Section 1 Lecture 1: Review of the Two-Dimensional Method of Characteristics

• Anderson, Chapter 11 pp. 377-403
Prandtl-Meyer Expansion Fan: Mathematical Analysis Revisited

- Consider flow expansion around an infinitesimal corner

\[ \frac{\pi}{2} - \mu - d\theta \]

**Infinitesimal Expansion Fan Flow Geometry**

\[ \frac{\pi}{2} + \mu \]

\[ V \]

\[ V + dV \]

\[ \sin\left(\frac{\pi}{2} - \mu - d\theta\right) \]

\[ \sin\left(\frac{\pi}{2} + \mu\right) \]

- From Law of Sines

\[ \frac{V}{\sin\left(\frac{\pi}{2} - \mu - d\theta\right)} = \frac{V + dV}{\sin\left(\frac{\pi}{2} + \mu\right)} \]
Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont’d)

• Expanding and collecting terms

\[ 1 + \frac{dV}{V} = \frac{\cos(\mu)}{\cos(\mu)\cos(d\theta) - \sin(\mu)\sin(d\theta)} \]

• Letting \( d\theta \) is considered be infinitesimal

\[ \cos(d\theta) = 1 \rightarrow \sin(d\theta) = d\theta \]

\[ 1 + \frac{dV}{V} = \frac{\cos(\mu)}{\cos(\mu) - \sin(\mu)(d\theta)} = \frac{1}{1 - \tan(\mu)(d\theta)} \]
Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont’d)

- Exploiting the form of the power series (expanded about $\frac{dV}{V}=0$)

\[
\frac{1}{1 - \frac{dV}{V}} = 1 + \frac{1}{\left(1 - \frac{dV}{V}\right)^2} \left(\frac{dV}{V} - 0\right) + O\left(\frac{dV^2}{V}\right)
\]

- … truncate after first order term

\[
\frac{1}{1 - \frac{dV}{V}} \approx 1 + \frac{dV}{V} \quad \rightarrow \quad \frac{1}{1 - \frac{dV}{V}} \approx \frac{1}{1 - \tan(\mu)(d\theta)}
\]
Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont’d)

• Solve for $d\theta$ in terms of $dV/V$

\[
\frac{1}{1 - \frac{dV}{V}} \approx 1 + \frac{dV}{V} \quad \Rightarrow \quad \frac{1}{1 - \frac{dV}{V}} \approx \frac{1}{1 - \tan(\mu)(d\theta)}
\]

\[
1 - \tan(\mu)(d\theta) = 1 - \frac{dV}{V} \quad \Rightarrow \quad d\theta = \frac{1}{\tan(\mu)} \frac{dV}{V}
\]

• Since disturbance is infinitesimal (mach wave)

\[
\sin(\mu) = \frac{1}{M}
\]
Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont’d)

• “Differential form” of Prandtl-Meyer wave

\[ d\theta = \frac{1}{\tan(\mu)} \frac{dV}{V} \]

• For an infinitesimal disturbance (mach wave)

\[ \sin(\mu) = \frac{1}{M} \]
Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont’d)

\[
\sin(\mu) = \frac{1}{M} \rightarrow \sin^2(\mu) = \frac{1}{M^2} = \frac{\sin^2(\mu) + \cos^2(\mu)}{M^2} \\
M^2 = \frac{\sin^2(\mu) + \cos^2(\mu)}{\sin^2(\mu)} = 1 + \frac{1}{\tan^2(\mu)} \rightarrow \frac{1}{\tan^2(\mu)} = M^2 - 1 \\
\frac{1}{\tan(\mu)} = \pm \sqrt{M^2 - 1}
\]

• and …. 

\[d\theta = \pm \sqrt{M^2 - 1} \frac{dV}{V}\]  

keep ± sign
Prandtl-Meyer Expansion Fan: Mathematical Analysis (cont’d)

• As demonstrated in section 6.2 of MAE 5420

\[
\theta = \int \pm \sqrt{M^2 - 1} \frac{dV}{V} = \pm \nu(M) + \text{Const} \rightarrow
\]

\[

\nu(M) = \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \left\{ \frac{\gamma - 1}{\gamma + 1} \left( M^2 - 1 \right) \right\} - \tan^{-1} \sqrt{M^2 - 1}
\]

\[
\theta \pm \nu(M) = \text{Const}
\]
Characteristic Lines

- Right running characteristic lines

\[ \theta + V(M) = K_\perp \]

Slope: \( \theta - \mu \)

- C_\perp “right running” characteristic Line is a Generalization For infinitesimal expansion corner flow

MAE 5540 – Propulsion Systems I
Characteristic Lines (cont’d)

• Left running characteristic lines

\[ \theta - V(M) = K_+ \]

Slope: \( \theta + \mu \)

• \( C_+ \) “left running” characteristic Line is a Generalization infinitesimal compression corner flow
Characteristic Lines

• Supersonic “compatibility” equations

\[ \theta + \nu(M) = Const \equiv K_- \]
\[ \theta - \nu(M) = Const \equiv K_+ \]

• Apply along “characteristic lines” in flow field
Physical Meaning of Characteristic Lines

• Schlieren Photo of Supersonic nozzle flow with roughened wall
Regions of Influence and Domains of Dependence

Characteristics Lines Determine Regions of Influence and Dependence in Supersonic Flow Field

A does not "feel" influence of \{B, C\}
Regions of Influence and Domains of Dependence (cont’d)

A does "feel" influence of \{D\}
Regions of Influence and Domains of Dependence (concluded)

$M_\infty > 1$

Domain of dependence for point A

Region of influence for point A

C+

C−
SR-71 Near Field Shock Wave Patterns

Speed of sound across each successive shock wave is Higher (temperature increases) … wave catch up and Reinforce each other

\[ c = \sqrt{\gamma R_g T} \]
“Method of Characteristics”

• Basic principle of Methods of Characteristics

-- If supersonic flow properties are known at two points in a flow field,

-- There is one and only one set of properties compatible* with these at a third point,

-- Determined by the intersection of characteristics, or mach waves, from the two original points.

*Root of term “compatibility equations”
“Method of Characteristics” (cont’d)

• Compatibility Equations relate the velocity magnitude and direction along the characteristic line.

• In 2-D and quasi 1-D flow, compatibility equations are independent of spatial position, in 3-D methods, space becomes a player and complexity goes up considerably.

• Computational Machinery for applying the method of Characteristics are the so-called “unit processes”

• By repeated application of unit processes, flow field can be solved in entirety.
Unit Process 1: Internal Flow Field

- Conditions Known at Points \{1, 2\}
- Point \{3\} is at intersection of \{C_+, C_-\} characteristics

\[ \theta + \sqrt{\nu(M)} = \text{Const} \equiv K_- \]
\[ \theta - \sqrt{\nu(M)} = \text{Const} \equiv K_+ \]
Unit Process 1: Internal Flow Field (cont’d)

Point \{1\} \rightarrow \{M_1, \theta_1\} known \rightarrow

\[ \nu_1 = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1}} \left( M_1^2 - 1 \right) \right\} - \tan^{-1} \sqrt{M_1^2 - 1} \]

Along \{C_−\} \rightarrow \theta_1 + \nu_1 = \text{const} = \left( K_− \right)_1
Unit Process 1: Internal Flow Field (cont’d)

Point \{2\} \rightarrow \{M_2, \theta_2\} known \rightarrow

\[ \nu_2 = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1}} \left( M_2^2 - 1 \right) \right\} - \tan^{-1} \sqrt{M_2^2 - 1} \]

Along \{C_+\} \rightarrow \theta_2 - \nu_2 = \text{const} = (K_+)_2
Unit Process 1: Internal Flow Field  (cont’d)

Mach and Flow Direction solved for at Point 3

\[
\theta + \mathbf{v}(M) = \text{Const} \equiv K_-
\]

\[
\theta - \mathbf{v}(M) = \text{Const} \equiv K_+
\]

**Point\{3\} →**

\[
\begin{align*}
\theta_1 + \nu_1 &= \theta_3 + \nu_3 \\
\theta_2 - \nu_2 &= \theta_3 - \nu_3
\end{align*}
\]

\[
\begin{align*}
\frac{\theta_3}{2} &= \frac{(\theta_1 + \nu_1) + (\theta_2 - \nu_2)}{2} = \frac{(K_-)_1 + (K_+)_2}{2} \\
\nu_3 &= \frac{(\theta_1 + \nu_1) - (\theta_2 - \nu_2)}{2} = \frac{(K_-)_1 - (K_+)_2}{2}
\end{align*}
\]

\[
M_3 = \text{Solve} \left[ \nu_3 = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1}} \left( M_3^2 - 1 \right) \right\} - \tan^{-1} \sqrt{M_3^2 - 1} \right]
\]

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Unit Process 1: Internal Flow Field (cont’d)

But where is Point \{3\}?

• \{M,\theta\} known at points \{1,2,3\}
  ---> \{\mu_1,\mu_2,\mu_3\} known
Unit Process 1: Internal Flow Field  (concluded)

• Slope of characteristics lines approximated by:

\[
\text{slope}\{C_-\} = \frac{(\theta_1 - \mu_1) + (\theta_3 - \mu_3)}{2}
\]

Intersection locates point 3

\[
\text{slope}\{C_+\} = \frac{(\theta_2 + \mu_2) + (\theta_3 + \mu_3)}{2}
\]
Unit Process 1: Internal Flow Example (cont’d)

- Solve for \( \{x_3, y_3\} \)

\[
\begin{align*}
\frac{y_3 - y_1}{x_3 - x_1} &= \tan\left[ \text{slope}\left\{ C_\text{-} \right\} \right] \\
\frac{y_3 - y_2}{x_3 - x_2} &= \tan\left[ \text{slope}\left\{ C_\text{+} \right\} \right]
\end{align*}
\]

\[
\begin{align*}
y_3 &= (x_3 - x_1)\tan\left[ \text{slope}\left\{ C_\text{-} \right\} \right] + y_1 \\
y_3 &= (x_3 - x_2)\tan\left[ \text{slope}\left\{ C_\text{+} \right\} \right] + y_2
\end{align*}
\]
Unit Process 1: Internal Flow Example (cont’d)

- Solve for \{x_3, y_3\}

\[
x_3 = \frac{x_1 \cdot \tan (\text{slope } C_-) - x_2 \cdot \tan (\text{slope } C_+) + (y_2 - y_1)}{\tan (\text{slope } C_-) - \tan (\text{slope } C_+)}
\]

\[
y_3 = \frac{\tan (\text{slope } C_-) \cdot \tan (\text{slope } C_+) \cdot (x_1 - x_2) + \tan (\text{slope } C_-) \cdot y_2 - \tan (\text{slope } C_+) \cdot y_1}{\tan (\text{slope } C_-) - \tan (\text{slope } C_+)}
\]
Unit Process 1: Internal Flow Example

\[ M_1 = 2.0, \ \theta_1 = 10^\circ, \ \{x_1,y_1\} = \{1.0,2.0\} \]

\[ M_2 = 1.75, \ \theta_2 = 5^\circ, \ \{x_2,y_2\} = \{1.5,1.0\} \]
Unit Process 1: Internal Flow Example (cont’d)

- Point 1, compute

\[
\left\{ \nu_1, \mu_1, \left( K_+ \right)_1 \right\}
\]

\[
\nu_1 = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1}} \left( 2.0^2 - 1 \right) \right\} - \tan^{-1} \sqrt{2.0^2 - 1} = 26.37976^\circ
\]

\[
\mu_1 = \frac{180}{\pi} \sin^{-1} \left[ \frac{1}{2.0} \right] = 30^\circ
\]

\[
\left( K_+ \right)_1 = \theta_1 + \nu_1 = 10^\circ + 26.37976^\circ = 36.37976^\circ
\]
Unit Process 1: Internal Flow Example (cont’d)

- Point 2, compute
  \[ \{ \nu_2, \mu_2, (K_+)_2 \} \]

\[
\nu_2 = \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1} \left( 1.75^2 - 1 \right)} \right\} - \tan^{-1} \sqrt{1.75^2 - 1} = 19.27319^\circ
\]

\[
\mu_2 = \frac{180}{\pi} \sin^{-1} \left[ \frac{1}{1.75} \right] = 34.84990^\circ
\]

\[
(K_+)_2 = \theta_2 - \nu_2 = 5^\circ - 19.27319^\circ = -14.27319^\circ
\]
Unit Process 1: Internal Flow Example (cont’d)

• Point 3 Solve for \[ \{ \theta_3, \nu_3 \} \]

\[
\theta_3 = \frac{(K_-)_1 + (K_+)_2}{2} = \frac{36.37976 + (-14.27319)}{2} = 11.0533 \text{ deg.}
\]

\[
\nu_3 = \frac{(K_-)_1 - (K_+)_2}{2} = \frac{36.37976 - (-14.27319)}{2} = 25.3265 \text{ deg.}
\]
Unit Process 1: Internal Flow Example (cont’d)

- Point 3 Solve for \( \{ M_3, \mu_3 \} \)

\[
M_3 = \text{Solve} \left[ 25.3265 \times \frac{\pi}{180} = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1}} \left( M_3^2 - 1 \right) \right\} - \tan^{-1} \sqrt{M_3^2 - 1} \right]
\]

\[
\sin(\mu) = \frac{1}{M}
\]

\[
M_3 = 1.96198 \\
\text{---} \mu_3 = 30.6431^\circ
\]
Unit Process 1: Internal Flow Example (cont’d)

• Locate Point 3

• Line Slope Angles

\[
slope\{C_-\} = \frac{(\theta_1 - \mu_1) + (\theta_3 - \mu_3)}{2} = \frac{(10 - 30) + (11.053 - 30.6431)}{2} = -19.795 \text{ deg}
\]

\[
slope\{C_+\} = \frac{(\theta_2 + \mu_2) + (\theta_3 + \mu_3)}{2} = \frac{(5 + 34.8499) + (11.053 + 30.6431)}{2} = 40.773 \text{ deg}
\]
Unit Process 1: Internal Flow Example (cont’d)

• Solve for \( \{x_3, y_3\} \)

\[
x_3 = \frac{-1 \tan \left( \frac{\pi}{180} \left(-19.794887\right) \right) + 1.5 \tan \left( \frac{40.773123 \pi}{180} \right) + 2 - 1}{\tan \left( \frac{\pi}{180} \left(-19.794887\right) \right) - \tan \left( \frac{\pi}{180} \left(40.773123\right) \right)}
\]

= 2.17091
Unit Process 1: Internal Flow Example (cont’d)

• Solve for \{x_3, y_3\}

\[
y_3 = \frac{\tan\left(\frac{\pi}{180} \cdot (-19.79489)\right) \cdot \tan\left(\frac{\pi}{180} \cdot 40.7731\right) \cdot (1.0 - 1.5) - 2 \tan\left(40.773123 \frac{\pi}{180}\right) + 1 \tan\left(\frac{\pi}{180} \cdot (-19.79489)\right)}{\tan\left(\frac{\pi}{180} \cdot (-19.79489)\right) - \tan\left(\frac{\pi}{180} \cdot 40.773123\right)} = 1.57856
\]
Unit Process 1: Internal Flow Example (concluded)

\[
\begin{bmatrix}
M_1 \\
\theta_1 \\
x_1 \\
y_1
\end{bmatrix} =
\begin{bmatrix}
2.0 \\
10^\circ \\
1.0 \\
2.0
\end{bmatrix} \rightarrow
\begin{bmatrix}
M_2 \\
\theta_2 \\
x_2 \\
y_2
\end{bmatrix} =
\begin{bmatrix}
1.75 \\
5^\circ \\
1.5 \\
1
\end{bmatrix} \rightarrow
\begin{bmatrix}
M_3 \\
\theta_3 \\
x_3 \\
y_3
\end{bmatrix} =
\begin{bmatrix}
1.96198 \\
11.0533 \\
2.17091 \\
1.57856
\end{bmatrix}
\]
Unit Process 2: Wall Point

\[ \theta + \mathcal{V}(M) = \text{Const} \equiv K_- \]

\[ \theta - \mathcal{V}(M) = \text{Const} \equiv K_+ \]

- Conditions Known at Points \{4\}, Wall boundary at point 5
Unit Process 2: Wall Point (cont’d)

\[ \theta_5 + \nu_5 = \theta_4 + \nu_4 \rightarrow \nu_5 = \theta_4 + \nu_4 - \theta_5 \]

\[ \text{slope}\{C_\cdot\} = \frac{(\theta_4 - \mu_4) + (\theta_5 - \mu_5)}{2} \]

• Iterative solution
Unit Process 2: Wall Point (cont’d)

• Iterative solution process

\[
\nu_4 = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1}} \left( M_4^2 - 1 \right) \right\} - \tan^{-1} \sqrt{M_4^2 - 1}
\]

Along \( \{C_-\} \rightarrow \theta_4 + \nu_4 = \text{const} = (K_-)_4 \)

• Pick \( \theta_5 \)

\[
\nu_5 = \theta_4 + \nu_4 - \theta_5
\]

\[
M_5 = \text{Solve} \left[ \nu_5 \frac{\pi}{180} = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1}} \left( M_5^2 - 1 \right) \right\} - \tan^{-1} \sqrt{M_5^2 - 1} \right]
\]
Unit Process 2: Wall Point (concluded)

• Solve for Mach angle, C\textsubscript{-} slope

\[
slope \{ C\textsubscript{-} \} = \frac{(\theta_4 - \mu_4) + (\theta_5 - \mu_5)}{2}
\]

\[
\sin(\mu_5) = \frac{1}{M_5}
\]

• In Similar manner as before find intersection of C\textsubscript{-} and surface mold line .. Get new \( \theta_5 \), repeat iteration
Unit Process 3: Shock Point

- Conditions Known at Points \{6\}, Shock boundary at point 7
  Freestream Mach Number Known

- Along \(C_+\) characteristic

\[
\theta_6 - \nu_6 = \theta_7 - \nu_7
\]

- Iterative Solution
Unit Process 3: Shock Point

• Pick $\theta_7$ ---> Oblique Shock wave solver

$M_\infty, \theta_7$ ---> $M_7$ (behind shock)

\[ \theta_7 = \theta_6 - \nu_6 + \nu_7(M_7) \]

• Iterative solution Repeat
  Using new $\theta_7$ until convergence
Unit Process 2: Shock Point (cont’d)

• Pick $\theta_7$ --> Oblique Shock wave solver

--> M7

• Iterative solution
Unit Process 2: Shock Point (cont’d)

- But

• Iterative solution
Using MOC for Supersonic Nozzle Design

In order to expand an internal steady flow through a duct from subsonic to supersonic speed, we established in Chap. 5 that the duct has to be convergent-divergent in shape.

Moreover, we developed relations for the local Mach number, and hence the pressure, density, and temperature, as functions of local area ratio $A/A^*$. However, these relations assumed quasi-one-dimensional flow, whereas, strictly speaking, the flow two-dimensional. Moreover, the quasi-one-dimensional theory tells us nothing about the proper contour of the duct, i.e., what is the proper variation of area with respect to the flow direction $A = A(x)$. If the nozzle contour is not proper, shock waves may occur inside the duct.

The method of characteristics provides a technique for properly designing the contour of a supersonic nozzle for shockfree, isentropic flow, taking into account the multidimensional flow inside the duct. The purpose of this section is to illustrate such an application.
Supersonic Nozzle Design

• Strategic contouring will “absorb” mach waves to give isentropic flow in divergent section
Supersonic Nozzle Design (cont’d)

• Rocket Nozzle (Minimum Length)

• Bell Nozzle (gradual expansion)

• Use compatibility eqs. to design boundary with shock free flow
Using Method of Characteristics to Design a Bell Nozzle

• This approach “prescribes” the expansion section of the nozzle, and then uses M.O.C to design turning section to achieve wave cancellation at wall …. And ensure isentropic flow
• Straightening section is designed to cancel all of the expansion waves generated by expansion section

• Wave generated at $g$, reflects at $h$ and is cancelled by wall at $i$
Using Method of Characteristics to Design a Bell Nozzle

“Running Method”

- Pick \( \theta_{\text{max}} \) and length of expansion section

- Use Prandtl-Meyer Expansion equations to calculate \( M, \nu, \mu \) at Each point along boundary

Wall unit process calculation

“\( \Delta \theta \)” increments
Initial Data Line

• Unit Processes must start somewhere .. Need a datum from which too start process

• Example nozzle flow … Throat
Using Method of Characteristics to Design a Bell Nozzle

\( \Delta \theta = \frac{\theta_{MAX}}{N} \) \\
\( \theta_0 = 0 \) \\
\( M_0 = 1 \) \\
\( \nu(1) = 0 \) \\
\( \theta_i = i \cdot \Delta \theta \) \\

\( \nu(M_i) - \theta_i = \nu(M_0) - \theta_0 \) \\
\( \nu(M_i) = \theta_i = i \cdot \Delta \theta \) \\

Initial Data Line

Extract \( M_i \rightarrow \nu(M_i) = \)

\( \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \left( \frac{\gamma-1}{\gamma+1} \left( M_i^2-1 \right) \right) - \tan^{-1} \left( M_i^2-1 \right) \) \\

\( i = \{1,...,N\} \) \\

\( \mu_i = \sin^{-1} \left( \frac{1}{M_i} \right) \)
Using Method of Characteristics to Design a Bell Nozzle (3)

1) Wall point: \( \theta_{\text{wall}} \rightarrow \nu_{\text{wall}} \rightarrow M_{\text{wall}} \rightarrow \mu_{\text{wall}} \)

2) Centerline point: \( \theta_{\text{cl}} = 0 \rightarrow \nu_{\text{cl}} = \theta_{\text{wall}} + \nu_{\text{wall}} \)
   \[ \nu_{\text{cl}} \rightarrow M_{\text{cl}} \rightarrow \mu_{\text{cl}} \]

3) Characteristic line slope: \( C_- = \frac{\theta_{\text{wall}} - \mu_{\text{wall}} + (0 - \mu_{\text{cl}})}{2} \)

Centerline Intercept calculation (see next slide)
Centerline Intercept Solution

1) Initial Point: \( \{x_1, y_1, \theta_1, M_1\} \rightarrow \left[ \begin{array}{c} v_1 \\ \mu_1 \end{array} \right] \)

2) Centerline Intercept: \( \theta_{cl} = 0 \)
   
   right running characteristic line \( \rightarrow \theta_{cl} + v_{cl} = \theta_1 + v_1 \)
   
   \( \rightarrow v_{cl} = \theta_1 + v_1 \rightarrow \left[ \begin{array}{c} M_{cl} \\ \mu_{cl} \end{array} \right] \rightarrow \text{Slope}(C_-) = \theta_1 - \mu_1 - \mu_{cl} \)
   
   \( y_{cl} = 0 \rightarrow \frac{0 - y_1}{x_{cl} - x_1} = \tan(\text{Slope}(C_-)) \rightarrow x_{cl} = -\frac{y_1}{\tan(\text{Slope}(C_-))} + x_1 \)
Using Method of Characteristics to Design a Bell Nozzle (4)

\[
\begin{align*}
\theta_3 &= \frac{(\theta_1 + \nu_1) + (\theta_2 - \nu_2)}{2} = \frac{(K_-)_1 + (K_+)_2}{2} \\
\nu_3 &= \frac{(\theta_1 + \nu_1) - (\theta_2 - \nu_2)}{2} = \frac{(K_-)_1 - (K_+)_2}{2}
\end{align*}
\]

“\(\Delta \theta\)” increments

Internal Flow unit process calculation (slides 24-26)
Using Method of Characteristics to Design a Bell Nozzle

Complete expansion section grid

“Δθ” increments
Nozzle Construction Example: Chamber and Expansion Section

Chamber section

Expansion Section

$R_c \text{ chamb}$

$R_c \text{ exp}$

$\{x_N, y_N\}$

$\theta_{\text{max}}$

$X, \text{ CM}$

$Y, \text{ CM}$
Nozzle Construction Example: Initial characteristic line

**Prandtl - Meyer Expansion**: \( \nu_1 = \nu_{\text{throat}} + \theta_1 \rightarrow M_1 \)

**C_ Characteristic Line**: \( \theta_{cl} = 0 \rightarrow \nu_{cl} = \theta_1 + \nu_1 \)

**Characteristic line slope**: 

\[
C_- = \frac{\theta_{\text{wall}} - \mu_{\text{wall}} + (0 - \mu_{cl})}{2}
\]
Nozzle Construction Example: Second characteristic line

*Prandtl - Meyer Expansion*: \( \nu_2 = \nu_1 + (\theta_2 - \theta_1) \rightarrow M_2 \)

\[
\begin{align*}
\theta_3 &= \frac{(\theta_1 + \nu_1) + (\theta_2 - \nu_2)}{2} = \frac{(K_-)_1 + (K_+)_2}{2} \\
\nu_3 &= \frac{(\theta_1 + \nu_1) - (\theta_2 - \nu_2)}{2} = \frac{(K_-)_1 - (K_+)_2}{2}
\end{align*}
\]

Internal Flow Solution
Nozzle Construction Example: Repeat Unit Processes to Complete Grid for Expansion Section

1. Prandtl-Meyer Expansion at Wall
2. Internal Flow
3. Centerline Intercept
Using Method of Characteristics to Design a Bell Nozzle (6)

Now begin straightening section

Wall Unit Process Calculation

\[ C_+ \rightarrow \theta_{\text{wall}} - \nu_{\text{wall}} = \theta_{\text{field}} - \nu_{\text{field}} \]

Characteristic line slope:

\[ C+ = \frac{\theta_{\text{wall}} + \mu_{\text{wall}} + (\theta_{\text{field}} + \mu_{\text{field}})}{2} \]
Wall Point Solution for Turning Section

Final Point on Expansion Grid: \( \{ x_{\text{field}}, y_{\text{field}}, \theta_{\text{field}}, M_{\text{field}} \} \rightarrow \begin{bmatrix} v_{\text{field}} \\ \mu_{\text{field}} \end{bmatrix} \)

Previous Point on Wall: \( \{ x_1, y_1, \theta_1, M_1 \} \rightarrow \begin{bmatrix} v_1 \\ \mu_1 \end{bmatrix} \)

Current Wall Slope: \( \theta_{\text{wall}} \)

Left running (C+) line

\( v_{\text{wall}} = (\theta_{\text{wall}} - \theta_{\text{field}}) + v_{\text{field}} \)

\( \rightarrow \begin{bmatrix} M_{\text{wall}} \\ \mu_{\text{wall}} \end{bmatrix} \)

\( (x_1, y_1) \rightarrow (x_{\text{wall}}, y_{\text{wall}}) \)
Wall Point Solution for Turning Section (2)

**Left running (C+) line**

\[
\mathbf{v}_{\text{wall}} = (\theta_{\text{wall}} - \theta_{\text{field}}) + \mathbf{v}_{\text{field}} \rightarrow \begin{bmatrix} M_{\text{wall}} \\ \mu_{\text{wall}} \end{bmatrix} \text{slope}\{C^+\} = \left(\frac{\theta_{\text{field}} + \mu_{\text{field}} + \theta_{\text{wall}} + \mu_{\text{wall}}}{2}\right)
\]

\[
\frac{y_{\text{wall}} - y_1}{x_{\text{wall}} - x_1} = \tan(\theta_{\text{wall}})
\]

\[
\frac{y_{\text{wall}} - y_{\text{field}}}{x_{\text{wall}} - x_{\text{field}}} = \tan(\text{slope}\{C^+\})
\]

\((x_1, y_1)\)  \(\rightarrow\)  \((x_{\text{wall}}, y_{\text{wall}})\)

\((x_{\text{field}}, y_{\text{field}})\)

\(\theta_{\text{wall}}\)
Wall Point Solution for Turning Section (3)

\[
\frac{y_{wall} - y_1}{x_{wall} - x_1} = \tan(\theta_{wall}) \\
\frac{y_{wall} - y_{field}}{x_{wall} - x_{field}} = \tan(\text{slope}\{C^+\})
\]

Regroup in Matrix Form
Solve via Cramer’s Rule

\[
\begin{bmatrix}
y_1 - x_1 \cdot \tan(\theta_{wall}) \\
y_{field} - x_{field} \cdot \tan(\text{slope}\{C^+\})
\end{bmatrix}
= \begin{bmatrix}
-tan(\theta_{wall}) & 1 \\
-tan(\text{slope}\{C^+\}) & 1
\end{bmatrix}
\begin{bmatrix}
x_{wall} \\
y_{wall}
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_{wall} \\
y_{wall}
\end{bmatrix}
= \begin{bmatrix}
1 & -1 \\
\tan(\text{slope}\{C^+\}) & -\tan(\theta_{w})
\end{bmatrix}
\begin{bmatrix}
y_1 - x_1 \cdot \tan(\theta_{w}) \\
y_{field} - x_{field} \cdot \tan(\text{slope}\{C^+\})
\end{bmatrix}
\]

\[
\frac{x_{wall}}{y_{wall}} = \frac{\tan(\text{slope}\{C^+\}) - \tan(\theta_{w})}{\tan(\text{slope}\{C^+\}) - \tan(\theta_{w})}
\]

MAE 5540 – Propulsion Systems I
Wall Point Solution for Turning Section (4)

\[
\frac{y_{\text{wall}} - y_1}{x_{\text{wall}} - x_1} = \tan(\theta_{\text{wall}})
\]

\[
\frac{y_{\text{wall}} - y_{\text{filed}}}{x_{\text{wall}} - x_{\text{field}}} = \tan\left(slope\{C^+\}\right)
\]

**Simplify Solution**

\[
x_{\text{wall}} = \frac{x_1 \cdot \tan(\theta_{\text{wall}}) - x_{\text{field}} \cdot \tan\left(slope\{C^+\}\right) + (y_{\text{field}} - y_1)}{\tan(\theta_{\text{wall}}) - \tan\left(slope\{C^+\}\right)}
\]

\[
y_{\text{wall}} = \frac{\tan(\theta_{\text{wall}}) \cdot \tan\left(slope\{C^+\}\right) \cdot (x_1 - x_{\text{field}}) + \tan(\theta_{\text{wall}}) \cdot y_{\text{field}} - \tan\left(slope\{C^+\}\right) \cdot y_1}{\tan(\theta_{\text{wall}}) - \tan\left(slope\{C^+\}\right)}
\]

\[
\rightarrow slope\{C^+\} = \frac{\theta_{\text{wall}} + \mu_{\text{wall}} + \theta_{\text{field}} + \mu_{\text{field}}}{2}
\]
Using Method of Characteristics to Design a Bell Nozzle

Complete Grid for Straightening Section, Solving along Final Expansion section characteristic line
Using Method of Characteristics to Design a Bell Nozzle

Completed Grid for Straightening Section
Nozzle Construction Example: Work from Final Characteristic Line to Complete Nozzle Profile

Assign “thetas” ….  
\[
\Delta \theta = \frac{\theta_{\text{max}} - \theta_{\text{exit}}}{N_{\text{characteristics}}} \quad \Rightarrow \quad \theta_j = \theta_{\text{max}} - j \cdot \Delta \theta
\]
Nozzle Construction Example: Code Layout

1. Prandtl-Meyer Expansion at Wall
2. Internal Flow
3. Centerline Intercept
4. Wall Intercept
Minimum Length Nozzle Design (cont’d)

- Find minimum length nozzle with shock-free flow $\theta_{exit} = 0$

$$\nu_{exit} \rightarrow M_{exit}$$

- Along $C_+$ characteristic $\{b,c\}$

$$\theta_{cl} - \nu_{cl} = \theta_{exit} - \nu_{exit}$$

$$\theta_{cl}, \theta_{exit} = 0 \rightarrow \nu_{cl} = \nu_{exit}$$

- At expansion corners, $\{a,d\}$

From Prandtl-Meyer Theory

$$\nu_{\{a,d\}} = \nu_{throat} + \theta_{w_{max}}$$

$$\nu_{throat} = \nu_{m=1} = 0$$

$$\nu_{\{a,d\}} = \theta_{w_{max}}$$

$$\nu(1.0) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1} (1.0^2 - 1)} \right\} - \tan^{-1} \sqrt{1.0^2 - 1} = 0$$
Minimum Length Nozzle Design (cont’d)

• Find minimum length nozzle with shock-free flow \( \theta_{exit} = 0 \)

\[ \nu_{exit} \rightarrow M_{exit} \]

\[ \nu_{exit} \rightarrow M_{exit} \]

• Along \( C_\text{−} \) characteristic \( \{a,c\} \) at point \( a \)

\[ \theta_{w_{\text{max}}} + \nu_a = \theta_{cl} + \nu_{cl} \rightarrow \theta_{cl} = 0 \]

\[ \theta_{w_{\text{max}}} + \nu_a = \nu_{cl} \]

• Along \( C_+ \) characteristic \( \{d,c\} \) at point \( a \)

\[ -\theta_{w_{\text{max}}} - \nu_d = \theta_{cl} - \nu_{cl} \rightarrow \theta_{cl} = 0 \]

\[ \theta_{w_{\text{max}}} + \nu_d = \nu_{cl} \rightarrow \nu_a = \nu_d \]
Minimum Length Nozzle Design (cont’d)

But from Previous discussion

\[ \nu_{\{a,d\}} = \theta_{\text{w,max}} \quad \rightarrow \quad \theta_{\text{w,max}} + \nu_a = \nu_{cl} \]

\[ \rightarrow \quad \nu_a = \nu_d \quad \rightarrow \quad \nu_{cl} = \nu_{\text{exit}} \]

\[ \rightarrow \quad \theta_{\text{w,max}} + \theta_{\text{w,max}} = \nu_{cl} \quad \rightarrow \quad \theta_{\text{w,max}} = \frac{\nu_{cl}}{2} = \frac{\nu_{\text{exit}}}{2} \]

- **Criterion for Minimum Length “Bell” Nozzle**

\[ \theta_{\text{w,Max}} = \frac{\nu_{\text{exit}}}{2} \]

- Length for a given expansion angle is more important than the precise shape of nozzle …
Minimum Length Nozzle: Construction Example

• Use Method of characteristics, compute and graph contour for two-D minimum length nozzle for a design exit mach number of 2.0

$M_{\text{exit}} = 2.0$
Minimum Length Nozzle: Construction Example (cont’d)

\[ M_{\text{exit}} = 2.0 \rightarrow \nu(M_{\text{exit}}) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \frac{\gamma - 1}{\gamma + 1} \left(2.0^2 - 1\right) \right\} - \tan^{-1} \sqrt{2.0^2 - 1} = \]

\[ \frac{180}{\pi} \left( \left( \frac{1.4 + 1}{1.4 - 1} \right)^{0.5} \tan \left( \left( \left( \frac{1.4 - 1}{1.4 + 1} \right) (2.0^2 - 1) \right)^{0.5} \right) \right) - \tan \left( \left( (2.0^2 - 1) \right)^{0.5} \right) = 26.3798^\circ \]

\[ \theta_{\text{w Max}} = \frac{\nu_{\text{exit}}}{2} = 13.1899^\circ \]
Minimum Length Nozzle: Construction Example (cont’d)

\[ \text{slope } \{ C_- \} = \frac{(\theta_1 - \mu_1) + (\theta_3 - \mu_3)}{2} \]

\[ \text{slope } \{ C_+ \} = \frac{(\theta_2 + \mu_2) + (\theta_3 + \mu_3)}{2} \]
Minimum Length Nozzle: Construction Example (cont’d)

Inflection Point

- Section of Nozzle with $\theta_{wall}$ increasing called “expansion section”
- Inflection point occurs at $\left(\theta_{wall}\right)_{max}$
- Downstream of inflection point called “straightening” section
Minimum Length Nozzle: Construction

Example (cont’d)

• Starting up the calculation …

Point a … Need to Kick “a” just a bit
Downstream of throat

\[
\begin{align*}
M &= 1.026 \\
\theta_{wall} &= 0.19
\end{align*}
\]

\[
\nu(1.026) = 0.19 \rightarrow \begin{cases} C_{-a} = \theta + \nu = 0.38 \\ \mu_a = 77.1^\circ \end{cases}
\]

\[
\theta_1 + \nu_1 = C_- = 0.38 \rightarrow \theta_1 = 0 \rightarrow \nu_1 = 0.38 \rightarrow \begin{cases} M_1 = 1.042 \\ \mu_a = 73.58^\circ \end{cases}
\]

\[
slope\{C_-\} \approx \frac{(\theta_a - \mu_a) + \theta_1 - \mu_1}{2} = -75.43^\circ
\]
Minimum Length Nozzle: Construction
Example (cont’d)

<table>
<thead>
<tr>
<th>Point #</th>
<th>C_ = θ + ν (deg),</th>
<th>C_+ = θ - ν (deg),</th>
<th>( \theta = \frac{C_ - C_+}{2} ) (deg),</th>
<th>( \nu = \frac{C_ - C_+}{2} ) (deg),</th>
<th>( M )</th>
<th>( \mu ) (deg)</th>
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<tbody>
<tr>
<td>a</td>
<td>0.38</td>
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<td>0.19</td>
<td>0.0</td>
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**Minimum Length Nozzle: Construction Example (cont’d)**

<table>
<thead>
<tr>
<th>Point #</th>
<th>$C_\theta = \theta + \nu$ (deg)</th>
<th>$C_\theta = \theta - \nu$ (deg)</th>
<th>$\theta = \frac{C_- + C_+}{2}$ (deg)</th>
<th>$\nu = \frac{C_- - C_+}{2}$ (deg)</th>
<th>$M$</th>
<th>$\mu$ (deg)</th>
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<td>14</td>
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<td><strong>26.38</strong></td>
<td><strong>2.000</strong></td>
<td><strong>30.00</strong></td>
</tr>
</tbody>
</table>

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MAE 5540 – Propulsion Systems I
Typical Conical Nozzle Contour

\[ D_e = \sqrt{\frac{A_{\text{exit}}}{A^*}} D_t \]

\varepsilon --> expansion ratio \( (A_{\text{exit}}/A^*) \)

- \( D_t \): Throat diameter
- \( R_1 \): Radius of curvature of nozzle contraction
- \( N \): Transition point from circular contraction to conical nozzle
- \( L_N \): Nozzle Length
- \( D_e \): Exit diameter

- \( R_1 \approx 0.75D_t \) is typical

\[ X_N = R_1 \sin(\theta_{\text{nozzle}}) \]

\[ Y_N = \frac{1}{2} D_{\text{throat}} + R_1 \left[ 1 - \cos(\theta_{\text{nozzle}}) \right] \]
Typical Conical Nozzle Contour

(Cont’d)

- Solve for Nozzle length in terms of other parameters

\[ \tan(\theta_{\text{nozzle}}) = \frac{\frac{1}{2} D_e - \left\{ \frac{1}{2} D_{\text{throat}} + R_1 \left[ 1 - \cos(\theta_{\text{nozzle}}) \right] \right\} }{L_N - R_1 \sin(\theta_{\text{nozzle}})} \]

\[ \frac{1}{2} \left[ D_e - D_{\text{throat}} \right] - R_1 \left[ 1 - \cos(\theta_{\text{nozzle}}) \right] \]

\[ \frac{L_N - R_1 \sin(\theta_{\text{nozzle}})}{L_N - R_1 \sin(\theta_{\text{nozzle}})} \]

\[ \rightarrow \left\{ L_N - R_1 \sin(\theta_{\text{nozzle}}) \right\} \tan(\theta_{\text{nozzle}}) = \frac{1}{2} \left[ \sqrt{\frac{A_{\text{exit}}}{A^*}} - 1 \right] D_{\text{throat}} - R_1 \left[ 1 - \cos(\theta_{\text{nozzle}}) \right] \]

\[ \rightarrow L_N \tan(\theta_{\text{nozzle}}) = \frac{1}{2} \left[ \sqrt{\frac{A_{\text{exit}}}{A^*}} - 1 \right] D_{\text{throat}} - R_1 \left[ 1 - \cos(\theta_{\text{nozzle}}) \right] - \tan(\theta_{\text{nozzle}}) \sin(\theta_{\text{nozzle}}) \]
Typical Conical Nozzle Contour

- Using trig identities

\[
1 - \cos\left(\theta_{\text{nozzle}}\right) - \tan(\theta_{\text{nozzle}})\sin(\theta_{\text{nozzle}}) = 1 - \cos(\theta_{\text{nozzle}}) - \frac{\sin^2(\theta_{\text{nozzle}})}{\cos(\theta_{\text{nozzle}})} = \\
1 - \frac{\cos^2(\theta_{\text{nozzle}}) + \sin^2(\theta_{\text{nozzle}})}{\cos(\theta_{\text{nozzle}})} = 1 - \frac{1}{\cos(\theta_{\text{nozzle}})}
\]

\[
L_N \tan(\theta_{\text{nozzle}}) = \frac{1}{2} \left[ \sqrt{\frac{A_{\text{exit}}}{A^*}} - 1 \right] D_{\text{throat}} + R_1 \left[ \frac{1}{\cos(\theta_{\text{nozzle}})} - 1 \right]
\]

\[
L_N = \frac{1}{2} \left[ \sqrt{\frac{A_{\text{exit}}}{A^*}} - 1 \right] D_{\text{throat}} + R_1 \left[ \frac{1}{\cos(\theta_{\text{nozzle}})} - 1 \right] \tan(\theta_{\text{nozzle}})
\]

- \(R_1 \sim 0.75D_t\) is typical
Minimum Length Conical Nozzle (1)

- Modify characteristic along C+ line from C₁ to exit plane for non-zero Exit angle

\[ \nu_{cl} = \nu_{exit} - \theta_{exit} = \nu_{exit} - \theta_{w_{max}} \]

- From earlier Minimum Length Nozzle derivation "","

\[ \theta_{w_{max}} = \frac{\nu_{cl}}{2} = \frac{\nu_{exit} - \theta_{w_{max}}}{2} \]
Minimum Length Conical Nozzle (2)

- Simplify

\[ \frac{3}{2} \theta_{w_{\text{max}}} = \frac{v_{\text{exit}}}{2} \]

\[ \theta_{w_{\text{max}}} = \frac{2}{3} \frac{v_{\text{exit}}}{2} = \frac{v_{\text{exit}}}{3} \]

"Two-thirds rule-of-thumb"

Applies strictly for conical nozzles

Generally applied as "safety factor"

for most nozzles
Minimum Length Conical Nozzle

• Example… given

\[ D_{\text{throat}} = 1 \text{ cm} \]
\[ \frac{A_e}{A^*} = 8 \]
\[ \gamma = 1.2 \]

\[
\frac{A}{A^*} = \frac{1}{M} \left[ \left( \frac{2}{\gamma + 1} \right) \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \right]^{\frac{\gamma + 1}{2(\gamma - 1)}} = 8.0 = \\
\left( \left( \frac{2}{1.2 + 1} \right) \left( 1 + \frac{1.2 - 1}{2} (3.122^2) \right) \right)^{\frac{1.2 + 1}{2 (1.2 - 1)}} \\
\frac{3.122}{3.122}
\]

\[ M_{\text{exit}} = 3.122 \]
Minimum Length Conical Nozzle

( cont’d )

\[ M_{\text{exit}} = 3.122 \]

\[ v(M_{\text{exit}}) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1}} \left( M_{\text{exit}}^2 - 1 \right) \right\} - \tan^{-1} \sqrt{M_{\text{exit}}^2 - 1} \]

\[ = \frac{180}{\pi} \left( \left( \frac{1.2 + 1}{1.2 - 1} \right)^{0.5} \tan \left( \frac{1.2 - 1}{1.2 + 1} \left( 3.122^2 - 1 \right) \right)^{0.5} \right) - \tan \left( \left( 3.122^2 - 1 \right)^{0.5} \right) \]

= 67.06°

\[ \theta_{w_{\text{Max}}} = \frac{v_{\text{exit}}}{2} = 33.53° \]

Apply 2/3’rds rule

\[ \theta_{w_{\text{max}}} = \frac{2}{3} \frac{v_{\text{exit}}}{2} = 22.35° \]
Minimum Length Conical Nozzle (cont’d)

• $R_1 \sim 0.75D_t$ is typical …. $R_1=0.75\text{ cm}$

\[
\theta_{w_{\text{max}}} = \frac{2}{3} \frac{\nu_{\text{exit}}}{2} = 22.35^\circ
\]

\[
[L_N]_{\text{min}} = \frac{1}{2} \left[ \frac{A_{\text{exit}}}{A^*} - 1 \right] D_{\text{throat}} + R_1 \left[ \frac{1}{\cos(\theta_{\text{nozzle}})} - 1 \right] \tan(\theta_{\text{nozzle}})
\]

\[
= \frac{1}{2} (8^{0.5} - 1) 1.0 + 0.75 \cdot 1 \left( \frac{1}{\cos\left(\frac{\pi}{180} 22.35\right)} - 1 \right) \tan\left(\frac{\pi}{180} 22.35\right)
\]

= 2.372 cm

• Any shorter and you have “problems”
Bell Nozzle Contour Design

approximate shape can be formed from a parabola

\[ Y' = PX' + Q + \left( SX' + T \right)^{1/2} \]

\[ R_1 = 1.5R_t \quad \text{Upstream of the throat} \]

\[ R_1 = 0.382R_t \quad \text{Downstream of the throat} \]

\[ X = X' + X_N \]

\[ Y = Y' + Y_N \]

\[ \varepsilon \rightarrow \text{expansion ratio } (A_{\text{exit}}/A^*) \]

\[ X_N = R_1 \sin(\theta_{\text{nozzle}}) \]

\[ Y_N = \frac{1}{2} D_{\text{throat}} + R_1 \left[ 1 - \cos(\theta_{\text{nozzle}}) \right] \]
Bell Nozzle Contour Design (cont’d)

- 4 unknowns in parabolic segment \((P, Q, S, T)\)
- 4 boundary conditions

\[
Y' = PX' + Q + \left(SX' + T\right)^{1/2}
\]

\[
\theta_e = \sqrt{e}R_t
\]

\[
X_N = R_1 \sin(\theta_{\text{nozzle}})
\]

\[
Y_N = \frac{1}{2}D_{\text{throat}} + R_1 \left[ 1 - \cos(\theta_{\text{nozzle}}) \right]
\]

@\(N\):
\[
X_N' = 0 \quad Y_N' = 0
\]

@exit:
\[
\begin{bmatrix}
X_e' = X_e - X_N \\
= L_N - X_N
\end{bmatrix}
\]

\[
\begin{bmatrix}
Y_e' = Y_e - Y_N \\
= \sqrt{e}R_t - Y_N
\end{bmatrix}
\]

- Boundary Conditions
  - \(\theta_e\)
  - \(\theta_N\)

\(e\) -> expansion ratio \((A_{\text{exit}}/A^*)\)
Bell Nozzle Contour Design (cont’d)

- Evaluate position boundary condition at $N$

$$Y' = PX' + Q + \left( SX' + T \right)^{1/2}$$

@ $N : e = P \times \left[ L_N - X_N \right] + Q + \left( S \times 0 + T \right)^{1/2} \rightarrow Q^2 = T$$

- Evaluate slope boundary condition at $N$

$$\tan \theta_N = \left( \frac{dY'}{dX'} \right)_N = P + \frac{1}{2} \frac{1}{\left( S \times X'_N + T \right)^{1/2}} \times S =$$

$$P + \frac{S}{2} \frac{1}{\left( S \times 0 + T \right)^{1/2}} \rightarrow Q = -T^{1/2} \rightarrow \tan \theta_N = P - \frac{S}{2Q}$$
Bell Nozzle Contour Design (cont’d)

\[ Y' = PX' + Q + \left( SX' + T \right)^{1/2} \]

• Rearranging slope boundary condition at \( N \)

\[ Q = -\frac{S}{2\left( \tan \theta_N - P \right)} \]

• Evaluate Slope Boundary condition at \( e \)

\[ \tan \theta_e \left( \frac{dY'}{dX'} \right)_e = P + \frac{1}{2} \frac{S}{\left( S \times X'_e + T \right)^{1/2}} \rightarrow \]

\[ \text{rearranging} \rightarrow \left( S \times X'_e + T \right)^{1/2} = \frac{S}{2\left( \tan \theta_e - P \right)} \]
Bell Nozzle Contour Design (cont’d)

\[ Y' = PX' + Q + \left( SX' + T \right)^{1/2} \]

• Evaluate Position Boundary Condition at \( e \)

\[ Y'_e = PX'_e + Q + \left( SX'_e + T \right)^{1/2} \rightarrow \]

\[ \left( SX'_e + T \right)^{1/2} = Y'_e - PX'_e \]

• And the Collection expressions are

\[ \left( SX'_e + T \right)^{1/2} = Y'_e - PX'_e \]

\[ Q = -T^{1/2} \]

\[ Q = -\frac{S}{2\left(\tan \theta_N - P\right)} \]

\[ \left( SX'_e + T \right)^{1/2} = \frac{S}{2\left(\tan \theta_e - P\right)} \]
Bell Nozzle Contour Design (cont’d)

1) \( (SX_e' + T)^{1/2} = Y_e' - PX_e' \)

2) \( Q = -T^{1/2} \)

3) \( Q = -\frac{S}{2(\tan \theta_N - P)} \)

4) \( (SX_e' + T)^{1/2} = \frac{S}{2(\tan \theta_e - P)} \)

\[ X = X' + X_N \]
\[ Y = Y' + Y_N \]

\( X_N = R_1 \sin(\theta_{nozzle}) \)

\( Y_N = \frac{1}{2} D_{throat} + R_1 \left[ 1 - \cos(\theta_{nozzle}) \right] \)

\[ Y' = PX_e' + Q + (SX_e' + T)^{1/2} \]

- 4 equations in 4 unknowns
- Analytical Solution is a Mess getting there .. But result is OK

\[ i) \rightarrow P = \frac{Y_e' \left( \tan \theta_N + \tan \theta_e \right) - 2X_e' \tan \theta_e \tan \theta_N}{2Y_e' - X_e' \tan \theta_N - X_e' \tan \theta_e} \]

\[ \frac{Y_e' - PX_e'}{X_e' \tan \theta_N - Y_e'} \]

\[ ii) \rightarrow S = \left( \frac{(Y_e' - PX_e')}{X_e' \tan \theta_N - Y_e'} \right)^2 \left( \tan \theta_N - P \right) \]

\[ iii) \rightarrow Q = -\frac{S}{2(\tan \theta_N - P)} \]

\[ iv) \rightarrow T = Q^2 \]
SSME Nozzle example
SSME Nozzle example (cont’d)

• Fit with Parabolic bell profile
SSME Nozzle example (cont’d)

• Fit with Parabolic bell profile

\[
Y(x) = \frac{1}{2} D_{\text{throat}} + R_1 \left[ 1 - \cos(\theta_N) \right] + P \left\{ X - R_1 \sin(\theta_N) \right\} + Q + \left( S \left\{ X - R_1 \sin(\theta_N) \right\} + T \right)^{1/2}
\]

BOUNDARY CONDITIONS

\[
\begin{align*}
\theta_e &= 10^\circ \\
\theta_N &= 35^\circ \\
D_{\text{throat}} &= 24.5 \text{ cm} \\
A_e/A^* &= 77.5 \\
R_1 &= 4.681 \text{ cm}
\end{align*}
\]

• Pretty good model
SSME Nozzle example (Cont’d)

- \( M_{exit} = 4.677 \)

\[
V(M_{exit}) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left\{ \sqrt{\frac{\gamma - 1}{\gamma + 1}} \left( M_{exit}^2 - 1 \right) \right\} - \tan^{-1} \sqrt{M_{exit}^2 - 1}
\]

\[
= \frac{180}{\pi} \left( \left( \frac{1.196 + 1}{1.196 - 1} \right)^{0.5} \tan \left( \left( \frac{1.196 - 1}{1.196 + 1} \right) \left( 4.677^2 - 1 \right) \right)^{0.5} \right) - \tan \left( (4.677^2 - 1)^{0.5} \right)
\]

\[
= 102.34° \quad \Rightarrow \quad \theta_{w_{Max}} = \frac{V_{exit}}{2} = 51.17°
\]
SSME Nozzle example (cont’d)

\[ \theta_{\text{w, Max}} = \frac{v_{\text{exit}}}{2} = 51.17^\circ \]

- **SSME is definitely not a minimum length nozzle**

\[ 35/51.7 = 0.677 \]

“two thirds rule”
SSME Nozzle example (cont’d)

\[ \theta_{W_{\text{Max}}} = \frac{\nu_{\text{exit}}^2}{2} = 51.17^\circ \]

- ~ “minimum length SSME Nozzle

Rule of Thumb
Use \( \theta_N < \frac{2}{3} \theta_{\text{max}} \)

“two thirds rule”
Comparison of Cone and Bell Nozzles

For the same $\varepsilon$, we would expect $\lambda_{bell} > \lambda_{cone}$

A bell nozzle, while more complex to build, will generally yield a more efficient exhaust than a cone in a shorter nozzle length.

Same nozzle efficiency factor can be reached with about 70% of the length of a cone nozzle.

Alternatively, efficiency factor can be increased from about 98% for a cone to about 99.2% for a bell of the same length
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow

• For 3-D flow using method of characteristics, Prandtl Meyer flow introduces an additional term to account for the ability of the flow to expand into the third (circumferential) dimension

\[ d\theta = \mp \sqrt{M^2 - 1} \cdot \frac{dV}{V} \pm \frac{1}{\sqrt{M^2 - 1} \mp \frac{1}{\tan \theta}} \cdot \frac{dr}{r} \]

• Since the first term on right hand side is just the differential form of the Prandtl-Meyer function,

\[ \theta = \int \pm \sqrt{M^2 - 1} \frac{dV}{V} = \pm \Psi(M) + \text{Const} \rightarrow \]

\[ dv = \sqrt{M^2 - 1} \cdot \frac{dV}{V} \]
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow \(^{(2)}\)

- Thus the revised axi-compatibility equations .. \((x, y)_{3D} \rightarrow (x, r)_{axi}\) are

\[
C_+ \text{ characteristic line } \rightarrow \partial(\theta - \nu) = -\frac{1}{\sqrt{M^2 - 1 + 1 / \tan \theta}} \frac{\partial r}{r}
\]

\[
C_- \text{ characteristic line } \rightarrow \partial(\theta + \nu) = \frac{1}{\sqrt{M^2 - 1 - 1 / \tan \theta}} \frac{\partial r}{r}
\]
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow

Along the C⁺ line

\[ C_+ \text{ characteristic line} \rightarrow \partial(\theta - \nu) = -\frac{1}{\sqrt{M^2 - 1 + 1}} \frac{\partial r}{\tan \theta} \]

\[ \rightarrow \sqrt{M^2 - 1} = \frac{1}{\tan \mu} \rightarrow \frac{\partial}{\partial K_+} (\theta - \nu) = -\frac{1}{1 / \tan \mu + 1 / \tan \theta} \frac{1}{r} \frac{dr}{r} \]

\[ = -\frac{\partial r}{\cos \mu \cdot \cos \theta \cdot \left( \frac{\sin \theta + \sin \mu}{\cos \theta \cos \mu} \right)} \left( \frac{\sin \mu \cdot \sin \theta}{r} \right) = -\frac{dr}{(\sin \theta \cdot \cos \mu + \sin \mu \cdot \cos \theta)} \left( \frac{\sin \mu \cdot \sin \theta}{r} \right) \]

\[ \partial(\theta - \nu) = -\partial K_+ \left( \frac{\sin \mu \cdot \sin \theta}{r} \right) \rightarrow \frac{\partial}{\partial K_+} \frac{\partial(\theta - \nu)}{\partial K_+} = -\left( \frac{\sin \mu \cdot \sin \theta}{r} \right) \]

\[ \partial K_+ = \frac{\partial r}{\sin(\theta + \mu)} \rightarrow dr = \left[ \sin(\theta + \mu) \right] \cdot \partial K_+ \]

\[ \frac{\partial r}{\partial x} = \tan(\theta + \mu) \rightarrow \left[ \sin(\theta + \mu) \right] \cdot \partial K_+ = \tan(\theta + \mu) \cdot dx \rightarrow \partial K_+ = \frac{\partial x}{\cos(\theta + \mu)} \]
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (4)

- Along the C⁻ line

\[ \frac{1}{\sqrt{M^2 - 1 - 1 / \tan \theta}} \frac{\partial r}{r} \]

\[ \to \sqrt{M^2 - 1} = 1 / \tan \mu \rightarrow \frac{\partial}{\partial K_\mu} (\theta + \nu) = \frac{1}{1 / \tan \mu - 1 / \tan \theta} \frac{dr}{r} = \frac{\tan \mu \cdot \tan \theta}{\tan \theta - \tan \mu} \frac{dr}{r} \]

\[ \frac{\partial r}{\cos \mu \cdot \cos \theta} \left( \frac{\sin \theta - \sin \mu}{\cos \theta - \cos \mu} \right) \]

\[ \frac{\partial r}{\sin (\theta - \mu)} \left( \frac{\sin \mu \cdot \sin \theta}{r} \right) \rightarrow \partial K_\mu = \frac{\partial r}{\sin (\theta - \mu)} \]

\[ \partial (\theta + \nu) = \partial K_\mu \left( \frac{\sin \mu \cdot \sin \theta}{r} \right) \rightarrow \frac{\partial (\theta + \nu)}{\partial K_\mu} = \left( \frac{\sin \mu \cdot \sin \theta}{r} \right) \]

\[ \partial K_- = \frac{\partial r}{\sin (\theta - \mu)} \rightarrow dr = \left[ \sin (\theta - \mu) \right] \cdot \partial K_- \]

\[ \frac{\partial r}{\partial x} = \tan (\theta - \mu) \rightarrow \left[ \sin (\theta - \mu) \right] \cdot \partial K_- = \tan (\theta - \mu) \cdot dx \rightarrow \partial K_- = \frac{\partial x}{\cos (\theta - \mu)} \]
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (4)

• Collecting terms, the revised compatibility relations are

\[
\frac{\partial (\theta - v)}{\partial K_+} = -\left( \frac{\sin \mu \cdot \sin \theta}{r} \right)
\]

\[C_+ \text{ characteristic line} \rightarrow \quad \frac{\partial K_+}{\partial r} = \frac{\partial \theta}{\sin (\theta + \mu)} = \frac{\partial x}{\cos (\theta + \mu)}\]

\[
\frac{\partial (\theta + v)}{\partial K_-} = \left( \frac{\sin \mu \cdot \sin \theta}{r} \right)
\]

\[C_- \text{ characteristic line} \rightarrow \quad \frac{\partial K_-}{\partial r} = \frac{\partial \theta}{\sin (\theta - \mu)} = \frac{\partial x}{\cos (\theta - \mu)}\]

\[
\text{Slope}(C_+) = \theta + \mu
\]

\[
\text{Slope}(C_-) = \theta - \mu
\]
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow \(^{(4)}\)

- Unit Processes
- Internal Flow
- Wall
- Centerline
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow

1) Internal Flow Field Solution Process

Assume Straight Initial Characteristic Line →

\[
\begin{align*}
\text{Slope}(C_+)^{(0)} &= \theta_2 + \mu_2 \\
\text{Slope}(C_-)^{(0)} &= \theta_1 - \mu_1
\end{align*}
\]

→→ Solve for \( \{x_3, r_3\} \) →

\[
\begin{align*}
r_3 &= \frac{\tan\left[\text{Slope}(C_-)^{(j)}\right] \cdot \tan\left[\text{Slope}(C_+)^{(j)}\right] (x_1 - x_2) + \tan\left[\text{Slope}(C_-)^{(j)}\right] \cdot r_2 - \tan\left[\text{Slope}(C_+)^{(j)}\right] \cdot r_1}{\tan\left[\text{Slope}(C_-)^{(j)}\right] - \tan\left[\text{Slope}(C_+)^{(j)}\right]} \\
x_3 &= \frac{(r_2 - r_1) + x_1 \cdot \tan\left[\text{Slope}(C_-)^{(j)}\right] - x_2 \cdot \tan\left[\text{Slope}(C_+)^{(j)}\right]}{\tan\left[\text{Slope}(C_-)^{(j)}\right] - \tan\left[\text{Slope}(C_+)^{(j)}\right]} \cdot x_3
\end{align*}
\]
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (6)

• Internal Flow Field, cont’d

.....Calculate increments in $K_+, K_-$

\[
\Delta K_+ = \frac{x_3 - x_2}{\cos[\text{Slope}(C_+)^{(j)}]}
\]

\[
\Delta K_- = \frac{x_3 - x_1}{\cos[\text{Slope}(C_-)^{(j)}]}
\]

.....Advance Characteristic Line Invariants

$C_+$ characteristic line $\rightarrow \theta_3 - \nu_3 = \theta_2 - \nu_2 - \frac{\sin \mu_2 \cdot \sin \theta_2}{r_2} \Delta K_+$

$C_-$ characteristic line $\rightarrow \theta_3 + \nu_3 = \theta_1 + \nu_1 + \frac{\sin \mu_1 \cdot \sin \theta_1}{r_1} \Delta K_-$
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (7)

....Solve for \( \theta_3 \)

\[
\theta_3 = \frac{\theta_2 - v_2 - \frac{\sin \mu_2 \cdot \sin \theta_2}{r_2} \Delta K_+ + \theta_1 + v_1 + \frac{\sin \mu_1 \cdot \sin \theta_1}{r_1} \Delta K_-}{2}\]

\[
(\theta_2 + \theta_1) + (v_1 - v_2) + \frac{\sin \mu_1 \cdot \sin \theta_1}{r_1} \Delta K_- - \frac{\sin \mu_2 \cdot \sin \theta_2}{r_2} \Delta K_+ \]

....Solve for \( \nu_3 \)

\[
\nu_3 = \frac{\theta_1 + v_1 + \frac{\sin \mu_1 \cdot \sin \theta_1}{r_1} \Delta K_- - \left( \theta_2 - v_2 - \frac{\sin \mu_2 \cdot \sin \theta_2}{r_2} \Delta K_+ \right)}{2}\]

\[
(\theta_1 - \theta_2) + (v_1 + v_2) + \frac{\sin \mu_1 \cdot \sin \theta_1}{r_1} \Delta K_+ + \frac{\sin \mu_2 \cdot \sin \theta_2}{r_2} \Delta K_- \]

\[\rightarrow \rightarrow \rightarrow \text{....Solve for } (\theta_3, \nu_3) \rightarrow M_3, \mu_3\]

.....Recalculate

\[Slope(C_+)^{(j)} = \frac{\theta_2 + \mu_2 + \theta_3 + \mu_3}{2}\]

\[Slope(C_-)^{(j)} = \frac{\theta_1 - \mu_1 + \theta_3 - \mu_3}{2}\]

Iterate from \( \rightarrow \rightarrow \rightarrow \text{to} \rightarrow \rightarrow \rightarrow \)
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (8)

2) Wall Point Solution

\[
\frac{\partial (\theta - \nu)}{\partial K_+} = - \left( \frac{\sin \mu \cdot \sin \theta}{r} \right) \quad \text{and} \quad \frac{\partial K_+}{\partial r} = \frac{\partial r}{\partial x} = \frac{\partial x}{\cos(\theta + \mu)}
\]

\[
\frac{\partial (\theta - \nu)}{\partial x} = - \left( \frac{\sin \mu \cdot \sin \theta \cdot \cos(\theta + \mu)}{r} \right)
\]

\[
\frac{\partial r}{\partial x} = \frac{\sin \mu \cdot \sin \theta}{\sin(\theta + \mu)}
\]

\[
\theta_{\text{wall}} - \nu_{\text{wall}} = \theta_{\text{field}} - \nu_{\text{field}} \left( \frac{\sin \mu \cdot \sin \theta}{\sin(\theta + \mu)} \right) \left( \ln r_{\text{wall}} - \ln r_{\text{field}} \right) = \theta_{\text{field}} - \nu_{\text{field}} \left( \frac{\sin \mu \cdot \sin \theta}{\sin(\theta + \mu)} \right) \left( \ln \frac{r_{\text{wall}}}{r_{\text{field}}} \right)
\]

\[
\nu_{\text{wall}} = \theta_{\text{wall}} - \theta_{\text{field}} + \nu_{\text{field}} \left( \frac{\sin \mu \cdot \sin \theta}{\sin(\theta + \mu)} \right) \left( \ln \frac{r_{\text{field}}}{r_{\text{wall}}} \right)
\]
Appendix I: Modification of M.O.C. for 3-D Axisymmetric Flow (9)

3) Centerline Solution

→ Initial Point: \( \{x_1, r_1, \theta_1, M_1\} \rightarrow \begin{bmatrix} v_1 \\ \mu_1 \end{bmatrix} \)

Initial Slope → Assume Straight Initial Characteristic Line → \( \text{Slope}(C_-)^{(0)} = \theta_1 - \mu_1 \)

→ Centerline Intercept: \( \theta_{cl} = 0 \)

\[
y_{cl} = 0 \rightarrow \frac{0 - y_1}{x_{cl} - x_1} = \tan\left(\text{Slope}(C_-)\right) \rightarrow x_{cl} = -\frac{y_1}{\tan\left(\text{Slope}(C_-)\right)} + x_1
\]

\[
\Delta K_- = \frac{x_{cl} - x_1}{\cos\left[\text{Slope}(C_-)^{(j)}\right]}
\]

∈ right running \( (C_-) \) characteristic line → \( \theta_{cl} + v_{cl} = \theta_1 + v_1 + \frac{\sin \mu_1 \cdot \sin \theta_1}{r_1} \Delta K_- \)

→ \( v_{cl} = \theta_1 + v_1 + \frac{\sin \mu_1 \cdot \sin \theta_1}{r_1} \Delta K_- \rightarrow \begin{bmatrix} M_{cl} \\ \mu_{cl} \end{bmatrix} \)

.....Recalculate → \( \text{Slope}(C_-)^{(0)} = \frac{\theta_1 - \mu_1 - \mu_{cl}}{2} \)

Iterate from \( \rightarrow \) to \( \rightarrow \rightarrow \)

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