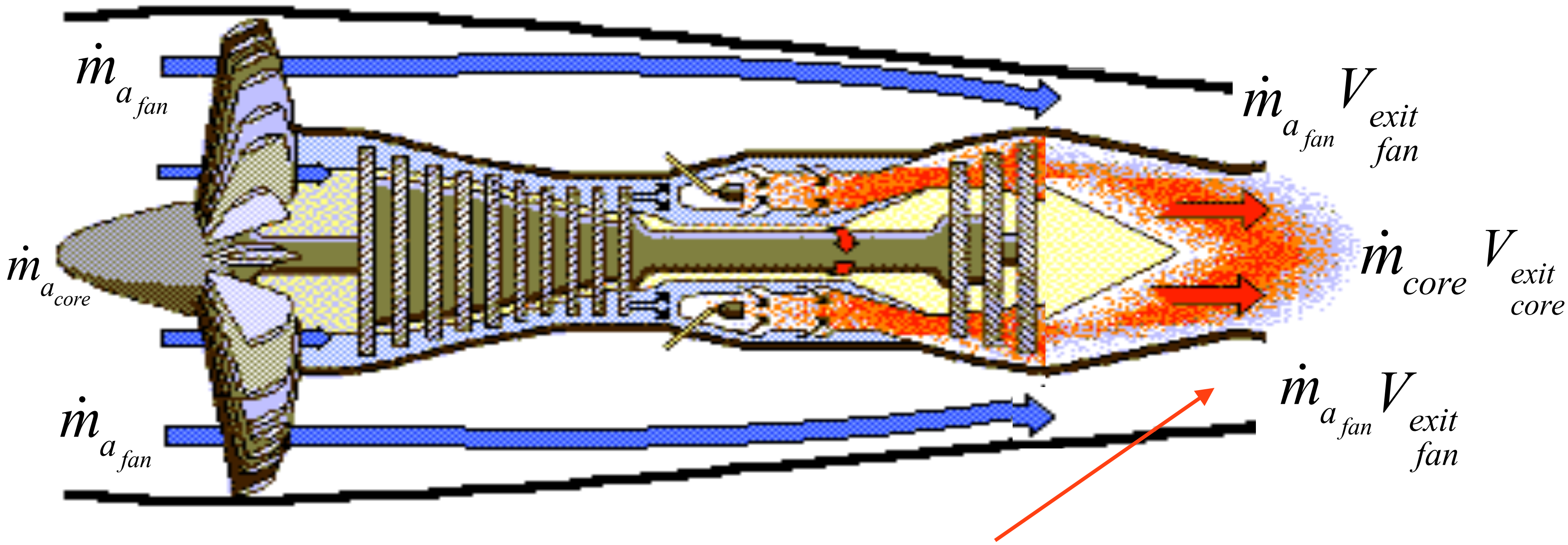
Section 6.2: Optimal TurboFan Bypass Ratio



Review 1: Normalized Thrust and Isp

- Fully expanded nozzle & $f >> 1$
- Inlet, fan, compressor, turbine, and fan /core nozzles are isentropic
- Combustor heat addition is as constant pressure and Low Mach

$$p_{\substack{exit\\fan}} = p_{\substack{exit\\core}} = p_\infty \rightarrow \pi_d = \pi_b = \pi_{n_{core}} = \pi_{\substack{n\\fan}} = 1$$

$$\pi_{\substack{c\\core}} = \left(\tau_{\substack{c\\core}}\right)^{\frac{\gamma}{\gamma-1}}, \pi_{\substack{c\\fan}} = \left(\tau_{\substack{c\\fan}}\right)^{\frac{\gamma}{\gamma-1}}, \pi_t = \left(\tau_t\right)^{\frac{\gamma}{\gamma-1}}.$$

$$\left(\mathbb{T}_{optimal}\right)_{turbofan} = \gamma \cdot M_\infty^2 \left[\left(\frac{1}{1+\beta}\right) \cdot \frac{V_{\substack{exit\\core}}}{V_\infty} + \left(\frac{\beta}{1+\beta}\right) \cdot \frac{V_{\substack{exit\\fan}}}{V_\infty} - 1\right] =$$

$$= \gamma \cdot M_\infty^2 \left[\left(\frac{1}{1+\beta}\right) \cdot \left(\frac{V_{\substack{exit\\core}}}{V_\infty} - 1\right) + \left(\frac{\beta}{1+\beta}\right) \cdot \left(\frac{V_{\substack{exit\\fan}}}{V_\infty} - 1\right)\right]$$

$$\left(\mathbb{I}_{optimal}\right)_{turbofan} = \left(\mathbb{T}_{optimal}\right)_{turbofan} \cdot \frac{f}{\gamma \cdot M_\infty}$$

Review 2:
Fan/Core Summary

- Fan

$$\left(\frac{V_{exit_{fan}}}{V_\infty}\right)^2 = \left(\frac{\tau_r \cdot \tau_{c_{fan}} - 1}{\tau_r - 1}\right)$$

$$\left(\mathbb{T}\right)_{fan} = \gamma \cdot M_\infty^2 \left[\left(\frac{\beta}{1+\beta}\right) \cdot \left(\frac{V_{exit_{fan}}}{V_\infty} - 1\right)\right] = \gamma \cdot M_\infty^2 \left[\left(\frac{\beta}{1+\beta}\right) \cdot \left(\sqrt{\frac{\tau_r \cdot \tau_{c_{fan}} - 1}{\tau_r - 1}} - 1\right)\right]$$

- Core

$$\left(\frac{V_{exit_{core}}}{V_\infty}\right)^2 = \left(\frac{\tau_r \cdot \tau_{c_{core}} \cdot \tau_t - 1}{\tau_r - 1}\right) \cdot \left(\frac{\tau_\lambda}{\tau_{c_{core}} \cdot \tau_r}\right)$$

$$\left(\mathbb{T}\right)_{core} = \gamma \cdot M_\infty^2 \left[\left(\frac{1 + \frac{1}{f}(1+\beta)}{(1+\beta)}\right) \cdot \left(\sqrt{\left(\frac{\tau_r \cdot \tau_{c_{core}} \cdot \tau_t - 1}{\tau_r - 1}\right) \cdot \left(\frac{\tau_\lambda}{\tau_{c_{core}} \cdot \tau_r}\right)} - 1\right)\right]$$

Review 3: TurboFan Matching Equations

- Turbine Work

$$\tau_t = 1 - \left(\frac{\tau_r}{\tau_\lambda}\right)\frac{\left(\tau_{c_{core}} - 1\right) + \beta\cdot\left(\tau_{c_{fan}} - 1\right)}{\left(1 + \frac{1+\beta}{f}\right)}$$

- Fuel/Air Flow

$$\frac{1}{f} = \frac{1}{1+\beta}\cdot\left(\frac{\tau_\lambda - \tau_r\cdot\tau_{c_{core}}}{\tau_{fuel} - \tau_\lambda}\right)$$

$$f = (1+\beta)\cdot\left(\frac{\tau_{fuel} - \tau_\lambda}{\tau_\lambda - \tau_r\cdot\tau_{c_{core}}}\right)$$

$$f = \frac{\dot{m}_a}{\dot{m}_{fuel}}$$

If the bypass ratio goes to zero the matching condition reduces to the usual turbojet formula.

Thermal Efficiency of an Ideal TurboFan

- Recall the Definition of Thermal Efficiency

$$\eta_{th} = \frac{K.E_{out} - K.E._{in}}{\dot{m}_{fuel} \cdot h_{fuel}} = 1 - \frac{heat\ rejected}{heat\ input} = 1 - \frac{\left(\dot{m}_{a_{core}}\right) \cdot \left(h_{\substack{exit\\core}} - h_\infty\right) + \dot{m}_{\substack{a\\fan}} \cdot \left(h_{\substack{exit\\fan}} - h_\infty\right) + \dot{m}_{fuel} \cdot h_{\substack{exit\\core}}}{\left(\dot{m}_{a_{core}} + \dot{m}_{fuel}\right) \cdot h_{0_4} - \left(\dot{m}_{a_{core}}\right) \cdot h_{0_3}}$$

Fan Flow Heat Rejection

Core Flow Heat Rejection

Fuel Flow Heat Rejection

Head Added in Combustor

- Rewrite efficiency by adding and subtracting

$\dot{m}_{fuel} \cdot h_\infty$ and dividing by $\dot{m}_{a_{core}}$

$$\eta_{th} = 1 - \frac{\left(1 + \frac{\dot{m}_{fuel}}{\dot{m}_{a_{core}}}\right) \cdot \left(h_{\substack{exit\\core}} - h_\infty\right) + \frac{\dot{m}_{\substack{a\\fan}}}{\dot{m}_{a_{core}}} \cdot \left(h_{\substack{exit\\fan}} - h_\infty\right) + \frac{\dot{m}_{fuel}}{\dot{m}_{a_{core}}} \cdot h_\infty}{\left(1 + \frac{\dot{m}_{fuel}}{\dot{m}_{a_{core}}}\right) \cdot h_{0_4} - h_{0_3}} = 1 - \frac{\left(1 + \frac{\dot{m}_{fuel}}{\dot{m}_{a_{core}}}\right) \cdot \left(h_{\substack{exit\\core}} - h_\infty\right) + \beta \cdot \left(h_{\substack{exit\\fan}} - h_\infty\right) + \frac{\dot{m}_{fuel}}{\dot{m}_{a_{core}}} \cdot h_\infty}{\left(1 + \frac{\dot{m}_{fuel}}{\dot{m}_{a_{core}}}\right) \cdot h_{0_4} - h_{0_3}}$$

Thermal Efficiency of an Ideal TurboFan (2)

- From the definition of bypass flow

$$\frac{\dot{m}_{a_{fan}}}{\dot{m}_{a_{core}}} = \beta$$

$$\frac{\dot{m}_{fuel}}{\dot{m}_{a_{core}}} = \frac{\dot{m}_{fuel}}{\dot{m}_{a_{total}}}\frac{\dot{m}_{a_{total}}}{\dot{m}_{a_{core}}} = \frac{\dot{m}_{fuel}}{\dot{m}_{a_{total}}}\frac{\dot{m}_{a_{core}} + \dot{m}_{a_{fan}}}{\dot{m}_{a_{core}}} = \frac{1}{f}\cdot\left(1+\beta\right)$$

- Substitution gives

$$\rightarrow \eta_{th} = 1 - \frac{\left(1+\frac{1}{f}\cdot\left(1+\beta\right)\right)\cdot\left(h_{\substack{exit\\core}} - h_{\infty}\right) + \beta\cdot\left(h_{\substack{exit\\fan}} - h_{\infty}\right) + \frac{1}{f}\cdot\left(1+\beta\right)\cdot h_{\infty}}{\left(1+\frac{1}{f}\cdot\left(1+\beta\right)\right)\cdot h_{0_4} - h_{0_3}}$$

Thermal Efficiency of an Ideal TurboFan (3)

- Collecting up h_∞

$$\eta_{th} = 1 - \frac{\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot h_{\substack{exit\\core}} + \beta\cdot h_{\substack{exit\\fan}} - \left[\left(1+\frac{1}{f}\cdot(1+\beta)\right) - \beta - \frac{1}{f}\cdot(1+\beta)\right]h_\infty}{\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot h_{0_4} - h_{0_3}} =$$

$$1 - \frac{\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot h_{\substack{exit\\core}} + \beta\cdot h_{\substack{exit\\fan}} - \left[1-\beta\right]h_\infty}{\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot h_{0_4} - h_{0_3}}$$

- *Ideal fan is quasi isentropic …*

$$h_{\substack{exit\\fan}} \approx h_\infty \rightarrow \eta_{th} = 1 - \frac{\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot h_{\substack{exit\\core}} - h_\infty}{\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot h_{0_4} - h_{0_3}}$$

Thermal Efficiency of an Ideal TurboFan (4)

- Factoring out

$$\frac{h_{0_3}}{h_\infty}$$

$$\eta_{th} = 1 - \frac{\left(1 + \frac{1}{f}\cdot\left(1+\beta\right)\right)\cdot\frac{h_{\substack{exit\\core}}}{h_\infty} - 1}{\left(1 + \frac{1}{f}\cdot\left(1+\beta\right)\right)\cdot\frac{h_{0_4}}{h_\infty} - \frac{h_{0_3}}{h_\infty}} = 1 - \frac{\left(1 + \frac{1}{f}\cdot\left(1+\beta\right)\right)\cdot\frac{h_{\substack{exit\\core}}}{h_\infty} - 1}{\frac{h_{0_3}}{h_\infty}\left[\left(1 + \frac{1}{f}\cdot\left(1+\beta\right)\right)\cdot\frac{h_{0_4}}{h_\infty}\frac{h_\infty}{h_{0_3}} - 1\right]}$$

- From enthalpy cascade

$$\frac{h_{0_3}}{h_\infty} = \frac{h_{0_3}}{h_{0_2}}\cdot\frac{h_{0_2}}{h_{0_1}}\cdot\frac{h_{0_1}}{h_{0_\infty}}\cdot\frac{h_{0_\infty}}{h_\infty} \rightarrow$$

$$\frac{h_{0_3}}{h_{0_2}} = \tau_{\substack{c\\core}}$$

$$\frac{h_{0_2}}{h_{0_1}} = 1$$

$$\frac{h_{0_1}}{h_{0_\infty}} = \tau_r$$

$$\frac{h_{0_4}}{h_\infty} = \tau_\gamma$$

$$\rightarrow \quad \frac{h_{0_3}}{h_\infty} = \tau_{\substack{c\\core}}\cdot\tau_r$$

$$\frac{h_{0_4}}{h_\infty}\frac{h_\infty}{h_{0_3}} = \frac{\tau_\gamma}{\tau_{\substack{c\\core}}\cdot\tau_r}$$

Thermal Efficiency of an Ideal TurboFan (5)

- Substituting

$$\eta_{th} = 1 - \frac{\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot\frac{h_{\substack{exit\\core}}}{h_\infty} - 1}{\tau_{\substack{c\\core}}\cdot\tau_r\left[\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot\frac{\tau_\gamma}{\tau_{\substack{c\\core}}\cdot\tau_r} - 1\right]}$$

- As shown before during turbojet analysis

$$\frac{h_{\substack{exit\\core}}}{h_\infty} = \frac{h_{0_{\substack{exit\\core}}}}{h_\infty}\cdot\frac{h_{\substack{exit\\core}}}{h_{0_{\substack{exit\\core}}}} = \tau_{lambda}\cdot\frac{1}{1+\frac{\gamma-1}{2}M^2_{\substack{exit\\core}}} \rightarrow 1+\frac{\gamma-1}{2}M^2_{\substack{exit\\core}} = \tau_{\substack{c\\core}}\cdot\tau_r \rightarrow \frac{h_{\substack{exit\\core}}}{h_\infty} = \frac{\tau_{lambda}}{\tau_{\substack{c\\core}}\cdot\tau_r}$$

Thermal Efficiency of an Ideal TurboFan (6)

- Substituting

$$\eta_{th} = 1 - \frac{\left(1 + \frac{1}{f}\cdot(1+\beta)\right)\cdot\frac{h_{\substack{exit\\core}}}{h_\infty} - 1}{\tau_{c_{core}}\cdot\tau_r\left[\left(1 + \frac{1}{f}\cdot(1+\beta)\right)\cdot\frac{\tau_\gamma}{\tau_{c_{core}}\cdot\tau_r} - 1\right]} = 1 - \frac{\left(1 + \frac{1}{f}\cdot(1+\beta)\right)\cdot\frac{\tau_\gamma}{\tau_{c_{core}}\cdot\tau_r} - 1}{\tau_{c_{core}}\cdot\tau_r\left[\left(1 + \frac{1}{f}\cdot(1+\beta)\right)\cdot\frac{\tau_\gamma}{\tau_{c_{core}}\cdot\tau_r} - 1\right]} \sim 1 - \frac{1}{\tau_{c_{core}}\cdot\tau_r}$$

1/f ~ 0

- This solution is identical to the turbo jet analysis with the core flow replacing the normal turbine flow path.

- This analysis shows for the ideal (isentropic) fan → $h_{exit_{fan}} = h_\infty$ --> *the* the heat rejected by the fan stream is zero.

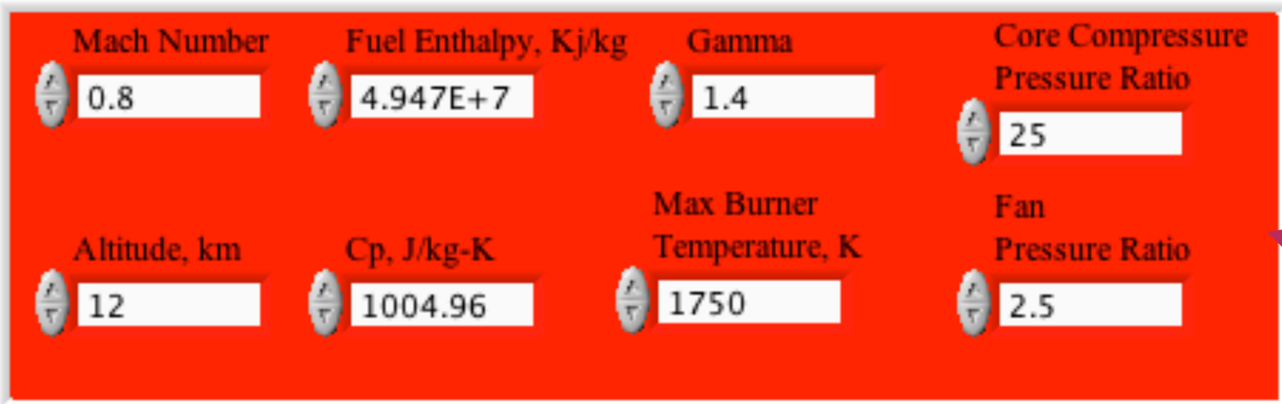
- Therefore the thermal efficiency of the ideal turbofan is independent of the parameters of the fan stream.

- Only the mechanical efficiency (Thrust, Specific Impulse) are altered by bypass flow.

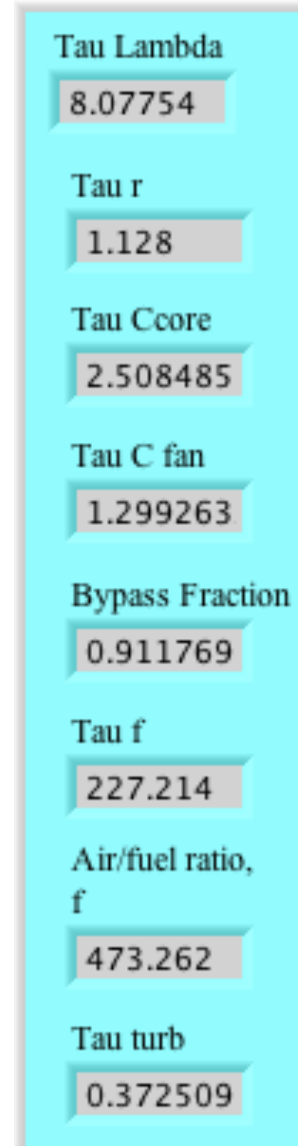
Example Turbofan Calculation

beta=10.3333

Freestream Conditions



NonDimensional Parameters



$$\frac{V_{exit\ core}}{V_\infty} = \sqrt{\left(\frac{\tau_r \cdot \tau_c \cdot \tau_t - 1}{\tau_r - 1}\right)} \cdot \sqrt{\frac{\tau_\lambda}{\tau_c \cdot \tau_r}}$$

$$\left(\frac{V_{exit\ fan}}{V_\infty}\right) = \sqrt{\frac{\tau_r \cdot \tau_{c\ fan} - 1}{\tau_r - 1}}$$

$$\left(\mathbb{T}_{optimal}\right)_{turbofan} = \gamma \cdot M_\infty^2 \left[\frac{1 + \frac{1}{f}(1+\beta)}{(1+\beta)} \cdot \frac{V_{exit\ core}}{V_\infty} + \frac{\beta}{1+\beta} \cdot \frac{V_{exit\ fan}}{V_\infty} - 1\right] =$$

$$\cong \gamma \cdot M_\infty^2 \left[\left(\frac{1}{1+\beta}\right) \cdot \left(\frac{V_{exit\ core}}{V_\infty} - 1\right) + \left(\frac{\beta}{1+\beta}\right) \cdot \left(\frac{V_{exit\ fan}}{V_\infty} - 1\right)\right]$$

Example Turbofan Calculation (2)

Freestream Conditions

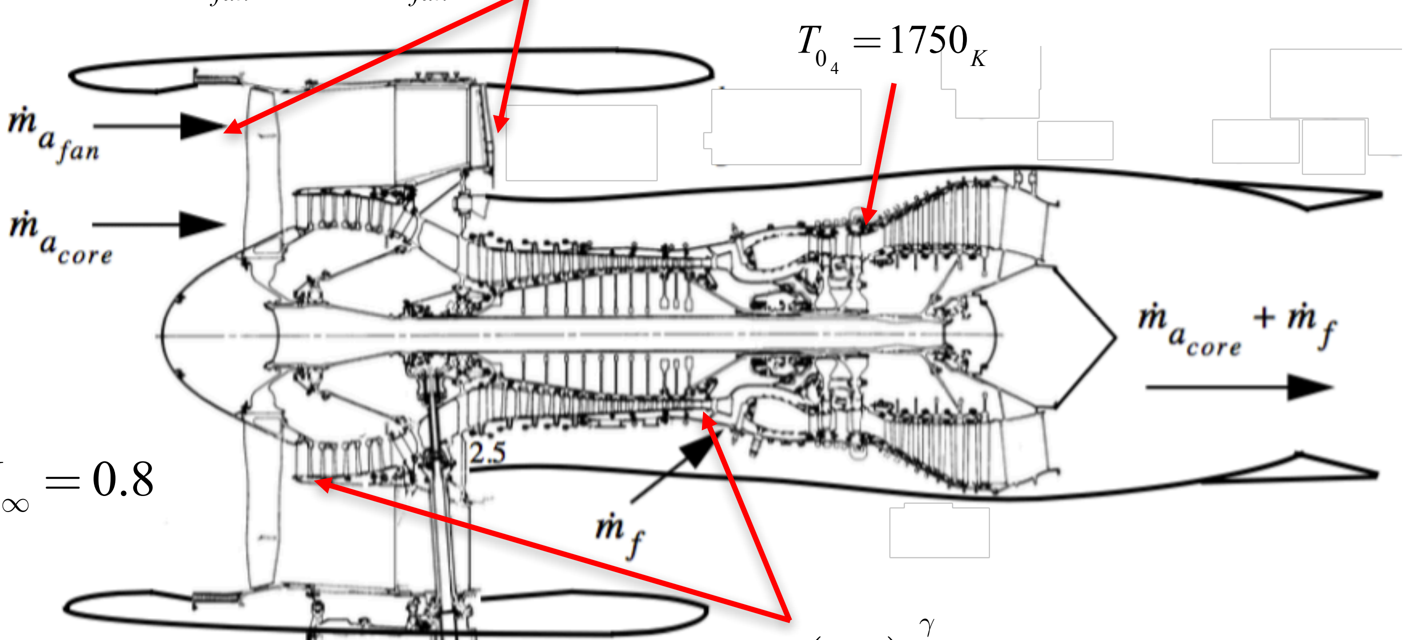
Mach Number: 0.8	Fuel Enthalpy, Kj/kg: 4.947E+7	Gamma: 1.4	Core Compressure Pressure Ratio: 25
Altitude, km: 12	Cp, J/kg-K: 1004.96	Max Burner Temperature, K: 1750	Fan Pressure Ratio: 2.5

beta=10.33333

NonDimensional Parameters

Parameter	Value
Tau Lambda	8.07754
Tau r	1.128
Tau Ccore	2.508485
Tau C fan	1.299263
Bypass Fraction	0.911769
Tau f	227.214
Air/fuel ratio, f	473.262
Tau turb	0.372509

$$\pi_{c_{fan}} = \left(\frac{P_{0_3}}{P_{0_2}}\right)_{fan} = \left(\frac{T_{0_3}}{T_{0_2}}\right)_{fan}^{\frac{\gamma}{\gamma-1}} = \left(\tau_{c_{fan}}\right)^{\frac{\gamma}{\gamma-1}}$$

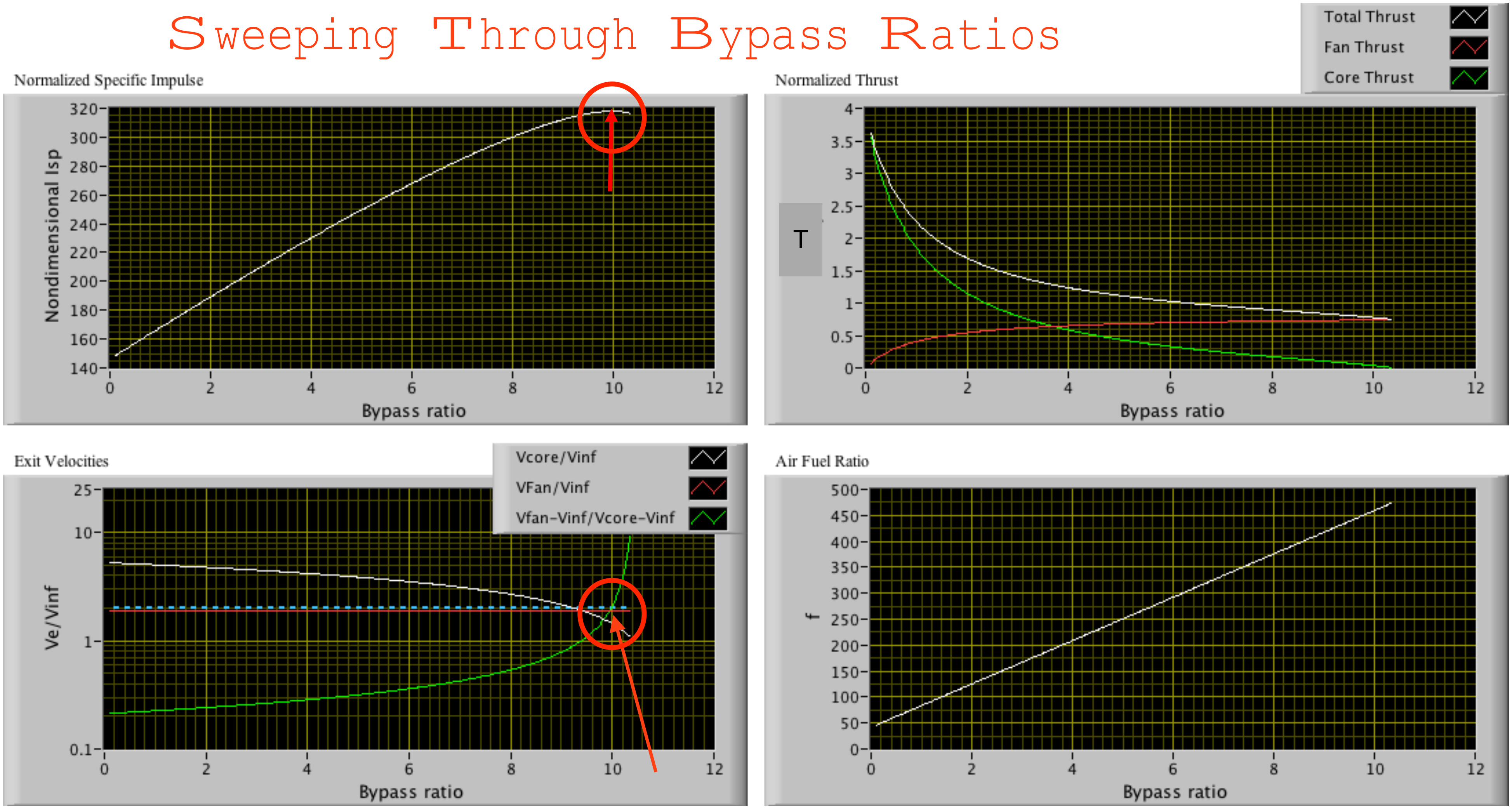


$M_\infty = 0.8$

$$\pi_{c_{core}} = \frac{P_{0_3}}{P_{0_2}} = \left(\frac{T_{0_3}}{T_{0_2}}\right)^{\frac{\gamma}{\gamma-1}} = \left(\tau_{c_{core}}\right)^{\frac{\gamma}{\gamma-1}}$$

Example Calculation (3)

Sweeping Through Bypass Ratios



- Optimal Efficiency (Isp) occurs at a Bypass Ratio that results in most of the thrust produced by Fan and not Core Flow (assumed *Nozzle exit pressure* = P_∞)

What Bypass Ratio Gives Optimized TurboFan Performance?

Normalized Specific Impulse $\rightarrow$

$$\left(\mathbb{I}\right)_{turbofan} = \frac{I_{sp}\cdot g_0}{c_\infty} = \left(\mathbb{T}\right)_{turbofan}\cdot\frac{f}{\gamma\cdot M_\infty}$$

fully expanded nozzle $\rightarrow$

$$\left(\mathbb{T}\right)_{turbofan} = \gamma\cdot M_\infty^2\left[\left(\frac{1}{1+\beta}\right)\cdot\left(\frac{V_{exit_{core}}}{V_\infty}-1\right)+\left(\frac{\beta}{1+\beta}\right)\cdot\left(\frac{V_{exit_{fan}}}{V_\infty}-1\right)\right]$$

Air / Fuel Ratio $\rightarrow$

$$f = \left(1+\beta\right)\cdot\left(\frac{\tau_{fuel}-\tau_\lambda}{\tau_\lambda-\tau_r\cdot\tau_{c_{core}}}\right)$$

Optimized TurboFan Performance (2)

Substitute Normalized Thrust and air / fuel ratio into Specific Impulse

$$\left(\mathbb{I}\right)_{turbofan} = \frac{I_{sp} \cdot g_0}{c_\infty} = \gamma \cdot M_\infty^2 \left[\left(\frac{1}{1+\beta}\right) \cdot \left(\frac{V_{\substack{exit\\core}}}{V_\infty} - 1 \right) + \left(\frac{\beta}{1+\beta}\right) \cdot \left(\frac{V_{\substack{exit\\fan}}}{V_\infty} - 1 \right) \right] \cdot \frac{\left(1+\beta\right) \cdot \left(\dfrac{\tau_{fuel} - \tau_\lambda}{\tau_\lambda - \tau_r \cdot \tau_{\substack{c\\core}}} \right)}{\gamma \cdot M_\infty}$$

Simplify $\rightarrow$

$$\left(\mathbb{I}\right)_{turbofan} = M_\infty \cdot \left(\frac{\tau_{fuel} - \tau_\lambda}{\tau_\lambda - \tau_r \cdot \tau_{\substack{c\\core}}} \right) \left[\left(\frac{V_{\substack{exit\\core}}}{V_\infty} - 1 \right) + \beta \cdot \left(\frac{V_{\substack{exit\\fan}}}{V_\infty} - 1 \right) \right]$$

- ***what value of β maximizes the specific impulse?***

Optimized TurboFan Performance (3)

$$\left(\mathbb{I}\right)_{turbofan} = M_\infty \cdot \left(\frac{\tau_{fuel} - \tau_\lambda}{\tau_\lambda - \tau_r \cdot \tau_{c_{core}}}\right)\left[\left(\frac{V_{exit_{core}}}{V_\infty} - 1\right) + \beta \cdot \left(\frac{V_{exit_{fan}}}{V_\infty} - 1\right)\right]$$

- ***What value of β maximizes the specific impulse?***

Necessary Condition

$$\rightarrow \frac{\partial\left(\mathbb{I}\right)_{turbofan}}{\partial \beta} = 0 \rightarrow \frac{\partial}{\partial \beta}\left[\left(\frac{V_{exit_{core}}}{V_\infty} - 1\right) + \beta \cdot \left(\frac{V_{exit_{fan}}}{V_\infty} - 1\right)\right] = 0$$

- Since fan is ~ Isentropic only bypass nozzle and not bypass massflow affects Bypass exit velocity ratio

$$\frac{\partial}{\partial \beta}\left(\frac{V_{exit_{fan}}}{V_\infty}\right) = 0$$

$$Thus \rightarrow \frac{\partial}{\partial \beta}\left(\frac{V_{exit_{core}}}{V_\infty}\right) = 1 - \frac{V_{exit_{fan}}}{V_\infty}$$

Optimized TurboFan Performance (4)

- ***Find …***

$$\frac{\partial}{\partial \beta}\left(\frac{V_{exit_{core}}}{V_{\infty}}\right)$$

- ***Start with***

$$\frac{\partial}{\partial \beta}\left[\left(\frac{V_{exit_{core}}}{V_{\infty}}\right)^2\right] = 2 \cdot \left(\frac{V_{exit_{core}}}{V_{\infty}}\right) \cdot \frac{\partial}{\partial \beta}\left(\frac{V_{exit_{core}}}{V_{\infty}}\right) \rightarrow \frac{\partial}{\partial \beta}\left[\left(\frac{V_{exit_{core}}}{V_{\infty}}\right)^2\right] = 2 \cdot \left(\frac{V_{exit_{core}}}{V_{\infty}}\right) \cdot \left(1 - \frac{V_{exit_{fan}}}{V_{\infty}}\right)$$

from earlier derivation

$$\left(\frac{V_{exit_{core}}}{V_{\infty}}\right)^2 = \left(\frac{\tau_r \cdot \tau_{c_{core}} \cdot \tau_t - 1}{\tau_r - 1}\right) \cdot \left(\frac{\tau_\lambda}{\tau_c \cdot \tau_r}\right) \rightarrow \frac{\partial}{\partial \beta}\left[\left(\frac{\tau_r \cdot \tau_{c_{core}} \cdot \tau_t - 1}{\tau_r - 1}\right) \cdot \left(\frac{\tau_\lambda}{\tau_c \cdot \tau_r}\right)\right] = 2 \cdot \left(\frac{V_{exit_{core}}}{V_{\infty}}\right) \cdot \left(1 - \frac{V_{exit_{fan}}}{V_{\infty}}\right)$$

Optimized TurboFan Performance (5)

- ***Also …*** *from earlier derivation*

$$\frac{\partial}{\partial \beta}\left[\left(\frac{\tau_r \cdot \tau_{c_{core}} \cdot \tau_t - 1}{\tau_r - 1}\right)\cdot\left(\frac{\tau_\lambda}{\tau_c \cdot \tau_r}\right)\right] = 2\cdot\left(\frac{V_{exit_{core}}}{V_\infty}\right)\cdot\left(1-\frac{V_{exit_{fan}}}{V_\infty}\right)$$

τ_t is a function of β
since turbine powers both fan
and compressor

$$\rightarrow \frac{\partial}{\partial \beta}\left[\left(\frac{\tau_r \cdot \tau_c \cdot \tau_t - 1}{\tau_r - 1}\right)\cdot\right] = \left(\frac{\tau_\lambda}{\tau_{c_{core}} \cdot \tau_r}\right)\cdot\left(\frac{\tau_r \cdot \tau_{c_{core}}}{\tau_r - 1}\right)\cdot\left(\frac{\partial \tau_t}{\partial \beta}\right) = \left(\frac{\tau_\lambda}{\tau_r - 1}\right)\cdot\left(\frac{\partial \tau_t}{\partial \beta}\right)$$

Optimized TurboFan Performance (6)

- ***Taking the derivtive***

$$\rightarrow \frac{\partial}{\partial \beta}\left[\left(\frac{\tau_r \cdot \tau_c \cdot \tau_t - 1}{\tau_r - 1}\right)\cdot\right] = \left(\frac{\tau_\lambda}{\tau_{c_{core}} \cdot \tau_r}\right)\cdot\left(\frac{\tau_r \cdot \tau_{c_{core}}}{\tau_r - 1}\right)\cdot\left(\frac{\partial \tau_t}{\partial \beta}\right) = \left(\frac{\tau_\lambda}{\tau_r - 1}\right)\cdot\left(\frac{\partial \tau_t}{\partial \beta}\right)$$

$$\left(\frac{\partial \tau_t}{\partial \beta}\right) = \frac{\partial}{\partial \beta}\left[1 - \left(\frac{\tau_r}{\tau_\lambda}\right)\frac{\left(\tau_{c_{core}} - 1\right) + \beta \cdot \left(\tau_{c_{fan}} - 1\right)}{\left(1 + \frac{1+\beta}{f}\right)}\right] = -\left(\frac{\tau_r}{\tau_\lambda}\right)\frac{\left(\tau_{c_{fan}} - 1\right)}{\left(1 + \frac{1+\beta}{f}\right)} + \left(\frac{\tau_r}{\tau_\lambda}\right)\frac{\left(\tau_{c_{fan}} - 1\right)\left(\tau_{c_{core}} - 1\right) + \beta \cdot \left(\tau_{c_{fan}} - 1\right)}{\left(1 + \frac{1+\beta}{f}\right)^2} \cdot \frac{1}{f}$$

$$\frac{1}{f} \approx 0 \rightarrow \left(\frac{\partial \tau_t}{\partial \beta}\right) = \left(\frac{\tau_r}{\tau_\lambda}\right)\left(\tau_{c_{fan}} - 1\right) \rightarrow \frac{\partial}{\partial \beta}\left[\left(\frac{\tau_r \cdot \tau_c \cdot \tau_t - 1}{\tau_r - 1}\right)\right] = -\left(\frac{\tau_\lambda}{\tau_r - 1}\right)\cdot\left(\frac{\tau_r}{\tau_\lambda}\right)\left(\tau_{c_{fan}} - 1\right) = \left(\frac{\tau_r}{\tau_r - 1}\right)\cdot\left(\tau_{c_{fan}} - 1\right)$$

$$substituting \rightarrow \boxed{\frac{\left(\frac{\tau_r}{\tau_r - 1}\right)\cdot\left(\tau_{c_{fan}} - 1\right)}{2 \cdot V_{exit_{core}} / V_\infty} = \left(\frac{V_{exit_{fan}}}{V_\infty} - 1\right)}$$

Optimized TurboFan Performance (7)

$$\text{from earlier} \rightarrow \left(\frac{V_{\substack{exit\\fan}}}{V_\infty}\right)^2 = \left(\frac{\tau_r \cdot \tau_{\substack{c\\fan}} - 1}{\tau_r - 1}\right) = \left(\frac{\tau_r \cdot \tau_{\substack{c\\fan}} - 1}{\tau_r - 1} - 1\right) + 1 = \left(\frac{\tau_r \cdot \tau_{\substack{c\\fan}} - 1 - (\tau_r - 1)}{\tau_r - 1} + 1\right) = \left(\frac{\tau_r \cdot \left(\tau_{\substack{c\\fan}} - 1\right)}{\tau_r - 1} + 1\right)$$

Simplifying

$$\rightarrow \left(\frac{V_{\substack{exit\\fan}}}{V_\infty}\right)^2 - 1 = \frac{\tau_r \cdot \left(\tau_{\substack{c\\fan}} - 1\right)}{\tau_r - 1} \rightarrow \text{Substituting} \rightarrow \frac{\left(V_{\substack{exit\\fan}} / V_\infty\right)^2 - 1}{2 \cdot V_{\substack{exit\\core}} / V_\infty} = \left(\frac{V_{\substack{exit\\fan}}}{V_\infty} - 1\right)$$

Re *arranging* and *expanding diffference of squares*

$$\left(\frac{V_{\substack{exit\\fan}}}{V_\infty}\right)^2 - 1 = 2 \cdot \left(\frac{V_{\substack{exit\\core}}}{V_\infty}\right)\left(\frac{V_{\substack{exit\\fan}}}{V_\infty} - 1\right) \rightarrow \left(\frac{V_{\substack{exit\\fan}}}{V_\infty}\right)^2 - 1 = \left(\frac{V_{\substack{exit\\fan}}}{V_\infty} + 1\right) \cdot \left(\frac{V_{\substack{exit\\fan}}}{V_\infty} - 1\right) = 2 \cdot \left(\frac{V_{\substack{exit\\core}}}{V_\infty}\right)\left(\frac{V_{\substack{exit\\fan}}}{V_\infty} - 1\right)$$

Cancelling Terms

$$\left(\frac{V_{\substack{exit\\fan}}}{V_\infty} + 1\right) = 2 \cdot \left(\frac{V_{\substack{exit\\core}}}{V_\infty}\right) \rightarrow \left(\frac{V_{\substack{exit\\fan}}}{V_\infty} + 1 - 2\right) = 2 \cdot \left(\frac{V_{\substack{exit\\core}}}{V_\infty}\right) - 2$$

$$\rightarrow \left(\frac{V_{\substack{exit\\fan}}}{V_\infty} - 1\right) = 2 \cdot \left(\frac{V_{\substack{exit\\core}}}{V_\infty} - 1\right) \rightarrow \boxed{\left(V_{\substack{exit\\fan}} - V_\infty\right) = 2 \cdot \left(V_{\substack{exit\\core}} - V_\infty\right)}$$

Optimized TurboFan Performance (8)

Optimization Criterion

$$\left(V_{\substack{exit\\fan}} - V_{\infty}\right) = 2 \cdot \left(V_{\substack{exit\\core}} - V_{\infty}\right)$$

Optimal turbofan design delivers twice the velocity increment across the fan compared to velocity increment across the turbine (core flow)

How does this criterion relate to the fan bypass ratio?

Optimized TurboFan Performance (9)

Optimization Criterion

$$2\cdot\left(V_{\substack{exit\\core}}-V_{\infty}\right)=\left(V_{\substack{exit\\fan}}-V_{\infty}\right)\rightarrow 2\cdot\left(\frac{V_{\substack{exit\\core}}}{V_{\infty}}-1\right)=\left(\frac{V_{\substack{exit\\fan}}}{V_{\infty}}-1\right)\rightarrow 2\cdot\frac{V_{\substack{exit\\core}}}{V_{\infty}}-2=\left(\frac{V_{\substack{exit\\fan}}}{V_{\infty}}-1\right)\rightarrow \boxed{\frac{V_{\substack{exit\\core}}}{V_{\infty}}=\frac{1}{2}\left(\frac{V_{\substack{exit\\fan}}}{V_{\infty}}+1\right)}$$

From Earlier Derivations

$$\left(\frac{V_{\substack{exit\\core}}}{V_{\infty}}\right)=\left[\sqrt{\left(\frac{\tau_r\cdot\tau_{\substack{c\\core}}\cdot\tau_t-1}{\tau_r-1}\right)\cdot\left(\frac{\tau_\lambda}{\tau_{\substack{c\\core}}\cdot\tau_r}\right)}\right]$$

$$\frac{V_{\substack{exit\\fan}}}{V_{\infty}}=\sqrt{\left(\frac{\tau_r\cdot\tau_{\substack{c\\fan}}-1}{\tau_r-1}\right)}$$

$$\rightarrow \textit{Substitute} \rightarrow \sqrt{\left(\frac{\tau_r\cdot\tau_{\substack{c\\core}}\cdot\tau_t-1}{\tau_r-1}\right)\cdot\left(\frac{\tau_\lambda}{\tau_{\substack{c\\core}}\cdot\tau_r}\right)}=\frac{1}{2}\left(\sqrt{\left(\frac{\tau_r\cdot\tau_{\substack{c\\fan}}-1}{\tau_r-1}\right)}+1\right)$$

Square Both Sides $\rightarrow$

$$\left(\frac{\tau_r\cdot\tau_{\substack{c\\core}}\cdot\tau_t-1}{\tau_r-1}\right)\cdot\left(\frac{\tau_\lambda}{\tau_{\substack{c\\core}}\cdot\tau_r}\right)=\frac{1}{4}\left(\frac{\tau_r\cdot\tau_{\substack{c\\fan}}-1}{\tau_r-1}\right)+\frac{1}{2}\sqrt{\left(\frac{\tau_r\cdot\tau_{\substack{c\\fan}}-1}{\tau_r-1}\right)}+1$$

Optimized TurboFan Performance (10)

- Solve for τ_t ….

$$\tau_t = \left\{ 1 + \left(\frac{1}{4} \left(\frac{\tau_r \cdot \tau_{c_{fan}} - 1}{\tau_r - 1} \right) + \frac{1}{2} \sqrt{\left(\frac{\tau_r \cdot \tau_{c_{fan}} - 1}{\tau_r - 1} \right)} + 1 \right) \cdot \left(\frac{\tau_{c_{core}} \cdot \tau_r}{\tau_\lambda} \right) \cdot (\tau_r - 1) \right\} \cdot \left(\frac{1}{\tau_r \cdot \tau_{c_{core}}} \right)$$

- From Earlier Derivation

$$\tau_t = 1 - \left(\frac{\tau_r}{\tau_\lambda} \right) \cdot \left[\left(\tau_{c_{core}} - 1 \right) + \beta \cdot \left(\tau_{c_{fan}} - 1 \right) \right]$$

- Substitute into above

$$\left(\frac{\tau_r}{\tau_\lambda} \right) \cdot \left[\left(\tau_{c_{core}} - 1 \right) + \beta \cdot \left(\tau_{c_{fan}} - 1 \right) \right] = 1 - \left\{ 1 + \left(\frac{1}{4} \left(\frac{\tau_r \cdot \tau_{c_{fan}} - 1}{\tau_r - 1} \right) + \frac{1}{2} \sqrt{\left(\frac{\tau_r \cdot \tau_{c_{fan}} - 1}{\tau_r - 1} \right)} + 1 \right) \cdot \left(\frac{\tau_{c_{core}} \cdot \tau_r}{\tau_\lambda} \right) \cdot (\tau_r - 1) \right\} \cdot \left(\frac{1}{\tau_r \cdot \tau_{c_{core}}} \right)$$

Optimized TurboFan Performance (10)

- Finally …. Solving for the optimal bypass ratio gives

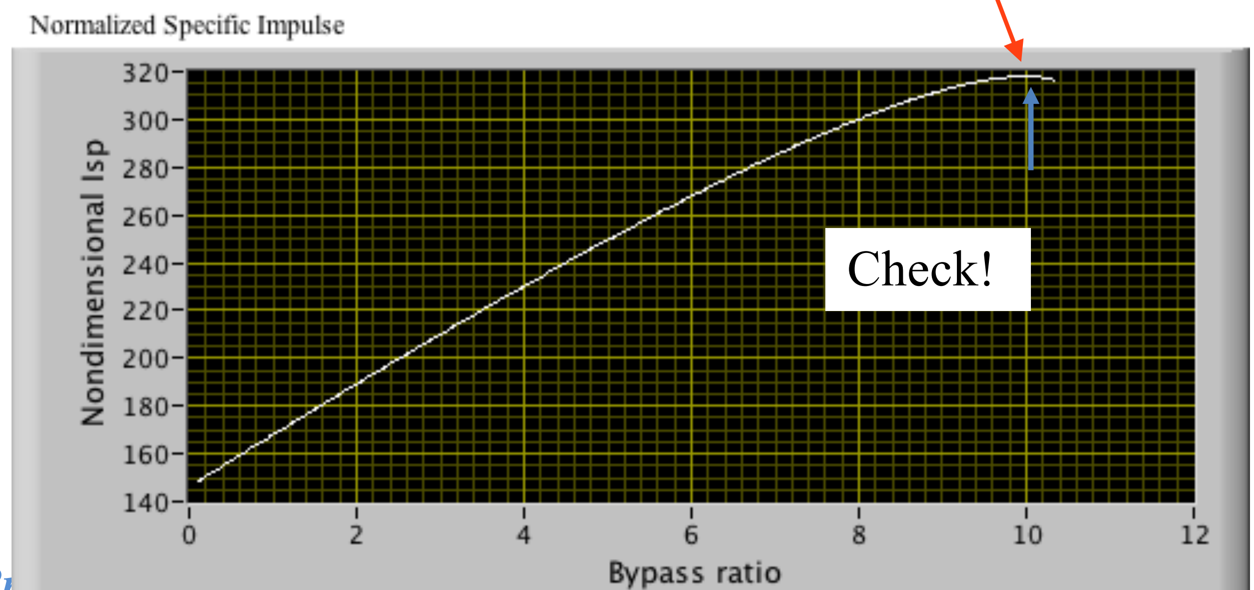
$$\beta_{optimal} = \frac{1}{\left(\tau_{c_{fan}} - 1\right)} \cdot \left\{ \left[\left(\frac{\tau_\lambda}{\tau_{c_{core}} \cdot \tau_r} - 1 \right) \cdot \left(\tau_{c_{core}} - 1 \right) + \frac{\tau_\lambda}{\tau_r^{\,2} \cdot \tau_{c_{core}}} \left(\tau_r - 1 \right) - \frac{1}{4} \left(\frac{\tau_r - 1}{\tau_r} \right) \left(\sqrt{\frac{\tau_r \cdot \tau_{c_{fan}} - 1}{\tau_r - 1}} + 1 \right) \right] \right\}$$

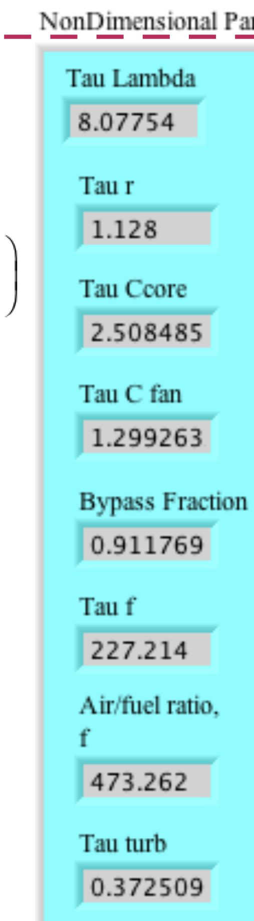
Whew! .. Check Numerical Result agans earlier simulation

$$\beta_{optimal} =$$

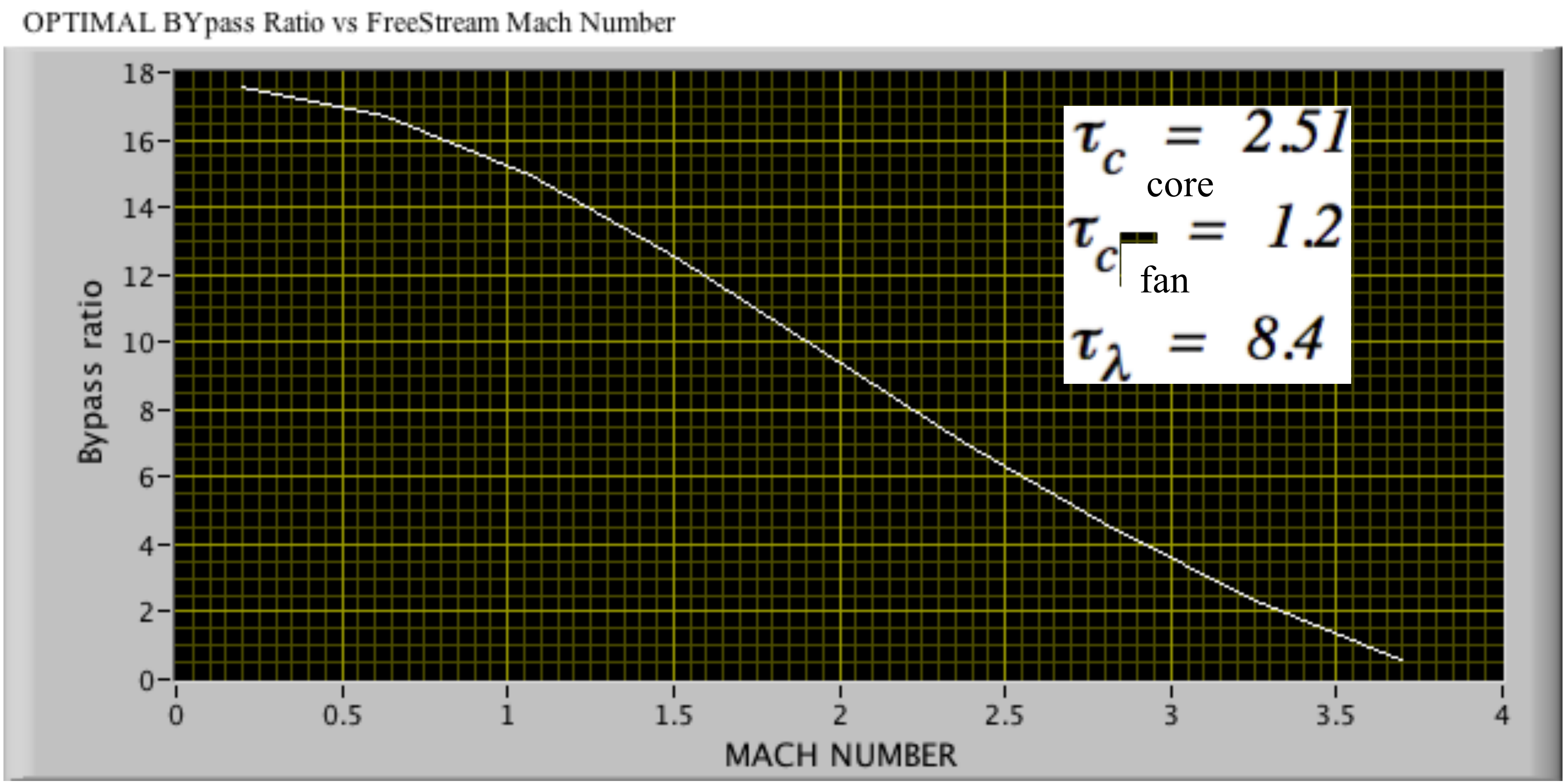
$$\left(\frac{1}{1.2993-1}\right)\left(\left(\left(\frac{8.0775}{2.5085 \cdot 1.128}\right)-1\right)(2.5085-1)+\frac{8.0775}{1.128^2 2.5085}(1.128-1)-\frac{1}{4}\left(\frac{1.128-1}{1.128}\right)\left(\left(\frac{1.128 \cdot 2.5085-1}{1.128-1}\right)^{0.5}+1\right)\right)$$

= 10.16



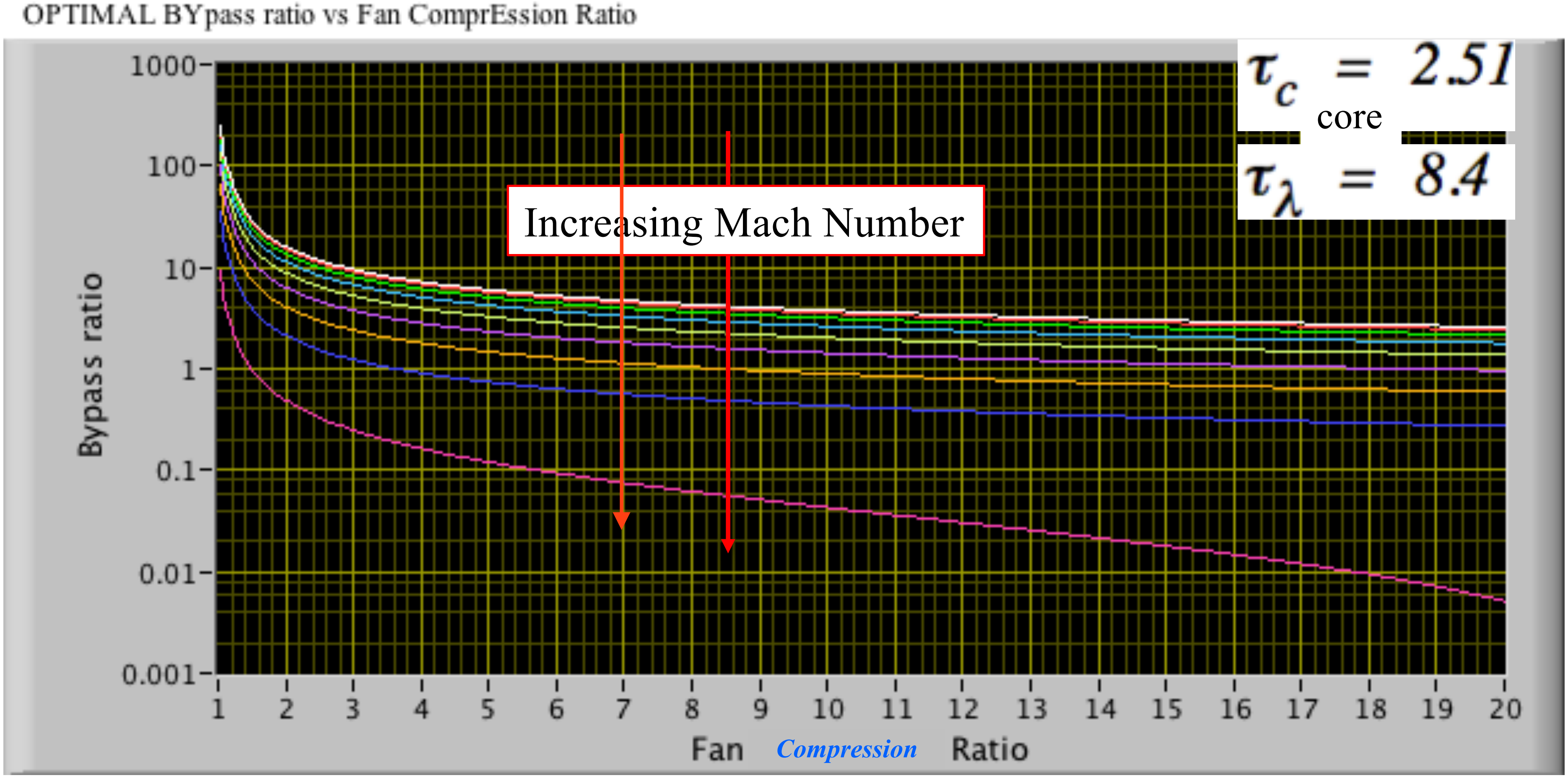


Discussion of Optimal TurboFan Performance



- Variation of Optimal Bypass Ratio as a Function of Freestream Mach Number
- At Higher Mach Numbers optimum bypass ratio decreases until the fan disappears altogether and we basically convert the engine to a turbojet.

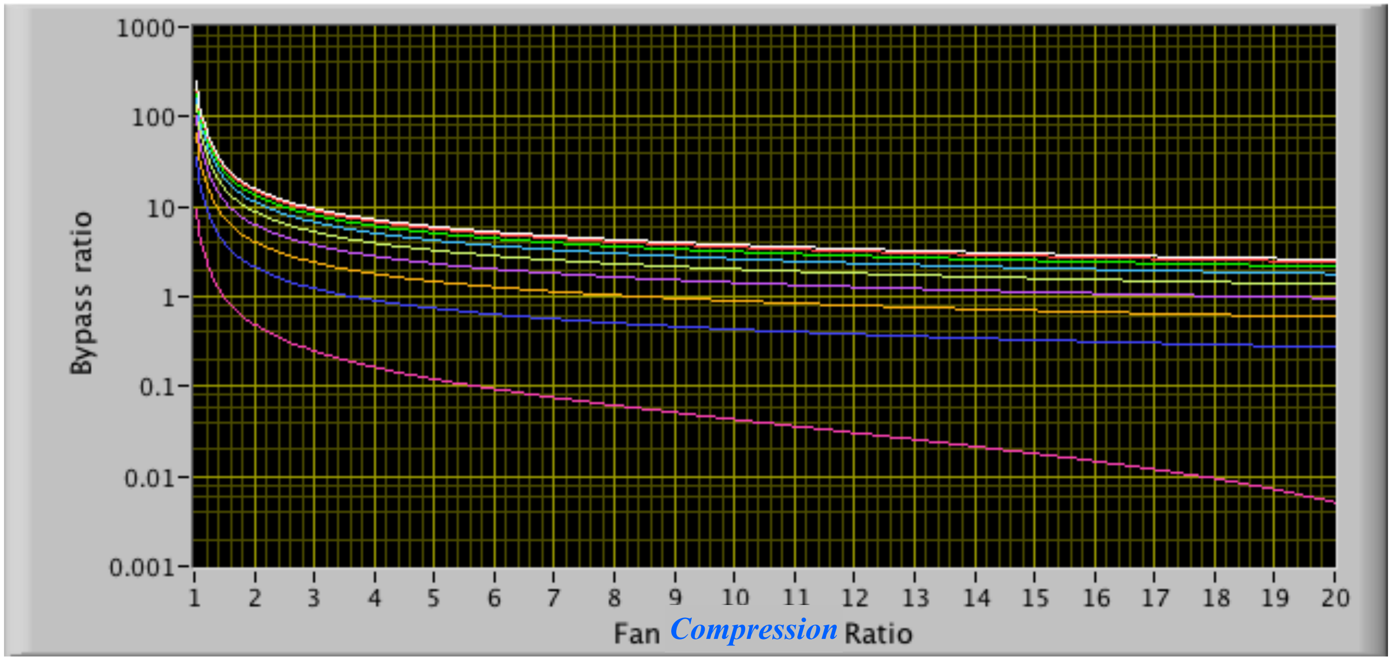
Discussion of Optimal TurboFan Performance (2)



Ideal turbofan bypass ratio for maximum specific impulse as a function of fan pressure ratio. Plot shown for several Mach numbers

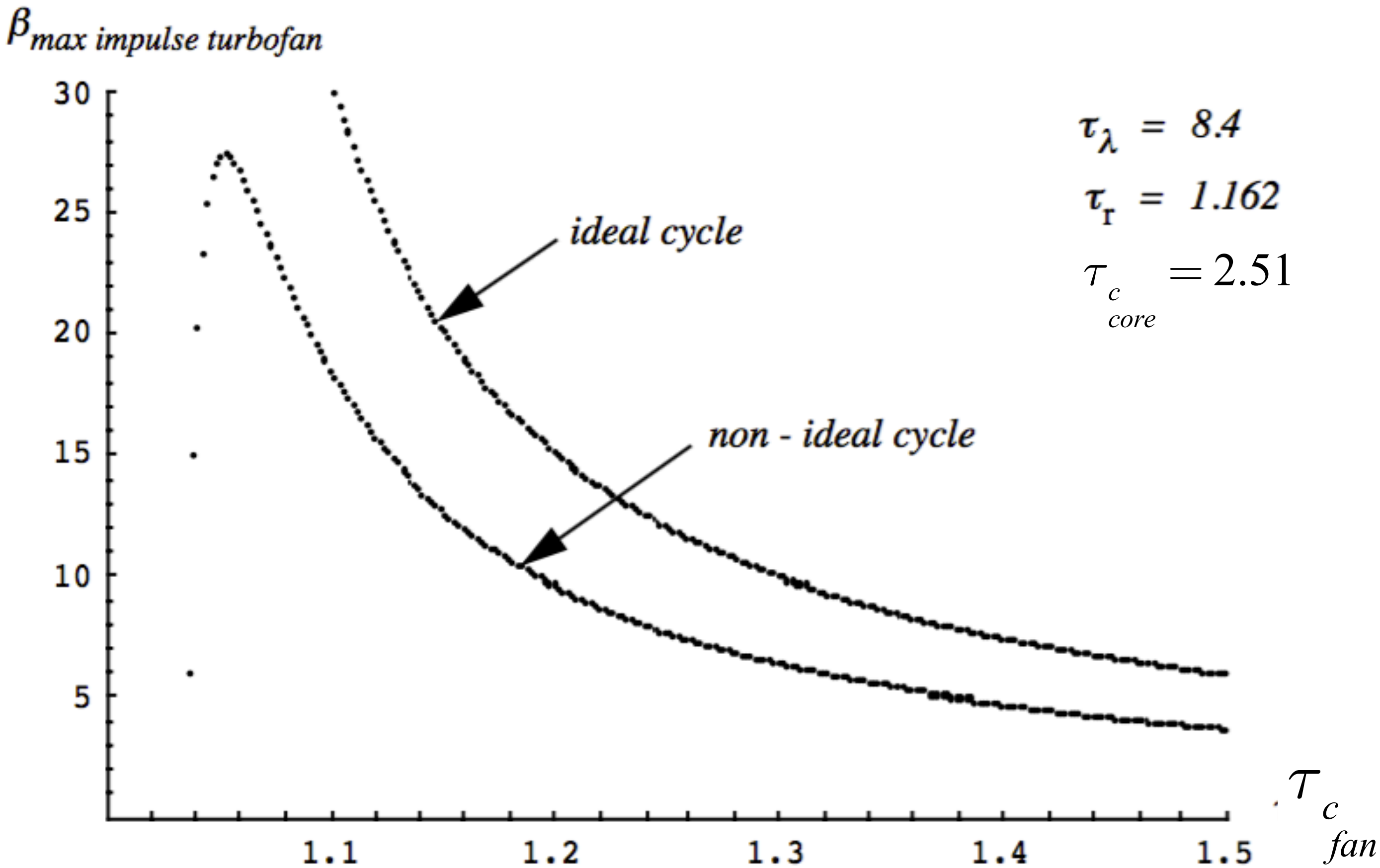
Discussion of Optimal TurboFan Performance (3)

OPTIMAL BYpass ratio vs Fan ComprEssion Ratio



- From this plot is is observed that increasing fan pressure ratio leads to an optimum at a lower bypass ratio.
- Curves all allow for optimum systems at very low fan pressure ratios and high bypass ratios.
- This result is an artifact of the assumptions underlying the ideal turbofan.
- Once non-ideal effects are included, low fan pressure ratio solutions reduce to much lower bypass ratios. …

Comparisons of Ideal and Non-Ideal TurboFan



Turbofan bypass ratio for maximum specific impulse as a function of fan temperature ratio comparing the ideal with a non-ideal cycle.

TurboFan Efficiencies

- Recall that

Propulsive Efficiency =

$$\eta_{propulsive} = \frac{\dot{W}_p}{\left(K.E._{exit} - K.E._{\infty}\right)}$$

Kinetic energy production rate

Thermal Efficiency = $\rightarrow$

$$\eta_{thermal} = \frac{\left(K.E._{exit} - K.E._{\infty}\right)}{\dot{m}_{fuel} \cdot h_{fuel}}$$

Kinetic energy production rate

Combustion Enthalpy of Fuel

TurboFan Efficiencies (2)

Propulsive Efficiency

$$\eta_{propulsive} = \frac{Thrust\ power}{Rate\ of\ kinetic\ enegy\ added\ to\ engine\ flow} \qquad (8)$$

Turbofan engines have two different streams that called hot stream which comes from the core of the engine and cold stream which passes through fan of the engine. The first expression of propulsive efficiency becomes:

$$\eta_{propulsive} = \frac{V_\infty\cdot\left(T_{fan}+T_{core}\right)}{\frac{1}{2}\dot{m}_{a_{core}}\cdot\left(\left(\frac{1+f}{f}\right)\cdot V^2_{exit_{core}} - V^2_\infty\right)+\frac{1}{2}\dot{m}_{a_{fan}}\cdot\left(V^2_{exit_{fan}}-V^2_\infty\right)} = \frac{V_\infty\cdot\left(T_{fan}+T_{core}\right)}{\frac{1}{2}\dot{m}_{a_{core}}\cdot\left[\left(\frac{1+f}{f}\right)\cdot V^2_{exit_{core}} - V^2_\infty+\beta\cdot\left(V^2_{exit_{fan}}-V^2_\infty\right)\right]} =$$

$$= \frac{2\cdot V_\infty\cdot\left(T_{fan}+T_{core}\right)}{\dot{m}_{a_{core}}\cdot\left[\left(\frac{1+f}{f}\right)\cdot V^2_{exit_{core}}+\beta\cdot V^2_{exit_{fan}}-\left(1+\beta\right)\cdot V^2_\infty\right]} = \frac{2\cdot\left(T_{fan}+T_{core}\right)/V_\infty}{\dot{m}_{a_{core}}\cdot\left[\left(\frac{1+f}{f}\right)\cdot\left(\frac{V_{exit_{core}}}{V_\infty}\right)^2+\beta\cdot\left(\frac{V_{exit_{fan}}}{V_\infty}\right)^2-\left(1+\beta\right)\right]}$$

TurboFan Efficiencies (3)

Propulsive Efficiency

$$\eta_{propulsive} = \frac{2\cdot\left(T_{fan}+T_{core}\right)\Big/\left(\dot{m}_{a_{core}}\cdot V_\infty\right)}{\left[\left(\frac{1+f}{f}\right)\cdot\left(\frac{V_{exit_{core}}}{V_\infty}\right)^2+\beta\cdot\left(\frac{V_{exit_{fan}}}{V_\infty}\right)^2-\left(1+\beta\right)\right]} = \frac{2\cdot\left(T_{fan}+T_{core}\right)\Big/\left(\rho_\infty\cdot A_\infty V_\infty^{\ 2}\right)}{\left[\left(\frac{1+f}{f}\right)\cdot\left(\frac{V_{exit_{core}}}{V_\infty}\right)^2+\beta\cdot\left(\frac{V_{exit_{fan}}}{V_\infty}\right)^2-\left(1+\beta\right)\right]} =$$

$$\frac{2\cdot\frac{R_g\cdot T_\infty}{V_\infty^{\ 2}}\left(\frac{T_{fan}+T_{core}}{p_\infty A_\infty}\right)}{\left[\left(\frac{1+f}{f}\right)\cdot\left(\frac{V_{exit_{core}}}{V_\infty}\right)^2+\beta\cdot\left(\frac{V_{exit_{fan}}}{V_\infty}\right)^2-\left(1+\beta\right)\right]} = \frac{\frac{2}{\gamma}\cdot\frac{\gamma\cdot R_g\cdot T_\infty}{V_\infty^{\ 2}}\left(\frac{T_{fan}+T_{core}}{p_\infty A_\infty}\right)}{\left[\left(\frac{1+f}{f}\right)\cdot\left(\frac{V_{exit_{core}}}{V_\infty}\right)^2+\beta\cdot\left(\frac{V_{exit_{fan}}}{V_\infty}\right)^2-\left(1+\beta\right)\right]} =$$

$$\frac{\frac{2}{\gamma\cdot M_\infty^2}\cdot\left(\frac{T_{fan}+T_{core}}{p_\infty A_\infty}\right)}{\left[\left(\frac{1+f}{f}\right)\cdot\left(\frac{V_{exit_{core}}}{V_\infty}\right)^2+\beta\cdot\left(\frac{V_{exit_{fan}}}{V_\infty}\right)^2-\left(1+\beta\right)\right]} = \frac{\frac{2}{\gamma\cdot M_\infty^2}\cdot\left(\mathbb{T}_{fan}+\mathbb{T}_{core}\right)}{\left[\left(\frac{1+f}{f}\right)\cdot\left(\frac{V_{exit_{core}}}{V_\infty}\right)^2+\beta\cdot\left(\frac{V_{exit_{fan}}}{V_\infty}\right)^2-\left(1+\beta\right)\right]}$$

TurboFan Efficiencies (4)

Propulsive Efficiency

$$\left(\mathbb{T}\right)_{turbofan} = \left(\mathbb{T}_{fan} + \mathbb{T}_{core}\right) = \gamma \cdot M_\infty^2 \left[\left(\frac{1}{1+\beta}\right)\cdot\left(\frac{V_{exit\ core}}{V_\infty} - 1\right) + \left(\frac{\beta}{1+\beta}\right)\cdot\left(\frac{V_{exit\ fan}}{V_\infty} - 1\right)\right] \rightarrow$$

$$\eta_{propulsive} = \frac{\dfrac{2}{\gamma \cdot M_\infty^2}\cdot\left(\mathbb{T}_{fan} + \mathbb{T}_{core}\right)}{\left[\left(\dfrac{1+f}{f}\right)\cdot\left(\dfrac{V_{exit_{core}}}{V_\infty}\right)^2 + \beta\cdot\left(\dfrac{V_{exit_{fan}}}{V_\infty}\right)^2 - (1+\beta)\right]} = \frac{2\cdot\left[\left(\dfrac{1}{1+\beta}\right)\cdot\left(\dfrac{V_{exit\ core}}{V_\infty} - 1\right) + \left(\dfrac{\beta}{1+\beta}\right)\cdot\left(\dfrac{V_{exit\ fan}}{V_\infty} - 1\right)\right]}{\left[\left(\dfrac{1+f}{f}\right)\cdot\left(\dfrac{V_{exit_{core}}}{V_\infty}\right)^2 + \beta\cdot\left(\dfrac{V_{exit_{fan}}}{V_\infty}\right)^2 - (1+\beta)\right]}$$

Check → for Turbojet

$$\beta = 0$$

$$\left(\frac{1+f}{f}\right) = 1 \quad \rightarrow \eta_{propulsive} = \frac{2\cdot\left(\dfrac{V_{exit\ core}}{V_\infty} - 1\right)}{\left[\left(\dfrac{V_{exit_{core}}}{V_\infty}\right)^2 - 1\right]} = \frac{2\cdot\left(\dfrac{V_{exit\ core}}{V_\infty} - 1\right)}{\left(\dfrac{V_{exit\ core}}{V_\infty} - 1\right)\cdot\left(\dfrac{V_{exit\ core}}{V_\infty} + 1\right)} = \frac{2}{\left(1 + \dfrac{V_{exit\ core}}{V_\infty}\right)} \surd$$

TurboFan Efficiencies (6)

Propulsive Efficiency *"Take Away"*

$$\eta_{propulsive} = \frac{2 \cdot \left[\left(\frac{1}{1+\beta} \right) \cdot \left(\frac{V_{exit_{core}}}{V_\infty} - 1 \right) + \left(\frac{\beta}{1+\beta} \right) \cdot \left(\frac{V_{exit_{fan}}}{V_\infty} - 1 \right) \right]}{\left[\left(\frac{1+f}{f} \right) \cdot \left(\frac{V_{exit_{core}}}{V_\infty} \right)^2 + \beta \cdot \left(\frac{V_{exit_{fan}}}{V_\infty} \right)^2 - (1+\beta) \right]}$$

$$\rightarrow \left(\frac{V_{exit_{core}}}{V_\infty} \right) = \sqrt{\left(\frac{\tau_r \cdot \tau_c \cdot \tau_t - 1}{\tau_r - 1} \right)} \cdot \sqrt{\frac{\tau_\lambda}{\tau_c \cdot \tau_r}} \rightarrow \left(\frac{V_{exit_{fan}}}{V_\infty} \right) = \sqrt{\frac{\tau_r \cdot \tau_{c_{fan}} - 1}{\tau_r - 1}}$$

TurboFan Efficiencies (7)

Thermal Efficiency *Take Away*

- Also, recall from earlier …

$$\eta_{th} = 1 - \frac{\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot\frac{h_{exit_{core}}}{h_\infty} - 1}{\tau_{c_{core}}\cdot\tau_r\left[\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot\frac{\tau_\gamma}{\tau_{c_{core}}\cdot\tau_r} - 1\right]} = 1 - \frac{\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot\frac{\tau_\gamma}{\tau_{c_{core}}\cdot\tau_r} - 1}{\tau_{c_{core}}\cdot\tau_r\left[\left(1+\frac{1}{f}\cdot(1+\beta)\right)\cdot\frac{\tau_\gamma}{\tau_{c_{core}}\cdot\tau_r} - 1\right]} = 1 - \frac{1}{\tau_{c_{core}}\cdot\tau_r}$$

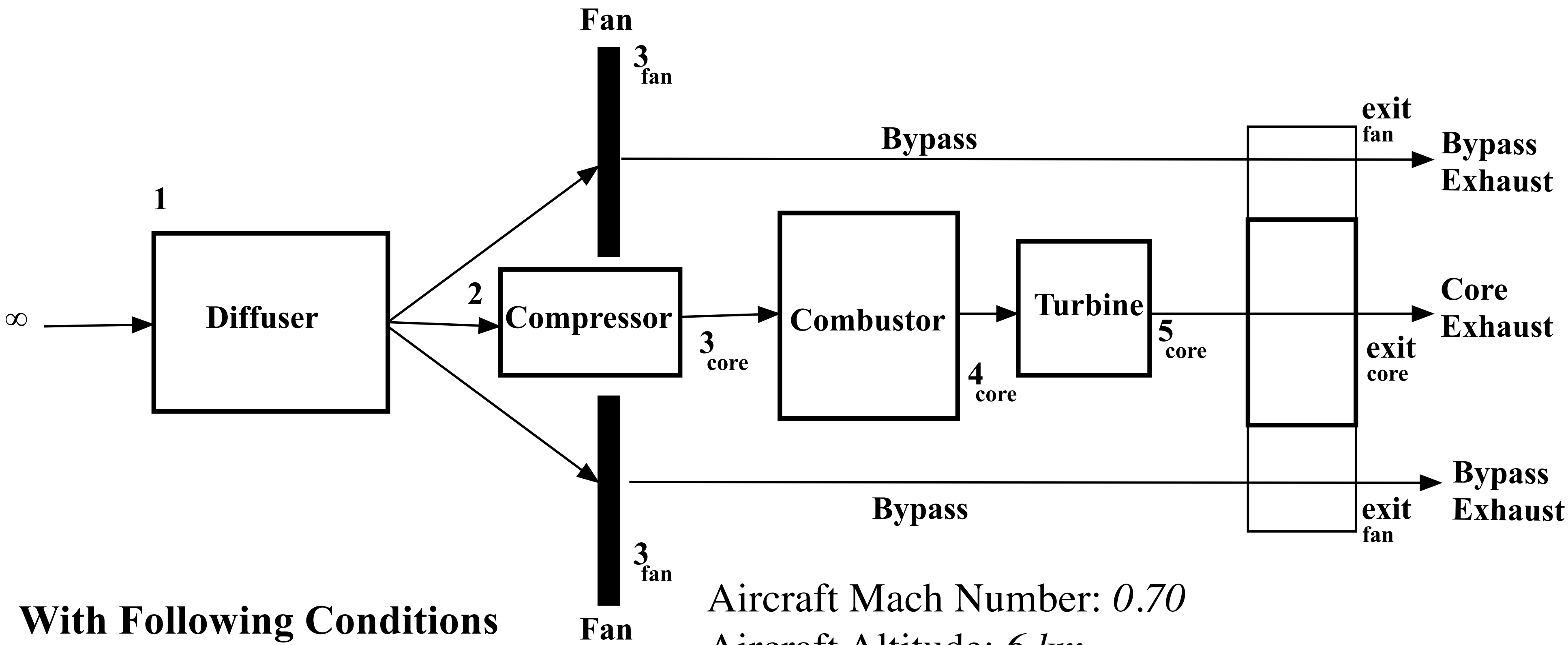
- This solution is identical to the turbo jet analysis with the core flow replacing the normal turbine flow path.

- This analysis shows for the ideal (isentropic) fan → $h_{exit_{fan}} = h_\infty$ --> *the* the heat rejected by the fan stream is zero.

- Therefore the thermal efficiency of the ideal turbofan is independent of the parameters of the fan stream.

Homework 6.2

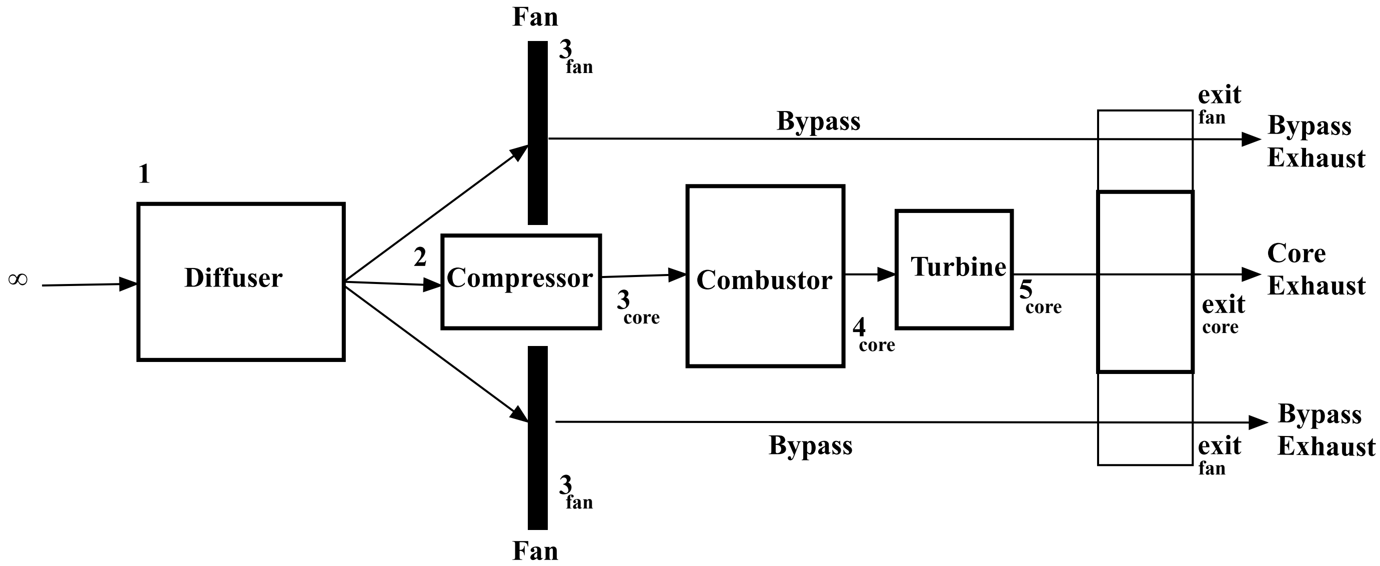
Consider the TurboFan Engine whose Block Diagram is Shown Below

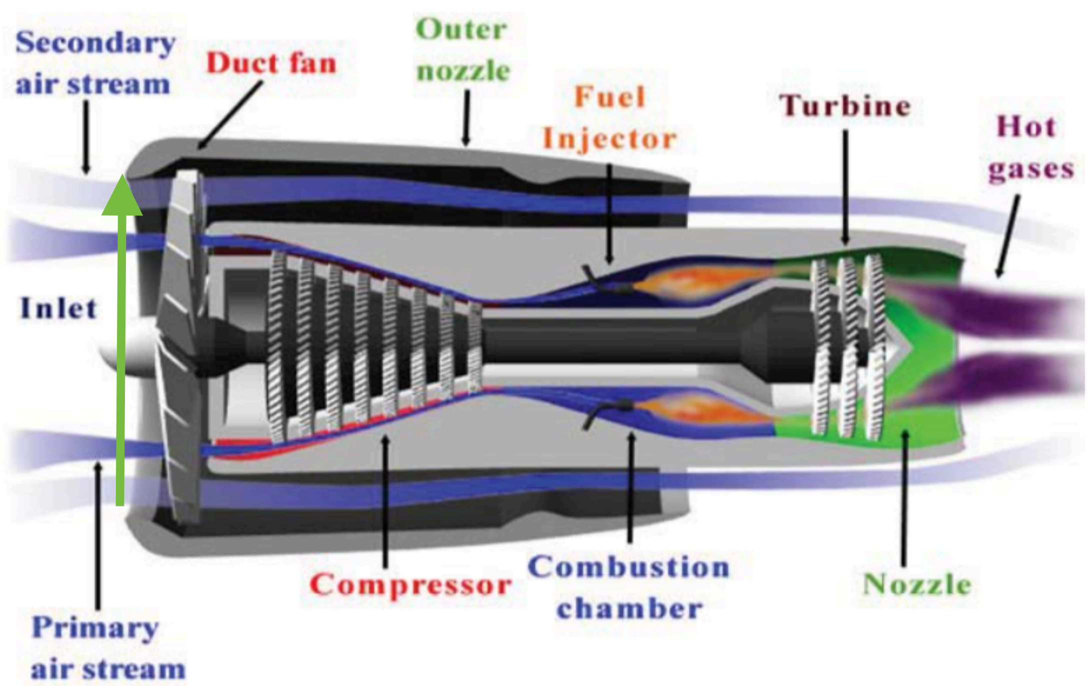


With Following Conditions

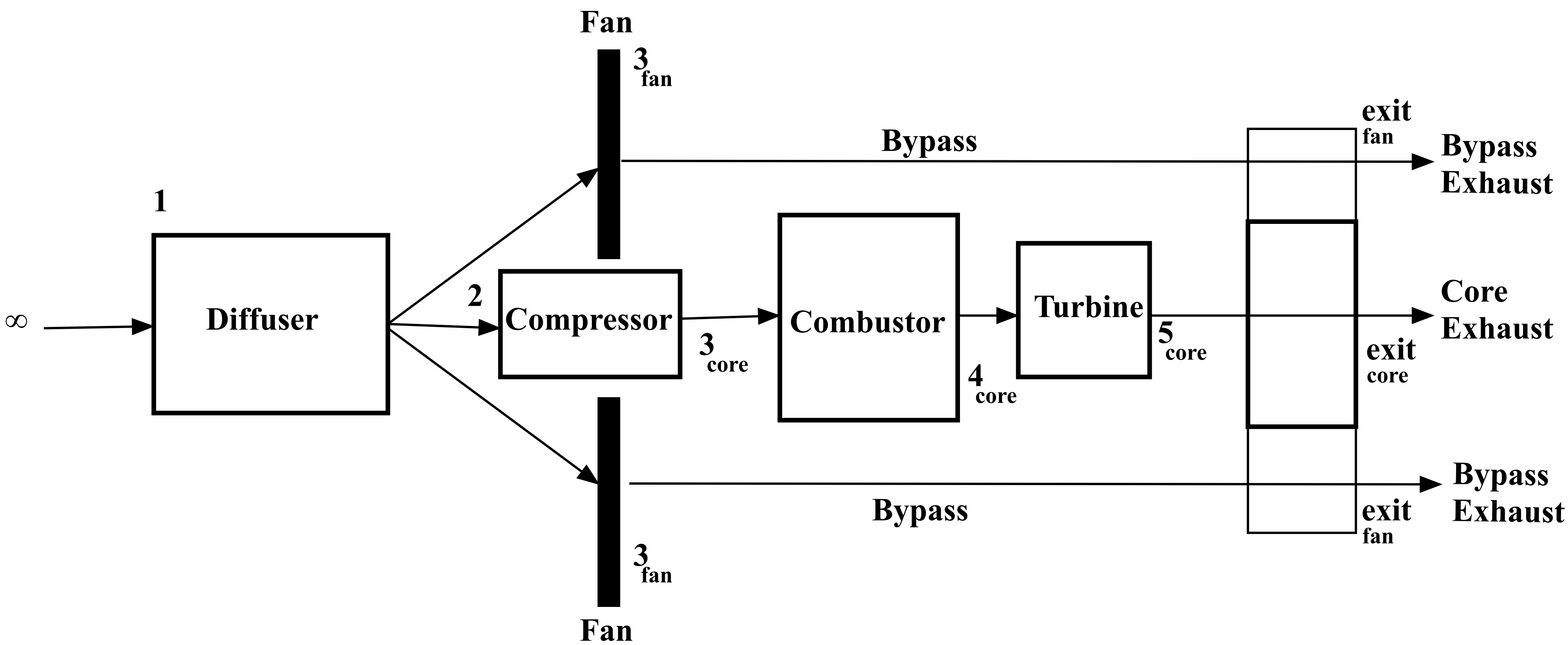
Aircraft Mach Number: *0.70*
Aircraft Altitude: *6 km*
Bypass Ratio: *2*
Fan Pressure Ratio: *2*
Compressor Ratio: *6*
Burner Outlet Temperature: *1700 K*
Fuel: *JP4* → $h_f = 42.68\ MJ/kg$

Homework 6.2 (2)





Homework 6.2 (3)



Assume the Following Component Properties

i) Diffuser, Compressor, Fan, Turbine, Nozzle ~ Isentropic

ii) Nozzle exit flow is NOT mixed

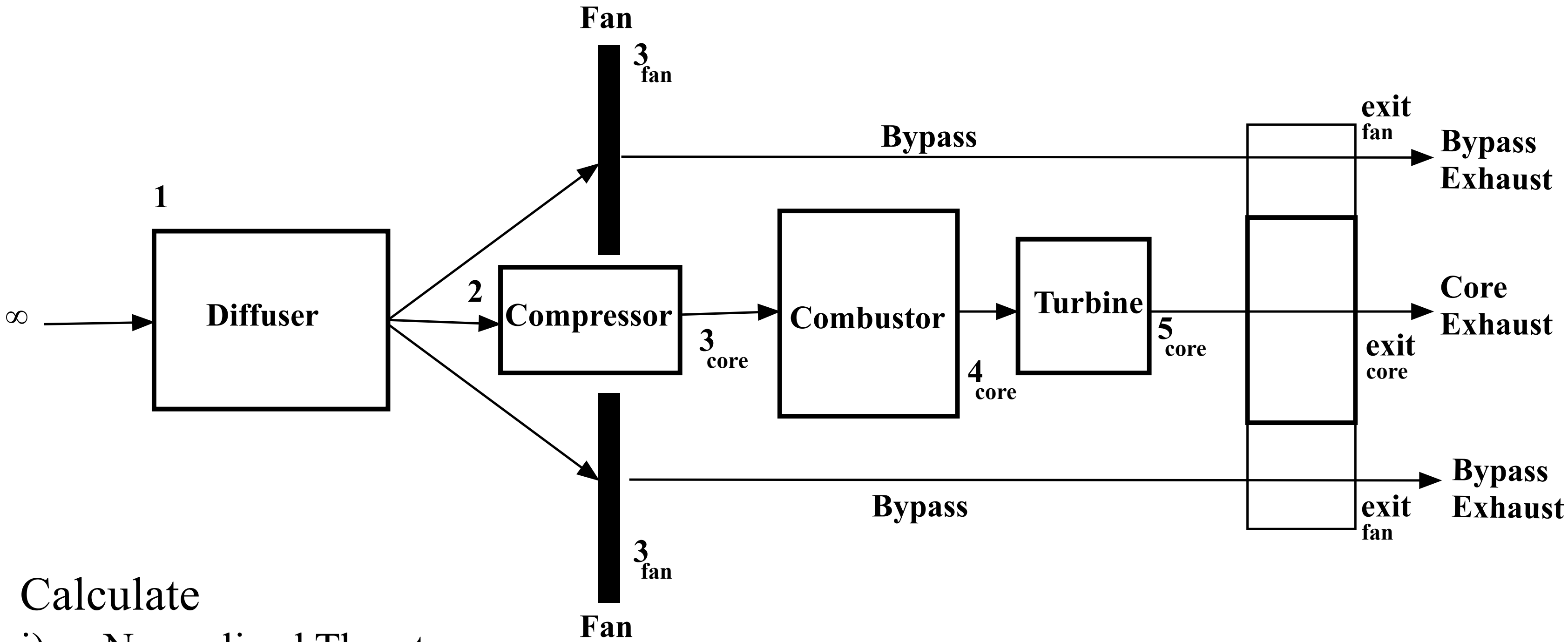
iii) Combustor is 35% efficient → $\eta_b = \frac{\dot{Q}_{3-4}}{\dot{m}_{fuel} \cdot h_f} = \tau_f \cdot \frac{C_p \cdot T_\infty}{h_f}$

iv) Fuel massflow is NOT negligible

v) Mean specific heats, gamma are constant across engine constant → $\gamma \approx 1.4$, $C_p \approx 1004.96_{J/kg-K}$

vi) Fan, Core Nozzle Exits Optimized for Altitude

Homework 6.2 (4)



Calculate

i) Normalized Thrust

ii) % of Thrust delivered by Core Flow

iii) % of Thrust delivered by Bypass Flow

ii) Ratio of Bypass Thrust to Core Thrust

iii) Normalized Specific Impulse

iv) TSFC *lbm/lbf-hr*

v) Bypass Ratio for Optimal Isp

vi) Optimal TSFC

vii) Thermal, Propulsive, and Total Efficiency

Verify Graphical peak is at optimal Bypass ratio

$$\mathbb{T} = \frac{F_{thrust}}{p_\infty \cdot A_\infty}$$

$$\mathbb{I} = \frac{I_{sp} \cdot g_0}{c_\infty}$$

$$TSFC = \frac{1}{g_0 \cdot I_{sp}}$$


Questions??