On the Flow of a Compressible Fluid through Orifices

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By making certain basic assumptions, the author has determined a theoretical expression for the contraction coefficient, C, appropriate to an orifice when transmitting a compressible fluid, either above or below the critical pressure ratio, provided that the corresponding value for incompressible flow, C_i, be known.

INTRODUCTION

When a fluid is discharged through a convergent nozzle, conditions across the exit section are generally assumed, with little error, to be sensibly uniform. The exit velocity may therefore be predicted from one-dimensional considerations, thus enabling the mass flow to be determined. This approach is not appropriate to the exit conditions across an orifice, since the streamlines are generally still contracting, so that the flow pattern is essentially two- or three-dimensional in character in this region. If, however, conditions may be assumed to be nearly uniform across some other section, such as at a *vena contracta* or at a throat, the principles of energy, momentum, and continuity may enable both the size of the jet and the conditions across it to be simply determined at this section.

Such principles, when applied to the discharge of an incompressible fluid through a Borda mouth-piece, for example, show that, if friction and gravity may be neglected, the jet has a contraction coefficient of 0·5. By the application of similar reasoning it will be shown that a theoretical expression for the contraction coefficient, C, may be determined, appropriate to an orifice when transmitting a compressible fluid, either above or below the critical pressure ratio, provided that the corresponding value for incompressible flow, C_i, be known.

Comparison with such experiments as those of Stanton (1926)†, Schiller (1933) and Perry (1949) indicates that the mass flow through a sharp-edged orifice may be predicted to within a few per cent at any pressure ratio. These tests were carried out on one type of orifice, transmitting either air or superheated steam, but the theory is more general in character, and further comparisons with tests are needed.

The present analysis considers cases in which the effect of friction is small, gravity and heat transfer may be neglected, and isentropic changes of state may be represented over the range considered by a law of the type:

$$p/\rho^n = \text{constant}$$

The above assumptions, which are adopted in most nozzle and orifice problems, are generally found in practice to give adequate accuracy, without making the analysis excessively tedious. The theory is divided into two parts; in the first, the pressure ratio across the orifice is assumed to be greater than the critical value, so that the flow is everywhere subsonic; in the second, choked flows are considered. The former case assumes constant conditions across the *vena contracta* which the jet may be expected ultimately to form, and the latter case makes a similar assumption concerning the first throat, where sonic conditions are postulated.

The contraction coefficient is defined as the ratio of the above minimum areas to the nominal projected area of the orifice (normal to the jet) and, since at the critical pressure ratio the *vena contracta* may be identified with the first throat, the two definitions are compatible for the borderline case dividing the two régimes. It is perhaps worth noting at this stage that although for supercritical conditions the flow is choked, in so far as conditions are sonic at the first throat, the mass flow does not become independent of the downstream pressure. This characteristic difference between the behaviour of the flow through a

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† An alphabetical list of references is given in Appendix II.

nozzle (which always flows full) as distinct from an orifice (through which contraction of the jet occurs), was noted and qualitatively explained by Stanton (1926), who indicated that the throat of the jet may be expected to increase in size as the back pressure is reduced.

By postulating, in addition, that compressibility effects may be neglected in the approach area to the orifice, quantitative expressions for these phenomena are deduced. Probably the most questionable of the above assumptions is that concerning conditions across the throat for supercritical flows, since the outermost layer must be at the back pressure. However, the curvature of the streamlines, which will be most marked towards the edge of the jet, implies a transverse pressure gradient in this region. This will give rise to a pressure distribution across the throat that is somewhat as indicated in Fig. 1.
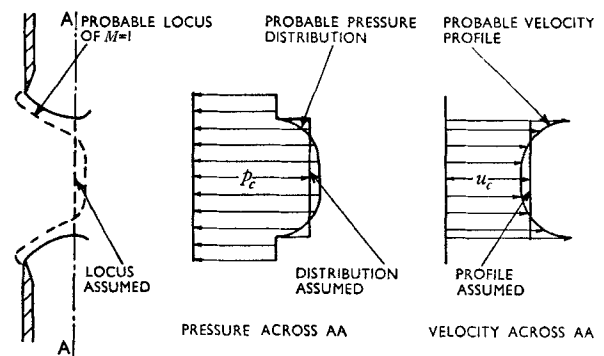

Fig. 1. Throat Conditions for Supercritical Flows

It is beyond the scope of a one-dimensional treatment to incorporate such refinements and, in the subsequent analysis, the throat will be assumed to be at the critical pressure. The error introduced by establishing the equation of motion on this basis will be largely offset by the corresponding assumption that the velocity is sonic across the whole section, whereas in practice it must be supersonic towards the edge of the jet, if the effect of viscosity is negligible. Since the latter factor must tend to retard the outer layers of the jet, some uncertainty, in any event, exists concerning conditions in this region.

Notation.

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A	Projected area of orifice.
a	Contracted area of jet.
C	Contraction coefficient of orifice.
C_i	Contraction coefficient of orifice for incompressible flow.
C_v	Viscosity correction factor, $\dot{m}_a/\dot{m}$.
F	Force defect on reservoir walls.
f	Force defect coefficient.
g	Gravitational acceleration.
H	Total hydraulic head in reservoir.
K	Theoretical mass-flow coefficient of orifice.
K_N	Theoretical mass-flow coefficient of nozzle.
$\dot{m}$	Theoretical mass flow $(= W/g)$.
$\dot{m}_a$	Actual mass flow.
n	Index of isentropic expansion $(= \gamma$ for a perfect gas$)$.

p_0 Pressure in reservoir.
p Back pressure.
r Pressure ratio, p/p_0.
r_c Critical pressure ratio.
u Velocity through contracted area.
u_c Critical velocity.
W Discharge, weight/time.
W_N Discharge through corresponding nozzle.
w_0 Specific weight in reservoir.
γ Ratio of specific heats, c_p/c_v.
ρ Density at contracted area.
ρ_0 Density in reservoir ($= w_0/g$).

SUBCRITICAL FLOW AND THE FORCE DEFECT COEFFICIENT

Suppose that a compressible fluid issues from a reservoir in which conditions are steady, through an orifice having a projected area, A, normal to the axis of the fluid jet, as indicated in Fig. 2a. By equating the resultant force acting on the fluid
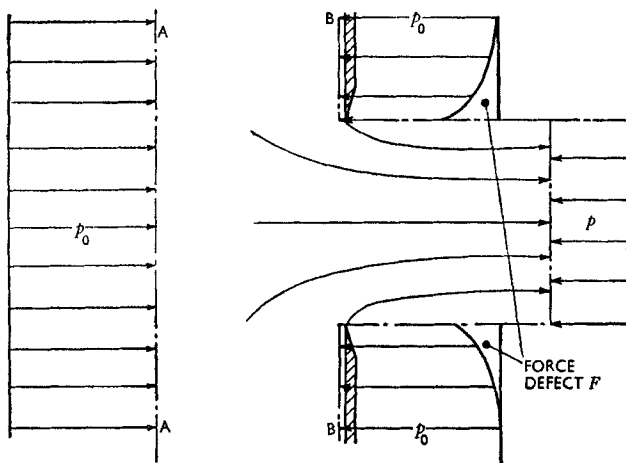
a For subcritical conditions.
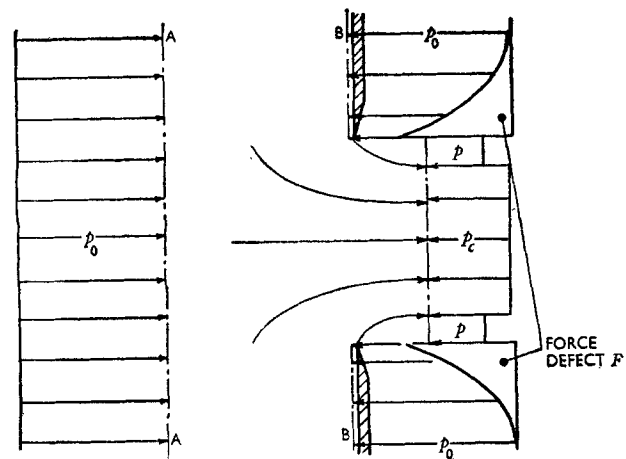
b For supercritical conditions.

Fig. 2. Forces on Control Volume

contained within the boundaries AA and BB to the corresponding flux of momentum through them, an equation of motion is obtained:

$$(p_0 - p)A + F = \dot{m}u \quad . \quad . \quad . \quad . \quad (1)$$

The first term represents the direct driving force across the orifice area, and the second term the integral of the resolved components of the defects of pressure along the walls of the reservoir surrounding the aperture. This latter force defect, F, is associated with the velocity of the fluid as it approaches the orifice, the increase of kinetic energy occurring at the expense

of its pressure energy. The tailing-off of pressure due to this sink effect adds to the driving force as indicated, enhancing the momentum flux through the boundaries.

For an orifice the fluid velocity will be comparatively small along the reservoir walls, and hence the effect of compressibility on the flow pattern actually within the reservoir may be neglected. This assumption will be increasingly valid as the orifice shape approaches that of a Borda mouth-piece from a nozzle-like form. If, however, the effect of compressibility on the force defect may be neglected, it should be possible to express it in terms of a Newtonian force coefficient which is a constant. That is, if the fluid may be assumed sensibly incompressible within the reservoir, the force defect, F, should be a unique function of the reservoir density ρ_0, the size of the orifice, as described by A for example, and the efflux, which is probably most conveniently expressed in terms of the mass flow, $\dot{m}$. This presupposes that the streamline pattern within the reservoir is not modified to any appreciable extent by changes in the boundary conditions downstream, associated with, for example, compressibility effects on the form of the emergent jet. If these assumptions are valid:

$$F = \phi(\dot{m}, A, \rho_0)$$

Dimensional considerations suggest that the relation between these variables must be of the form:

$$F = f\frac{\dot{m}^2}{\rho_0 A} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where f is a dimensionless coefficient having a value depending only on the form of the orifice. It will subsequently be referred to as the force defect coefficient. Since f may be expected to have the same value, whatever the Mach number of the flow, its value may be inferred from low-speed or incompressible-flow theory or tests. If the contraction coefficient of the orifice is denoted by C_i for incompressible flow, the corresponding volumetric flow rate:

$$Q = C_i A u \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where u is the velocity of the contracted jet which, neglecting friction, is given by:

$$u = \sqrt{(2gH)}$$

This equation, when re-expressed in terms of the notation of the paper, becomes:

$$u = \sqrt{\frac{2(p_0 - p)}{\rho_0}} \quad . \quad . \quad . \quad . \quad (4)$$

In practice, it is found that the resulting expression for the discharge requires factoring by a so-called coefficient of velocity to align theory with experiment, but at high Reynolds numbers the value of this coefficient is only slightly less than unity. Throughout the subsequent analysis of compressible flows the effect of friction will be neglected, it being recognized that the theoretical mass flow may require a correction for the effect of viscosity. This will be presumed to be of the same order as the velocity coefficient for incompressible flow.

Equation (3) may be re-expressed in terms of the mass flow as:

$$\dot{m} = \rho_0 C_i A u$$

and, when the theoretical expression for the velocity is substituted into this from equation (4), it becomes:

$$\dot{m} = \rho_0 C_i A \sqrt{\frac{2(p_0 - p)}{\rho_0}}$$

These equations enable the first and last terms in the equation of motion (1) to be expressed for incompressible flow as:

$$(p_0 - p)A = \frac{1}{2C_i^2} \cdot \frac{\dot{m}^2}{\rho_0 A}$$

$$\dot{m}u = \frac{1}{C_i} \cdot \frac{\dot{m}^2}{\rho_0 A}$$

On making these substitutions, together with that for the force defect given by equation (2), f is found to be uniquely related to C_i:

$$\frac{1}{2C_i^2} \cdot \frac{\dot{m}^2}{\rho_0 A} + f \cdot \frac{\dot{m}^2}{\rho_0 A} = \frac{1}{C_i} \cdot \frac{\dot{m}^2}{\rho_0 A}$$

that is,

$$f = \frac{1}{C_i} - \frac{1}{2C_i^2} \quad . \quad . \quad . \quad (5)$$

Since the incompressible-flow contraction coefficient lies between the values for a Borda mouth-piece, for which $C_i = 0.5$, and for a nozzle, for which $C_i = 1$, it follows that the force defect coefficient will lie between the limits $0 < f < 0.5$. That f is zero for a Borda mouth-piece might have been anticipated from physical considerations, since in this case the velocity along the walls of the reservoir tends to zero everywhere (except along the re-entrant portion, which gives rise to no additional force along the axis of the jet). The above expression for f provides the key to the remainder of the analysis since it has been argued that it may, with little error, be carried forward unchanged to an analysis of compressible flow through the orifice.

In order to use the equation of motion to determine the discharge for compressible flows in terms of f, which will now be presumed known, an expression is required for the velocity developed by the contracted jet. For the ideal case of a reversible adiabatic expansion, the Saint-Venant and Wanzel equation is appropriate, and this is, for subcritical flows:

$$u = \sqrt{\left\{ \frac{p_0}{\rho_0} \cdot \frac{2n}{n-1} \cdot \left(1 - r^{\frac{n-1}{n}} \right) \right\}} \quad . \quad . \quad . \quad (6)$$

where r denotes the pressure ratio, p/p_0, and n is the index which best represents isentropic expansions by a law of the type:

$$p/\rho^n = \text{constant}$$

For a near-perfect gas n may, of course, be identified with γ, the ratio of the specific heats. If the contraction coefficient be denoted by C, the mass flow is then:

$$\dot{m} = \rho C A u$$
$$= CA \sqrt{\left\{ p_0 \rho_0 \frac{2n}{n-1} \cdot r^{2/n} \cdot \left(1 - r^{\frac{n-1}{n}} \right) \right\}} \quad . \quad (7)$$

By substituting the value $C = 1$ the corresponding widely used expression for a nozzle is obtained. This suggests that the corresponding mass-flow coefficient for a nozzle, K_N, may be used as a convenient substitution:

$$K_N = \sqrt{\left\{ \frac{2n}{n-1} r^{2/n} \left(1 - r^{\frac{n-1}{n}} \right) \right\}} \quad . \quad . \quad . \quad (8)$$

Hence, for a nozzle:

$$\dot{m} = K_N A \sqrt{(p_0 \rho_0)}$$

and, for an orifice,

$$\dot{m} = C K_N A \sqrt{(p_0 \rho_0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The velocity through the contracted area of the jet is, in either case, given by equation (6) which on substituting for K_N reduces to:

$$u = \frac{K_N}{r^{1/n}} \sqrt{\frac{p_0}{\rho_0}} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence, for subcritical compressible flows, the last term in the equation of motion (1) for an orifice becomes:

$$\dot{m}u = \frac{C(K_N)^2 A p_0}{r^{1/n}}$$

The force defect may similarly be expressed in terms of K_N as:

$$F = f \cdot \dot{m}^2 / p_0 A$$
$$= f C^2 (K_N)^2 A p_0$$

so that the equation of motion gives, with those substitutions, a quadratic expression for C:

$$(1-r)A p_0 + f C^2 (K_N)^2 A p_0 = \frac{C(K_N)^2 A p_0}{r^{1/n}}$$

that is,

$$f C^2 - \frac{1}{r^{1/n}} C + \frac{1-r}{(K_N)^2} = 0$$

Therefore

$$C = \frac{1}{2 f r^{1/n}} \left[1 - \sqrt{\left\{ 1 - \frac{(2r^{1/n})^2 (1-r) f}{(K_N)^2} \right\}} \right] \quad . \quad . \quad (11)$$

Since it will be noted that K_N depends only on r and n, the contraction coefficient is thus determined for subcritical flows as a function of r, n, and f, the latter being found from the incompressible-flow contraction coefficient, C_i.

The theoretical mass flow may then be deduced by factoring the Saint-Venant and Wanzel equation for the mass flow by the contraction coefficient. It remains to develop an expression corresponding with equation (11), to cover those cases in which the local Mach number reaches unity in the jet. Such flows, which will be referred to as supercritical, are investigated in the next section.

SUPERCRITICAL FLOWS

It has been assumed above that the flow was both reversible and adiabatic during its expansion between the reservoir and the downstream pressures. This implied that friction and heat transfer could be neglected, and experiments with real fluids support this assumption for high-speed compressible flows. If, however, the downstream pressure is less than a certain critical pressure, p_c, so that the stream may become supersonic, the assumption of isentropic flow cannot be valid a priori, even though the flow be both frictionless and thermally insulated. This is due to the fact that shock waves may form as the fluid recompresses after over-expanding downstream of the first throat. Evidence of this alternative expansion and compression is provided by schlieren and shadowgraph photographs of fluid jets, and these wave phenomena might also be foreseen from theoretical considerations (for example, Stanton 1926).

It is therefore preferable to analyse supercritical flows in terms of a reference section across the first throat, rather than one across the section at which the fluid has expanded to the downstream pressure. This corresponds to the use of the throat as the control section in convergent–divergent nozzle problems, where conditions at exit are examined subsequently. The equations of motion will therefore be established in terms of the boundaries indicated in Fig. 2b. The pressure and velocity across the jet are considered to have values corresponding to sonic conditions. If these critical values are denoted by suffix c the equation of motion is:

$$p_0 A - p A(1-C) - p_c C A + F = \dot{m} u_c$$

The left-hand side of this equation consists of the direct driving force across the area of the orifice, together with the force defect, as before. In this case, however, although the force on boundary AA is unchanged, the back pressure across the area of the orifice, A, is now considered to consist of two parts, the critical pressure, p_c, being assumed to act across the throat area ($a = CA$) and the downstream pressure, p, to act over the area remaining.

For supercritical conditions the expressions for the throat velocity and mass flow are obtained by substituting the critical value of the pressure ratio into equations (6) and (7), as for a choked nozzle:

$$u_c = \sqrt{\left\{ \frac{p_0}{\rho_0} \cdot \frac{2n}{n-1} \cdot \left(1 - r_c^{\frac{n-1}{n}} \right) \right\}} = \sqrt{\left\{ \frac{p_0}{\rho_0} \cdot \frac{2n}{n+1} \right\}}$$

$$\dot{m} = CA \sqrt{\left\{ p_0 \rho_0 \cdot \frac{2n}{n-1} \cdot r_c^{2/n} \cdot \left(1 - r_c^{\frac{n-1}{n}} \right) \right\}}$$

$$= CA \sqrt{\left\{ p_0 \rho_0 n \left(\frac{2}{n+1} \right)^{\frac{n+1}{n-1}} \right\}}$$

since $r_c = \left(\frac{2}{n+1} \right)^{\frac{n}{n-1}}$

If the substitution K_N is introduced as before to denote the mass-flow coefficient of the corresponding nozzle which, being choked, has the value:

$$K_N = \sqrt{\left\{ \frac{2n}{n-1} \cdot r_c^{2/n} \left(1 - r_c^{\frac{n-1}{n}} \right) \right\}} = \sqrt{\left\{ n \left(\frac{2}{n+1} \right)^{\frac{n+1}{n-1}} \right\}} \quad (12)$$

the expressions for u_c and $\dot{m}$ become:

$$u_c = \frac{K_N}{r_c^{1/n}} \sqrt{\frac{p_0}{\rho_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\dot{m} = C K_N A \sqrt{(p_0 \rho_0)} \quad . \quad . \quad . \quad . \quad (14)$$

These equations are analogous to equations (9) and (10) although in this case r is replaced by the constant value r_c so that K_N is now independent of the pressure ratio r. The flux of momentum and the force defect may now be expressed as:

$$\dot{m} u_c = \frac{C(K_N)^2 A p_0}{r_c^{1/n}}$$

$$F = f C^2 (K_N)^2 A p_0$$

so that the equation of motion for choked flows is:

$$(1-r)Ap_0+(r-r_c)CAp_0+fC^2(K_N)^2Ap_0 = \frac{C(K_N)^2Ap_0}{r_c^{1/n}}$$

This again yields a quadratic expression for C:

$$fC^2-\frac{1}{r_c^{1/n}}\Big[1+\frac{(r_c-r)r_c^{1/n}}{(K_N)^2}\Big]C+\frac{1-r}{(K_N)^2} = 0$$

Therefore

$$C = \frac{1}{2fr_c^{1/n}}\Big[\Big\{1+\frac{(r_c-r)r_c^{1/n}}{(K_N)^2}\Big\}$$
$$-\sqrt{\Big\{\Big(1+\frac{(r_c-r)r_c^{1/n}}{(K_N)^2}\Big)-\Big(\frac{(2r_c^{1/n})^2(1-r)f}{(K_N)^2}\Big)\Big\}}\Big] \quad . \quad . \quad . \quad (15)$$

This expression, which determines the contraction coefficient of a choked orifice, is seen to reduce to equation (11) when the overall pressure ratio, r, is equal to the critical value, r_c. The two equations together determine the contraction coefficient throughout the whole range of pressure ratios. The basic assumption is that the effect of compressibility on the force defect along the reservoir walls may be neglected; this may be expected to be valid so long as the orifice does not approach a nozzle form. For a Borda mouth-piece the force defect is zero and in this case the theory is, in this sense, exact. For such a mouth-piece it is convenient to return to the original quadratic expressions for C, since these reduce to linear equations in the limiting case when f is zero. For a Borda mouth-piece the expressions for its contraction coefficient, C_B, thus reduce to:

$$C_B = \frac{r^{1/n}(1-r)}{(K_N)^2} \quad . \quad . \quad . \quad . \quad (16)$$

for subcritical flows, that is, when $r_c < r < 1$, and

$$C_B = \frac{r_c^{1/n}(1-r)}{r_c^{1/n}(r_c-r)+(K_N)^2} \quad . \quad . \quad . \quad (17)$$

for choked flows, that is, when $0 < r < r_c$.

By substituting the choked value of K_N from equation (12) in equation (17), the latter may be expressed more simply, after some algebraic manipulation, as:

$$C_B = \frac{1-r}{r_c(1+n)-r}$$

This theoretical expression presupposes that the mouth-piece is short enough for the emergent jet to clear the outer lip of the orifice, although, of course, not so short that there is an appreciable approach velocity along the reservoir wall.

SUMMARY AND THEORETICAL CURVES

The purpose of the paper is to present a method of finding the discharge through any orifice, when transmitting a compressible fluid, either above or below the critical pressure ratio. It is presumed that the incompressible-flow contraction coefficient, C_i, is known either from theory or experiment; for example, $C_i = 0.5$ for a Borda mouth-piece and $C_i = \pi/(\pi+2) = 0.611$ for a plane slit. From C_i the strength of the sink effect along the reservoir walls may be inferred. Its intensity determines the magnitude of a force defect coefficient, f, such that:

$$f = \frac{1}{C_i}-\frac{1}{2C_i^2}$$

On the assumption that the flow pattern within the reservoir is very little influenced by the effect of compressibility, f enables theoretical expressions for the contraction coefficient, C, to be deduced for compressible flows, both above and below the critical pressure ratio. This contraction coefficient enables the mass flow to be determined from the expression:

$$\dot{m} = CK_NA\sqrt{(p_0\rho_0)}$$

For a nozzle $C = 1$, and hence K_N is the widely used mass-flow coefficient for a nozzle, based on the Saint-Venant and Wanzel equation. This mass-flow coefficient is shown dotted in Fig. 3 for the typical case of $n = 1.4$. The contraction coefficient at any particular pressure ratio may be found by first determining f and K_N and substituting their values into either equation (11) or equation (15) according to whether r is greater or less than the critical value r_c. Some representative curves have been

obtained in this way and plotted in Fig. 4. They indicate that, as the back pressure is reduced, the jet expands progressively, thus causing a continuous increase in the flow rate. This process continues into the supercritical flow range (although at a decreasing rate) so that even though the flow is choked, in the sense that conditions at the throat remain constant, the mass flow continues to increase. This fact is emphasized by defining a theoretical mass-flow coefficient for an orifice K such that:

$$K = CK_N = \frac{\dot{m}}{A\sqrt{(p_0\rho_0)}}$$

Curves showing the theoretical variation of K with pressure ratio have been plotted in Fig. 3, which includes the corresponding curve for a nozzle (K_N). Comparison of the orifice
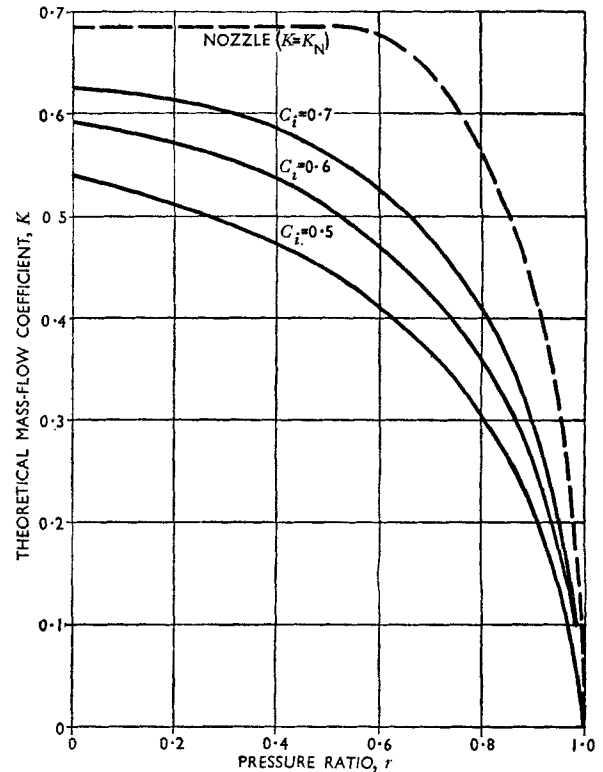

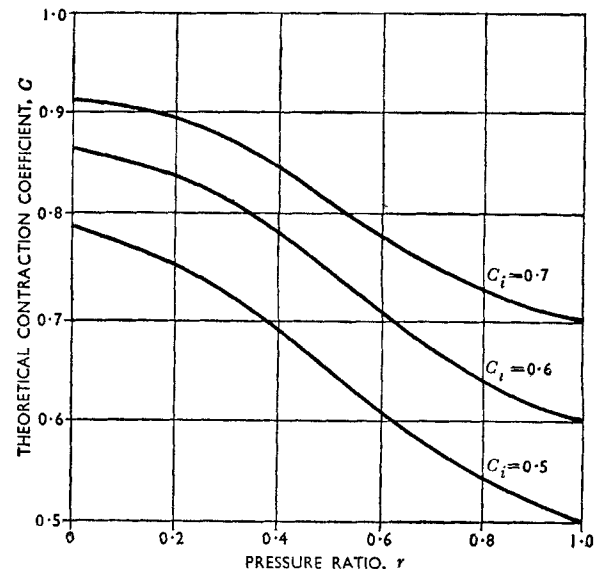Fig. 3. Theoretical Mass-flow Coefficient Plotted as a Function of Pressure Ratio for $n = 1.4$

Fig. 4. Theoretical Contraction Coefficient Plotted as a Function of Pressure Ratio for $n = 1.4$

and nozzle curves indicates the characteristic difference between their behaviour, the choking of the former being completely masked by the variation of C with r. A numerical example is given in Appendix I.

COMPARISON WITH THE HODOGRAPH SOLUTION FOR A PLANE SLIT

Theoretical results for incompressible flow through orifices can, under certain conditions, be extended to compressible flows, using a hodograph method (Howarth 1953). The results shown in Table 1 for the contraction coefficient of a slit in a plane wall

TABLE 1. CONTRACTION COEFFICIENT OF SLIT IN PLANE WALL

r	C
1·000	0·611
0·932	0·623
0·867	0·636
0·805	0·650
0·747	0·665
0·692	0·681
0·639	0·699
0·590	0·717
0·543	0·738
0·528	0·745

have been computed, using tables due to Ferguson and Lighthill (1947).

The results in Table 1 are appropriate to $\gamma = 1·4$ and the method applies to subsonic flows only ($r > 0·528$).

The corresponding values obtained by the method outlined in the paper are as given in Table 2, at roughly comparable

TABLE 2. CORRESPONDING VALUES FOR CONTRACTION COEFFICIENT OBTAINED BY METHOD OUTLINED IN THE PAPER

r	C	Error
1·00	0·611	—
0·932	0·622	−0·001
0·865	0·635	−0·001
0·805	0·648	−0·002
0·745	0·663	−0·002
0·690	0·679	−0·002
0·640	0·696	−0·003
0·590	0·714	−0·003
0·545	0·734	−0·004
0·528	0·741	−0·004

values of r, together with the 'error' treating the hodograph solution as 'exact'.

It can be seen that the present method slightly underestimates the mass flow, the discrepancy increasing progressively from zero to about 0·5 per cent at the critical pressure ratio.

COMPARISON WITH EXPERIMENT, AND CONCLUSIONS

Probably the most exhaustively tested orifice is that of the sharp-edged type. The results of Perry (1949) and Schiller (1933) using such an orifice to transmit air and steam, respectively, have been plotted in Fig. 5a and b. Since the incompressible-flow discharge coefficient is considered to be 0·6 in each instance, this corresponds to a velocity coefficient, as used in hydraulics, of 0·982, if the contraction coefficient be given the datum value of 0·611. It would appear logical to retain the correction factor C_v for the theoretical curves throughout the range of pressure ratios. The actual mass flow for a real fluid, $\dot{m}_a$, is then predicted by:

$$\frac{\dot{m}_a}{A\sqrt{(p_0\rho_0)}} = C_v C K_N = C_v K$$

It is perhaps preferable to consider C_v in the general case as

an overall correction for viscosity effects, rather than simply as a velocity coefficient. In Fig. 5a and b the theoretical curves have been drawn, both with and without the factor $C_v = 0·982$. It is seen that the experimental results in each instance lie between the two curves, making it problematic as to whether such a correction is worth while. In either instance the agreement appears satisfactory for most purposes for sharp-edged orifices transmitting either air or steam.
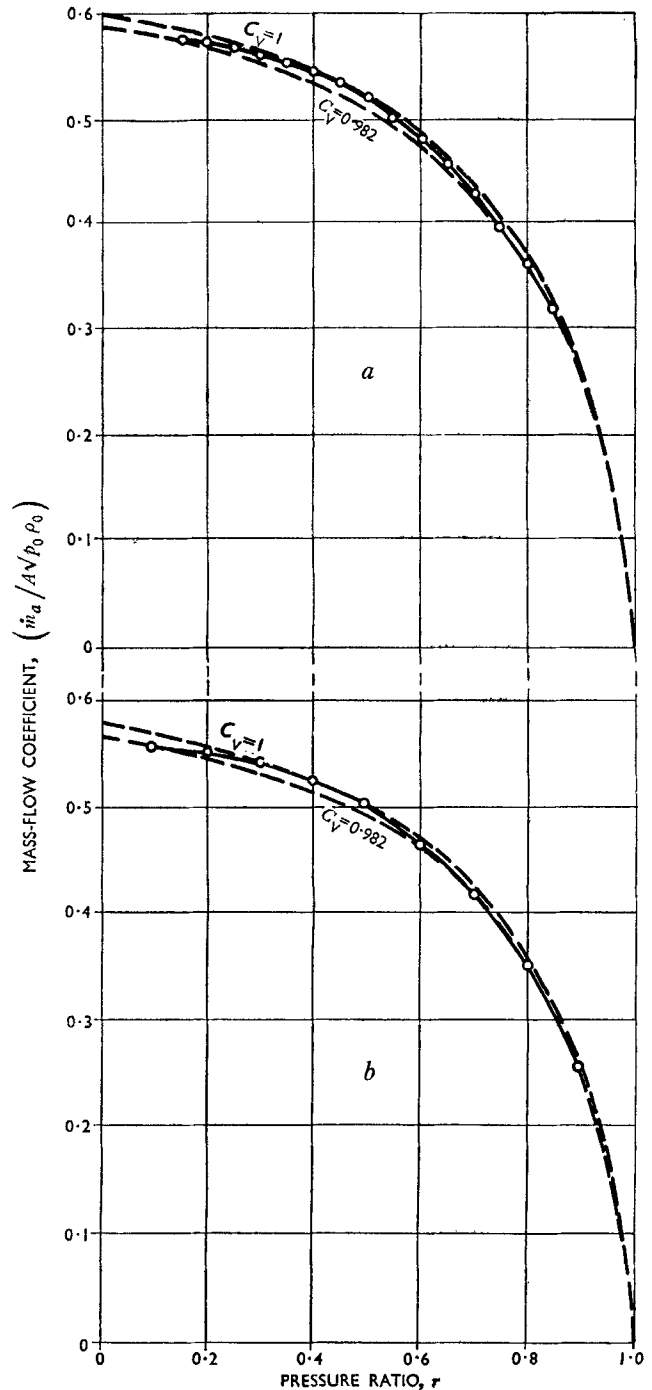

Fig. 5. Mass-flow Coefficients Plotted as a Function of Pressure Ratio

a For $n = 1·4$.
——○—— Perry's tests, using air.
— — — — Theory ($n = 1·4$, $C_i = \pi/(\pi+2)$).

b For $n = 1·3$.
——○—— Schiller's tests, using steam.
— — — — Theory ($n = 1·3$, $C_i = \pi/(\pi+2)$).

Further comparisons of the theory with experiment are required to establish the full range over which the assumptions of the theory are valid. Since the neglect of compressibility within the reservoir cannot be justified when the orifice tends to a nozzle-like form, an upper limit of $C_i = 0.7$ for application of the theory is tentatively suggested.

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APPENDIX I*

NUMERICAL EXAMPLE

Many engineers prefer to work in terms of the weight flow, W, and specific weight, w, rather than the mass flow, $\dot{m}$, and density, ρ, to which they are related by:

$$W = \dot{m}g$$
$$w = \rho g$$

From equation (9) the discharge in terms of W becomes:

$$W = CK_N Ag\sqrt{(p_0\rho_0)}$$

that is,

$$W = CK_N A\sqrt{(gp_0 w_0)} \quad . \quad . \quad . \quad . \quad (18)$$

For a nozzle $C = 1$, so that the corresponding nozzle discharge is:

$$W_N = K_N A\sqrt{(gp_0 w_0)} \quad . \quad . \quad . \quad . \quad (19)$$

A number of methods, using charts, tables, or working rules, have been developed by engineers for calculating W_N. As a typical example may be quoted Napier's equation for a choked nozzle discharging steam:

$$W_N = Ap_0/70 \quad . \quad . \quad . \quad . \quad (20)$$

in which W_N is the discharge in Lb. per sec; A is the throat area in in^2; and p_0 is the reservoir pressure in Lb/in^2.

The discharge through an orifice is related to the corresponding nozzle discharge by:

$$W = CW_N \quad . \quad . \quad . \quad . \quad . \quad (21)$$

C is the contraction coefficient, which may be obtained from Fig. 4 for a specified pressure ratio, r, and incompressible-flow contraction coefficient, C_i, as used in hydraulics. Fig. 4 refers strictly to an expansion index $n = 1.4$, but the value of C does not appear to be critically dependent on n. The corresponding algebraic expressions for C are equations (11) and (15) for subcritical and choked flows, respectively. In the former expression K_N is given by equation (8) and in the latter by equation (12). If the equivalent nozzle discharge has been determined without reference to these equations, K_N may alternatively be deduced from W_N by use of equation (19).

As a typical numerical example the leakage rate through an annular slit which is 6 inches in diameter and 0.01 inch wide will be estimated. It has an area:

$$A = \pi \times 6 \text{ in} \times 0.01 \text{ in} = 0.1885 \text{ in}^2$$

and, if it were one stage of a labryrinth seal, for example, it would probably have a hydraulic (that is, incompressible-flow) contraction coefficient, $C_i = 0.6$. Thus the corresponding force defect coefficient is, by equation (5):

$$f = \frac{1}{C_i} - \frac{1}{2C_i^2}$$

$$= \frac{1}{0.6} - \frac{1}{2 \times 0.6^2} = 0.278$$

* The nomenclature used in this appendix was adopted at the special request of the author.

Neglecting any velocity of approach or 'carry over' effects, the nozzle mass-flow coefficient is, for subcritical flows:

$$K_N = \sqrt{\left\{\frac{2n}{n-1}r^{2/n}\left(1 - r^{\frac{n-1}{n}}\right)\right\}}$$

For superheated steam at the critical pressure ratio ($r = r_c = 0.546$, if $n = 1.3$):

$$K_N = 0.667$$

For values of r less than 0.546, K_N remains constant at this value.

Suppose that the discharge corresponding to a pressure ratio of 0.546 is required. From equation (8) (or equation (12)):

$$C = \frac{1}{2fr^{1/n}}\left[1 - \sqrt{\left\{1 - \frac{(2r^{1/n})^2(1-r)f}{(K_N)^2}\right\}}\right]$$

$$= \frac{1}{2 \times 0.278 \times 0.546^{\frac{1}{1.3}}} \times$$

$$\left[1 - \sqrt{\left\{1 - \frac{\left(2 \times 0.546^{\frac{1}{1.3}}\right)^2 \times 0.454 \times 0.278}{0.667^2}\right\}}\right]$$

$$= 2.865[1 - 0.744]$$

$$= 0.732$$

The equivalent nozzle discharge may be obtained from:

$$W_N = K_N A\sqrt{(gp_0 w_0)}$$

or by any other preferred method. If the steam is initially at, for example, 200 Lb/in^2 and has 100 °F of superheat, its total heat, or enthalpy, is 1,258 B.t.u./Lb. Its corresponding specific volume is thus, according to Callendar (1939):

$$\frac{1}{w_0} = 1.253 \times \frac{1,258 - 835}{200} \text{ ft}^3/\text{Lb}$$

that is, $w_0 = 0.377 \text{ Lb/ft}^3$

Therefore,

$$W_N = 0.667 \times 0.1885 \text{ in}^2 \sqrt{\{32.2 \text{ ft/sec}^2}$$
$$\times 200 \text{ Lb/in}^2 \times 0.377 \text{ Lb/ft}^3\}[1 \text{ ft}/12 \text{ in}]$$
$$= 0.516 \text{ Lb/sec}$$

Hence, the discharge through the orifice is, finally:

$$W = CW_N$$
$$= 0.732 \times 0.516 \text{ Lb/sec}$$
$$= 0.378 \text{ Lb/sec}$$

This flow rate corresponds with a reservoir pressure of 200 Lb/in^2 and a back pressure of 0.546×200 Lb/in^2 that is, 109 Lb/in^2. Any reduction in the latter does not affect W_N but it does increase C, thus increasing the discharge, W, even though the orifice is choked.

APPENDIX II

REFERENCES

CALLENDAR, H. L.   1939 'Abridged Callendar Steam Tables, Fahrenheit Units' (Edward Arnold and Co.).

FERGUSON, D. F., and LIGHTHILL, M. J.   1947 Proc. Roy. Soc., vol. A192, p. 135, 'The Hodograph Transformation in Trans-sonic Flow'.

HOWARTH, E. L.   1953 'Modern Developments in Fluid Dynamics: High Speed Flow', vol. 1, p. 222 (Oxford University Press).

PERRY, J. A., jun.   1949 Trans. A.S.M.E., vol. 71, p. 757, 'Critical Flow Through Sharp-edged Orifices'.

SCHILLER, W.   1933 Forschung auf dem Gebiete des Ingenieurwesens, vol. 4, p. 128, 'Uberkritische Entspannung kompressibler Flüssigkeiten'.

STANTON, T. E.   1926 Proc. Roy. Soc., vol. A111, p. 306, 'The Flow of Gases at High Speeds'.

Communications

Mr. C. H. BOSANQUET (Billingham) wrote that the author's method of calculating discharge coefficients depended entirely on the assumption of the constancy of f, using his formula

$$\frac{dC}{df} = \frac{C^3}{1-C}$$

Values of C approaching unity must therefore be very sensitive to small variations of f.

For a perfect nozzle $C = 1$ by definition but if $f = 0.5$ and $n = 1.4$ then C, which was initially unity, fell rapidly with decreasing r and passed through a minimum value of 0.852 when $r = 0.7$. It then rose to 0.888 at the critical ratio and 0.959 for discharge into a vacuum.

If C was assumed constant then $f = 0.5$ for small pressure differences and rose to 0.571 at and beyond the critical ratio.

For intermediate conditions it varied nearly linearly with ρu^2.

Critical data for other values of n were given in Table 3.

TABLE 3. CRITICAL DATA FOR VARIOUS VALUES OF n

n	r_c	f_c/f_0
1·0	0·6065	1·1584
1·1	0·5847	1·1541
1·2	0·5645	1·1500
1·3	0·5457	1·1462
1·4	0·5283	1·1427
1·667	0·4871	1·1341

A suggested modification of the method was to multiply f_0 by a factor which depended on the mean mass flow in the plane of the orifice. The modified value of f was given by

$$\frac{f}{f_0} = 1 + \left(\frac{f_c}{f_0} - 1\right)\frac{\rho_c}{\rho}\left(\frac{\rho u}{\rho_c u_c}\right)^2$$

Densities and velocities were assumed the same as for a nozzle with the same mass flow. The value of ρu in the orifice plane depended on C so that it was necessary to solve by trial. The calculation of f was facilitated by employing the useful and adequately accurate approximation

$$\left(\frac{\rho u}{\rho_c u_c}\right)^2 + \left(\frac{r - r_c}{1 - r_c}\right)^2 = 1$$

At and beyond the critical ratio the first term was simply equal to C^2.

Table 1 gave values which fell almost exactly half-way between those calculated by the two methods so that the comparison was inconclusive. For small discharge coefficients and subsonic flows the modification did not make much difference. For values of C approaching unity the modified method gave results which were at least possible; constant f did not. The modified method therefore appeared preferable in spite of its greater complexity.

Mr. R. P. FRASER and Dr. P. N. ROWE (London) wrote that as a result of experimental work carried out in the High Speed Fluid Kinetics Laboratory of the Imperial College of Science and Technology (Fraser and Rowe 1954*, Coulter, Fraser, and Rowe (to be published)†, and Coulter 1949‡) it was possible to contribute some data to compare with the theoretical predictions of the paper in the supersonic range.

* FRASER, R. P., and ROWE, P. N. 1954 Jl. of the Imperial College Chem. Eng. Soc., vol. 8, p. 1, 'A Method of Measuring Very Large Gas Flow Rates'.
† COULTER, M. O., FRASER, R. P., and ROWE, P. N. (to be published), 'A Research into the Design of Supersonic Nozzles for Rockets'.
‡ COULTER, M. O. 1949 Ph.D. Thesis, London University, June, 'On the Efficiency of Supersonic Nozzles'.

Since a velocity coefficient of 1·00 was assumed, the contraction coefficient referred to in the paper was equivalent to a discharge coefficient. They had measured the discharge coefficient of various nozzles discharging air at pressure ratios from r equal to 0·012 to 0·033. All the nozzles had a throat of a nominal diameter of $\frac{1}{2}$ inch. Comparative discharge coefficients for various nozzles had been obtained by timing the rate of fall of reservoir pressure under similar conditions.

Fig. 6 showed the effect of the entry radius on the discharge coefficient of a supersonic convergent-divergent nozzle. Since the stream was everywhere supersonic downstream of the throat, it could not influence conditions upstream. In other words, in any supersonic nozzle the upstream flow was unaffected by the divergent section and, in particular, the discharge coefficient was unaltered by the supersonic expansion section. That was indeed
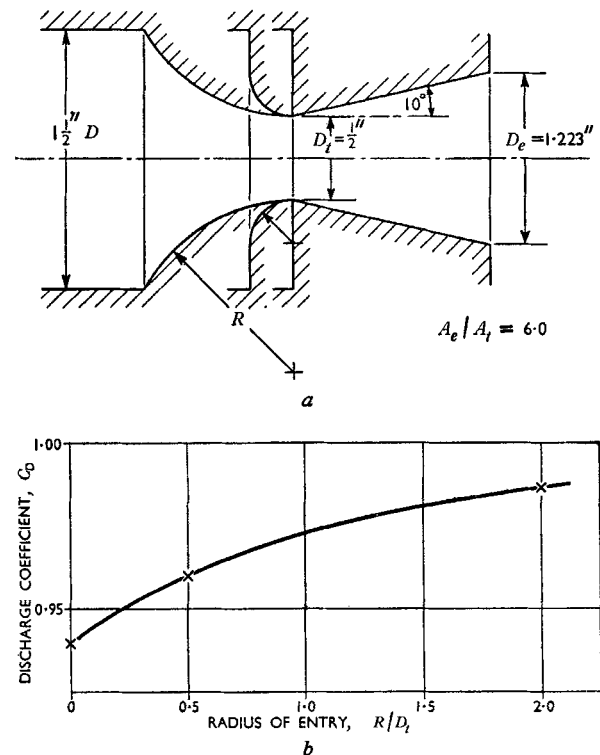
$A_e/A_t = 6.0$

a

b

Fig. 6. Effect of Entry Radius on Discharge Coefficient of Supersonic Convergent-divergent Nozzle

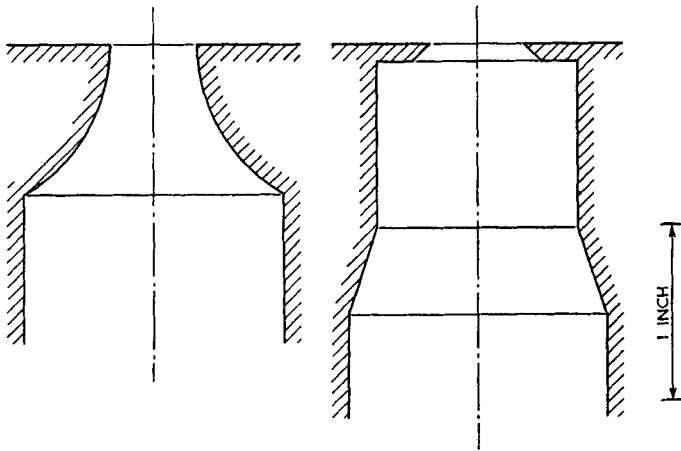| R/D	0	0·5	2·0
C_D	0·939	0·960	0·986

the case in practice for experiments had shown that the discharge coefficient was independent of the divergent section within the limits of experimental error (± 0.01). Thus, the nozzle with $R/D_t = 0$ was, in the supersonic (i.e., the author's supercritical) region, equivalent to an orifice where $C_i = 0.6$. One with $R/D_t = 2.0$ corresponded to a nozzle where $C_i = 1.0$. The author had predicted a difference in discharge coefficient between those two nozzles of about 0·14 (Fig. 4) whereas they had found 0·047. However, an increased contraction coefficient for incompressible flow (which they had not measured) would bring those results more into line.

In addition to measuring the discharge coefficient, they had measured the thrust reaction generated by the jets from various nozzles. Since the thrust was proportional to the product of discharge rate and the acceleration occurring in the nozzle, the thrust coefficient was equal to the product of discharge coefficient and velocity coefficient. For the case of a convergent-

a Radiused nozzle.   b Orifice with modified approach.

Fig. 7. Two Nozzles

divergent nozzle cut off at the throat (i.e., without a supersonic expansion section), the velocity coefficient corresponded to that of the author which he had assumed to be unity.

Thrust measurements made on the two nozzles of Fig. 7 showed that the thrust coefficient for that with the radiused approach ($R/D_t = 2 \cdot 0$) was $0 \cdot 975$ and that for the orifice with a

modified approach was $0 \cdot 908$ ($\pm 0 \cdot 007$). (Unfortunately they had no data for the conventional orifice treated in the paper.) The discharge coefficient for the radiused nozzle was known from the data of Fig. 6 ($0 \cdot 986$) and so its velocity coefficient was $0 \cdot 975/0 \cdot 986 = 0 \cdot 989$ which justified the author's assumption, in that instance, that the velocity coefficient was unity. If for the orifice with the modified approach the discharge coefficient for a nozzle of $R/D_t = 0$ was used ($0 \cdot 939$), the velocity coefficient for that was $0 \cdot 908/0 \cdot 0939 = 0 \cdot 967$ so that in that instance the assumption of a velocity coefficient of unity was much less accurate.

The author appeared to have assumed that there was a *vena contracta* in compressible flow at small pressure ratios (i.e. high reservoir pressures). Expansion at the exit plane according to the Prandtl–Meyer theory would produce immediate divergence as

a

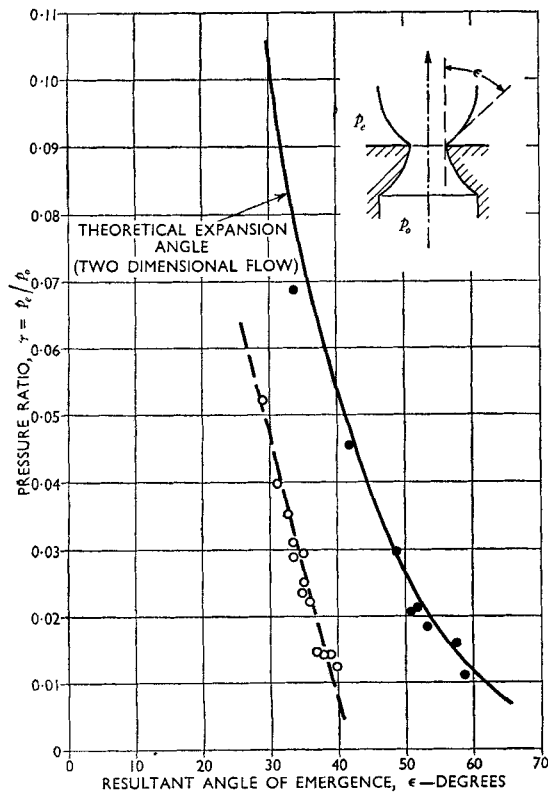

b

Fig. 9. Shadow Photographs
$r = 0 \cdot 024$.

Fig. 8. Pressure Ratio Plotted Against Resultant Angle of Emergence

● Experimental points for radiused nozzle.
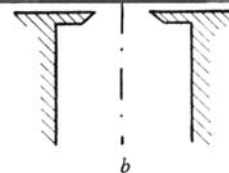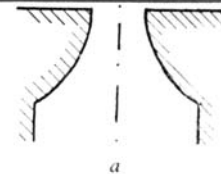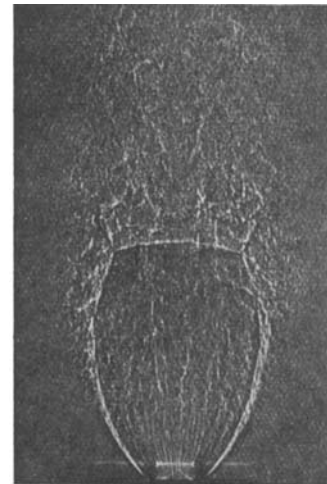○ Experimental points for orifice with modified approach.

shown in Fig. 8, in which the full line showed the theoretical expansion angle that would result from flow through a nozzle subjected to a pressure ratio, r. The treatment was for two-dimensional flow, $\gamma = 1\cdot40$, and it was assumed that the flow approached the throat perpendicularly. Those angles were achieved in practice for the three-dimensional flow of air through a radiused nozzle at different pressure ratios as was seen from the shadow photograph in Fig. 9a. The emergence angle had been measured from photographs of the jet at different pressure ratios and, for a radiused nozzle, they agreed closely with the theoretical angle as would be seen in Fig. 8.

Unfortunately, in the standard orifice the throat plane was not normally visible, which prevented photographic determination of the emergence angle. For the orifice with a modified approach, it would be seen that the emergence angle had been reduced (Figs. 8 and 9b) because the fluid now had radial velocity directed towards its flow axis which produced a contracting

a For radiused nozzle with
$r = 0\cdot23$.

b For orifice with modified
approach and $r = 0\cdot23$.

c For radiused nozzle with
$r = 0\cdot13$.

d For orifice with modified
approach and $r = 0\cdot13$.

Fig. 10. Emergence of a Substantially Parallel Jet

effect. That was precisely the effect the author had been considering but the result was a net enlargement, not a contraction.

The emergence angles for their orifice indicated a reduction of the Prandtl–Meyer angle. Thus, a parallel jet would emerge at some pressure ratio and at higher ratios the jet might contract outside the nozzle and the sonic plane would move downstream. Fig. 10 showed that a substantially parallel jet emerged at a pressure ratio of about $0\cdot13$ and that above that pressure ratio contraction occurred. It was not possible to measure those emergence angles very accurately but, undoubtedly, the order of the effect was as they had suggested.

They were of the opinion that the author's theoretical treatment must be modified for supersonic flow, particularly at high reservoir pressures, to take into account the expansion effect. They hoped in the near future to publish a comprehensive account of their experimental work in that field (Coulter, Fraser, and Rowe (to be published)).

Mr. D. H. TANTAM (*Associate Member*) wrote that the mass flow of a compressible fluid through an orifice had been given in equation (7) as

$$m = CA\sqrt{\left/ \left\{ p_0\rho_0\frac{2n}{n-1} \cdot r^{2/n} \cdot \left(1 - r^{\frac{n-1}{n}}\right)\right\}\right.}$$

In that an infinitely large reservoir to orifice area ratio was considered.

Many applications were concerned with an orifice in a pipeline when the ratio was much smaller and an equation which took that into account was given by

$$Q = \Omega A_2 \left[\frac{2gp_1 W_1 \frac{\gamma}{\gamma-1}\left\{1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right\}\left\{\frac{p_2}{p_1}\right\}^{\frac{2}{\gamma}}}{1 - \frac{1}{n^2}\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}} \right]^{\frac{1}{2}}$$

where Q was the mass flow; Ω, the coefficient of discharge; A_2, the area of orifice; n, the ratio of area of pipeline to area of orifice; p_1, the upstream pressure; p_2, the downstream pressure; W_1, the specific weight of fluid; γ, the ratio of specific heats; and g, the acceleration due to gravity.

Equation (7) was obtained from it of course when $n = \infty$.

However, in the practical application, the mass flow would depend on the contraction coefficient C, theoretical values of which had been given in Fig. 4. He would like to know, however, whether any recent information had been established on those coefficients in addition to the work of Perry (1949) and Schiller (1933) and how those results agreed with theoretical values.
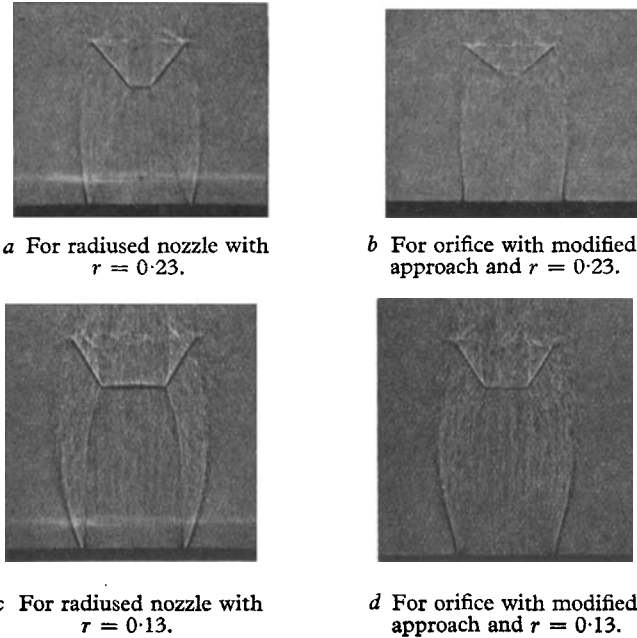
Author's Reply

Mr. D. A. JOBSON wrote, in reply to the communications, that he agreed that the force defect coefficient f could not, in general, be considered to be a constant. It was, in some respects, similar to a drag coefficient; just as the latter would depend on the Mach number, so f depended on the mass-flow coefficient and, hence, the pressure ratio r. It was only for orifice-like openings that the influence of compressibility on f might be expected to be small. The Borda mouth-piece represented one extreme for which f was always zero, whatever the value of r. At the other end of the scale, as Mr. Bosanquet had pointed out, f for a nozzle varied almost linearly with r from f_0 equal to $0 \cdot 5$ to the typical value of f equal to $0 \cdot 571$ for n equal to $1 \cdot 4$, thereafter it remained constant for choked flows. Hence the ratio f/f_0 might always be expected to lie between the limits of $1 \cdot 0$ and $1 \cdot 14$ for that value of n, and it could therefore always be estimated with reasonable certainty by some such method as he had suggested. That would enable the theory to be applied beyond the limit C_i equal to $0 \cdot 7$ which he himself had suggested, but he doubted whether the added complication was justifiable below that value.

He noted that Mr. Fraser and Dr. Rowe had measured discharges on an orifice fitted with an exit cone, which discharges were somewhat higher than those obtained and predicted for a simple orifice. It was his own belief that that interesting anomaly might be traced to the fact that the initial contraction of the jet, followed by the subsequent overexpansion downstream, might well, if enclosed in an exit cone, trap a region of dead air. The pressure in that would be different from the pressure at exit, and would therefore modify the contraction coefficient and, hence, the mass flow. He had referred to that possibility in the paper under Supercritical Flows, in connexion with the Borda mouth-piece; in that instance also the emergent jet might not always clear the outer lip downstream.

In regard to the shadowgraphs he was interested to note that they had recorded jet contraction at r equal to $0 \cdot 23$, even with an opening which was heavily countersunk upstream. They had also shown that the emergent angle of a nozzle jet, measured from the axis of the latter, had correlated well with the theoretical Prandtl–Meyer angle. For an orifice, sonic conditions at the lip (i.e. M equal to 1 in Fig. 1) were reached by fluid which was moving radially inwards. Hence, the theoretical curve of Fig. 8 would, in that instance, be shifted back 90 deg., indicating contraction, even at pressure ratios lower than $0 \cdot 01$. The corresponding shift for the modified orifice tested by them would be much less, as the countersink implied that the fluid would approach the lip obliquely. That angle would be even less than that of the countersink, owing to local separation at the shoulder, so that the upstream conditions were approaching those for a nozzle.

For the extreme case of an orifice discharging into a vacuum, Prandtl–Meyer theory suggested an inner contracting jet, surrounded by a very small mass of fluid which splayed outwards as in Fig. 46 of Howarth (1953). The motion of the latter would in practice be considerably modified by growth of boundary layer along the reservoir wall and he was unable to suggest any method of allowing for the expansion effect in that very limited region.

In the matter of recent information on contraction coefficients raised by Mr. Tantam, he hoped shortly to publish a further paper, which would enable velocity of approach effects to be accounted for.