K_+ = \frac{\partial r}{\left[\sin(\theta-\mu)\right]} \rightarrow$$

$$\partial(\theta+\nu) = \partial K_+\left(\frac{\sin\mu \cdot \sin\theta}{r}\right) \rightarrow \boxed{\frac{\partial(\theta+\nu)}{\partial K_+} = \left(\frac{\sin\mu \cdot \sin\theta}{r}\right)}$$

$$\partial K_- = \frac{\partial r}{\left[\sin(\theta-\mu)\right]} \rightarrow dr = \left[\sin(\theta-\mu)\right]\cdot\partial K_-$$

$$\frac{\partial r}{\partial x} = \tan(\theta-\mu) \rightarrow \left[\sin(\theta-\mu)\right]\cdot\partial K_- = \tan(\theta-\mu)\cdot dx \rightarrow \boxed{\partial K_- = \frac{\partial x}{\cos(\theta-\mu)}}$$

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- Collecting terms, the revised compatibility relations are

$$C_+ \ characteristic \ line \rightarrow \boxed{\begin{array}{l} \dfrac{\partial(\theta - \nu)}{\partial K_+} = -\left(\dfrac{\sin\mu \ \cdot \sin\theta}{r}\right) \\ \partial K_+ = \dfrac{\partial r}{\left[\sin(\theta+\mu)\right]} = \dfrac{\partial x}{\cos(\theta+\mu)} \end{array}}$$

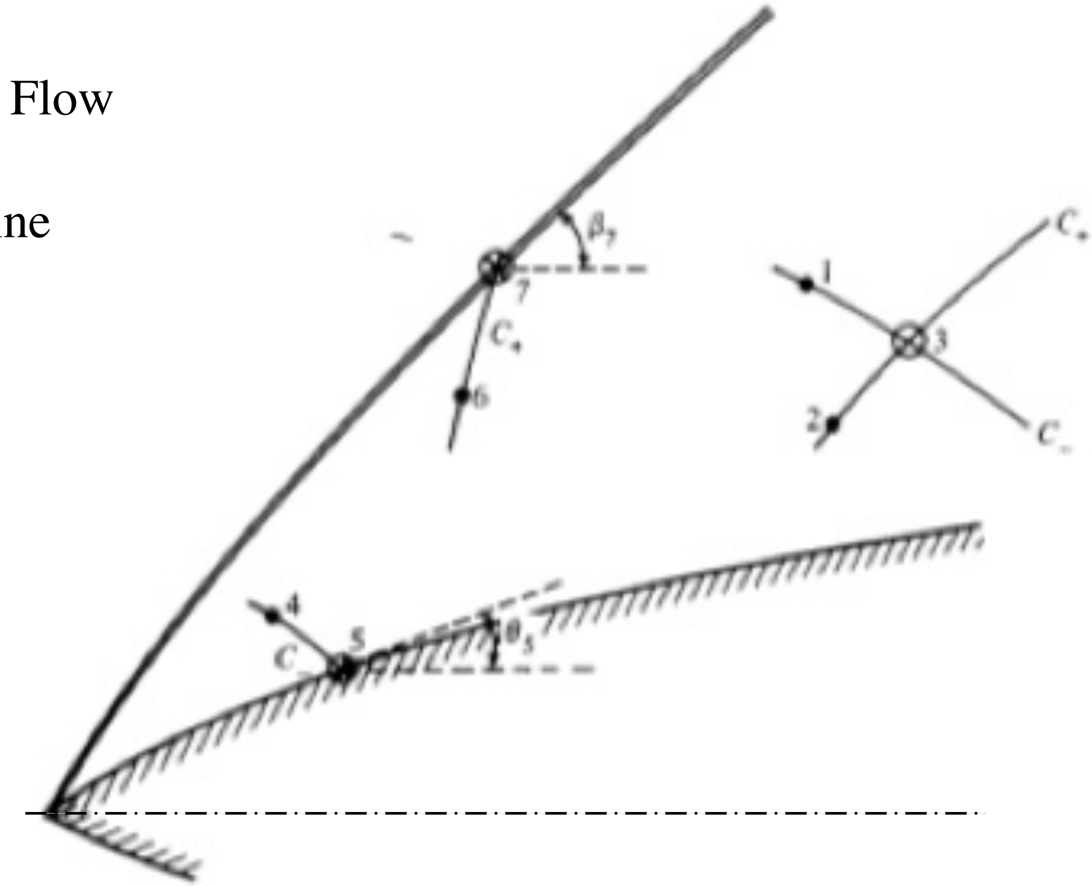
$$C_- \ characteristic \ line \rightarrow \boxed{\begin{array}{l} \dfrac{\partial(\theta + \nu)}{\partial K_-} = \left(\dfrac{\sin\mu \ \cdot \sin\theta}{r}\right) \\ \partial K_- = \dfrac{\partial r}{\left[\sin(\theta-\mu)\right]} = \dfrac{\partial x}{\cos(\theta-\mu)} \end{array}}$$

$$Slope(C_+) = \theta + \mu$$
$$Slope(C_-) = \theta - \mu$$

Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (4)

• Unit Processes

- Internal Flow
- Wall
- Centerline



Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (5)

1) *Internal Flow Field Solution Process*

Assume Straight Initial *Characteristic Line* $\rightarrow$

$$Slope(C_+)^{(0)} = \theta_2 + \mu_2$$
$$Slope(C_-)^{(0)} = \theta_1 - \mu_1$$

$\rightarrow\rightarrow$ *Solve for* $\{x_3, r_3\} \rightarrow$

$$r_3 = \frac{\tan\left[Slope(C_-)^{(j)}\right]\cdot\tan\left[Slope(C_+)^{(j)}\right](x_1 - x_2) + \tan\left[Slope(C_-)^{(j)}\right]\cdot r_2 - \tan\left[Slope(C_+)^{(j)}\right]\cdot r_1}{\tan\left[Slope(C_-)^{(j)}\right] - \tan\left[Slope(C_+)^{(j)}\right]}$$

$$x_3 = \frac{(r_2 - r_1) + x_1\cdot\tan\left[Slope(C_-)^{(j)}\right] - x_2\cdot\tan\left[Slope(C_+)^{(j)}\right]}{\tan\left[Slope(C_-)^{(j)}\right] - \tan\left[Slope(C_+)^{(j)}\right]}$$

Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (6)

- *Internal Flow Field, cont'd*

.....Calculate increments in K_+, K_-

$$\Delta K_+ = \frac{x_3 - x_2}{\cos\left[Slope\left(C_+\right)^{(j)}\right]}$$

$$\Delta K_- = \frac{x_3 - x_1}{\cos\left[Slope\left(C_-\right)^{(j)}\right]}$$

.....Advance Characteric Line Invariants

$$C_+ \; characteristic\; line \rightarrow \theta_3 - \nu_3 = \theta_2 - \nu_2 - \frac{\sin\mu_2 \cdot \sin\theta_2}{r_2}\Delta K_+$$

$$C_- \; characteristic\; line \rightarrow \theta_3 + \nu_3 = \theta_1 + \nu_1 + \frac{\sin\mu_1 \cdot \sin\theta_1}{r_1}\Delta K_-$$

Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (7)

....*Solve for* θ_3

$$\theta_3 = \frac{\theta_2 - \nu_2 - \frac{\sin\mu_2 \cdot \sin\theta_2}{r_2}\Delta K_+ + \theta_1 + \nu_1 + \frac{\sin\mu_1 \cdot \sin\theta_1}{r_1}\Delta K_-}{2} = \frac{(\theta_2 + \theta_1) + (\nu_1 - \nu_2) + \frac{\sin\mu_1 \cdot \sin\theta_1}{r_1}\Delta K_- - \frac{\sin\mu_2 \cdot \sin\theta_2}{r_2}\Delta K_+}{2}$$

....*Solve for* ν_3

$$\nu_3 = \frac{\theta_1 + \nu_1 + \frac{\sin\mu_1 \cdot \sin\theta_1}{r_1}\Delta K_- - \left(\theta_2 - \nu_2 - \frac{\sin\mu_2 \cdot \sin\theta_2}{r_2}\Delta K_+\right)}{2} = \frac{(\theta_1 - \theta_2) + (\nu_1 + \nu_2) + \frac{\sin\mu_1 \cdot \sin\theta_1}{r_1}\Delta K_+ + \frac{\sin\mu_2 \cdot \sin\theta_2}{r_2}\Delta K_-}{2}$$

$\rightarrow\rightarrow\rightarrow$ ....*Solve for* $(\theta_3, \nu_3) \rightarrow M_3, \mu_3$

....Re*calculate*

$$Slope(C_+)^{(j)} = \frac{\theta_2 + \mu_2 + \theta_3 + \mu_3}{2}$$

$$Slope(C_-)^{(j)} = \frac{\theta_1 - \mu_1 + \theta_3 - \mu_3}{2}$$

Iterate from $\rightarrow\rightarrow$ *to* $\rightarrow\rightarrow\rightarrow$

Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (8)

2) *Wall Point Solution*

$$\frac{\partial(\theta-\nu)}{\partial K_+} = -\left(\frac{\sin\mu \cdot \sin\theta}{r}\right)$$

$$\partial K_+ = \frac{\partial r}{\left[\sin(\theta+\mu)\right]} = \frac{\partial x}{\cos(\theta+\mu)}$$

$$\rightarrow \frac{\partial(\theta-\nu)}{\partial x} = -\left(\frac{\sin\mu \cdot \sin\theta \cdot \cos(\theta+\mu)}{r}\right)$$

$$\partial(\theta-\nu) = -\frac{\partial r}{\left[\sin(\theta+\mu)\right]}\left(\frac{\sin\mu \cdot \sin\theta}{r}\right) = -\left(\frac{\sin\mu \cdot \sin\theta}{\sin(\theta+\mu)}\right)\frac{\partial r}{r}$$

$$\theta_{wall} - \nu_{wall} = \theta_{field} - \nu_{field} - \left(\frac{\sin\mu \cdot \sin\theta}{\sin(\theta+\mu)}\right)\left(\ln r_{wall} - \ln r_{field}\right) = \theta_{field} - \nu_{field} - \left(\frac{\sin\mu \cdot \sin\theta}{\sin(\theta+\mu)}\right)\left(\ln\frac{r_{wall}}{r_{field}}\right)$$

$$\nu_{wall} = \theta_{wall} - \theta_{field} + \nu_{field} - \left(\frac{\sin\mu \cdot \sin\theta}{\sin(\theta+\mu)}\right)\left(\ln\frac{r_{field}}{r_{wall}}\right)$$

Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (9)

3) *Centerline Solution*

$$\rightarrow \textit{Initial Point}: \ \{x_1, r_1, \theta_1, M_1\} \rightarrow \begin{bmatrix} \nu_1 \\ \mu_1 \end{bmatrix}$$

$$\textit{Initial Slope} \rightarrow \textit{Assume Straight } \text{Initial } \textit{Characteristic Line} \rightarrow Slope(C_-)^{(0)} = \theta_1 - \mu_1$$

$$\rightarrow \textit{Centerline Intercept}: \ \theta_{cl} = 0$$

$$y_{cl} = 0 \rightarrow \frac{0 - y_1}{x_{cl} - x_1} = \tan\left(Slope(C_-)\right) \rightarrow \boxed{x_{cl} = -\frac{y_1}{\tan\left(Slope(C_-)\right)} + x_1}$$

$$\Delta K_- = \frac{x_{cl} - x_1}{\cos\left[Slope(C_-)^{(j)}\right]}$$

$$\in \textit{right running } (C_-) \textit{ characteristic line} \rightarrow \theta_{cl} + \nu_{cl} = \theta_1 + \nu_1 + \frac{\sin\mu_1 \cdot \sin\theta_1}{r_1}\Delta K_-$$

$$\rightarrow \nu_{cl} = \theta_1 + \nu_1 + \frac{\sin\mu_1 \cdot \sin\theta_1}{r_1}\Delta K_- \rightarrow\rightarrow \begin{bmatrix} M_{cl} \\ \mu_{cl} \end{bmatrix}$$

$$.....\text{Re}\,calculate \rightarrow Slope(C_-)^{(0)} = \frac{\theta_1 - \mu_1 - \mu_{cl}}{2} \qquad \textit{Iterate from } \rightarrow\rightarrow \textit{ to } \rightarrow\rightarrow\rightarrow$$