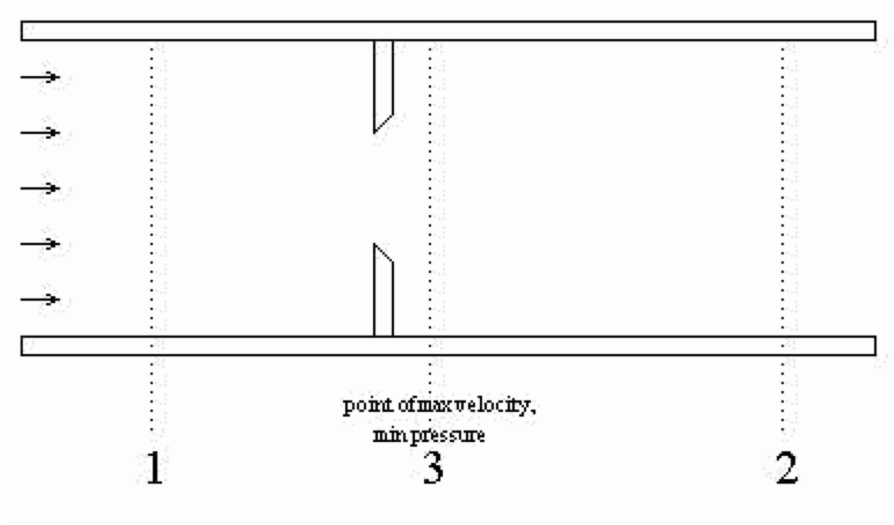
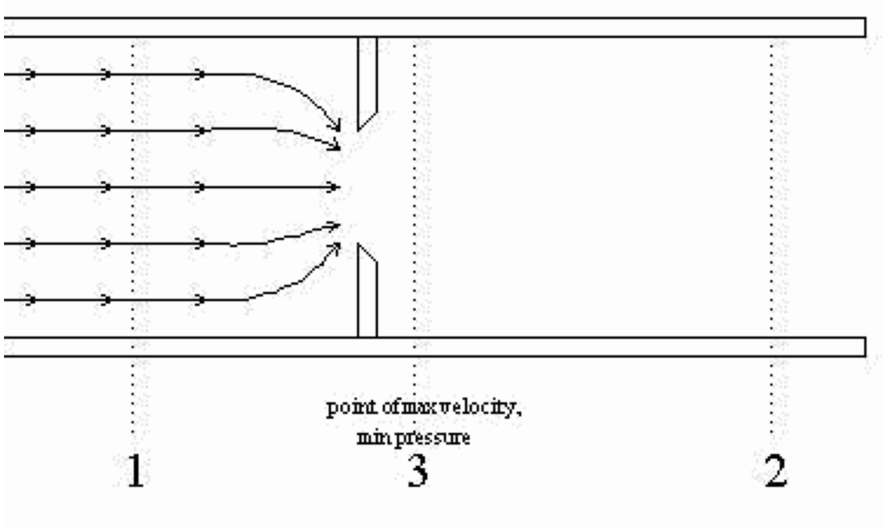
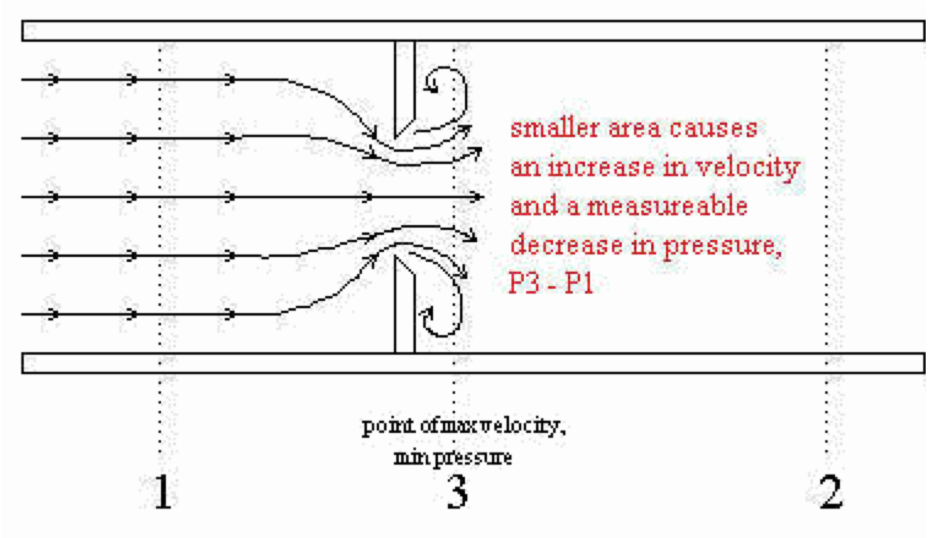
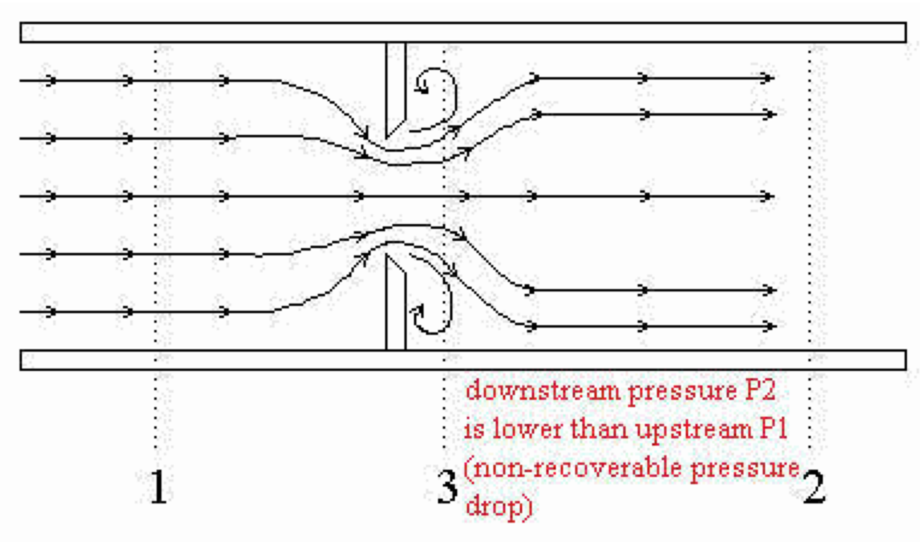
Injector Notes









Incompressible Injector Design

- Many Liquid Propellants are Essentially Incompressible Fluids

- Incompressible Fuel Examples:

  Kerosene (RP-1, RP-4), Ethanol, Methanol, UDMH (Unsymmetrical Dimthyl Hydrazine), MMH (Mono Methyl Hydrazine), Ammonia, Hydrazine, ~Liquid Hydrogen, etc.

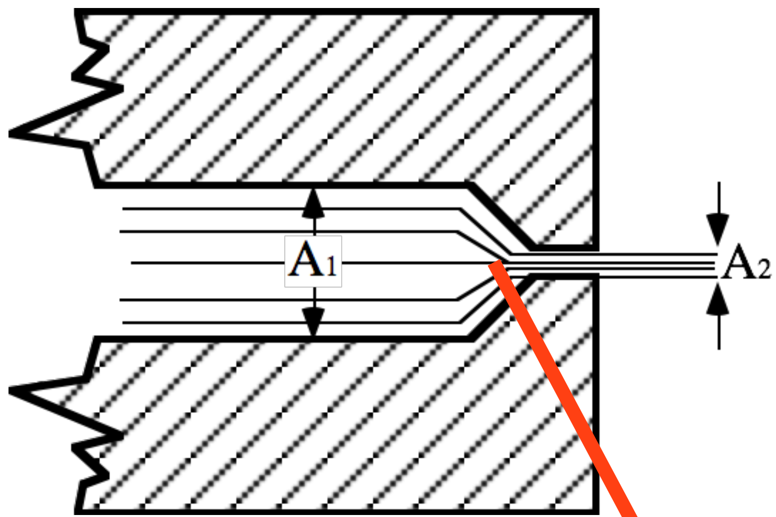
- Incompressible Oxidizer Examples:

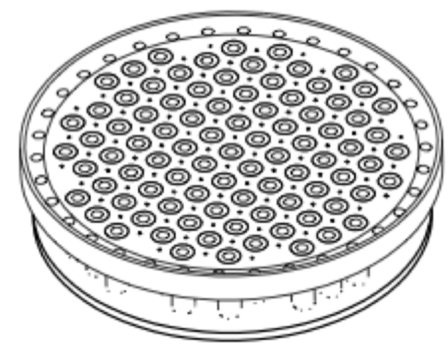
  Hydrogen Peroxide, Liquid Flourine, Nitrogen Tetraoxide, Nitric Acid, ~Liquid Oxygen, etc.

- Incompressible Assumption Allows Simplified Form of Injector Equations

Incompressible Injector Design

- Injector Geometry





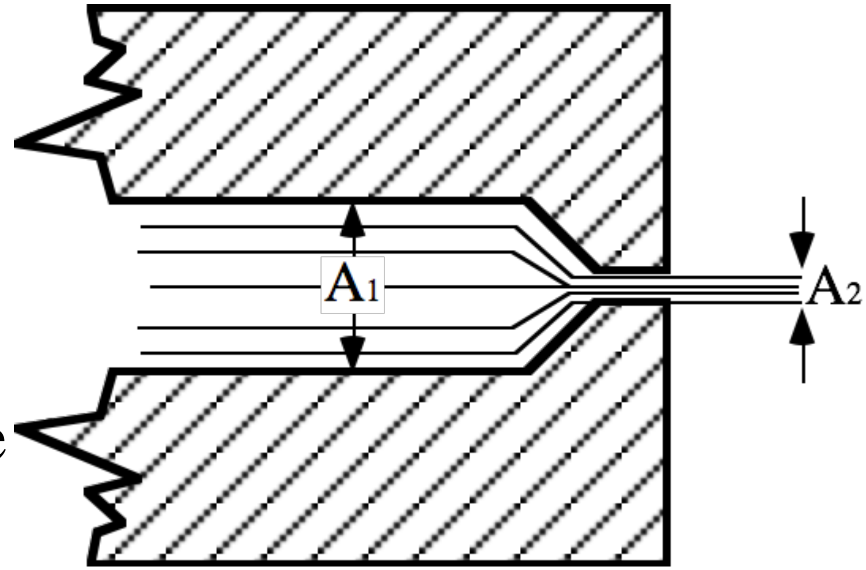
- Assume Liquid Propellants are incompressible ($\rho = const$)
- Momentum $p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$
- Continuity $\rho A_1 V_1 = \rho A_2 V_2$

$$\rightarrow \quad p_1 - p_2 = \frac{1}{2}\rho V_2^2 \left[1 - \left(\frac{A_2}{A_1}\right)^2\right]$$

Incompressible Injector Design (cont'd)

- Solve for V_2

$$V_2 = \frac{1}{\left[1-\left(\frac{A_2}{A_1}\right)^2\right]^{\frac{1}{2}}}\sqrt{2\left(\frac{p_1-p_2}{\rho}\right)}$$



- Friction effects in orifice will cause

$$V_{2_{actual}} < V_{2_{ideal}} \rightarrow V_{2_{actual}} \equiv C_v V_{2_{ideal}} \rightarrow$$

$$V_{2_{actual}} \equiv \frac{C_v}{\left[1-\left(\frac{A_2}{A_1}\right)^2\right]^{\frac{1}{2}}}\sqrt{2\left(\frac{p_1-p_2}{\rho}\right)}$$

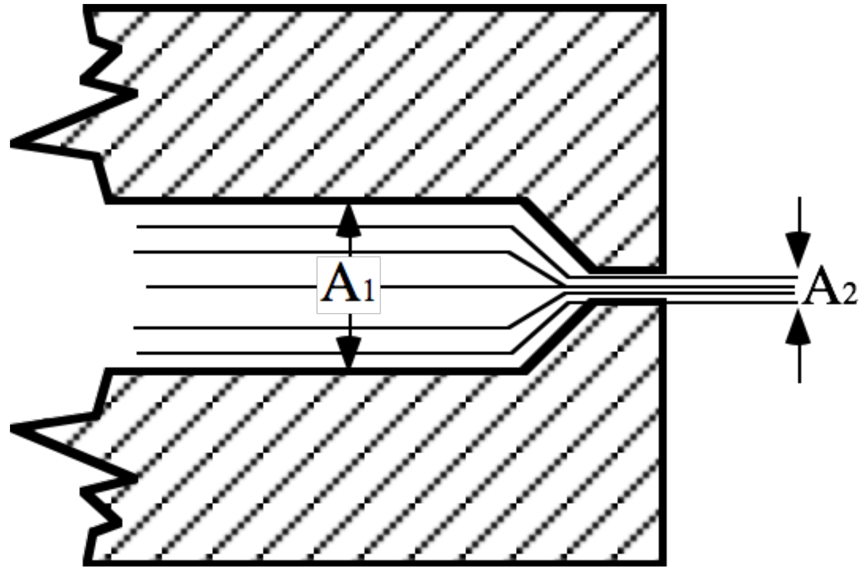
- Define "Discharge Coefficient"

$$C_d \equiv \frac{C_v}{\left[1-\left(\frac{A_2}{A_1}\right)^2\right]^{\frac{1}{2}}}$$

C_v --> "velocity coefficient"

Incompressible Injector Design (cont'd)

$$V_2 = \frac{1}{\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]^{\frac{1}{2}}} \sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)}$$

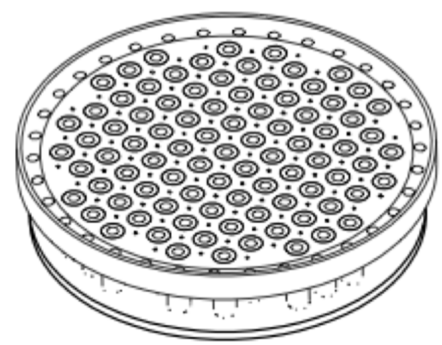


- Define Volumetric Flow as

$$Q_v = A_2 V_{2_{actual}} = A_2 C_d \sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)}$$

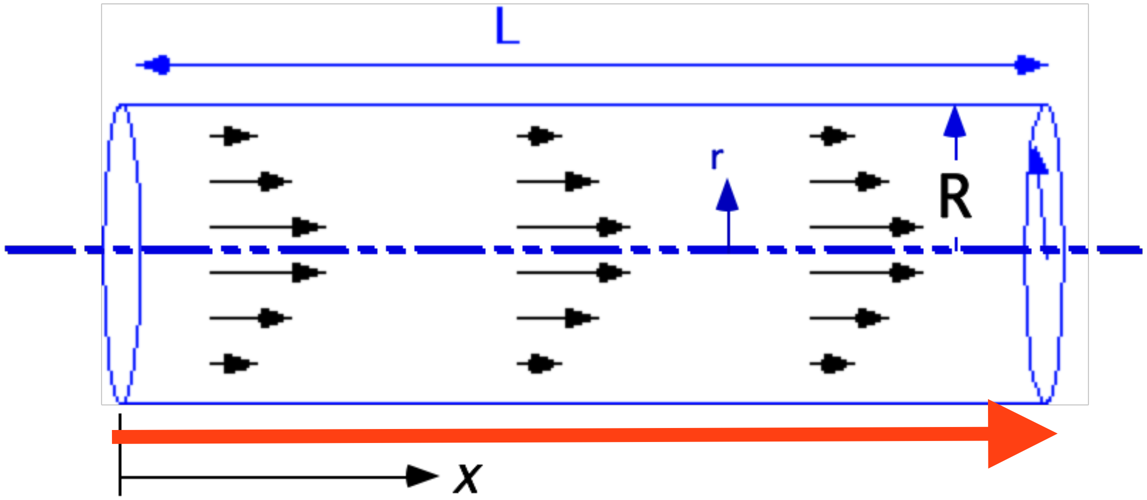
- Finally Massflow is

$$\dot{m} = \rho Q_v = A_2 C_d \sqrt{2\rho\left(p_1 - p_2\right)}$$



Injector Design (cont'd)

- How do we measure "discharge coefficient" for a particular Orifice design? … approximate by cylindrical pipe flow

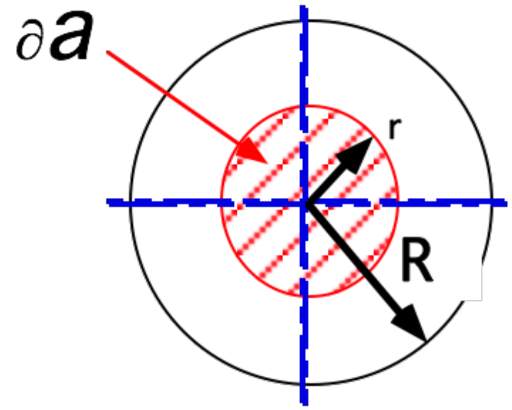


- **Laminar Flow**

- Incompressible pipe flow …. **Poiseuille flow**

$$Velocity\ profile \rightarrow u(r) = -\frac{1}{4\mu}\frac{\partial p}{dx}\left(R^2 - r^2\right)$$

Injector Design (cont'd)



- Calculate Volumetric flow rate thru orifice?

$$q(r) = u(r)\partial a = -\left(\pi r^2\right)\frac{1}{4\mu}\frac{\partial p}{dx}\left(R^2 - r^2\right) \rightarrow \text{total volumetric rate}$$

$$Q_v = \int_{-R}^{R} q(r)dr = 2\int_{0}^{R} q(r)dr = -\frac{1}{2\mu}\frac{\partial p}{dx}\int_{0}^{R}\pi r^2\left(R^2 - r^2\right)dr =$$

$$-\frac{\pi}{2\mu}\frac{\partial p}{dx}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = -\frac{\pi R^4}{8\mu}\frac{\partial p}{dx}$$

- **Laminar Flow**

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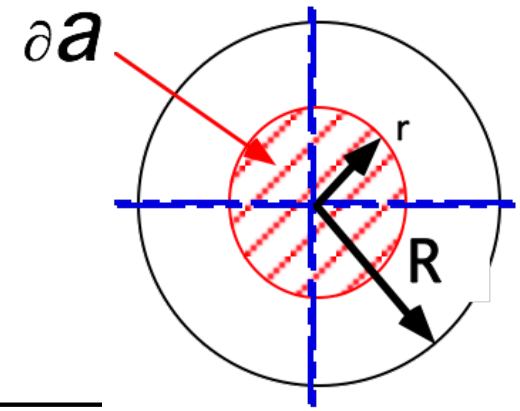
Injector Design (cont'd)



- Equate terms

$$Q_v = -\frac{\pi R^4}{8\mu}\frac{\partial p}{dx}$$

- **Poiselle Flow**

$$Q_v = A_2 C_d \sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)} = \pi R_2{}^2 C_d \sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)}$$

- **Discharge massflow (incompressible)**

$$Q_v = \pi R_2{}^2 C_d \sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)} = -\frac{\pi R_2{}^4}{8\mu}\frac{\partial p}{dx} \approx \frac{\pi R_2{}^4}{8\mu}\frac{p_1 - p_2}{L_d}$$

- **Laminar Flow**

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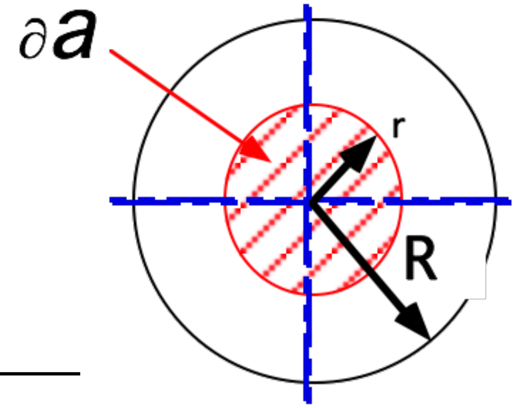
Injector Design (cont'd)

- Collect terms, **Solve for Cd**



$$C_d \sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)} = \frac{R_2{}^2}{8\mu L_d}\left(p_1 - p_2\right) \rightarrow$$

$$C_d = \frac{R_2{}^2}{L_d}\frac{\left(p_1 - p_2\right)}{8\mu}\sqrt{\frac{\rho}{2\left(p_1 - p_2\right)}} = \frac{D^2}{32\mu L_d}\sqrt{\frac{1}{2}\rho\left(p_1 - p_2\right)}$$

- From the basic definition of discharge coefficient

- **Laminar Flow**

$$V_{actual} \equiv C_d \sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)} \rightarrow \left[\frac{1}{2}\frac{\rho V_{actual}}{C_d}\right] \equiv \sqrt{\frac{1}{2}\rho\left(p_1 - p_2\right)}$$

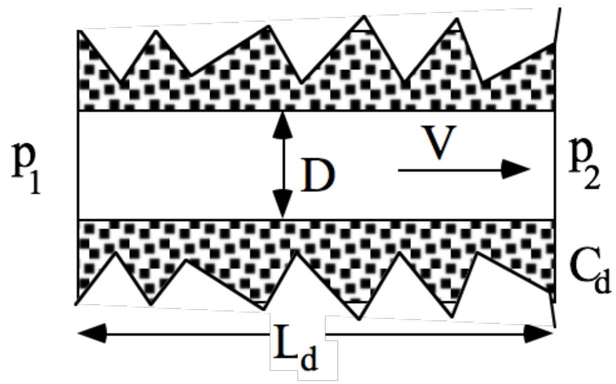
Injector Design (cont'd)

- Collect terms

$$\rightarrow \left[\frac{1}{2}\frac{\rho V_{actual}}{C_d}\right] \equiv \sqrt{\frac{1}{2}\rho\left(p_1 - p_2\right)} \rightarrow C_d = \frac{D^2}{64\mu L_d}\left[\frac{\rho V_{2\,actual}}{C_d}\right]$$

$$\rightarrow C_d^{\;2} = \frac{D}{64 L_d}\frac{\rho V_{2\,actual} D}{\mu} \rightarrow \boxed{C_d = \frac{1}{8}\sqrt{\frac{D}{L_d}}\sqrt{\mathrm{R}_{e_D}}}$$

$$\rightarrow\rightarrow C_d^{\;2} = \frac{D^2}{64 L_d^{\;2}}\frac{\rho V_{2\,actual} L_d}{\mu} \rightarrow \boxed{C_d = \frac{1}{8}\frac{D}{L_d}\sqrt{\mathrm{R}_{e_L}}}$$



"Cylindrical Injector Port Design"

- **Laminar Discharge coefficient**

-- directly proportional to diameter of injector port,

-- inversely proportional to square root of injector port length

-- Reynolds number dependent

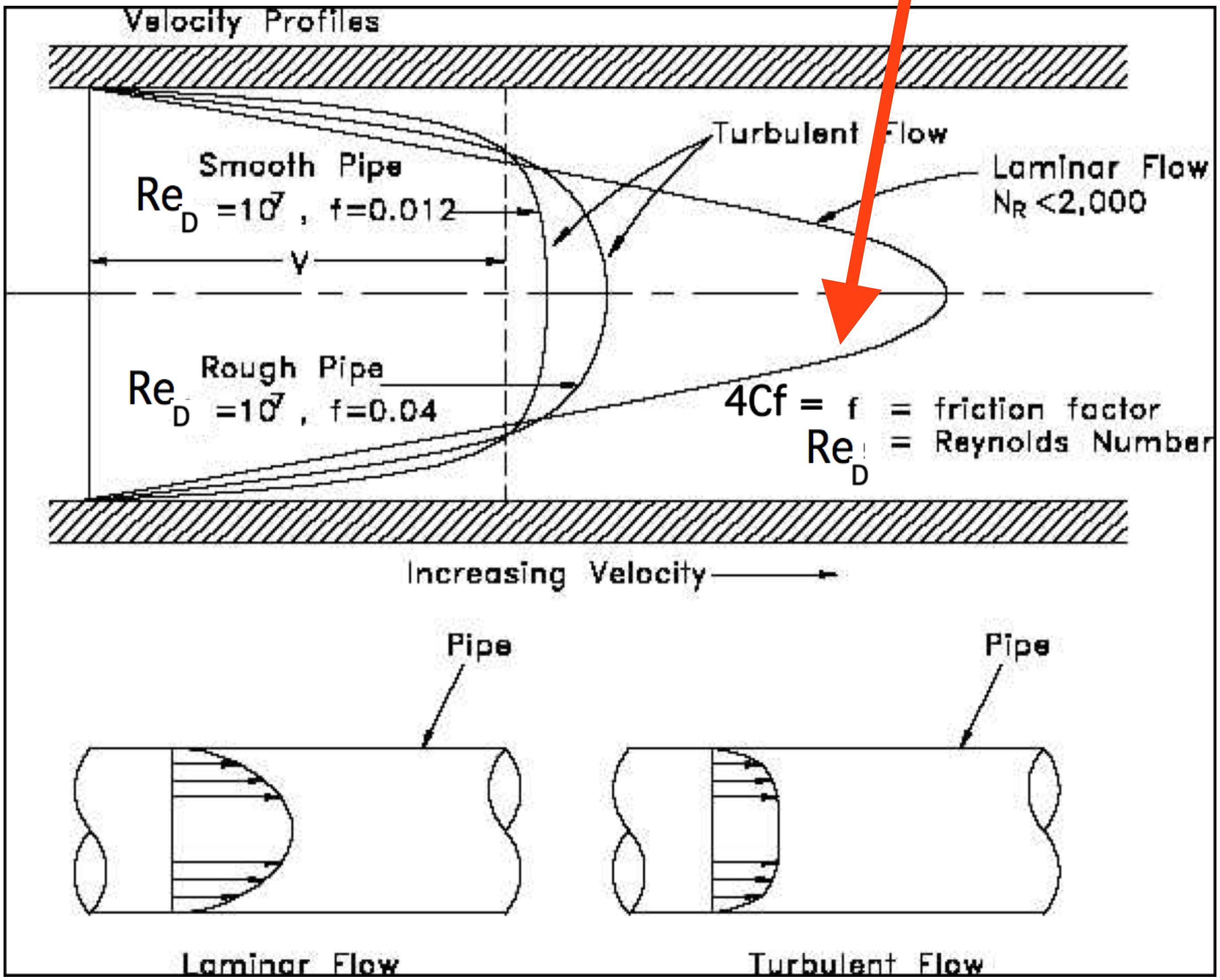
-- *provides mechanism for scaling from one injector design (diameter/length) to another*

-- *and from one working fluid to another*

- **Laminar Flow**

Injector Design for Turbulent Flow

- For Turbulent Flow the Velocity profile is considerably different than laminar



- **Turbulent Flow**
- **Pressure gradient proportional to skin friction**

Injector Design (cont'd)

- For Turbulent Flow the Velocity profile is considerably different than laminar

  - **Turbulent Flow**

$$\frac{\partial p}{dx} = -\frac{4\tau_0}{D} = -4\frac{1}{2}\rho\bar{U}^2\frac{C_f}{D} = -2\rho\bar{U}^2\frac{C_f}{D} \rightarrow$$

  - **Pressure gradient proportional to skin friction**

$$\bar{U} = mean\ velocity\ in\ channel$$

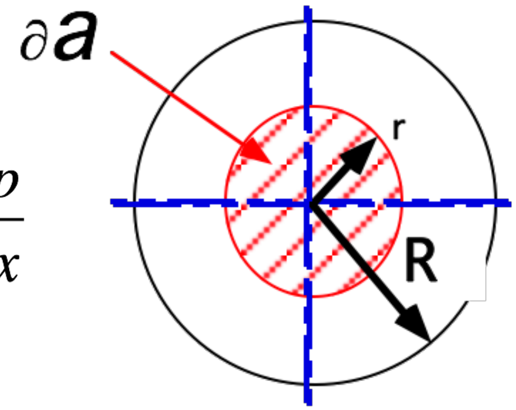
  - Solve for mean velocity

$$\bar{U} = -\frac{1}{2\rho\bar{U}\frac{C_f}{D}}\frac{\partial p}{dx} = -\frac{\mu}{\mu}\frac{1}{2\rho\bar{U}D\frac{C_f}{D^2}}\frac{\partial p}{dx} = -\frac{1}{2\,\mathrm{Re}_D C_f \frac{\mu}{D^2}}\frac{\partial p}{dx}$$

Injector Design (cont'd)

- Calculate Volumetric flow rate thru orifice?

$$Q_v = \frac{\pi D^2}{4}\bar{U} = -\frac{\frac{\pi D^2}{4}}{2\,\mathrm{Re}_D C_f \frac{\mu}{D^2}}\frac{\partial p}{dx} = -\frac{\pi D^4}{8\mu}\frac{1}{\mathrm{Re}_D C_f}\frac{\partial p}{dx}$$



- *Equate Volumetric Flow Rate with Volumetric Rate from Injector Eqn.*

- **Turbulent Flow**

$$\frac{\pi}{4}D^2 C_d\sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)} = -\frac{\pi D^4}{8\mu}\frac{1}{\mathrm{Re}_D C_f}\frac{\partial p}{dx}$$

$$Q_v = -\frac{\pi D^4}{8\mu}\frac{1}{\mathrm{Re}_D C_f}\frac{\partial p}{dx}$$

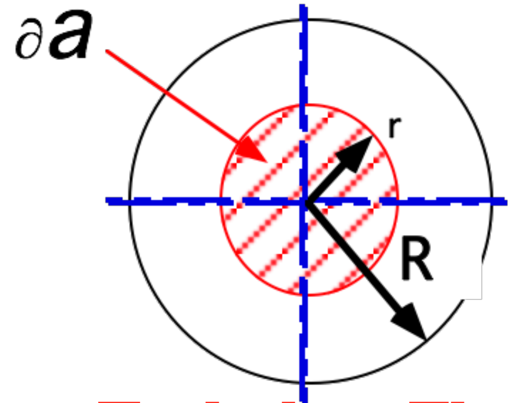
$$Q_v = A_2 C_d\sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)} = \frac{\pi}{4}D^2 C_d\sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)}$$

Injector Design (cont'd)

- Collect terms

Volumetric Rate Eqn.

$$\frac{\pi}{4} D^2 C_d \sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)} = -\frac{\pi D^4}{8\mu} \frac{1}{\mathrm{Re}_D C_f} \frac{\partial p}{\partial x}$$



- $Approximate: \frac{\partial p}{\partial x} \approx \frac{p_2 - p_1}{L_d}$ *(short injector)* & Solve for C_d

- **Turbulent Flow**

$$C_d = \frac{D^2}{2\mu L_d} \frac{1}{\mathrm{Re}_D C_f} (p_1 - p_2) \sqrt{\frac{\rho}{2(p_1 - p_2)}} = \frac{D^2}{2\mu L_d} \frac{1}{\mathrm{Re}_D C_f} \sqrt{\frac{1}{2}\rho(p_1 - p_2)}$$

- *From the basic definition of discharge coefficient*

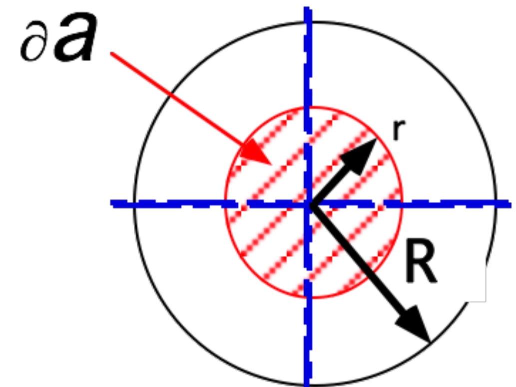
$$V_{actual} \equiv C_d \sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)} \rightarrow \left[\frac{1}{2} \frac{\rho V_{actual}}{C_d}\right] \equiv \sqrt{\frac{1}{2}\rho(p_1 - p_2)}$$

- *Equate terms*

$$C_d = \frac{D^2}{2\mu L_d} \frac{1}{\mathrm{Re}_D C_f} \sqrt{\frac{1}{2}\rho(p_1 - p_2)}$$

Injector Design (cont'd)

$$\rightarrow C_d = \frac{D^2}{2\mu L_d}\frac{1}{\mathrm{Re}_D C_f}\cdot\left[\frac{1}{2}\frac{\rho V_{actual}}{C_d}\right]$$



- *Simplify*

$$\rightarrow C_d^{\,2} = \frac{D^2}{4\mu L_d}\frac{\rho V_{actual}}{\mathrm{Re}_D C_f} = \frac{1}{4C_f}\left(\frac{D}{L}\right)\frac{1}{\mathrm{Re}_D}\frac{\rho V_{actual}\cdot D}{\mu} = \frac{1}{4C_f}\left(\frac{D}{L}\right)$$

- **Turbulent Flow**

- *Solve for C_d*

$$C_d = \frac{1}{2}\sqrt{\frac{D}{L}}\sqrt{\frac{1}{C_f}}$$

Injector Design (cont'd)

• Collect terms

$$C_d = \frac{1}{2}\sqrt{\frac{D}{L}\sqrt{\frac{1}{C_f}}}$$

• **Geometry & Reynold's Number Dependent**

• **Turbulent Flow**

• Turbulent Flow Discharge coefficient

-- Typically injector flow is turbulent when $Re_D > 4000$

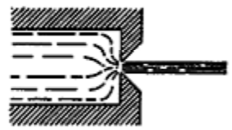
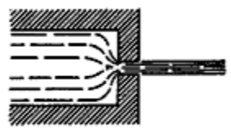
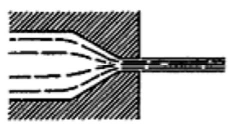
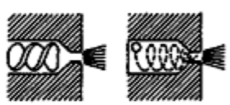
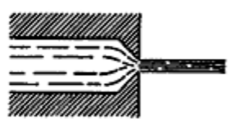
• Turbulent flow skin friction formulae ,... smooth orifice

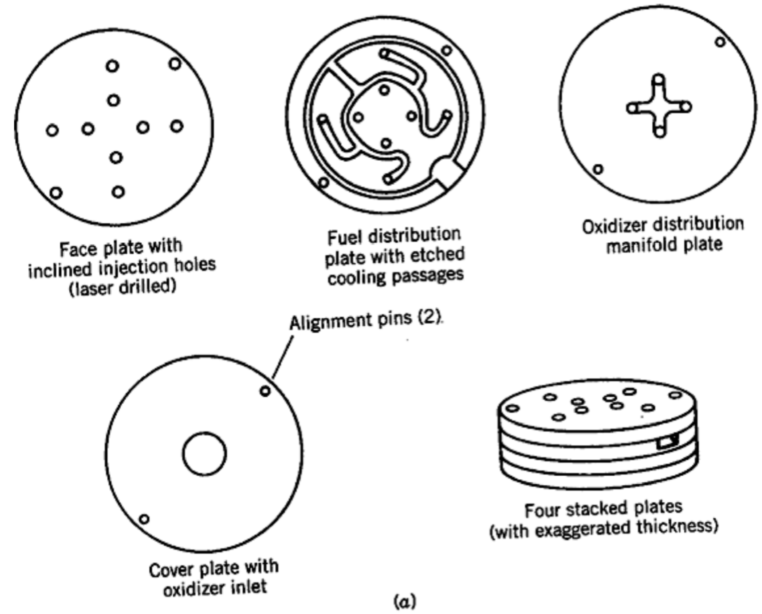
$$Prandtl: \ \left(4C_f\right)^{-\frac{1}{2}} = 2\log_{10}\left[\frac{Re_D\left(4C_f\right)^{\frac{1}{2}}}{2.51}\right]$$

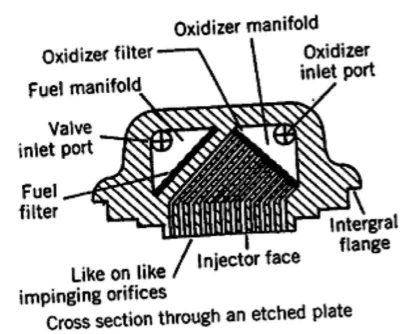
$$Blasius: \ C_f = \frac{1}{4}\left[\frac{0.3164}{\left[Re_D\right]^{\frac{1}{4}}}\right]$$

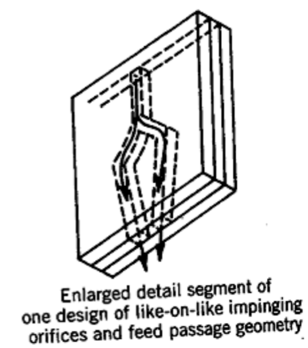
Injector Design (cont'd)

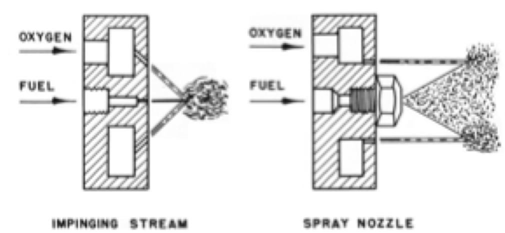
TABLE 8–2. Injector Discharge Coefficients

Orifice Type	Diagram	Diameter (mm)	Discharge Coefficient
Sharp-edged orifice		Above 2.5 Below 2.5	0.61 0.65 approx.
Short-tube with rounded entrance $L/D > 3.0$		1.00 1.57 1.00 (with $L/D \sim 1.0$)	0.88 0.90 0.70
Short tube with conical entrance		0.50 1.00 1.57 2.54 3.18	0.7 0.82 0.76 0.84–0.80 0.84–0.78
Short tube with spiral effect		1.0–6.4	0.2–0.55
Sharp-edged cone		1.00 1.57	0.70–0.69 0.72









Injector Design (cont'd)

$$C_d = \frac{1}{2}\sqrt{\frac{D}{L}}\sqrt{\frac{1}{C_f}}$$

• **Turbulent Flow**

• **Turbulent Flow Discharge coefficient**

-- Typically injector flow is turbulent when $Re_D > 4000$

• Turbulent flow skin friction formulae ,… rough orifice

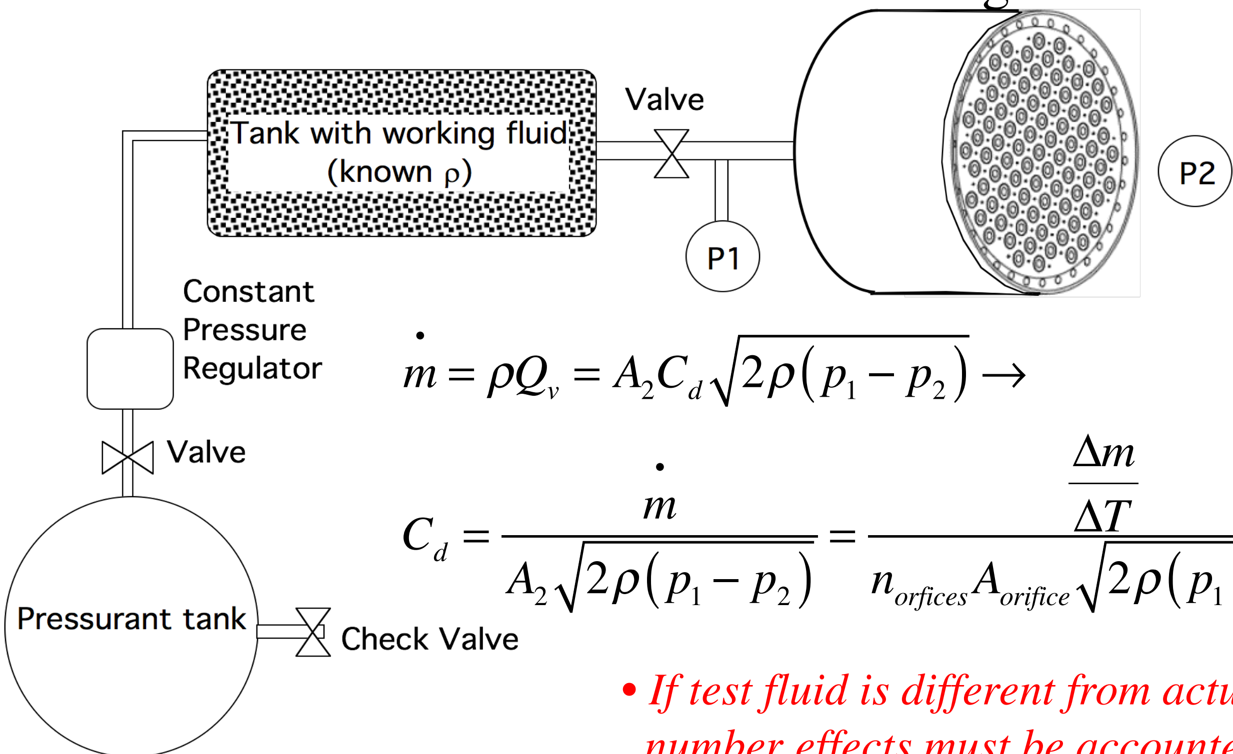
$$Colebrook: \left(4C_f\right)^{-\frac{1}{2}} = -2\log_{10}\left[\left(\frac{\varepsilon}{3.7D}\right) + \frac{2.51}{Re_D\left(4C_f\right)^{\frac{1}{2}}}\right]$$

$$Haaland: \ C_f = \frac{1}{12.96\left\{\log_{10}\left[\left(\frac{\varepsilon}{3.7D}\right)^{1.11} + \frac{6.9}{Re_D}\right]\right\}^2}$$

ε =average wall roughness height

Incompressible Injector Calibration

- Best bet is to measure the orifice discharge coefficient for your particular design…



$$\dot{m} = \rho Q_v = A_2 C_d \sqrt{2\rho\left(p_1 - p_2\right)} \rightarrow$$

$$C_d = \frac{\dot{m}}{A_2\sqrt{2\rho\left(p_1 - p_2\right)}} = \frac{\dfrac{\Delta m}{\Delta T}}{n_{orfices} A_{orifice}\sqrt{2\rho\left(p_1 - p_2\right)}}$$

- *If test fluid is different from actual oxidizer … Reynolds number effects must be accounted for ….*

Laminar Flow:

$$C_{d_{laminar}} = \frac{1}{8}\sqrt{\frac{D}{L_d}} \cdot \sqrt{R_{e_D}} \rightarrow \frac{\left(C_d\right)_{Oxidizer}}{\left(C_d\right)_{test\ fluid}} = \sqrt{\frac{\mathrm{R_e}_{Oxidizer}}{\mathrm{R_e}_{test\ fluid}}} = \sqrt{\frac{\rho_{Oxidizer}}{\rho_{test\ fluid}} \frac{\mu_{test\ fluid}}{\mu_{Oxidizer}}}$$

Turbulent Flow:

$$C_{d_{turbulent}} = 2\sqrt{\frac{D}{L_d}} \cdot \sqrt{\frac{1}{C_f}} \rightarrow Blasius \rightarrow C_f = \frac{1}{4}\frac{0.3164}{\left(R_{e_D}\right)^{1/4}} \rightarrow \frac{\left(C_d\right)_{Oxidizer}}{\left(C_d\right)_{Test\ Fluid}} = \sqrt{\frac{\left(C_f\right)_{Test\ Fluid}}{\left(C_f\right)_{Oxidizer}}} = \sqrt{\left\{\frac{\left(R_{e_D}\right)_{Oxidizer}}{\left(R_{e_D}\right)_{Test\ Fluid}}\right\}^{1/4}} = \left\{\frac{\rho_{Oxidizer}}{\rho_{Test\ Fluid}} \cdot \frac{\mu_{Test\ Fluid}}{\mu_{Oxidizer}}\right\}^{1/8}$$

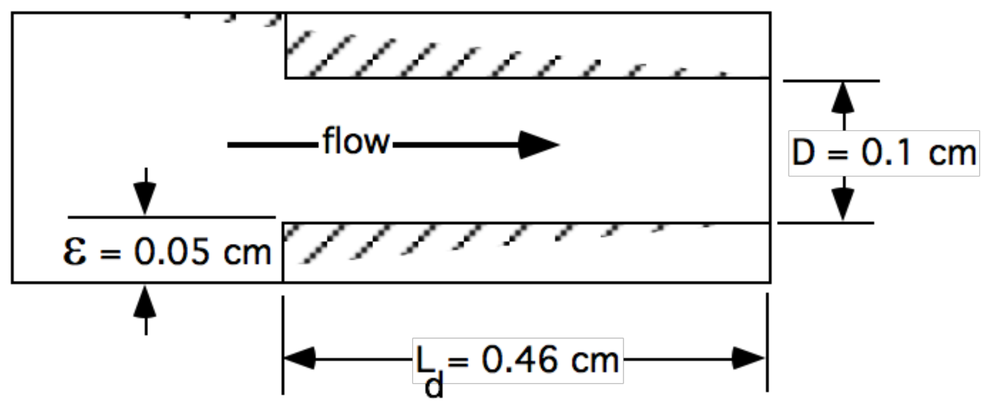
Example Calculation for Turbulent Discharge Coefficient

- LOX @ 80°K , D = **0.1** cm, L_d =.46 cm

Liquid Oxydizer Density, kg/M^3
1140

Oxidizer Viscosity, Nt-sec/m^2
0.0002799

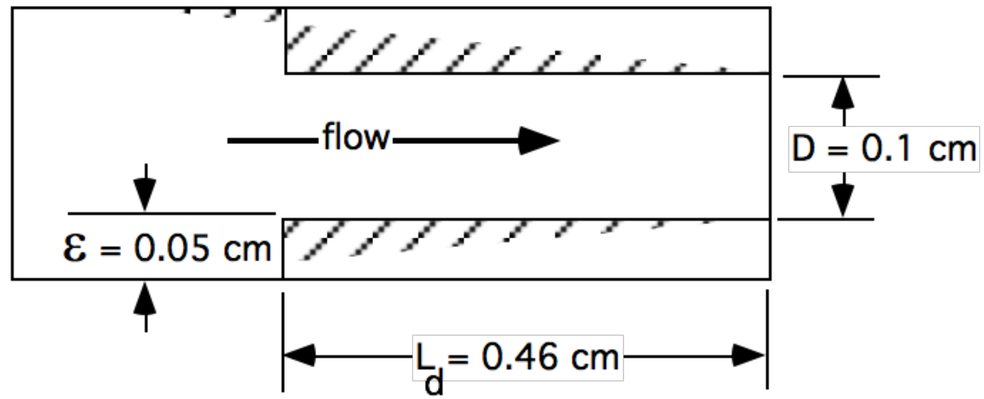
- Assumed combustor pressure = 2600 kpa
- Assumed injector pressure = 3800 kpa



$$RSS_{roughness} \rightarrow \frac{\varepsilon}{D} = \frac{\sqrt{\sum bumps^2}}{0.1} = \frac{\sqrt{0.05^2 + 0.05^2}}{0.1} = 0.7071$$

Example Calculation for Incompressible Turbulent Discharge Coefficient

- Assume initial C_d=0.81



$$V \approx C_d \sqrt{2\left(\frac{p_1 - p_2}{\rho}\right)} =$$

$$0.81\left(2\frac{(3800-2600)}{1140}1000\right)^{0.5} = 37.165 \text{ m/sec}$$

$$\mathrm{Re}_D = \frac{\rho V D}{\mu} = \frac{37.165 \cdot 1140 \frac{0.1}{100}}{0.0002799} = 151{,}369$$

Example Calculation for Turbulent Discharge Coefficient

- LOX @ 80°K , D = 0.08 cm, L_d =.4 cm

- *Haaland Formula for rough wall*

$$C_f = \frac{1}{12.96\left\{log_{10}\left[\left(\frac{\varepsilon}{3.7D}\right)^{1.11} + \frac{6.9}{Re_D}\right]\right\}^2} = \frac{1}{12.96\left(\log\left(\left(\frac{0.7071}{3.7}\right)^{1.11} + \frac{6.9}{151369}\right)\right)^2} = 0.121273$$

- *Colebrook Formula for rough wall*

$$\frac{1}{\sqrt{4\cdot C_f}} = -2\log 10\left\{\left(\frac{\varepsilon}{3.7\cdot D}\right) + \frac{2.51}{\mathrm{Re}_D\sqrt{4\cdot C_f}}\right\}$$

$$\left(-2\log\left(\frac{0.07071}{3.7\cdot 0.1} + \frac{2.51}{151369\,(4\cdot 0.121011)^{0.5}}\right)\right)^{-2} 0.25 \quad = 0.121011$$

Good agreement, take Average Value

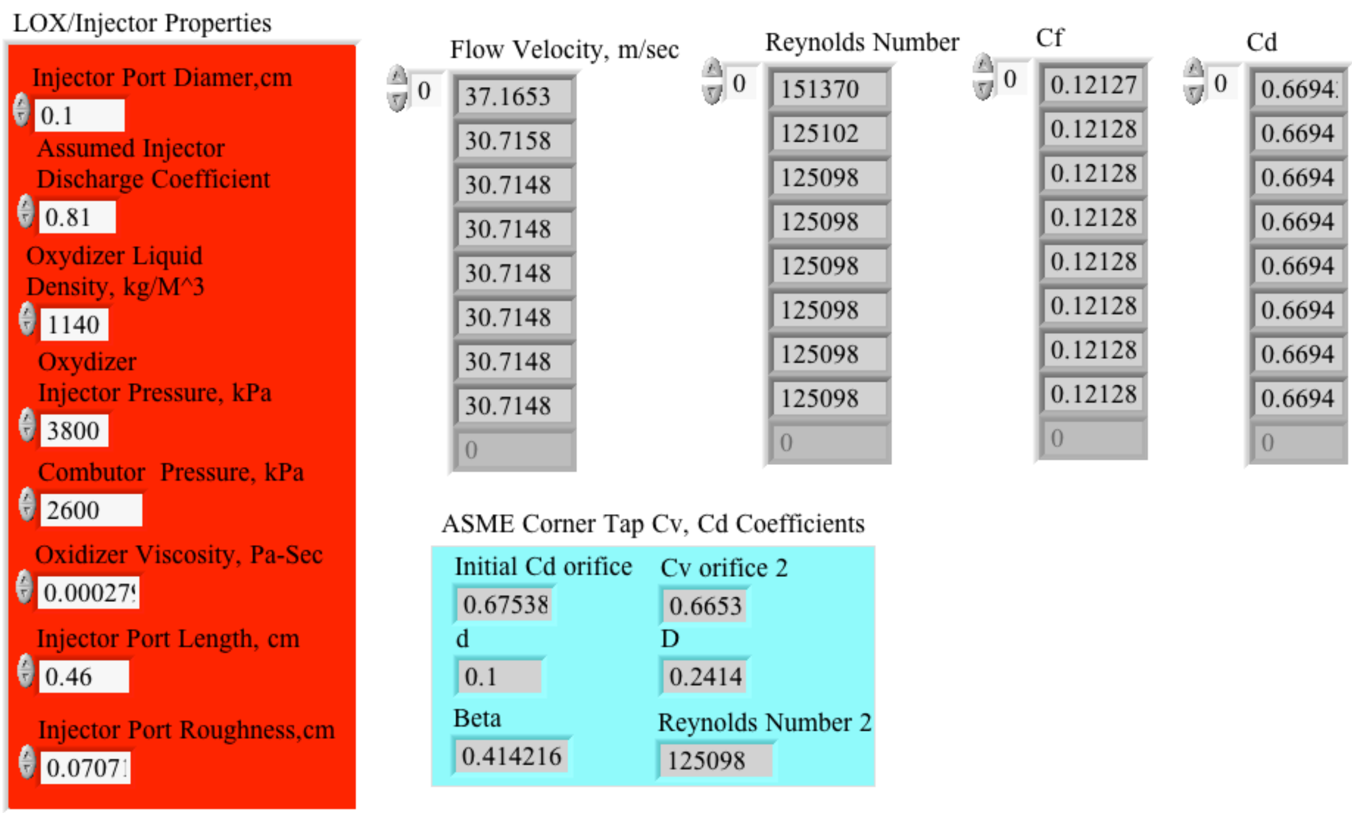
$C_f \simeq 0.121142$

Example Calculation Turbulent Discharge

- Re-compute discharge coefficient

$$C_d = \frac{1}{2}\sqrt{\frac{D}{L_d}}\sqrt{\frac{1}{C_f}} = \frac{1}{2}\left(\frac{0.1}{0.46 \cdot 0.082916}\right)^{0.5} = 0.6694$$

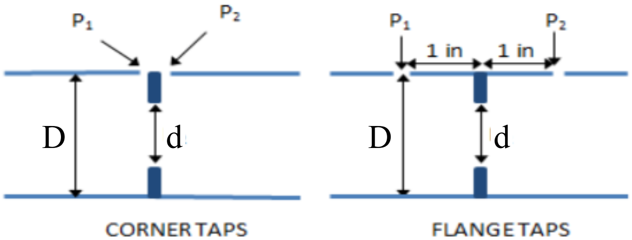
- *Assumed initial C_d=0.81*
- For arbitrary initial guess .. Calculations can be iterated



Injector Design (cont'd)

ASME "STANDARD MODEL"

DIMENSIONS IN INCHES



Corner tap:

$$C_v = \left[0.5991 + \frac{0.0044}{D} + \left(0.3155 + \frac{0.0175}{D}\right)\left(\left(\frac{d}{D}\right)^4 + 2\left(\frac{d}{D}\right)^{16}\right)\right]\sqrt{1-(d/D)^4}$$

$$+ \left[\frac{0.52}{D} - 0.192 + \left(16.48 - \frac{1.16}{D}\right)\left(\left(\frac{d}{D}\right)^4 + 4\left(\frac{d}{D}\right)^{16}\right)\right]\sqrt{\frac{1-(d/D)^4}{Re_D}}$$

Simple orifice

***ASME MFC-14M- 2003**

Flange tap:

$$C_v = \left[0.598 + 0.468\left(\left(\frac{d}{D}\right)^4 + 10\left(\frac{d}{D}\right)^{12}\right)\right]\sqrt{1-(d/D)^4} + \left(0.87 + 8.1\left(\frac{d}{D}\right)^4\right)\sqrt{\frac{1-(d/D)^4}{Re_D}}$$

$$C_d \equiv \frac{C_v}{\left[1-\left(\frac{A_2}{A_1}\right)^2\right]^{\frac{1}{2}}} = \frac{C_v}{\left[1-\left(\frac{d}{D}\right)^4\right]^{\frac{1}{2}}} = \frac{C_v}{\sqrt{1-\beta^4}}$$

$\beta \text{-->} \text{"contraction ratio"}$ "d/D"

- **need to take area change into account**

The American Society of
Mechanical Engineers

AN AMERICAN NATIONAL STANDARD

MEASUREMENT OF FLUID FLOW USING SMALL BORE PRECISION ORIFICE METERS

ASME MFC-14M–2003
(Revision of ASME MFC-14M–2001)

ASME MFC-14 2003 Summary

Flange Taps

$$C = \left[0.598 + 0.468\cdot\left(\beta^4 + 10\cdot\beta^{12}\right)\right]\cdot\sqrt{1 - \beta^4} + \left(0.87 + 8.1\cdot\beta^4\right)\cdot\sqrt{\frac{1 - \beta^4}{Re_1}}$$

Corner Taps

$$C = \left[0.5991 + \frac{0.0044\cdot 0.0254}{D_1} + \left(0.3155 + \frac{0.0175}{D_1}\right)\cdot\left(\beta^4 + 2\cdot\beta^{16}\right)\right]\cdot\sqrt{1 - \beta^4}\ \ldots$$

$$+\left[\frac{0.52\cdot 0.0254}{D_1} - 0.192 + \left(16.48 - \frac{1.16\cdot 0.0254}{D_1}\right)\cdot\left(\beta^4 + 4\cdot\beta^{16}\right)\right]\cdot\sqrt{\frac{1 - \beta^4}{Re_1}}$$

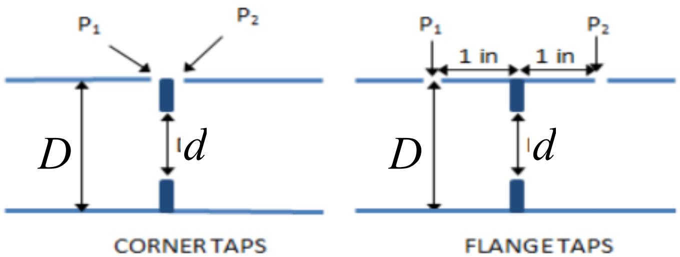
$$e = 1 - \frac{\Delta P}{\gamma\cdot P_1}\cdot\left(0.41 + 0.35\cdot\beta^4\right) \qquad Re_1 = \frac{D_1\cdot V_1\cdot\rho}{\mu} \qquad V_1 = \frac{Q}{A_1}$$

$$\rho = \frac{P_1\cdot MW}{R\cdot T} \qquad A_2 = \frac{\pi\cdot D_2^{\ 2}}{4} \qquad A_1 = \frac{\pi\cdot D_1^{\ 2}}{4}$$

$$Q = e\cdot C\cdot A_2\cdot\frac{\sqrt{\frac{2\Delta P}{\rho}}}{\sqrt{1 - \beta^4}} \qquad Q_{std} = Q\cdot\frac{P_1}{P_{std}}\cdot\frac{T_{std}}{T}$$

Injector Design (cont'd)

Simple orifice



***ASME MFC-14M-2001.**

Corner tap:

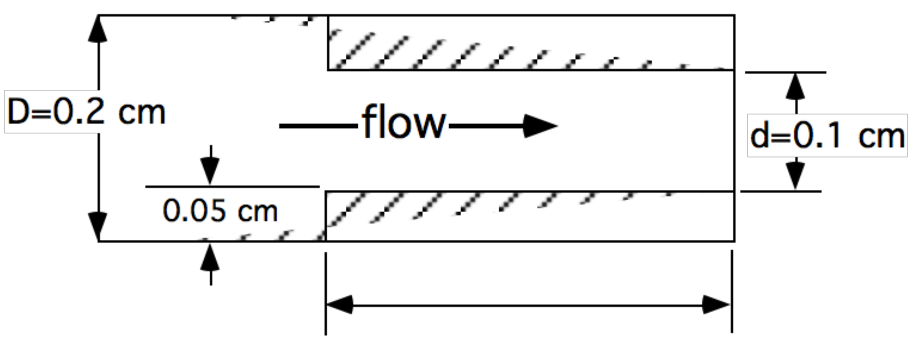
$$C_v = \left[0.5991 + \frac{0.0044}{D} + \left(0.3155 + \frac{0.0175}{D}\right)\left(\left(\frac{d}{D}\right)^4 + 2\left(\frac{d}{D}\right)^{16}\right)\right]\sqrt{1-(d/D)^4}$$

$$+\left[\frac{0.52}{D} - 0.192 + \left(16.48 - \frac{1.16}{D}\right)\left(\left(\frac{d}{D}\right)^4 + 4\left(\frac{d}{D}\right)^{16}\right)\right]\sqrt{\frac{1-(d/D)^4}{Re_D}}$$

Flange tap:

$$C_v = \left[0.598 + 0.468\left(\left(\frac{d}{D}\right)^4 + 10\left(\frac{d}{D}\right)^{12}\right)\right]\sqrt{1-(d/D)^4} + \left(0.87 + 8.1\left(\frac{d}{D}\right)^4\right)\sqrt{\frac{1-(d/D)^4}{Re_D}}$$

- BASED ON GEOMETRY OF PREVIOUS EXAMPLE …



d = 0.1 cm

D = 0.1 + 2(0.07071) = 0.24142 cm

$Re_D = 125{,}098 \times 0.24142/0.1 = 302{,}012$

$\beta = d/D = 0.1/0.24142 = 0.414216$

CV, CD VALUES Corner Tap

Cvi Value 2: 0.665367

Cdi Value: 0.675382

CV, CD VALUES Flange Tap

Cvi Value 2: 0.60591

Cdi Value: 0.61503

Good Comparison with Previous

$$C_d = \frac{1}{2}\sqrt{\frac{D}{L_d}}\sqrt{\frac{1}{C_f}} = \frac{1}{2}\left(\frac{0.1}{0.46\cdot 0.082916}\right)^{0.5} = 0.6694$$

Effects of Flow Compressibility on Injector Flow

- Some Common Propellants are in Gaseous Form
- Compressible Fuel/Oxidizer Examples:

Gaseous Hydrogen, Methane, Ethane, Gaseous oxygen (GOX

Accurate Injector Model requires Modeling of Flow Compressibility Effects

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R_g}}\frac{p_0}{\sqrt{T_0}}\frac{M}{\left[1+\frac{(\gamma-1)}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

General Massflow Equation for Compressible Flow

$$\frac{P_{out}}{P_{in}} = \frac{1}{\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}}} \rightarrow M = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_{in}}{P_{out}}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}$$

General Pressure Ratio Equation for Compressible Flow

- Substitute and collect terms

Effects of Flow Compressibility on Injector Flow (2)

$$\dot{m} = A\sqrt{\frac{\gamma}{R_g}\cdot\frac{P_{in}}{T_0}}\,\frac{\sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_{in}}{P_{out}}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}}{\left(\frac{P_{in}}{P_{out}}\right)^{\frac{\gamma+1}{2\gamma}}} = AP_{in}\sqrt{\frac{\gamma}{R_gT_0}\frac{2}{\gamma-1}\left[\left(\frac{P_{in}}{P_{out}}\right)^{\frac{\gamma-1}{\gamma}}\left(\frac{P_{in}}{P_{out}}\right)^{-\frac{\gamma+1}{\gamma}}-\left(\frac{P_{in}}{P_{out}}\right)^{-\frac{\gamma+1}{\gamma}}\right]}$$

Simplify

$$\dot{m} = A\sqrt{\frac{P_{in}}{R_gT_0}\frac{2\gamma}{\gamma-1}P_{in}\left[\left(\frac{P_{in}}{P_{out}}\right)^{\frac{-2}{\gamma}}-\left(\frac{P_{out}}{P_{in}}\right)^{\frac{\gamma+1}{\gamma}}\right]} = A\sqrt{\frac{2\gamma}{\gamma-1}\rho_{in}P_{in}\left[\left(\frac{P_{out}}{P_{in}}\right)^{\frac{2}{\gamma}}-\left(\frac{P_{out}}{P_{in}}\right)^{\frac{\gamma+1}{\gamma}}\right]}$$

Allow for Non-isentropic pressure losses (C_d)

$$\dot{m} = C_dA\sqrt{\frac{2\gamma}{\gamma-1}\rho_{in}P_{in}\left[\left(\frac{P_{out}}{P_{in}}\right)^{\frac{2}{\gamma}}-\left(\frac{P_{out}}{P_{in}}\right)^{\frac{\gamma+1}{\gamma}}\right]}$$

Effects of Flow Compressibility on Injector Flow (3)

1-D "Lossy" Compressible Mass Flow Equations

Unchoked Flow

$$\left(\frac{P_{in}}{P_{out}}\right)_{critical} < \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow \boxed{\begin{matrix}1.8929\\ \gamma=1.4\end{matrix}}$$

$$\dot{m} = C_d A \sqrt{\frac{2\gamma}{\gamma-1}\rho_{in}P_{in}\left[\left(\frac{P_{out}}{P_{in}}\right)^{\frac{2}{\gamma}} - \left(\frac{P_{out}}{P_{in}}\right)^{\frac{\gamma+1}{\gamma}}\right]}$$

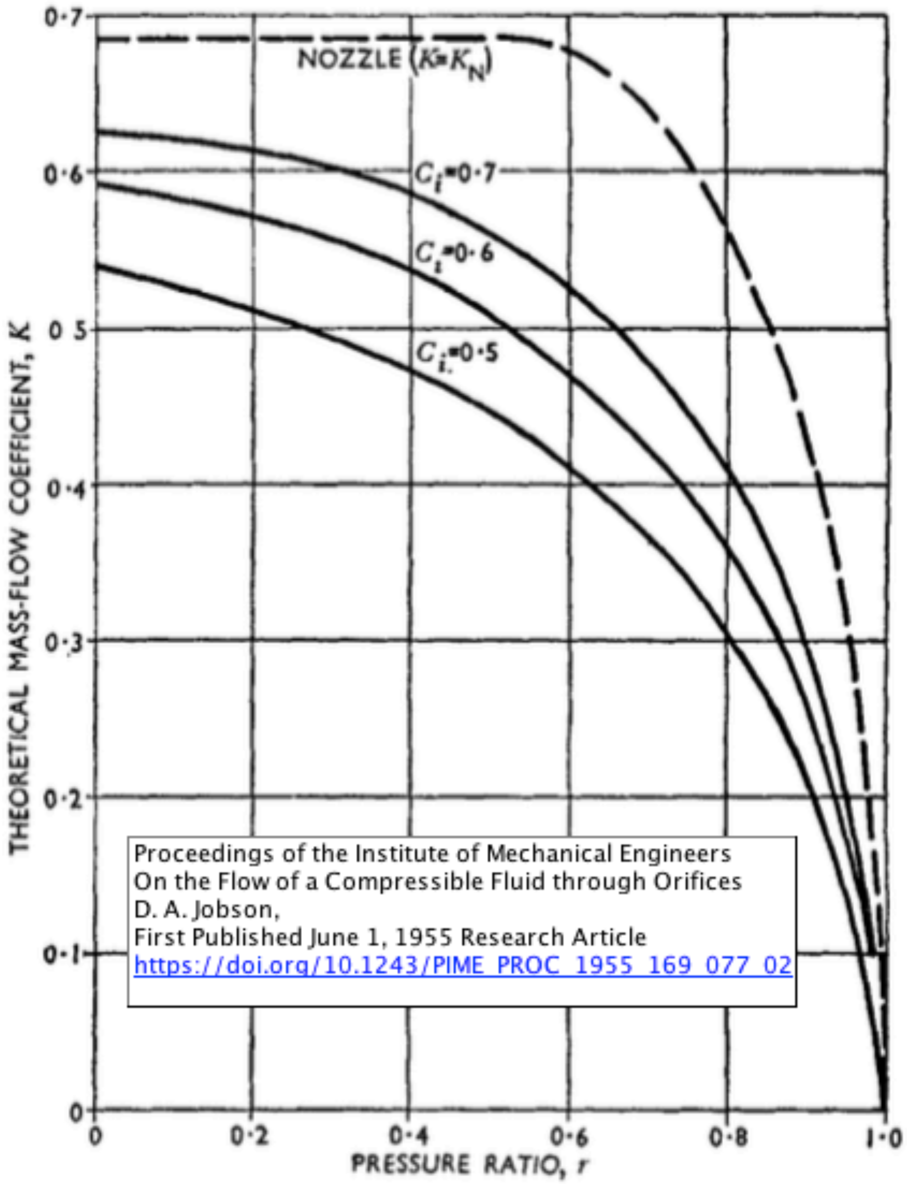
Choked Flow:

$$\left(\frac{P_{in}}{P_{out}}\right)_{critical} \geq \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow \boxed{\begin{matrix}1.8929\\ \gamma=1.4\end{matrix}}$$

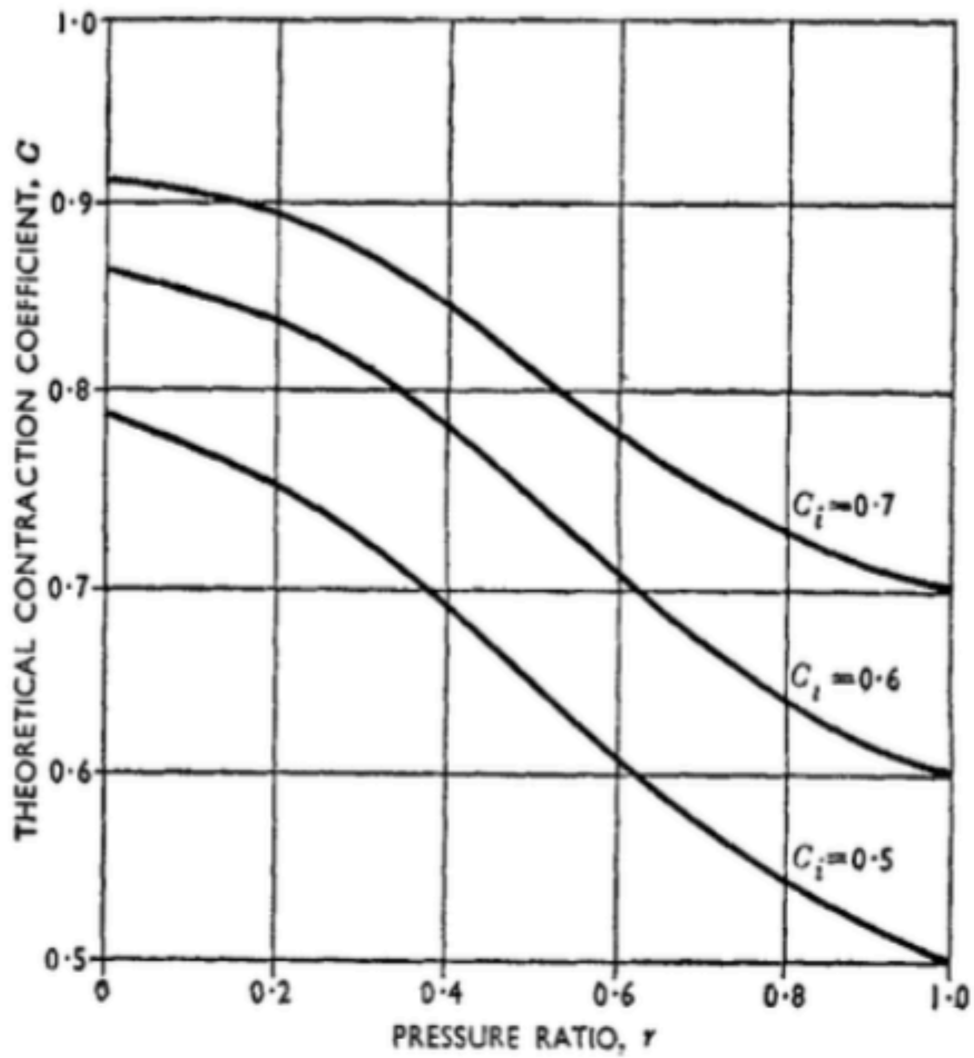
$$\dot{m} = C_d A_{in}^{*} \cdot \sqrt{\gamma P_{in} \cdot \rho_{in}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

C_d not a constant but depends on
Both port geometry and pressure ratio

Effects of Flow Compressibility on Injector Flow (4)



Theoretical Mass-flow Coefficient Plotted as a Function of Pressure Ratio for $n = 1{\cdot}4$



Theoretical Contraction Coefficient Plotted as a Function of Pressure Ratio for $n = 1{\cdot}4$

Injector Compressibility Analysis

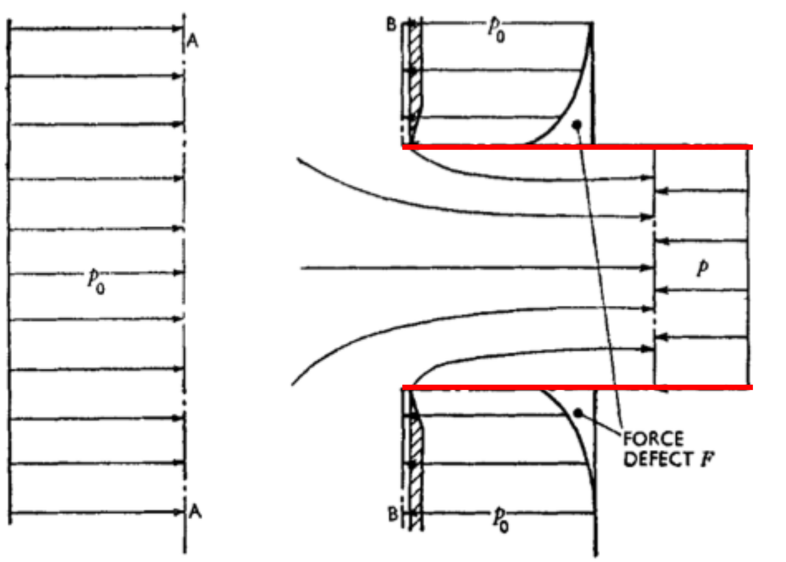
- *Rewrite Compressible Injector Equations*

Subcritical Flow: $\left(\frac{p}{P_0}\right) > \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$

$$\dot{m} = C_d \cdot A \sqrt{\frac{2\gamma}{\gamma-1} \cdot P_0 \cdot \rho_0 \cdot \left(\frac{p}{P_0}\right)^{\frac{2}{\gamma}} \left[1 - \left(\frac{p}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

$$Define \rightarrow r = \frac{p}{P_0} \rightarrow K_n = \sqrt{\frac{2\gamma}{\gamma-1} \cdot r^{\frac{2}{\gamma}} \left[1 - r^{\frac{\gamma-1}{\gamma}}\right]}$$

$$\rightarrow \dot{m} = K_n \cdot C_d \cdot A \cdot \sqrt{P_0 \cdot \rho_0}$$



a For subcritical conditions.

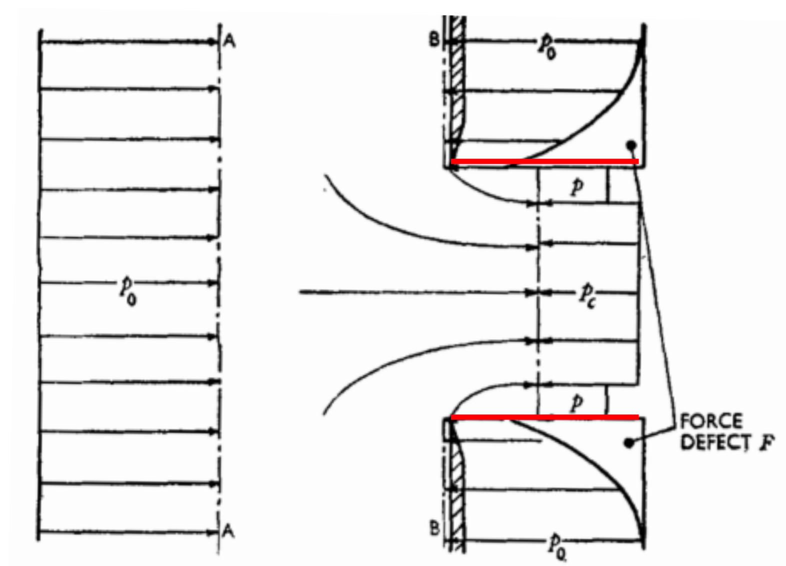
Supercritical Flow:

$$\left(\frac{p}{P_0}\right) \le \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\dot{m} = C_d \cdot A \sqrt{\gamma \cdot P_0 \cdot \rho_0 \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

$$Define \rightarrow r^* = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \rightarrow K_n = \sqrt{\gamma \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} = \sqrt{\gamma \cdot (r^*)^{\frac{\gamma+1}{\gamma}}}$$

$$\rightarrow \dot{m} = K_n \cdot C_d \cdot A \cdot \sqrt{P_0 \cdot \rho_0}$$



b For supercritical conditions.

Injector Compressibility Analysis (2)

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On the Flow of a Compressible Fluid through Orifices
D. A. Jobson,
First Published June 1, 1955 Research Article
https://doi.org/10.1243/PIME_PROC_1955_169_077_02

$$\text{Subcritical Flow}: \left(\frac{p}{P_0}\right) > \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

Force Balance of Port Interface

$$\left(P_0 - p\right)\cdot A + F = \dot{m}\cdot U_e$$

Define f → incompressible loss coefficient

$$F \equiv f\,\frac{\dot{m}^2}{\rho_0\cdot A}$$

Incompressible Injector Equation

$$\dot{m} = \left(C_d\right)_{inc}\cdot A\cdot\sqrt{2\cdot\rho_0\left(P_0 - p\right)}$$

Re-order as

$$\left(\frac{\dot{m}^2}{\rho_0\cdot A}\right)\cdot\frac{1}{2\left(C_d\right)^2_{inc}} = A\cdot\left(P_0 - p\right)$$

Write Momentum Flow in terms of massflow

$$\rightarrow \dot{m}\cdot U_e = \dot{m}\cdot\frac{Q_e}{\left(C_d\right)_{inc} A} = \dot{m}\cdot\frac{\rho_0\cdot Q_e}{\left(C_d\right)_{inc}\rho_0\cdot A} = \frac{\dot{m}^2}{\left(C_d\right)_{inc}\rho_0\cdot A}$$

Substitute into Force Eqn.

$$\left(\frac{\dot{m}^2}{\rho_0\cdot A}\right)\cdot\frac{1}{2\left(C_d\right)^2_{inc}} + f\,\frac{\dot{m}^2}{\rho_0\cdot A} = \dot{m}\cdot U_e$$

Force Balance Eq. Becomes

$$\left(\frac{\dot{m}^2}{\rho_0\cdot A}\right)\cdot\frac{1}{2\left(C_d\right)^2_{inc}} + f\,\frac{\dot{m}^2}{\rho_0\cdot A} = \frac{\dot{m}^2}{\left(C_d\right)_{inc}\rho_0\cdot A}$$

Dividing thru by $\dfrac{\dot{m}^2}{\rho_0\cdot A}$

$$\frac{1}{2\left(C_d\right)^2_{inc}} + f = \frac{1}{\left(C_d\right)_{inc}} \rightarrow \boxed{f = \frac{1}{\left(C_d\right)_{inc}} - \frac{1}{2\left(C_d\right)^2_{inc}}}$$

Injector Compressibility Analysis (3)

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$$Subcritical\ Flow: \left(\frac{p}{P_0}\right) > \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

Look at Port Interface Fluid Velocity

→ Flow Entering Port is Isentropic

$$\frac{P_0}{p} = \left(1+\left(\frac{\gamma-1}{2}\right)M^2\right)^{\frac{\gamma}{\gamma-1}} \rightarrow \left(\frac{P_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = \left(1+\left(\frac{\gamma-1}{2}\right)M^2\right) = \left(1+\left(\frac{\gamma-1}{2}\right)\frac{V^2}{\gamma R_g T}\right)$$

$$\left(\frac{P_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = \frac{T_0}{T} \rightarrow substitute \rightarrow \frac{T_0}{T} = \left(1+\left(\frac{\gamma-1}{2}\right)\frac{V^2}{\gamma R_g T}\right)$$

$$Solve \rightarrow \frac{1}{2}V^2 = \frac{\gamma}{\gamma-1}R_g T_0 - \frac{\gamma}{\gamma-1}R_g T$$

$$Substitute\ Gas\ Law \rightarrow \frac{p}{\rho} = R_g T$$

$$\frac{V^2}{2} + \left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho} = \frac{\gamma}{\gamma-1}\left(\frac{P_0}{\rho_0}\right) \rightarrow \text{"Compressible Bernoulli Equation"}$$

$$V^2 = \left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{P_0}{\rho_0} - \frac{p}{\rho}\right) = \left(\frac{2\gamma}{\gamma-1}\frac{P_0}{\rho_0}\right)\left(1 - \frac{p}{P_0}\cdot\frac{\rho_0}{\rho}\right)$$

$$\frac{\rho_0}{\rho} = \left(\frac{P_0}{p}\right)^{\frac{1}{\gamma}} = \left(\frac{p}{P_0}\right)^{-\frac{1}{\gamma}} \rightarrow$$

$$V^2 = \left(\frac{2\gamma}{\gamma-1}\frac{P_0}{\rho_0}\right)\left(1 - \frac{p}{P_0}\cdot\left(\frac{p}{P_0}\right)^{-\frac{1}{\gamma}}\right) = \left(\frac{2\gamma}{\gamma-1}\cdot\frac{P_0}{\rho_0}\right)\left(1-\left(\frac{p}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right)$$

$$V = \sqrt{\left(\frac{2\gamma}{\gamma-1}\cdot\frac{P_0}{\rho_0}\right)\left(1-\left(\frac{p}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right)}$$

Alternate Form of Compressible Bernoulli Equation

→ ***Flow is Isentropic***

$$\frac{P_0}{p} = \left(1 + \left(\frac{\gamma - 1}{2}\right) M^2\right)^{\frac{\gamma}{\gamma-1}} \rightarrow \left(\frac{P_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = \left(1 + \left(\frac{\gamma - 1}{2}\right) M^2\right) = \left(1 + \left(\frac{\gamma - 1}{2}\right) \frac{V^2}{\gamma R_g T}\right)$$

$$\left(\frac{P_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = \frac{T_0}{T} \rightarrow substitute \rightarrow \frac{T_0}{T} = \left(1 + \left(\frac{\gamma - 1}{2}\right) \frac{V^2}{\gamma R_g T}\right)$$

$$Solve \rightarrow \frac{1}{2} V^2 = \frac{\gamma}{\gamma - 1} R_g T_0 - \frac{\gamma}{\gamma - 1} R_g T$$

$$Substitute\ Gas\ Law \rightarrow \frac{p}{\rho} = R_g T$$

Compressible Bernoulli Eqn. in "Standard Form"

$$\frac{V^2}{2} + \left(\frac{\gamma}{\gamma - 1}\right) \frac{p}{\rho} = \frac{\gamma}{\gamma - 1} \left(\frac{P_0}{\rho_0}\right)$$

Compare to Incompressible Bernoulli Equation

$$p + \rho \cdot \frac{V^2}{2} = P_0 \rightarrow \boxed{\frac{V^2}{2} + \frac{p}{\rho} = \frac{P_0}{\rho}}$$

Compare to Incompressible Bernoulli Equation

$$p + \rho \cdot \frac{V^2}{2} = P_0 \rightarrow \boxed{\frac{V^2}{2} + \frac{p}{\rho} = \frac{P_0}{\rho}}$$

$$Sonic\ Velocity \rightarrow c = \sqrt{\gamma \cdot R_g \cdot T} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_{\Delta s = 0}}$$

$$Incompressible\ Flow \rightarrow \partial \rho = 0 \rightarrow c_{incr} = \infty$$

$$c_{incr} = \infty \rightarrow \gamma = \infty$$

$$\rightarrow \lim_{\gamma \rightarrow \infty} \left[\frac{V^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{\gamma}{\gamma - 1} \frac{P_0}{\rho}\right] = \boxed{\frac{V^2}{2} + \frac{p}{\rho} = \frac{P_0}{\rho}}!$$

At Mach zero, Compressible Bernoulli Reduces to Incompressible

Injector Compressibility Analysis (4)

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$$Subcritical\ Flow: \left(\frac{p}{P_0}\right) > \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

Apply to Injector Port Exit (Sometimes called Saint-Venant/Wanzel Eqn.)

$$V = U_e = \sqrt{\left(\frac{2\gamma}{\gamma-1}\cdot\frac{P_0}{\rho_0}\right)\left(1-\left(\frac{p}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right)} = \sqrt{\left(\frac{2\gamma}{\gamma-1}\cdot\frac{P_0}{\rho_0}\right)\left(1-r^{\frac{\gamma-1}{\gamma}}\right)}$$

Write U_e, exit momentum in terms of K_n

$$K_n = \sqrt{\frac{2\gamma}{\gamma-1}\cdot r^{\frac{2}{\gamma}}\left[1-r^{\frac{\gamma-1}{\gamma}}\right]}$$

$$U_e = \sqrt{\left(\frac{2\gamma}{\gamma-1}\cdot\frac{P_0}{\rho_0}\right)\left(1-r^{\frac{\gamma-1}{\gamma}}\right)} \rightarrow U_e = \frac{K_n}{r^{\frac{1}{\gamma}}}\sqrt{\frac{P_0}{\rho_0}}$$

$$\dot{m} = K_n\cdot C_d\cdot A\cdot\sqrt{P_0\cdot\rho_0} \rightarrow \dot{m}U_e = \frac{C_d\cdot K_n^{\ 2}\cdot A\cdot P_0}{r^{\frac{1}{\gamma}}}$$

Write F in terms of K_n

$$F \equiv f\frac{\dot{m}^2}{\rho_0\cdot A} = f\frac{\left(K_n\cdot C_d\cdot A\cdot\sqrt{P_0\cdot\rho_0}\right)^2}{\rho_0\cdot A} = f\cdot K_n^{\ 2}\cdot C_d^{\ 2}\cdot A\cdot P$$

Substitute into Force Balance Equation

$$\left(P_0-p\right)\cdot A + F - \dot{m}\cdot U_e = 0 \rightarrow P_0\left(1-r\right)\cdot A + f\cdot K_n^{\ 2}\cdot C_d^{\ 2}\cdot A\cdot P_0 - \frac{C_d\cdot K_n^{\ 2}\cdot A\cdot P_0}{r^{\frac{1}{\gamma}}} = 0$$

Injector Compressibility Analysis (5)

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$$Subcritical\ Flow: \left(\frac{p}{P_0}\right) > \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

Force Balance Equation

$$P_0(1-r)\cdot A + f\cdot K_n^{\,2}\cdot C_d^{\,2}\cdot A\cdot P_0 - \frac{C_d\cdot K_n^{\,2}\cdot A\cdot P_0}{r^{\frac{1}{\gamma}}} = 0$$

Simplify

$$f\cdot C_d^{\,2} - C_d\cdot r^{-\frac{1}{\gamma}} + \left(\frac{1-r}{K_n^{\,2}}\right) = 0$$

Solve Quadratic Equation

$$C_d = \frac{r^{-\frac{1}{\gamma}} \pm \sqrt{r^{-\frac{2}{\gamma}} - 4\cdot f\cdot\left(\frac{1-r}{K_n^{\,2}}\right)}}{2\cdot f} = \frac{1-\sqrt{1-f\cdot\left(2\cdot r^{\frac{1}{\gamma}}\right)^2\left(\frac{1-r}{K_n^{\,2}}\right)}}{2\cdot f\cdot r^{\frac{1}{\gamma}}} \rightarrow C_d < 1$$

$$C_d = \frac{1-\sqrt{1-f\cdot\left(2\cdot r^{\frac{1}{\gamma}}\right)^2\left(\frac{1-r}{K_n^{\,2}}\right)}}{2\cdot f\cdot r^{\frac{1}{\gamma}}}$$

- K_n depends only on **r** and ***γ***, … contraction coefficient for subcritical flows is a function of ***r***, ***γ***, and ***f***,

- ***f*** calculated from the incompressible-flow discharge coefficient, $(C_d)_{inc}$.

Injector Compressibility Analysis (6)

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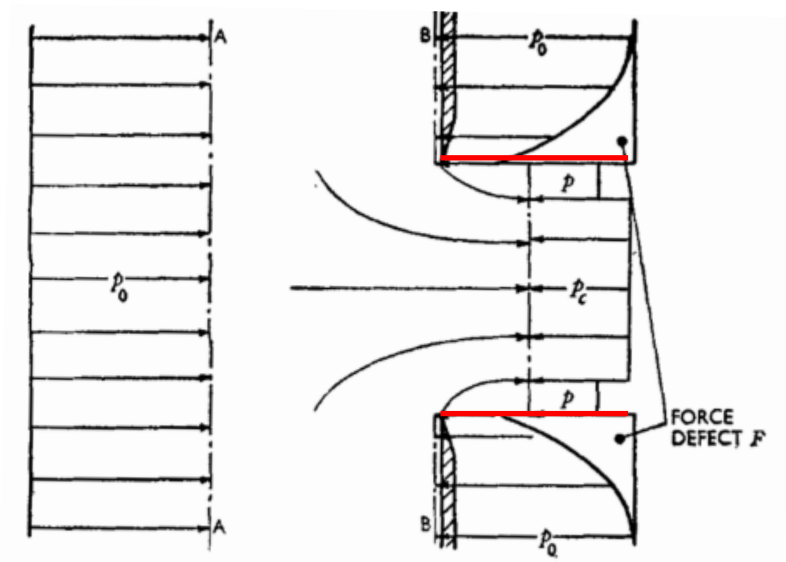
Supercritical Flow:

$$\left(\frac{p}{P_0}\right) \le \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

• Pressure p^* and Velocity U^* at Sonic Conditions

Force Balance Equation

$$\left(P_0 - p\right)\cdot A + C_{d\,inc}\cdot A\cdot\left(p - p^*\right) + F = \dot{m}\cdot U_e$$



b For supercritical conditions.

• As before, left-hand side of force equation consists of direct driving force across the area of the orifice, together with the force defect,

• In this case, however, although force on boundary AA is unchanged, back pressure across the area of the orifice, A , now consists of two parts,

• Critical pressure, p^*, being acts across the throat area ($A^* = C_d \cdot A$) and the downstream pressure, p, to act over the area remaining, 1- $C_d \cdot A$.

Injector Compressibility Analysis (6)

Proceedings of the Institute of Mechanical Engineers
On the Flow of a Compressible Fluid through Orifices
D. A. Jobson,
First Published June 1, 1955 Research Article
https://doi.org/10.1243/PIME_PROC_1955_169_077_02

Supercritical Flow:

$$\left(\frac{p}{P_0}\right) \leq \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

- Pressure p^* and Velocity U^* at Sonic Conditions

Sub $r^* = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$ ***into the previous expressions to derive critical values for*** $\left\{U_e, \dot{m}, K_n\right\}$

$$K_n = \sqrt{\frac{2\gamma}{\gamma-1} \cdot r^{*\frac{2}{\gamma}} \left[1 - r^{*\frac{\gamma-1}{\gamma}}\right]} = \sqrt{\gamma \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

$$U_e = \sqrt{\left(\frac{2\gamma}{\gamma-1} \cdot \frac{P_0}{\rho_0}\right)\left(1 - r^{*\frac{\gamma-1}{\gamma}}\right)} = \sqrt{\left(\frac{2\gamma}{\gamma+1} \cdot \frac{P_0}{\rho_0}\right)}$$

$$\dot{m} = K_n \cdot C_d \cdot A \cdot \sqrt{P_0 \cdot \rho_0} = C_d \cdot A \cdot \sqrt{\left(\gamma \cdot \frac{P_0}{\rho_0} \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right)}$$

- **Substitute into the Force Balance Equation and Collect Terms**

$$\left(1-r\right) \cdot A \cdot P_0 + \left(r - r^*\right) \cdot C_d \cdot A \cdot P_0 + f \cdot C_d{}^2 K_n{}^2 \cdot A \cdot P_0 - \frac{C_d \cdot K_n{}^2 \cdot A \cdot P_0}{\left(r^*\right)^{\frac{1}{\gamma}}} = 0$$

Injector Compressibility Analysis (7)

Proceedings of the Institute of Mechanical Engineers
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D. A. Jobson,
First Published June 1, 1955 Research Article
https://doi.org/10.1243/PIME_PROC_1955_169_077_02

Supercritical Flow:

$$\left(\frac{p}{P_0}\right) \le \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

• Pressure p^* and Velocity U^* at Sonic Conditions

Sub $r^* = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$ ***into the previous expressions to derive critical values for*** $\{U_e, \dot{m}, K_n\}$

$$K_n = \sqrt{\frac{2\gamma}{\gamma-1}\cdot r^{*\frac{2}{\gamma}}\left[1-r^{*\frac{\gamma-1}{\gamma}}\right]} = \sqrt{\gamma\cdot\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

$$U_e = \sqrt{\left(\frac{2\gamma}{\gamma-1}\cdot\frac{P_0}{\rho_0}\right)\left(1-r^{*\frac{\gamma-1}{\gamma}}\right)} = \sqrt{\left(\frac{2\gamma}{\gamma+1}\cdot\frac{P_0}{\rho_0}\right)}$$

$$\dot{m} = K_n\cdot C_d\cdot A\cdot\sqrt{P_0\cdot\rho_0} = C_d\cdot A\cdot\sqrt{\left(\gamma\cdot\frac{P_0}{\rho_0}\cdot\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right)}$$

• Substitute into the Force Balance Equation and Collect Terms

$$\left(1-r\right)\cdot A\cdot P_0 + \left(r-r^*\right)\cdot C_d\cdot A\cdot P_0 +$$

$$f\cdot C_d^{\,2} K_n^{\,2}\cdot A\cdot P_0 - \frac{C_d\cdot K_n^{\,2}\cdot A\cdot P_0}{\left(r^*\right)^{\frac{1}{\gamma}}} = 0$$

• Divide Through by $K_n^{\,2}\cdot A\cdot P_0$

$$\left(1-r\right)\cdot\frac{A\cdot P_0}{K_n^{\,2}\cdot A\cdot P_0} + \left(r-r^*\right)\cdot C_d\cdot\frac{A\cdot P_0}{K_n^{\,2}\cdot A\cdot P_0} - \frac{C_d}{\left(r^*\right)^{\frac{1}{\gamma}}} = 0$$

• Simplify ..

$$f\cdot C_d^{\,2} - \frac{1}{\left(r^*\right)^{\frac{1}{\gamma}}}\left[1-\left(\frac{r-r^*}{K_n^{\,2}}\right)\left(r^*\right)^{\frac{1}{\gamma}}\right]\cdot C_d + \left(\frac{1-r}{K_n^{\,2}}\right) = 0$$

Injector Compressibility Analysis (8)

Proceedings of the Institute of Mechanical Engineers
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D. A. Jobson,
First Published June 1, 1955 Research Article
https://doi.org/10.1243/PIME_PROC_1955_169_077_02

Supercritical Flow:

$$\left(\frac{p}{P_0}\right) \le \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

- Pressure p^* and Velocity U^* at Sonic Conditions

- **Quadratic Equation ..** $r^* = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$

$$f \cdot C_d^{\,2} - \frac{1}{\left(r^*\right)^{\frac{1}{\gamma}}}\left[1 - \left(\frac{r - r^*}{K_n^{\,2}}\right)\left(r^*\right)^{\frac{1}{\gamma}}\right] \cdot C_d + \left(\frac{1-r}{K_n^{\,2}}\right) = 0$$

- **Solution**

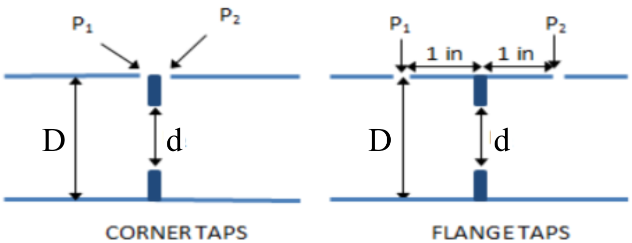
$$C_d = \left(\frac{1}{2 \cdot f \cdot r^{*\frac{1}{\gamma}}}\right) \cdot \left[\left\{1 + \frac{\left(r^* - \left(\frac{p}{P_0}\right)\right) \cdot r^{*\frac{1}{\gamma}}}{K_n^2}\right\} - \sqrt{\left(1 + \frac{\left(r^* - \left(\frac{p}{P_0}\right)\right) \cdot r^{*\frac{1}{\gamma}}}{K_n^2}\right)^2 - \frac{\left(2 \cdot r^{*\frac{1}{\gamma}}\right)^2 \cdot \left(1 - \left(\frac{p}{P_0}\right)\right) \cdot f}{K_n^2}}\right]$$

ASME "STANDARD MODEL"

DIMENSIONS IN INCHES

Corner tap:

$$C_v = \left[0.5991 + \frac{0.0044}{D} + \left(0.3155 + \frac{0.0175}{D}\right)\left(\left(\frac{d}{D}\right)^4 + 2\left(\frac{d}{D}\right)^{16}\right)\right]\sqrt{1-(d/D)^4}$$
$$+\left[\frac{0.52}{D} - 0.192 + \left(16.48 - \frac{1.16}{D}\right)\left(\left(\frac{d}{D}\right)^4 + 4\left(\frac{d}{D}\right)^{16}\right)\right]\sqrt{\frac{1-(d/D)^4}{Re_D}}$$



Simple orifice

***ASME MFC-14M-**

Flange tap:

$$C_v = \left[0.598 + 0.468\left(\left(\frac{d}{D}\right)^4 + 10\left(\frac{d}{D}\right)^{12}\right)\right]\sqrt{1-(d/D)^4} + \left(0.87 + 8.1\left(\frac{d}{D}\right)^4\right)\sqrt{\frac{1-(d/D)^4}{Re_D}}$$

$$C_d \equiv \frac{C_v}{\left[1-\left(\frac{A_2}{A_1}\right)^2\right]^{\frac{1}{2}}} = \frac{C_v}{\left[1-\left(\frac{d}{D}\right)^4\right]^{\frac{1}{2}}} = \frac{C_v}{\sqrt{1-\beta^4}}$$

β --> "contraction ratio" "d/D"

- **need to take area change into account**

Calculating $C_d i$ from A_1, A_2, P_1, P_2

Incompressible Injector Discharge Coefficient

$\rightarrow$ *Calculate* P_0

$$P_0 = \left[\frac{\left(\frac{A_1}{A_2}\right)^2 \cdot (p_1)^{\frac{\gamma+1}{\gamma}} - (p_2)^{\frac{\gamma+1}{\gamma}}}{\left(\frac{A_1}{A_2}\right)^2 \cdot (p_1)^{\frac{2}{\gamma}} - (p_2)^{\frac{2}{\gamma}}}\right]^{\frac{\gamma}{\gamma-1}} = \left[\frac{\left(\frac{D_1}{D_2}\right)^4 \cdot (p_1)^{\frac{\gamma+1}{\gamma}} - (p_2)^{\frac{\gamma+1}{\gamma}}}{\left(\frac{D_1}{D_2}\right)^4 \cdot (p_1)^{\frac{2}{\gamma}} - (p_2)^{\frac{2}{\gamma}}}\right]^{\frac{\gamma}{\gamma-1}}$$

if Subcritical

.. else

$$P_0 \sim p_1$$

**** *Begin Iteration*

$\rightarrow$ *Calculate* $\dot{m}$, *Assume* C_{d0} *Value*

Compressible, Subcritical

$$\dot{m} = C_{d0} \cdot P_0 \cdot A_1 \sqrt{\frac{2\gamma}{(\gamma-1)(R_g \cdot T_0)}\left[\left(\frac{p_1}{P_0}\right)^{\frac{2}{\gamma}} - \left(\frac{p_1}{P_0}\right)^{\frac{\gamma+1}{\gamma}}\right]}$$

Compressible, Supercritical

$$\dot{m}_{outlet} = C_{d0} \cdot P_0 \cdot A_1 \cdot \sqrt{\frac{\gamma}{R_g \cdot T_0}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

Option 1 Start with Compressible massflow

$\rightarrow$ *Calculate Reynolds number*

$$\mathrm{Re}_{D_1} = \frac{\rho_1 \cdot V_1 \cdot D_1}{\mu_1} = \frac{\frac{\dot{m}}{A_1} \cdot D_1}{\mu_1} = \frac{\frac{\dot{m}}{\frac{\pi}{4}D_1^2} \cdot D_1}{\mu_1} = \frac{4 \cdot \dot{m}}{\pi \cdot D_1 \cdot \mu_1}$$

$\rightarrow$ *Calculate* $C_v i$ *(ASME MFC−14M−2001)*

Corner Tap: (assume D_1, D_2 *−−> inches)*

$$C_v i = \left[0.5991 + \frac{0.0044}{D_1} + \left(0.3155 + \frac{0.0175}{D_1}\right) \cdot \left(\frac{D_2}{D_1}\right)^4 \left(1 + 2 \cdot \left(\frac{D_2}{D_1}\right)^{12}\right)\right] \cdot \sqrt{1 - \left(\frac{D_2}{D_1}\right)^4} + \left[\frac{0.52}{D_1} - 0.192 + \left(16.48 - \frac{1.16}{D_1}\right) \cdot \left(\frac{D_2}{D_1}\right)^4 \left(1 + 4 \cdot \left(\frac{D_2}{D_1}\right)^{12}\right)\right] \cdot \sqrt{\left(1 - \left(\frac{D_2}{D_1}\right)^4\right)\frac{1}{\mathrm{Re}_{D_1}}}$$

Flange Tap:

$$C_v i = \left[0.598 + 0.468 \cdot \left(\frac{D_2}{D_1}\right)^4 \left(1 + 10 \cdot \left(\frac{D_2}{D_1}\right)^8\right)\right] \cdot \sqrt{1 - \left(\frac{D_2}{D_1}\right)^4} + \left(0.87 + 8.1 \cdot \left(\frac{D_2}{D_1}\right)^4\right) \cdot \sqrt{\left(1 - \left(\frac{D_2}{D_1}\right)^4\right)\frac{1}{\mathrm{Re}_{D_1}}}$$

$\rightarrow$ *Calculate* $C_d i \rightarrow \beta = \sqrt{1 - \left(\frac{D_2}{D_1}\right)^4}$

$$C_d i = \frac{C_v i}{\sqrt{1 - \left(\frac{D_2}{D_1}\right)^4}}$$

Return to ****, *Iterate Until* $\left|\frac{(C_d i)_{j+1} - (C_d i)_j}{(C_d i)_j}\right| < \varepsilon$

Calculating $C_d i$ from A_1, A_2, P_1, P_2

Start with Incompressible analysis

Option 2 Start with Incompressible massflow

$$\rho_{inc} = \frac{P_1}{R_g \cdot T_1} \rightarrow \dot{m} = \left(\frac{C_d \cdot A_2}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}} \right) \sqrt{2 \cdot \rho \cdot (p_1 - p_2)}$$

**** *Begin Iteration*

$\rightarrow$ *Calculate Reynolds number*

$$\mathrm{R}_{e_{D_1}} = \frac{\rho_1 \cdot V_1 \cdot D_1}{\mu_1} = \frac{\frac{\dot{m}}{A_1} \cdot D_1}{\mu_1} = \frac{\frac{\dot{m}}{\frac{\pi}{4} D_1^2} \cdot D_1}{\mu_1} = \frac{4 \cdot \dot{m}}{\pi \cdot D_1 \cdot \mu_1}$$

$\rightarrow$ *Calculate* $C_v i$ (*ASME MFC* − 14*M* − 2001)

Flange Tap: (*assume D,d* −− > *inches*)

$$C_v i = \left[0.5991 + \frac{0.0044}{D_1} + \left(0.3155 + \frac{0.0175}{D_1}\right) \cdot \left(\frac{D_2}{D_1}\right)^4 \left(1 + 2 \cdot \left(\frac{D_2}{D_1}\right)^{12}\right) \right] \cdot \sqrt{1 - \left(\frac{D_2}{D_1}\right)^4} +$$
$$\left[\frac{0.52}{D_1} - 0.192 + \left(16.48 - \frac{1.16}{D_1}\right) \cdot \left(\frac{D_2}{D_1}\right)^4 \left(1 + 4 \cdot \left(\frac{D_2}{D_1}\right)^{12}\right) \right] \cdot \sqrt{\left(1 - \left(\frac{D_2}{D_1}\right)^4\right) \frac{1}{\mathrm{R}_{e_{D_1}}}}$$

Flange Tap: (*D,d* −− > *inches*)

$$C_v i = \left[0.598 + 0.468 \cdot \left(\frac{d}{D}\right)^4 \left(1 + 10 \cdot \left(\frac{d}{D}\right)^8\right) \right] \cdot \sqrt{1 - \left(\frac{d}{D}\right)^4} + \left(0.87 + 8.1 \cdot \left(\frac{d}{D}\right)^4\right) \cdot \sqrt{\left(1 - \left(\frac{d}{D}\right)^4\right) \frac{1}{\mathrm{R}_{e_D}}} =$$

$$\rightarrow \text{Calculate } C_d i \rightarrow \beta = \sqrt{1 - \left(\frac{D_2}{D_1}\right)^4} \rightarrow C_d i = \frac{C_v i}{\sqrt{1 - \left(\frac{D_2}{D_1}\right)^4}} =$$

Return to ****, *Iterate Until* $\left| \frac{(C_d i)_{j+1} - (C_d i)_j}{(C_d i)_j} \right| < \varepsilon$

UtahState UNIVERSITY

Mechanical & Aerospace Engineering

Collected Compressible Injector Equations

Proceedings of the Institute of Mechanical Engineers
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D. A. Jobson,
First Published June 1, 1955 Research Article
https://doi.org/10.1243/PIME_PROC_1955_169_077_02

$$Subcritical\ Flow: \left(\frac{p}{P_0}\right) > \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$r = \frac{p}{P_0}$$

$$f = \frac{1}{\left(C_d\right)_{inc}} - \frac{1}{2\left(C_d\right)^2_{inc}}$$

$$K_n = \sqrt{\frac{2\gamma}{\gamma-1} \cdot r^{\frac{2}{\gamma}}\left[1 - r^{\frac{\gamma-1}{\gamma}}\right]}$$

$$C_d = \frac{1 - \sqrt{1 - f \cdot \left(2 \cdot r^{\frac{1}{\gamma}}\right)^2 \left(\frac{1-r}{K_n^{\,2}}\right)}}{2 \cdot f \cdot r^{\frac{1}{\gamma}}}$$

$$\rightarrow \dot{m} = K_n \cdot C_d \cdot A \cdot \sqrt{P_0 \cdot \rho_0}$$

$$Supercritical\ Flow:$$

$$\left(\frac{p}{P_0}\right) \le \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \qquad r^* = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$f = \frac{1}{\left(C_d\right)_{inc}} - \frac{1}{2\left(C_d\right)^2_{inc}}$$

$$K_n = \sqrt{\frac{2\gamma}{\gamma-1} \cdot r^{*\frac{2}{\gamma}}\left[1 - r^{*\frac{\gamma-1}{\gamma}}\right]} = \sqrt{\gamma \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

$$C_d = \left(\frac{1}{2 \cdot f \cdot r^{*\frac{1}{\gamma}}}\right) \cdot \left[\left\{1 + \frac{\left(r^* - \left(\frac{p}{P_0}\right)\right) \cdot r^{*\frac{1}{\gamma}}}{K_n^2}\right\} - \sqrt{\left(1 + \frac{\left(r^* - \left(\frac{p}{P_0}\right)\right) \cdot r^{*\frac{1}{\gamma}}}{K_n^2}\right)^2 - \frac{\left(2 \cdot r^{*\frac{1}{\gamma}}\right)^2 \cdot \left(1 - \left(\frac{p}{P_0}\right)\right) \cdot f}{K_n^2}}\right]$$

$$\rightarrow \dot{m} = K_n \cdot C_d \cdot A \cdot \sqrt{P_0 \cdot \rho_0}$$

UtahState UNIVERSITY

Mechanical & Aerospace Engineering

Compressible Injector Equations, Plotted Results

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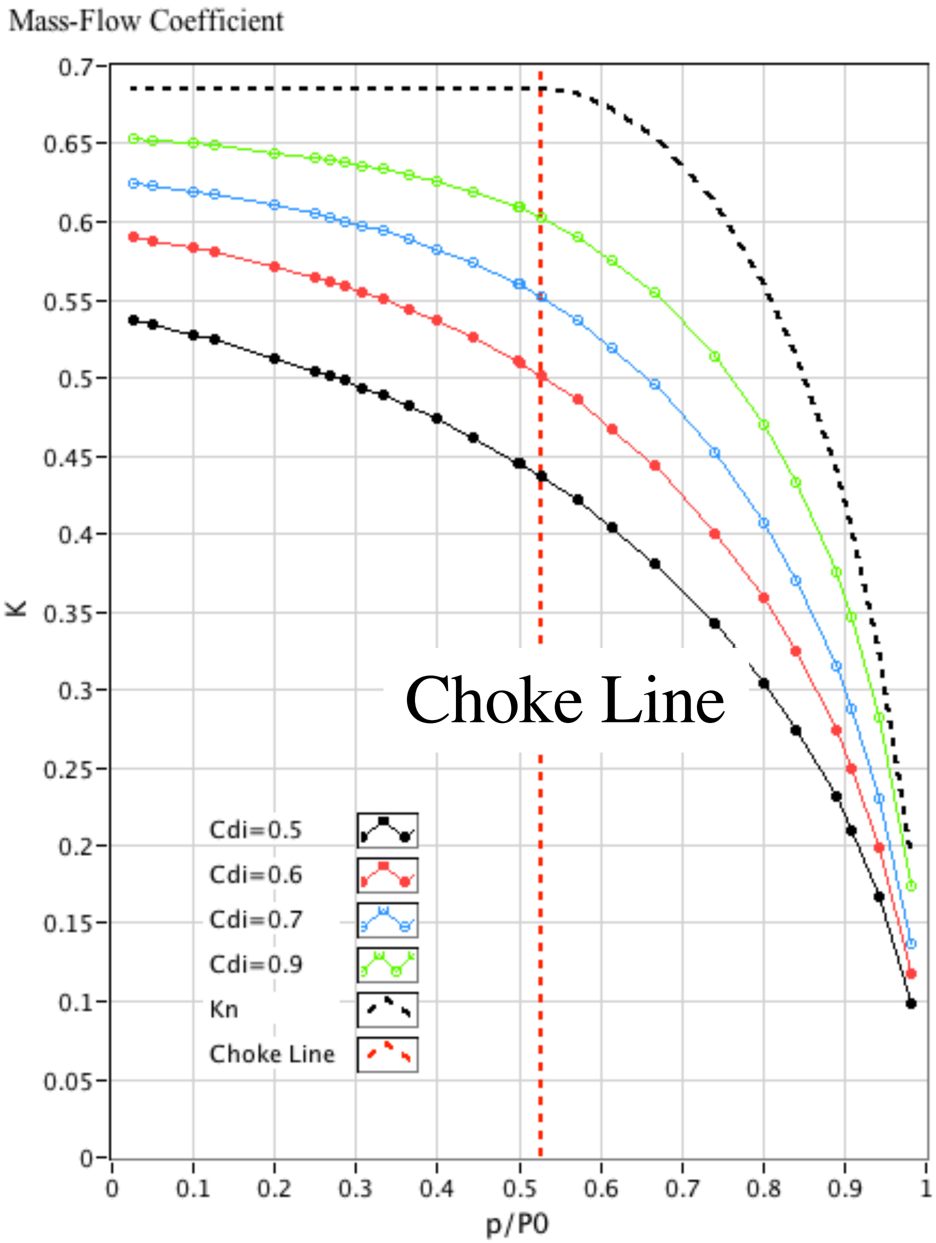
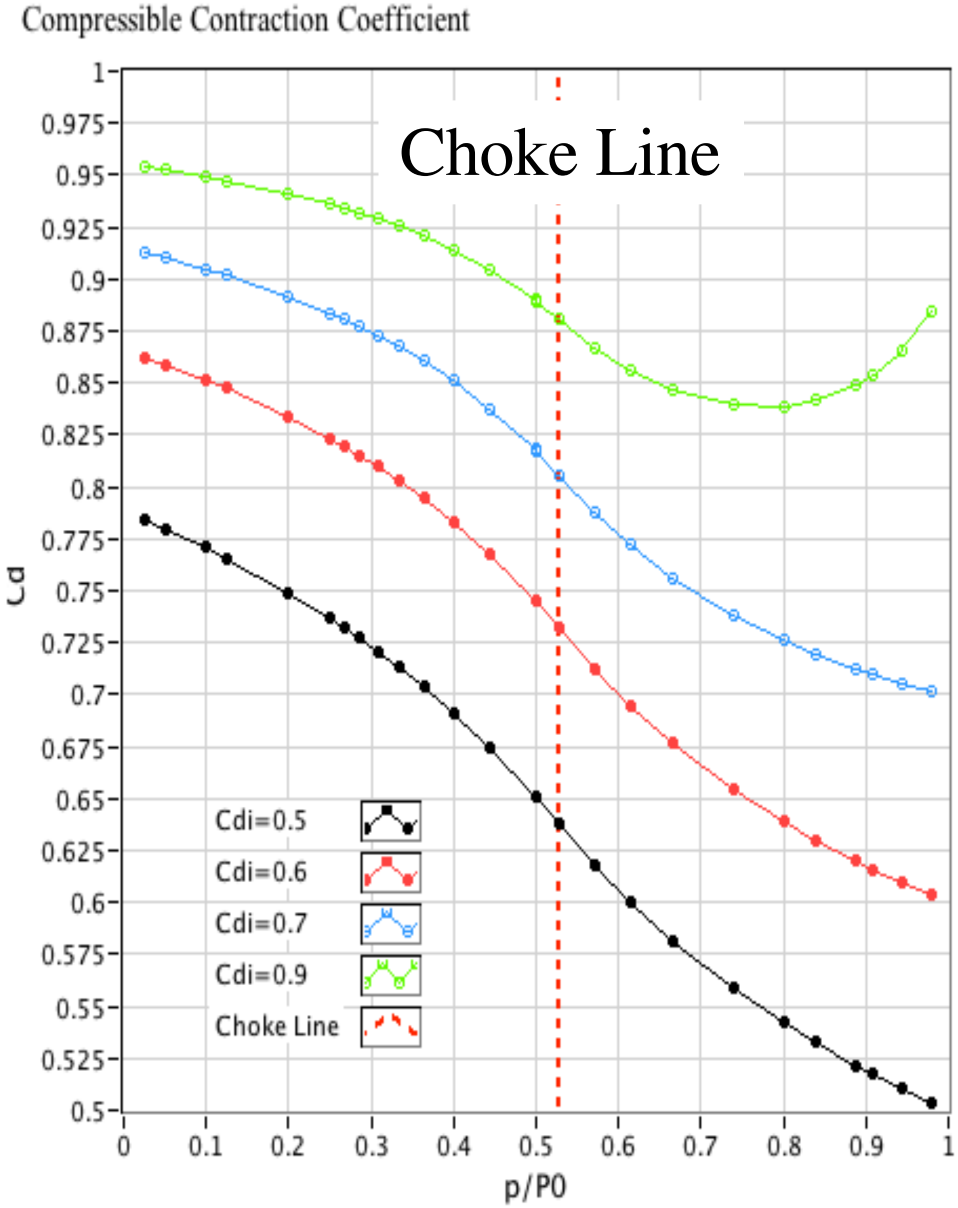
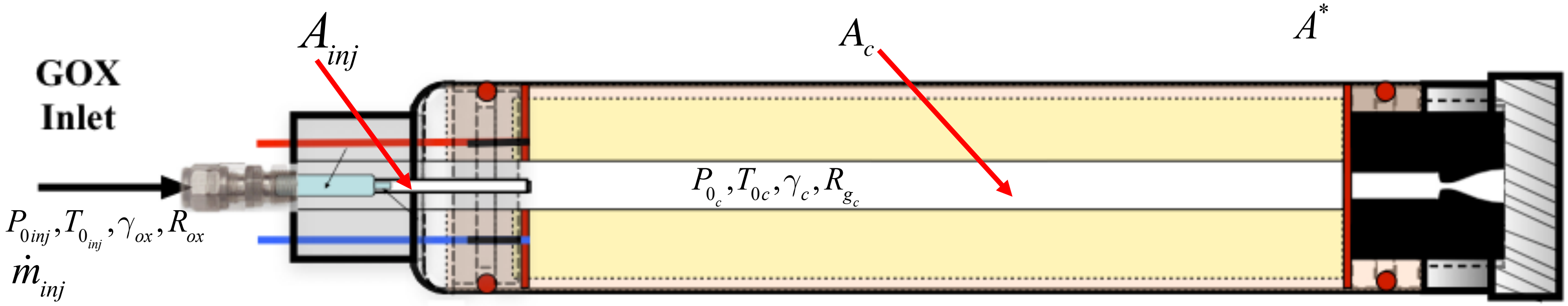
D. A. Jobson,
First Published June 1, 1955 Research Article
https://doi.org/10.1243/PIME_PROC_1955_169_077_02





Hybrid Ballistic Equations for Compressible Oxidizer



$$Subcritical: \left(\frac{P_{0inj}}{P_{0_c}}\right) < \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

$\rightarrow$ *Injector Not Choked*

$$K_n = \sqrt{\frac{2\cdot\gamma_{ox}}{\gamma_{ox}-1}\cdot\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)^{\frac{2}{\gamma_{ox}}}\left[1-\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)^{\frac{\gamma_{ox}-1}{\gamma_{ox}}}\right]} \rightarrow \dot{m}_{ox} = \left(K_n\cdot C_d\cdot A\right)_{inj}\cdot\frac{P_{0inj}}{\sqrt{R_{g_{ox}}T_{0_{inj}}}}$$

$$C_d = \frac{1}{2\cdot f\cdot\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)^{\frac{1}{\gamma_{ox}}}}\cdot\left[1-\sqrt{\left\{1-\left(2\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)^{\frac{1}{\gamma_{ox}}}\right)^2\left(1-\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)\right)\cdot f \Big/ K_n^2\right\}}\right]$$

Chamber Pressure :

$$\frac{\partial P_{0c}}{\partial t} = \frac{A_{burn}\dot{r}_{fuel}}{V_c}\left[\rho_{fuel}R_{g_c}T_{0c}-P_{0c}\right]-P_{0c}\left[\frac{A^*}{V_c}\sqrt{\gamma_c R_{g_c}T_0\left(\frac{2}{\gamma_c+1}\right)^{\frac{\gamma_c+1}{(\gamma_c-1)}}}\right]+\frac{R_{g_c}T_{0_c}}{V_c}\cdot\left\{\left(K_n\cdot C_d\cdot A_{inj}\right)_{inj}\cdot\frac{P_{0inj}}{\sqrt{R_{g_{ox}}\cdot T_{0_{inj}}}}\right\}$$

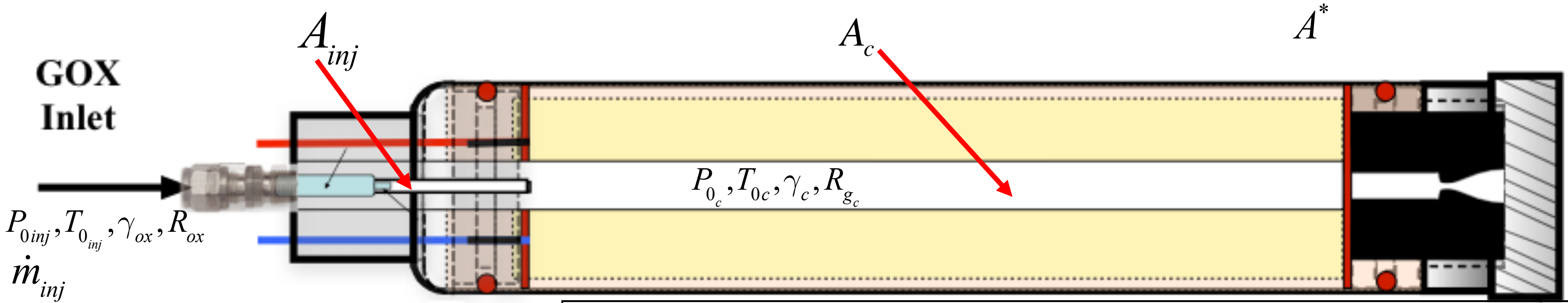
Regression :

0.23

$$\dot{r}_{fuel} = \left(\frac{0.047}{\rho_{fuel}\cdot\left(P_{r_c}\right)^{2/3}}\right)\cdot\left(\frac{C_{P_c}\cdot\left(T_{0_c}-T_{fuel_{surf}}\right)}{h_{v_{fuel}}}\right)\cdot\left(\frac{\dot{m}_{ox}}{A_c}\right)^{4/5}\cdot\left(\frac{\mu_c}{L}\right)^{1/5}$$

$$\dot{m}_{fuel} = \rho_{fuel}\cdot A_{burn}\cdot\dot{r}_{fuel} = \rho_{fuel}\cdot\pi\cdot\left(D\cdot L\right)_{port}\cdot\dot{r}_{fuel} \rightarrow O/F = \frac{\dot{m}_{ox}}{\dot{m}_{fuel}} = \frac{\left(K_n\cdot C_d\cdot A\right)_{inj}\cdot\frac{P_{0inj}}{\sqrt{R_{g_{ox}}T_{0_{inj}}}}}{\rho_{fuel}\cdot\pi\cdot\left(D\cdot L\right)_{port}\cdot\dot{r}_{fuel}}$$

Hybrid Ballistic Equations for Compressible Oxidizer



$$Supercritical: \left(\frac{P_{0inj}}{P_{0_c}}\right) \geq \left(\frac{\gamma_{ox}+1}{2}\right)^{\frac{\gamma_{ox}}{\gamma_{px}-1}}$$

$$\rightarrow Injector\ Choked \rightarrow r_c = \left(\frac{2}{\gamma_{0x}+1}\right)^{\frac{\gamma_{ox}}{\gamma_{ox}-1}}$$

$$K_n = \sqrt{\gamma_{ox}\cdot\left(\frac{2}{\gamma_{ox}+1}\right)^{\frac{\gamma_{ox}+1}{\gamma_{px}-1}}} \rightarrow \dot{m}_{ox} = \left(K_n\cdot C_d\cdot A\right)_{inj}\cdot\frac{P_{0inj}}{\sqrt{R_{g_{ox}}T_{0_{inj}}}}$$

$$C_d = \left(\frac{1}{2\cdot f\cdot r_c^{\frac{1}{\gamma_{ox}}}}\right)\cdot\left[\left\{1+\frac{\left(r_c-\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)\right)\cdot r_c^{\frac{1}{\gamma_{ox}}}}{K_n^2}\right\}-\sqrt{1+\frac{\left(r_c-\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)\right)\cdot r_c^{\frac{1}{\gamma_{ox}}}}{K_n^2}-\frac{\left(2\cdot r_c^{\frac{1}{\gamma_{ox}}}\right)^2\cdot\left(1-\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)\right)\cdot f}{K_n^2}}\right]$$

Chamber Pressure:

$$\frac{\partial P_{0c}}{\partial t} = \frac{A_{burn}\dot{r}_{fuel}}{V_c}\left[\rho_{fuel}R_{g_c}T_{0c}-P_{0c}\right]-P_{0c}\left[\frac{A^*}{V_c}\sqrt{\gamma_c R_{g_c}T_0\left(\frac{2}{\gamma_c+1}\right)^{\frac{\gamma_c+1}{(\gamma_c-1)}}}\right]+\frac{R_{g_c}T_{0_c}}{V_c}\cdot\left\{\left(K_n\cdot C_d\cdot A_{inj}\right)_{inj}\cdot\frac{P_{0inj}}{\sqrt{R_{g_{ox}}\cdot T_{0_{inj}}}}\right\}$$

Regression:

$$\dot{r}_{fuel} = \left(\frac{0.047}{\rho_{fuel}\cdot\left(P_{r_c}\right)^{2/3}}\right)\cdot\left(\frac{C_{P_c}\cdot\left(T_{0_c}-T_{fuel_{surf}}\right)}{h_{v_{fuel}}}\right)^{0.23}\cdot\left(\frac{\dot{m}_{ox}}{A_c}\right)^{4/5}\cdot\left(\frac{\mu_c}{L}\right)^{1/5}$$

$$\dot{m}_{fuel} = \rho_{fuel}\cdot A_{burn}\cdot\dot{r}_{fuel} = \rho_{fuel}\cdot\pi\cdot(D\cdot L)_{port}\cdot\dot{r}_{fuel} \rightarrow O/F = \frac{\dot{m}_{ox}}{\dot{m}_{fuel}} = \frac{\left(K_n\cdot C_d\cdot A\right)_{inj}\cdot\frac{P_{0inj}}{\sqrt{R_{g_{ox}}T_{0_{inj}}}}}{\rho_{fuel}\cdot\pi\cdot(D\cdot L)_{port}\cdot\dot{r}_{fuel}}$$

Injector Feed Coupling

Combustion Chamber Ballistic Equations

$\rightarrow$ *Incompressible*

Source for Injector Feed Back Coupling

Chamber Pressure:

$$\frac{\partial P_{0c}}{\partial t} = \frac{A_{burn} r_{fuel}}{V_c}\left[\rho_{fuel} R_{g_c} T_{0c} - P_{0c}\right] - P_{0c}\left[\frac{A^*}{V_c}\sqrt{\gamma_c R_{g_c} T_0 \left(\frac{2}{\gamma_c+1}\right)^{\frac{\gamma_c+1}{(\gamma_c-1)}}}\right] + \frac{R_{g_c} T_{0_c}}{V_c}\cdot\left(C_d \cdot A_{inj}\cdot\sqrt{2\cdot\rho_{ox}\cdot\left(P_{inj} - P_{0c}\right)}\right)$$

Oxidizer Massflow

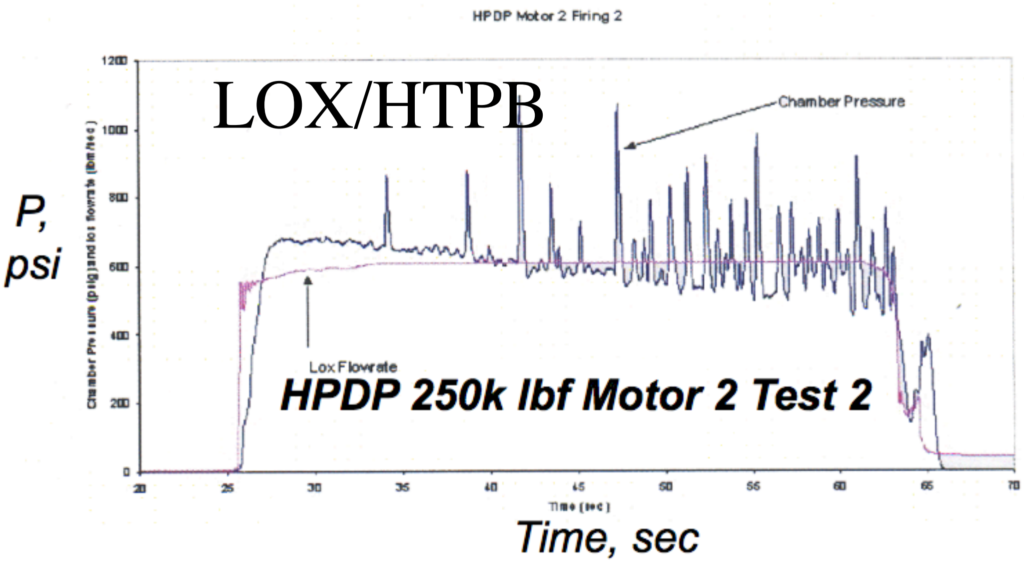
$$\dot{m}_{ox} = C_d \cdot A_{inj}\cdot\sqrt{2\cdot\rho_{ox}\cdot\left(P_{0inj} - P_{0c}\right)}$$

$$P_{0_c} \rightarrow drops \rightarrow \frac{\partial P_{0c}}{\partial t} \rightarrow increases \rightarrow P_{0_c} \rightarrow increases \rightarrow \frac{\partial P_{0c}}{\partial t} \rightarrow drops...etc$$

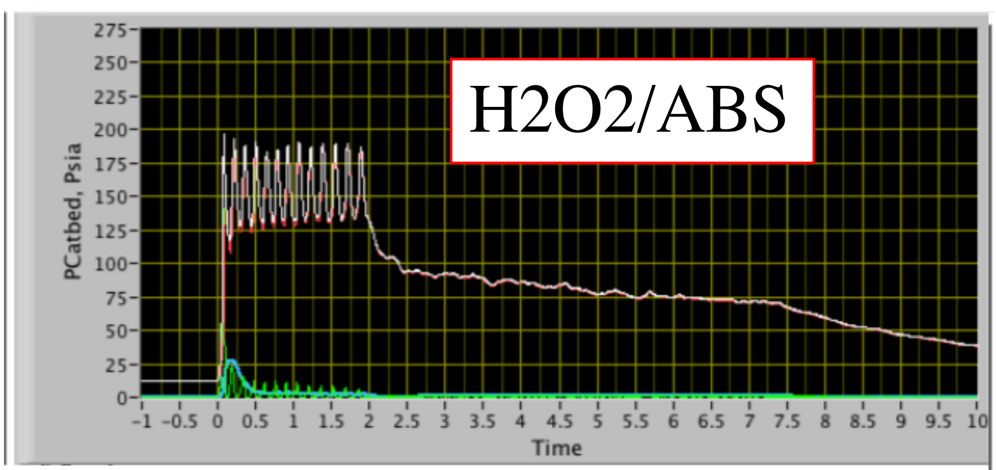
- Feed-Coupling Instability Especially Prevalent with Highly Incompressible Oxidizers, Strong Feedback Mechanism

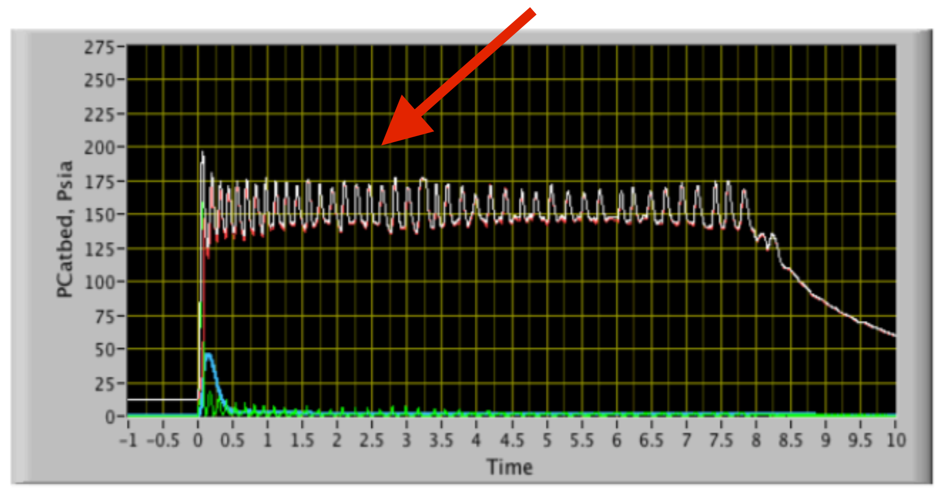
Injector Feed Coupling

Low Frequency Instabilities (2-100 Hz)



- Especially prevalent with Highly Incompressible Oxidizers





Injector Choking

- For Constant C_d →

$$\rightarrow Compressible, Choked\ Injector \rightarrow \left(\frac{P_{inj}}{P_{0_c}}\right) > \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

Chamber Pressure:

$$\frac{\partial P_{0c}}{\partial t} = \frac{A_{burn}\dot{r}_{fuel}}{V_c}\left[\rho_{fuel} R_{g_c} T_{0c} - P_{0c}\right] - P_{0c}\left[\frac{A^*}{V_c}\sqrt{\gamma_c R_{g_c} T_0 \left(\frac{2}{\gamma_c+1}\right)^{\frac{\gamma_c+1}{(\gamma_c-1)}}}\right] + \frac{R_{g_c} T_0}{V_c}\cdot\left(C_d \cdot A_{inj} \cdot P_{inj}\sqrt{\frac{\gamma_{ox}}{R_{g_{ox}} \cdot T_{0_{inj}}}\cdot\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma_{ox}+1}{\gamma_{ox}}}}\right)$$

Oxidizer Massflow

$$\dot{m}_{ox} = C_d \cdot A_{inj} \cdot P_{inj}\sqrt{\frac{\gamma_{ox}}{R_{g_{ox}} \cdot T_{0_{inj}}}\cdot\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma_{ox}+1}{\gamma_{ox}}}}$$

No Injector Feedback Mechanism

Injector Choking

- Allowing for Compressibility Effects on Cd→

$$Supercritical: \left(\frac{P_{0inj}}{P_{0_c}}\right) \geq \left(\frac{\gamma_{ox}+1}{2}\right)^{\frac{\gamma_{ox}}{\gamma_{px}-1}}$$

$$\rightarrow Injector\ Choked \rightarrow r_c = \left(\frac{2}{\gamma_{0x}+1}\right)^{\frac{\gamma_{ox}}{\gamma_{ox}-1}}$$

Chamber Pressure:

$$\frac{\partial P_{0c}}{\partial t} = \frac{A_{burn}\dot{r}_{fuel}}{V_c}\left[\rho_{fuel}R_{g_c}T_{0c} - P_{0c}\right] - P_{0c}\left[\frac{A^*}{V_c}\sqrt{\gamma_c R_{g_c} T_0 \left(\frac{2}{\gamma_c+1}\right)^{\frac{\gamma_c+1}{(\gamma_c-1)}}}\right] + \frac{R_{g_c}T_{0_c}}{V_c}\cdot\left\{\left(K_n\cdot C_d\cdot A_{inj}\right)_{inj}\cdot\frac{P_{0inj}}{\sqrt{R_{g_{ox}}\cdot T_{0_{inj}}}}\right\}$$

$$K_n = \sqrt{\gamma_{ox}\cdot\left(\frac{2}{\gamma_{ox}+1}\right)^{\frac{\gamma_{ox}+1}{\gamma_{px}-1}}} \rightarrow \dot{m}_{ox} = \left(K_n\cdot C_d\cdot A\right)_{inj}\cdot\frac{P_{0inj}}{\sqrt{R_{g_{ox}}T_{0_{inj}}}}$$

$$C_d = \left(\frac{1}{2\cdot f\cdot r_c^{\frac{1}{\gamma_{ox}}}}\right)\cdot\left[\left\{1+\frac{\left(r_c-\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)\right)\cdot r_c^{\frac{1}{\gamma_{ox}}}}{K_n^2}\right\} - \sqrt{1+\frac{\left(r_c-\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)\right)\cdot r_c^{\frac{1}{\gamma_{ox}}}}{K_n^2} - \frac{\left(2\cdot r_c^{\frac{1}{\gamma_{ox}}}\right)^2\cdot\left(1-\left(\frac{P_{0_c}}{P_{0_{inj}}}\right)\right)\cdot f}{K_n^2}}\right]$$

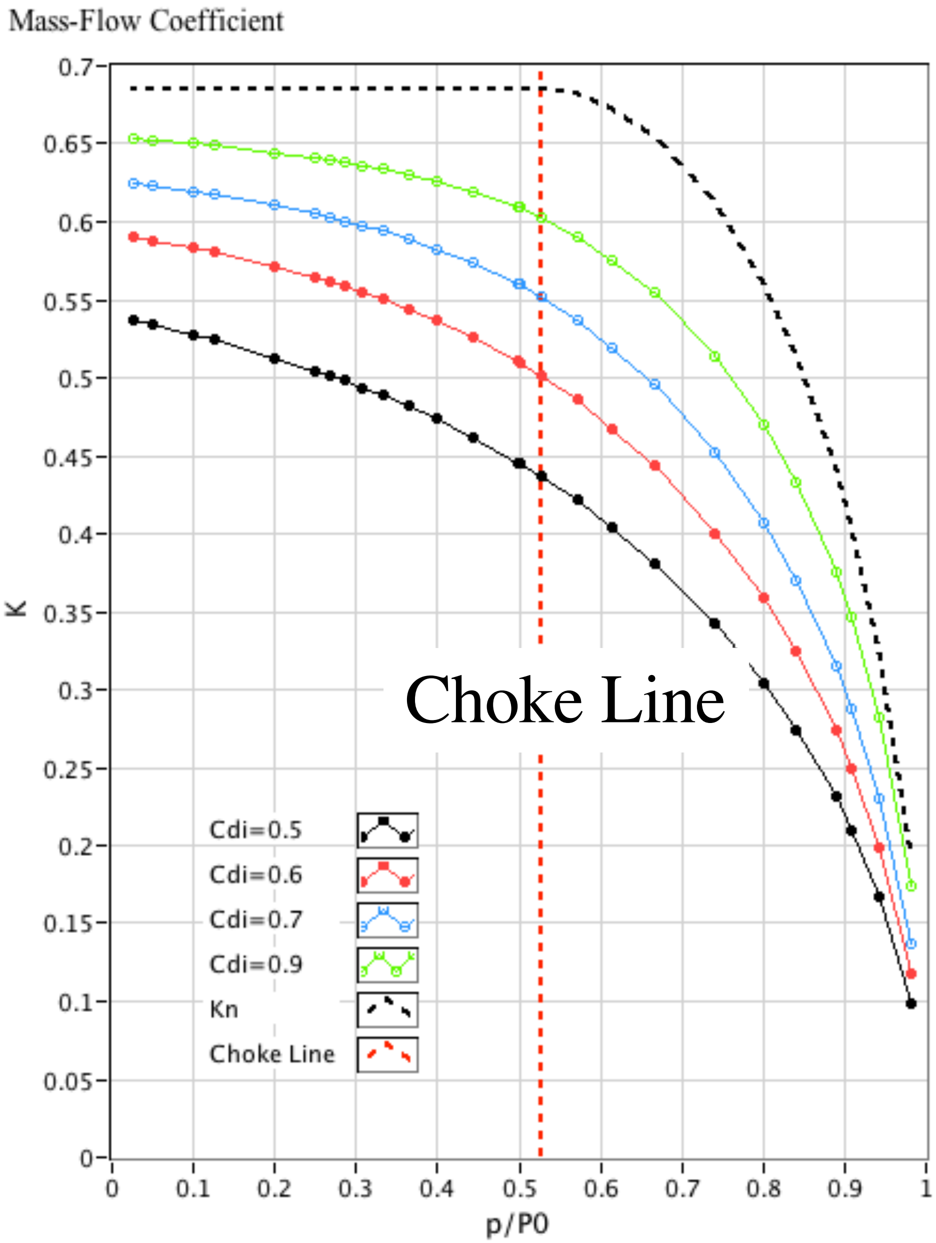
Much Weaker Feedback Mechanism

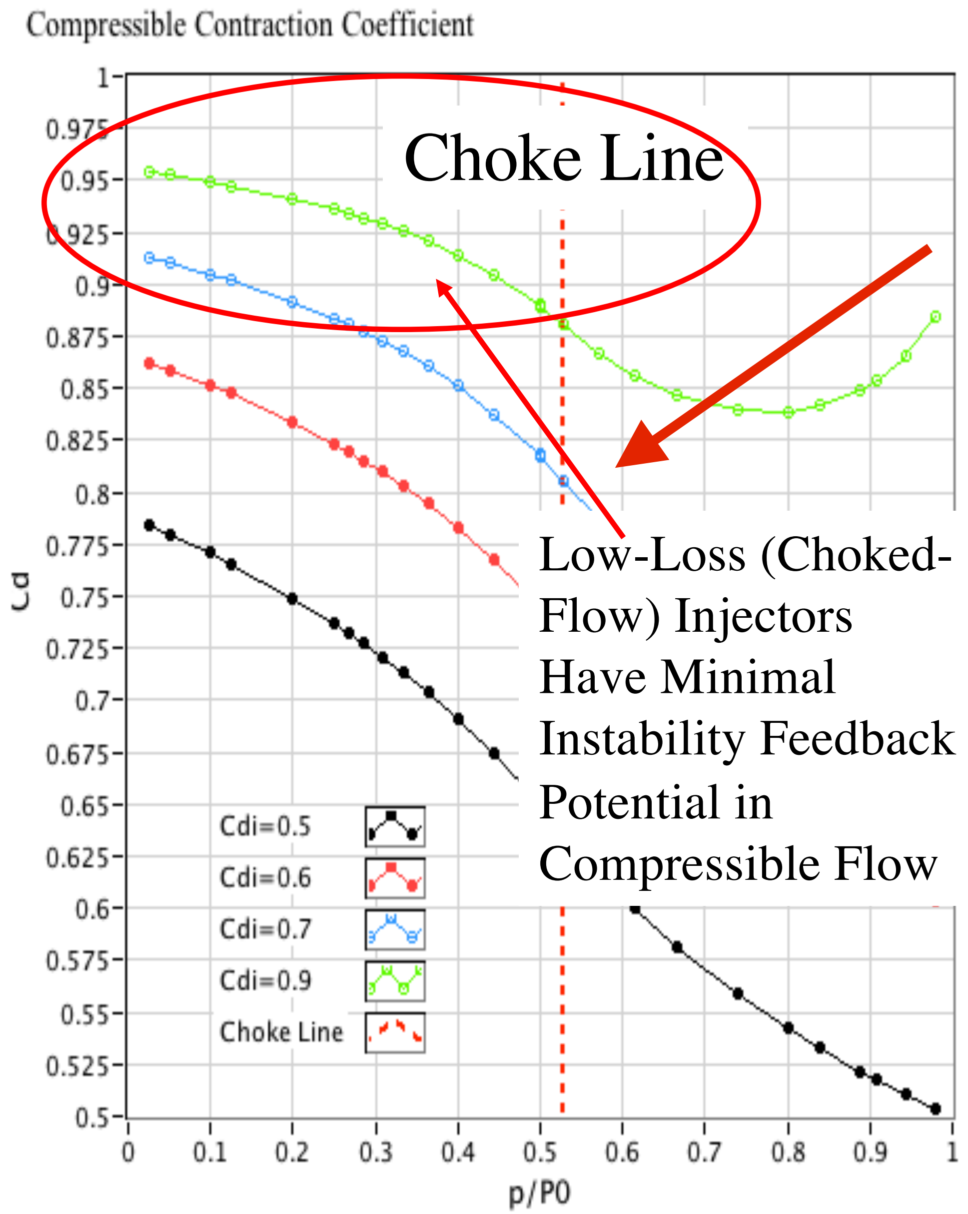
UtahState UNIVERSITY

Mechanical & Aerospace Engineering

Compressible Injector Equations, Revisited

Proceedings of the Institute of Mechanical Engineers
On the Flow of a Compressible Fluid through Orifices
D. A. Jobson,
First Published June 1, 1955 Research Article
https://doi.org/10.1243/PIME_PROC_1955_169_077_02





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GOX/ABS Combustion

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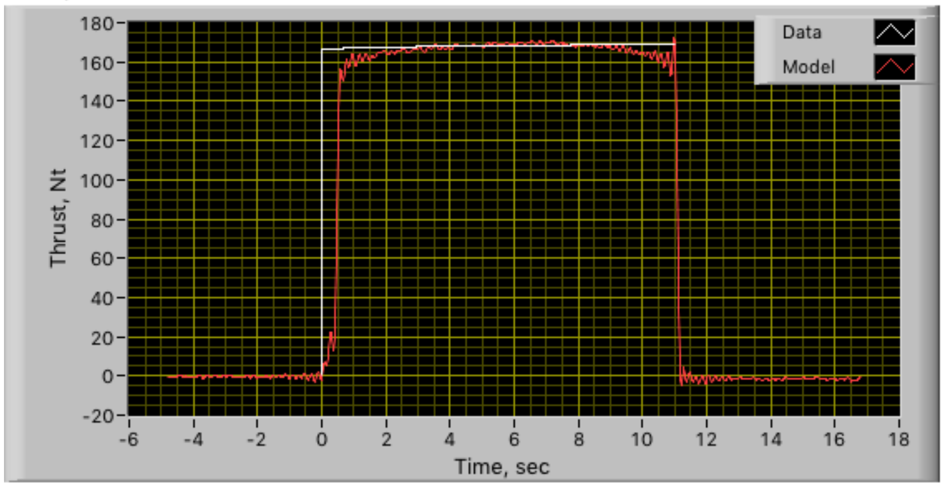
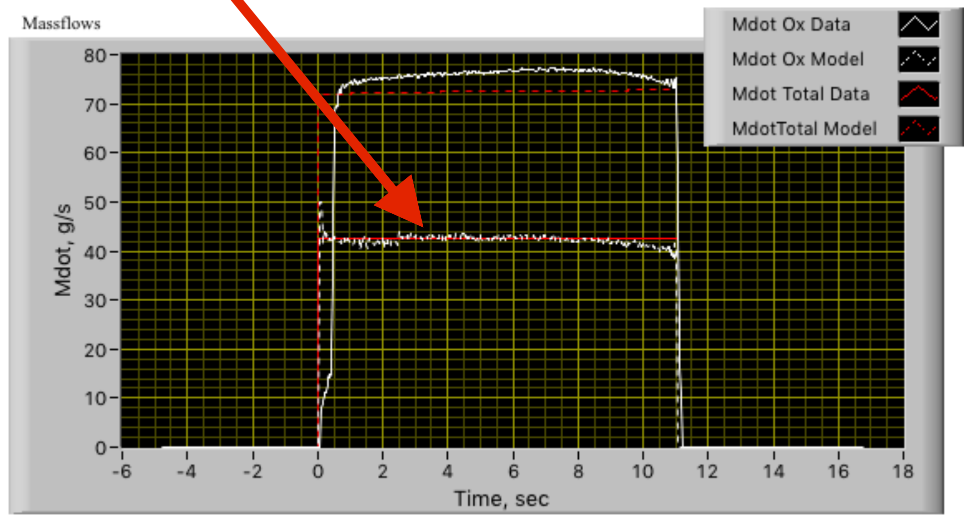
https://doi.org/10.1243/PIME_PROC_1955_169_077_02

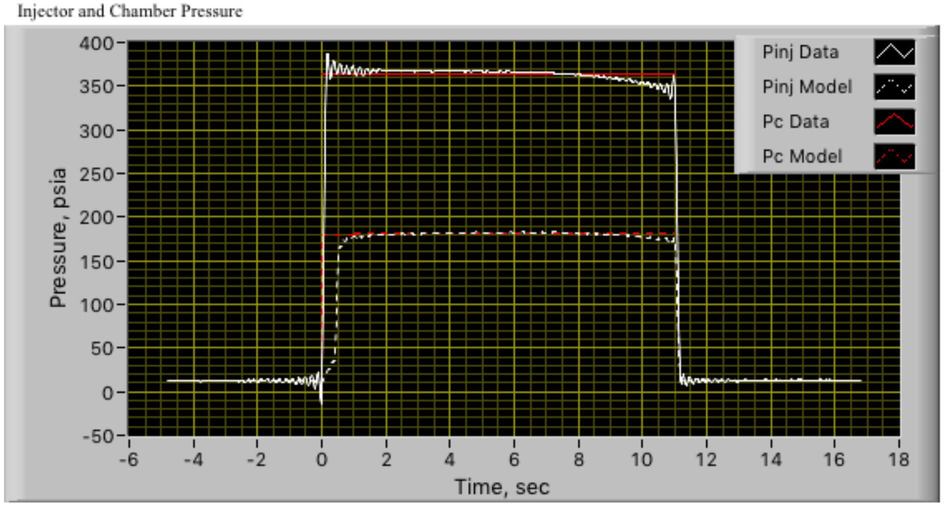
Thrust, Newtons



Massflows



Injector and Chamber Pressure



Consumed Mass