6.6 The Reynolds analogy

The Reynolds analogy

- Integral momentum eq. of flat surface flow without ∇p

$$\frac{d}{dx}\left[\delta\int_0^1 [\frac{u}{u_\infty}(\frac{u}{u_\infty}-1)]d\frac{y}{\delta}\right]=-\frac{C_f}{2}$$

- Integral energy eq. for the case of T_w=const

$$\frac{d}{dx}\left[\phi\delta\int_0^1 \frac{u}{u_\infty}(\frac{T-T_\infty}{T_w-T_\infty})d(\frac{y}{\delta_t})\right]=\frac{q_w}{\rho c_p u_\infty(T_w-T_\infty)}$$

- Distribution of u and T

$$\frac{u}{u_\infty}=\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad \Theta=1-\frac{3}{2}\frac{y}{\delta_t}+\frac{1}{2}\left(\frac{y}{\delta_t}\right)^3$$

The Reynolds analogy

- Similarity of temperature and flow boundary layers, Pr=1

$$\delta=\delta_t \quad \text{then} \quad \frac{T-T_w}{T_\infty-T_w}=\frac{u}{u_\infty} \;\Rightarrow\; \frac{T-T_\infty}{T_w-T_\infty}=-(\frac{u}{u_\infty}-1)$$

- Substituting the result in integral energy eq. by notice $\phi=\frac{\delta_t}{\delta}=1$

$$\frac{d}{dx}\left[\delta\int_0^1 [\frac{u}{u_\infty}(\frac{u}{u_\infty}-1)]d\frac{y}{\delta}\right]=-\frac{q_w}{\rho c_p u_\infty(T_w-T_\infty)}$$

- By comparing it with the integral momentum eq.

$$\frac{C_f}{2}=\frac{q_w}{\rho c_p u_\infty(T_w-T_\infty)}=\frac{h}{\rho c_p u_\infty}\equiv St$$

Stanton number

Reynolds analogy

The Reynolds analogy

- For the case of $\Pr\neq1$, Reynolds-Colburn analogy

$$\frac{h}{\rho c_p u_\infty}\Pr^{2/3}=\frac{C_f}{2} \;\Rightarrow\; \mathrm{St}\,\Pr^{2/3}=\frac{C_f}{2}$$

$$0.6\le\Pr\le50$$

- Stanton number

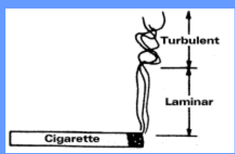
$$\mathrm{St}\equiv\frac{h}{\rho c_p u_\infty}=\frac{\mathrm{Nu}_x}{\mathrm{Re}_x\Pr}$$

$$\frac{h\Delta T}{\rho c_p u_\infty\Delta T}=\frac{\text{actual heat flux to the fluid}}{\text{heat flux capacity of the fluid flow}}$$

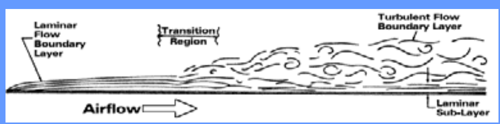
6.7 Turbulent boundary layers

Turbulence

- Boundary layer transition from laminar to turbulent
  - the smoke rise from a lighted cigarette in a draft-free room

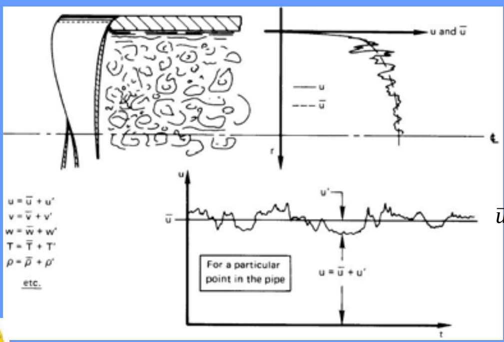


  - flow along a flat plat



Turbulence

- Fluctuation of u and other quantities in a turbulent pipe flow



$$\bar{u} \equiv \frac{1}{\mathbf{T}}\int_0^{\mathbf{T}} u\,dt$$

$$\overline{u'} = 0$$

Turbulence

- Mixing length l
  - Average length that a parcel of fluid moves between interactions (similar to that of the molecular mean free path)
  - For the laminar shear stress

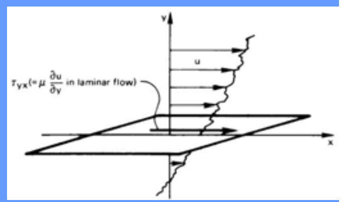
$$\tau_{yx} = (\text{constant})(\rho \bar{C})\underbrace{\left(l\frac{\partial u}{\partial y}\right)}_{=u'}$$

  - In turbulent case

$$\tau_{yx} = \mu\frac{\partial \bar{u}}{\partial y} + \tau'_{yx}$$

  - l become to mixing length
  - $\bar{C} \to v = \bar{v} + v'$

$$\tau'_{yx} = (\text{constant})[\rho(\bar{v} + v')]u'$$



The shear stress, τ_{yx}, in a laminar or turbulent flow.

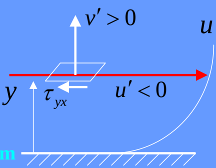
- By Navier-Stokes eq. Const=-1

$$\overline{\tau'_{yx}} = -\frac{\rho}{\mathbf{T}}\int_0^{\mathbf{T}}[(\bar{v}u' + v'u')]dt = -\rho\bar{v}\underbrace{\overline{u'}}_{0} - \rho\overline{v'u'}$$

$$\boxed{\overline{\tau'_{yx}} = -\rho\overline{v'u'}}$$

Turbulence

- Shear stress of turbulent flow



$$\tau_{yx} = \mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{v'u'}$$

  - Introduce **eddy diffusivity** ε_m for momentum

$$-\rho\overline{v'u'} = \rho\varepsilon_m\frac{\partial \bar{u}}{\partial y} \Rightarrow \tau_{yx} = \rho(\nu + \varepsilon_m)\frac{\partial \bar{u}}{\partial y}$$

$$\rho\varepsilon_m\frac{\partial \bar{u}}{\partial y} = -\rho\overline{v'u'} = -\rho(\text{cnstant})\left(\pm l\left|\frac{\partial \bar{u}}{\partial y}\right|\right)\left(\mp l\frac{\partial \bar{u}}{\partial y}\right)$$

$$= \rho(\text{cnstant})l^2\left|\frac{\partial \bar{u}}{\partial y}\right|\frac{\partial \bar{u}}{\partial y}$$

$$\boxed{\varepsilon_m = l^2\left|\frac{\partial \bar{u}}{\partial y}\right|}$$

Eddy diffusivity for momentum, Boussinesq in 1877

Turbulence near walls

- Average momentum eq. in the near wall region

$$\underbrace{\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}}_{\text{neglect very near the wall}} = \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{v'u'}\right) = \frac{\partial}{\partial y}\tau_{yx} = \frac{\partial}{\partial y}\left(\rho(\nu + \varepsilon_m)\frac{\partial \bar{u}}{\partial y}\right)$$

  - $\partial\tau_{yx}/\partial y \cong 0$ means that shear stress is constant in **y** and equal to the value on the wall

$$\tau_w \cong \tau_{yx} = \rho(\nu + \varepsilon_m)\frac{\partial \bar{u}}{\partial y} \Rightarrow \boxed{\underbrace{\int_0^{\bar{u}} d\bar{u}}_{=\bar{u}(y)} = \frac{\tau_w}{\rho}\int_0^y \frac{dy}{\nu + \varepsilon_m}}$$

$$\bar{u} = \text{fn}(\tau_w, \rho, \nu, y)$$

Near-wall velocity profile dose not depend directly upon x

  - 5 variables in 3 dimensions
  - 2 pi-groups

$$\frac{\bar{u}}{u^*} = \text{fn}\left(\frac{u^* y}{\nu}\right) \quad \text{or} \quad u^+ = \text{fn}(y^+)$$

Where $u^* \equiv \sqrt{\tau_w/\rho}$ Friction velocity

Turbulence near walls

- The viscous sublayer $(u^* y/\nu) \le 7$
  - Very near the wall, the eddies must become tiny, l and thus ε_m will tend to zero, so that $\nu >> \varepsilon_m$

$$\bar{u}(y) = \frac{\tau_w}{\rho}\int_0^y \frac{dy}{\nu + \varepsilon_m} \quad \boxed{\to 0}$$

$$\bar{u}(y) = \frac{\tau_w}{\rho}\int_0^y \frac{dy}{\nu} = \frac{\tau_w}{\rho}\frac{y}{\nu} \quad \xrightarrow{u^* \equiv \sqrt{\tau_w/\rho}} \quad \boxed{\frac{\bar{u}(y)}{u^*} = \frac{u^* y}{\nu}}$$

$$u^+ = y^+$$

  - This region is called as the viscous sublayer
    - Thickness ≈10~100μm
    - Turbulent mixing is ineffective in the region, it is responsible for a major fraction of thermal resistance

Turbulence near walls

- The log layer $(u^* y/\nu) \ge 30$ and $y/\delta \le 0.2$
  - Farther away from the wall, l is larger and turbulent shear stress is dominant: $\varepsilon_m >> \nu$

$$\tau_w \cong \rho\varepsilon_m\frac{\partial \bar{u}}{\partial y} = \rho l^2\left|\frac{\partial \bar{u}}{\partial y}\right|\frac{\partial \bar{u}}{\partial y}$$

Where $l = \kappa y$ for $y/\delta \le 0.2$, $\kappa = 0.41$

  - Assuming the velocity gradient to be positive

$$\int d\bar{u} = \sqrt{\frac{\tau_w}{\rho}}\int\frac{dy}{l} \Rightarrow \bar{u}(y) = u^*\int\frac{dy}{\kappa y} + C = \frac{u^*}{\kappa}\ln y + C$$

  - To fix the constant with experimental data

$$\boxed{\frac{\bar{u}(y)}{u^*} = \frac{1}{\kappa}\ln\left(\frac{u^* y}{\nu}\right) + B} \quad B \cong 5.5 \quad \textbf{Log law}$$

Turbulence near walls

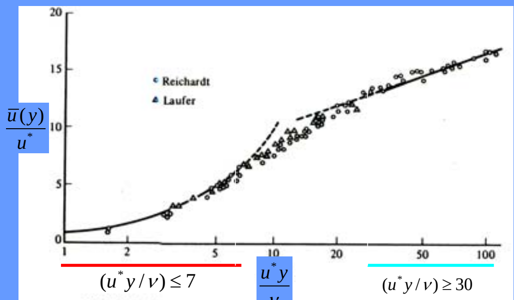


FIGURE 7-10
Velocity distribution in the buffer region of a turbulent boundary layer, according to experimental data by Reichardt for channel flow and by Laufer for pipe flow. (Reichardt, H.,[11] Laufer, J.[41])

Turbulence near walls

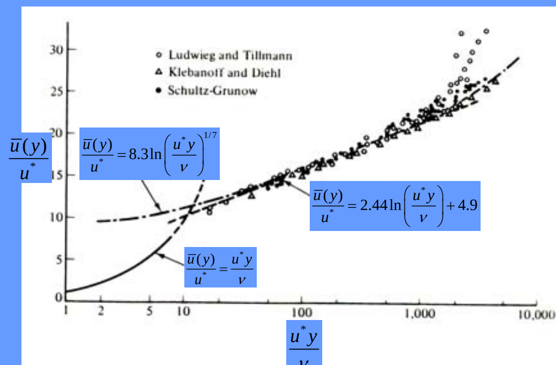


FIGURE 7-9
Mean-velocity distribution near smooth walls.

Turbulence near walls

- Buffer layer $7 < (u^* y/\nu) < 30$
  - more complicated equations for l, ε_m and u are used to connect the viscous sublayer to the log layer

$$l = \kappa y^{3/2}$$

- The outer part of the b.l ($y/\delta \gtrsim 0.2$)

$$l = 0.09\delta$$

Where $\frac{\delta(x)}{x} = \frac{0.16}{\mathrm{Re}_x^{1/7}}$

Lead to $C_f(x) = \frac{0.027}{\mathrm{Re}_x^{1/7}}$ for $\mathrm{Re}_x = 10^6 \sim 10^9$

More accurate formula valid for all Re_x $C_f(x) = \frac{0.455}{[\ln(0.06\,\mathrm{Re}_x)]^2}$

6.8 Heat transfer in turbulent boundary layers

The Reynolds-Colburn analogy for turbulent flow

- Eddy diffusivity of heat ε_h

$$q = -k\frac{\partial \bar{T}}{\partial y} - \underbrace{\begin{pmatrix}\text{another constant, which}\\ \text{reflects turbulence mixing}\end{pmatrix}}_{\equiv \rho c_p \varepsilon_h}\frac{\partial \bar{T}}{\partial y}$$

$$q = -\rho c_p(\alpha + \varepsilon_h)\frac{\partial \bar{T}}{\partial y}$$

- Turbulent Prandtl number, $\mathrm{Pr}_t \equiv \frac{\varepsilon_m}{\varepsilon_h}$

$$q = -\rho c_p\left(\frac{\nu}{\mathrm{Pr}} + \frac{\varepsilon_m}{\mathrm{Pr}_t}\right)\frac{\partial \bar{T}}{\partial y}$$

  - Pr is a physical property of the fluid
  - Pr_t is a property of the flow field more than of the fluid

The Reynolds-Colburn analogy for turbulent flow

- Average b.l energy eq.

$$\underbrace{\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}}_{\text{neglect very near wall}} = -\frac{\partial}{\partial y}q = \frac{\partial}{\partial y}\left[\rho c_p\left(\frac{\nu}{\mathrm{Pr}} + \frac{\varepsilon_m}{\mathrm{Pr}_t}\right)\frac{\partial \bar{T}}{\partial y}\right]$$

  - $\partial q/\partial y \cong 0$ means that heat flux is constant in y and equal to the wall heat flux

$$q = q_w = -\rho c_p\left(\frac{\nu}{\mathrm{Pr}} + \frac{\varepsilon_m}{\mathrm{Pr}_t}\right)\frac{\partial \bar{T}}{\partial y}$$

  - By integrating,

$$\frac{T_w - \bar{T}(y)}{q_w/(\rho c_p u^*)} = \begin{cases}\mathrm{Pr}\left(\frac{u^* y}{\nu}\right) & \text{thermal sublayer}\\ \frac{1}{\kappa}\ln\left(\frac{u^* y}{\nu}\right) + A(\mathrm{Pr}) & \text{thermal log layer}\end{cases}$$

For $\mathrm{Pr}_t = 1$

The Reynolds-Colburn analogy for turbulent flow

- For Pr>0.5 (experimental data)

$$A(\mathrm{Pr}) = 12.8\,\mathrm{Pr}^{0.68} - 7.3$$

- subtract the dimensionless **log-law** from its thermal counterpart

$$\frac{T_w - \overline{T}(y)}{q_w/(\rho c_p u^*)} - \frac{\overline{u}(y)}{u^*} = A(\mathrm{Pr}) - B$$

- In the outer part of the boundary

$$\frac{T_w - T_\infty}{q_w/(\rho c_p u^*)} - \frac{u_\infty}{u^*} = A(\mathrm{Pr}) - B$$

- recall

$$\frac{u^*}{u_\infty} \equiv \sqrt{\frac{\tau_w}{\rho u_\infty^2}} = \sqrt{\frac{C_f}{2}} \Rightarrow \frac{T_w - T_\infty}{q_w/(\rho c_p u_\infty)}\sqrt{\frac{C_f}{2}} - \sqrt{\frac{2}{C_f}} = A(\mathrm{Pr}) - B$$

The Reynolds-Colburn analogy for turbulent flow

- Rearrangement

$$\frac{q_w}{(\rho c_p u_\infty)(T_w - T_\infty)} = \frac{C_f/2}{1 + [A(\mathrm{Pr}) - B]\sqrt{C_f/2}}$$

- Substituting B=5.5 and $A(\mathrm{Pr}) = 12.8\,\mathrm{Pr}^{0.68} - 7.3$

$$\mathrm{St}_x \equiv \frac{h}{\rho c_p u_\infty} = \frac{C_f/2}{1 + 12.8[\mathrm{Pr}^{0.68} - 1]\sqrt{C_f/2}} \qquad \mathrm{Pr} \ge 0.5$$

- Works for either uniform T_w or uniform q_w
- C_f can be determined by using equation (6.102), p15 in this lecture note

Other equations for heat transfer in turbulent b.l.

- **For Pr ≈1, Re not too far from transition**

$$\mathrm{St}_x = \left(\frac{C_f}{2}\right)\mathrm{Pr}^{-2/3} \quad \text{for} \quad \mathrm{Pr} \approx 1$$

- **For wide-range (Žukauakas)**

$$\mathrm{St}_x = \left(\frac{C_f}{2}\right)\mathrm{Pr}^{-0.57} \quad \text{for} \quad 0.7 \le \mathrm{Pr} \le 380$$

- Flat plate flow with low-Re

$$C_f \cong \frac{0.0592}{\mathrm{Re}_x^{1/5}} \qquad 5\times10^5 \le \mathrm{Re}_x \le 10^7$$

$$\mathrm{Nu}_x = 0.0296\,\mathrm{Re}_x^{0.8}\,\mathrm{Pr}^{0.43}$$

- More accurate

$$\mathrm{Nu}_x = 0.032\,\mathrm{Re}_x^{0.8}\,\mathrm{Pr}^{0.43} \qquad 2\times10^5 \le \mathrm{Re}_x \le 5\times10^6$$

Other equations for heat transfer in turbulent b.l.

- **Average Nusselt number for uniform T_w**

$$\overline{h} = \frac{1}{L\Delta T}\int_0^L q\,dx = \frac{1}{L}\int_0^L h_x\,dx = \frac{1}{L}\int_0^L \frac{k\mathrm{Nu}_x}{x}\,dx$$

$$\overline{\mathrm{Nu}}_L = \frac{\overline{h}L}{k} = \int_0^L \frac{\mathrm{Nu}_x}{x}\,dx = 0.0296\,\mathrm{Pr}^{0.43}\left[\int_0^L \frac{1}{x}\mathrm{Re}_x^{0.8}\,dx\right]$$

$$\overline{\mathrm{Nu}}_L = 0.0370\,\mathrm{Re}_L^{0.8}\,\mathrm{Pr}^{0.43}$$

- It may be used for either T_w=const or q_w=const and for $\mathrm{Re}_L \le 3\times10^7$

Other equations for heat transfer in turbulent b.l.

- **Laminar and turbulent b.l**

$$\overline{h} = \frac{1}{L\Delta T}\int_0^L q\,dx = \frac{1}{L}\left[\int_0^{x_{\mathrm{trans}}} h_{\mathrm{laminar}}\,dx + \int_{x_{\mathrm{trans}}}^L h_{\mathrm{turbulent}}\,dx\right]$$

$$\overline{\mathrm{Nu}}_L = 0.0370\,\mathrm{Pr}^{0.43}\left\{\mathrm{Re}_L^{0.8} - \left[\mathrm{Re}_{\mathrm{trans}}^{0.8} - 17.95\,\mathrm{Pr}^{0.097}(\mathrm{Re}_{\mathrm{trans}})^{1/2}\right]\right\}$$

- **For liquid, Whitaker suggested**

$$\overline{\mathrm{Nu}}_L = 0.0370\,\mathrm{Pr}^{0.43}\left(\mathrm{Re}_L^{0.8} - 9200\right)\left(\frac{\mu_\infty}{\mu_w}\right)^{1/4}$$

A correlation for laminar, transitional, and turbulent flow

- **Churchill suggests**

$$\mathrm{Nu}_x = 0.45 + (0.3387\phi^{1/2})\left(1 + \frac{(\phi/2600)^{3/5}}{[1 + (\phi_u/\phi)^{7/2}]^{2/5}}\right)^{1/2}$$

- Where

$$\phi \equiv \mathrm{Re}_x\,\mathrm{Pr}^{2/3}\left[1 + \left(\frac{0.0468}{\mathrm{Pr}}\right)^{2/3}\right]^{-1/2}$$

$$\phi_u \approx 10^5 \sim 10^7 \quad \phi_u \approx \phi(\mathrm{Re}_x = \mathrm{Re}_u)$$

- Re_u is the Reynolds number at the end of the turbulent transition region
- It is for uniform T_w
- May be used for uniform q_w, if

0.3387 → 0.4637, 0.0468 → 0.02052

A correlation for laminar, transitional, and turbulent flow

- Average Nusselt number

$$\overline{\mathrm{Nu}}_L = 0.45 + (0.6774\phi^{1/2})\left(1 + \frac{(\phi/12500)^{3/5}}{[1+(\phi_{um}/\phi)^{7/2}]^{2/5}}\right)^{1/2}$$

  - Where $\phi_{um} \approx 1.875\phi(\mathrm{Re}_x = \mathrm{Re}_u)$
  - Applicable for either uniform T_w or uniform q_w

Homework

- 补充题

  水以2kg/s的质量流量流过直径为40mm，长为4m的圆管，管壁温度保持在90℃，水的进口温度为30℃。求水的出口温度和管子对水的散热量。水的物性参数按40℃的水查取。不考虑由温差引起的修正。
- 6.30
- 6.31