Section 2.1 : An Introduction to the Differential Conservation for Inviscid Flows

$$Continuity : \nabla \bullet \left(\rho \vec{V} \right) = 0$$

- Anderson,
  Chapter 6 pp. 239-260

$$Momentum : \rho \left(\vec{V} \bullet \nabla \right) \vec{V} = -\nabla P$$

$$Energy : \nabla \bullet \left[\left(e + \frac{\left\| V^2 \right\|}{2} \right) \rho \vec{V} \right] = -\nabla \bullet \left(p \vec{V} \right) + \rho \left(\dot{q} \right)$$

$$Crocco(entropy) : T \nabla s = \nabla h_o - \vec{V} \times \left(\nabla \times \vec{V} \right)$$

Differential Form of Continuity Equation

- Recall from section 2, integral form of continuity equation is:

$$-\iint_{C.S.}\left(\rho \vec{V} \bullet \vec{ds}\right) = \frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho dv\right)$$

"continuity equation"

- Look at first term

$$\iint_{C.S.}\left(\rho \vec{V} \bullet \vec{ds}\right)$$

- Can we write this surface integral in terms of a volume integral?

Divergence Theorem

- From advanced calculus recall that (Gauss Theorem)

$$\iiint_v \nabla \bullet \vec{V} dv = \iint_s \left(\vec{V} \bullet \vec{ds} \right)$$

- Applying to mass flux across control surface

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) = \iiint_v \nabla \bullet \left(\rho \vec{V} \right) dv$$

- The divergence theorem is a mathematical statement of the physical fact that, in the absence of the creation or destruction of matter, the density within a region of space can change only by having it flow into or away from the region through its boundary.

Divergence Theorem

(cont'd)

- Applying divergence theorem to continuity equation

$$-\iiint_{v} \nabla \bullet \left(\rho \vec{V} \right) dv = \frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho dv \right) \rightarrow \iiint_{c.v.} \left[\frac{\partial}{\partial t}\rho + \nabla \bullet \left(\rho \vec{V} \right) \right] dv = 0 \rightarrow$$

$$\frac{\partial}{\partial t}\rho + \nabla \bullet \left(\rho \vec{V} \right) = 0$$

$$\frac{\partial}{\partial t}\rho + \nabla \bullet \left(\rho \vec{V} \right) = 0$$

Differential form of continuity equation

Differential Form of Momentum Equation

- Recall from section 2, integral form of momentum equation for inviscid flow is

$$\iiint_{C.V.} \rho \vec{f}_b \, dv - \iint_{C.S.} (p) \, d\vec{S} = \iiint_{C.V.} \frac{\partial}{\partial t}\left(\rho \vec{V}\right) dv + \iint_{C.S.} \left(\rho \vec{V} \bullet d\vec{s}\right) \vec{V}$$

- Look at term $\iint_{C.S.} (p) \, d\vec{S}$

- Can we write the surface integral in terms of volume integral?

Analog of Divergence Theorem

- Start with Gauss theorem $\iiint_v \nabla \bullet \vec{V} dv = \iint_s \left(\vec{V} \bullet \vec{ds} \right)$

$$Let \rightarrow \vec{V} = \vec{\lambda} P \rightarrow \iiint_v \nabla \bullet \left(\vec{\lambda} P \right) dv = \iint_s \left(\left(\vec{\lambda} P \right) \bullet \vec{ds} \right)$$

$$\iiint_v \vec{\lambda} \bullet \nabla P dv = \iint_s \left(\vec{\lambda} \bullet \left(P \vec{ds} \right) \right) \rightarrow \vec{\lambda} \bullet \left(\iiint_v \nabla P dv \right) = \vec{\lambda} \bullet \left(\iint_s P \vec{ds} \right) \rightarrow$$

$$\boxed{\iiint_v \nabla P dv = \iint_s P \vec{ds}}$$

Momentum Equation

(revisited)

- Applying analog of divergence theorem to momentum equation

$$\iint_{C.S.} (p) \vec{dS} = \iiint_{C.V} \nabla P dv$$

- Momentum equation becomes

$$\iiint_{C.V.} \rho \vec{f_b} dv - \iiint_{C.V} \nabla P dv = \iiint_{C.V.} \frac{\partial}{\partial t}\left(\rho \vec{V} \right) dv + \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V}$$

- Now look at $\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V}$

Momentum Equation

(revisited, cont'd)

- Apply Gauss Theorem

$$\iint_{C.S.}\left(\rho\vec{V}\bullet\vec{ds}\right)\vec{V}=\left(\iiint_{v}\nabla\bullet\left(\rho\vec{V}\right)dv\right)\vec{V}$$

- Momentum equation becomes

$$\iiint_{C.V.}\rho\vec{f_b}\,dv-\iiint_{C.V}\nabla P dv=\iiint_{C.V.}\frac{\partial}{\partial t}\left(\rho\vec{V}\right)dv+\iiint_{v}\nabla\bullet\left(\rho\vec{V}\right)dv\,\vec{V}$$

Momentum Equation

(revisited, cont'd)

- Look at scalar components

-- Longitudinal (x) direction $\vec{V} \rightarrow u$

$$\iiint_{C.V.} \rho f_x \, dv - \iiint_{C.V} \frac{\partial P}{\partial x} dv = \iiint_{C.V.} \frac{\partial}{\partial t}(\rho u)\, dv + \iiint_{v} \nabla \bullet \left(\rho u \vec{V} \right) dv$$

$$\iiint_{C.V.} \left(\rho f_x - \frac{\partial P}{\partial x} - \frac{\partial}{\partial t}(\rho u) - \nabla \bullet \left(\rho u \vec{V} \right) \right) dv = 0$$

Momentum Equation

(revisited, cont'd)

- Integrand must be zero for all cases

$$\rho f_x - \frac{\partial P}{\partial x} - \frac{\partial}{\partial t}(\rho u) - \nabla \bullet \left(\rho u \overset{->}{V} \right)$$

- Rearranging

$$\frac{\partial}{\partial t}(\rho u) + \nabla \bullet \left(\rho u \overset{->}{V} \right) = -\frac{\partial P}{\partial x} + \rho f_x$$

X-COMPONENT … differential momentum equation

Momentum Equation

(revisited, cont'd)

- Lateral component, y

$$\frac{\partial}{\partial t}(\rho \mathrm{v}) + \nabla \bullet \left(\rho \mathrm{v} \overset{->}{V} \right) = -\frac{\partial P}{\partial y} + \rho f_y$$

- Normal component, z

$$\frac{\partial}{\partial t}(\rho w) + \nabla \bullet \left(\rho w \overset{->}{V} \right) = -\frac{\partial P}{\partial z} + \rho f_z$$

Momentum Equation (cont'd)

- Re-visit longitudinal momentum equation

$$\frac{\partial}{\partial t}(\rho u) + \nabla \bullet \left(\rho u \vec{V} \right) = -\frac{\partial P}{\partial x} + \rho f_x$$

- Expand derivative and gradient operator

$$\rho \frac{\partial}{\partial t} u + u \frac{\partial}{\partial t} \rho + u \nabla \bullet \left(\rho \vec{V} \right) + \left(\rho \vec{V} \right) \bullet \nabla u = -\frac{\partial P}{\partial x} + \rho f_x$$

- Multiply differential form of continuity equation by u

$$u \frac{\partial}{\partial t} \rho + u \nabla \bullet \left(\rho \vec{V} \right) = 0$$

Momentum Equation (cont'd)

- Subtract from expanded longitudinal momentum equation

$$\frac{\partial}{\partial t}(\rho u) + \nabla \bullet \left(\rho u \vec{V} \right) = -\frac{\partial P}{\partial x} + \rho f_x$$

- Result is

$$\rho \frac{\partial}{\partial t} u + \left(\rho \vec{V} \right) \bullet \nabla u = -\frac{\partial P}{\partial x} + \rho f_x$$

Momentum Equation (cont'd)

- Performing same operations on y, and z momentum equations we get collected set

$$\begin{bmatrix} \rho \frac{\partial}{\partial t} u + \left(\rho \vec{V} \right) \bullet \nabla u = -\frac{\partial P}{\partial x} + \rho f_x \\ \rho \frac{\partial}{\partial t} \mathrm{v} + \left(\rho \vec{V} \right) \bullet \nabla \mathrm{v} = -\frac{\partial P}{\partial y} + \rho f_y \\ \rho \frac{\partial}{\partial t} w + \left(\rho \vec{V} \right) \bullet \nabla w = -\frac{\partial P}{\partial z} + \rho f_x \end{bmatrix}$$

Momentum Equation (concluded)

- But

$$\left(\rho \vec{V} \bullet \nabla\right) V = \begin{bmatrix} \left(\rho \vec{V}\right) \bullet \nabla u \\ \left(\rho \vec{V}\right) \bullet \nabla v \\ \left(\rho \vec{V}\right) \bullet \nabla w \end{bmatrix}$$

- And momentum equation can be written in vector form as

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho \left(\vec{V} \bullet \nabla \right) \vec{V} = -\nabla P + \rho \vec{f}$$

Euler's Equation

- If we ignore body forces then

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho \left(\vec{V} \bullet \nabla \right) \vec{V} = -\nabla P$$

- Look at steady 1-D form of above

$$dp = -\rho V dV \qquad \text{(“euler's equation”)}$$

Euler's Equation (concluded)

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho \left(\vec{V} \bullet \nabla \right) \vec{V} = -\nabla P$$

- 3-D form of Euler equation derived back in section 3

("euler's equation")

Energy Equation

- From section 2 derivation, integral form of energy equation is:

$$\frac{\partial}{\partial t}\iiint_{C.V.}\left(\rho\left(e+\frac{\left\|V^2\right\|}{2}\right)dv\right)+\iint_{C.S.}\rho\vec{V}\bullet d\vec{S}\left(e+\frac{\left\|V^2\right\|}{2}\right)=$$

$$\iiint_{C.V.}(\rho\vec{f}\,dv)\bullet\vec{V}-\iint_{C.S.}(pd\vec{S})\bullet\vec{V}+\iiint_{C.V.}\left(\rho\left(\dot{q}\right)dv\right)$$

- Control surface integrals are converted to control volume using the derived forms of the gauss theorem

$$\iint_{C.S.}\rho\vec{V}\bullet d\vec{S}\left(e+\frac{\left\|V^2\right\|}{2}\right)=\iint_{C.S.}\left(e+\frac{\left\|V^2\right\|}{2}\right)\rho\vec{V}\bullet d\vec{S}=\iiint_{C.V.}\nabla\bullet\left(e+\frac{\left\|V^2\right\|}{2}\right)\rho\vec{V}\,dv$$

Energy Equation (concluded)

$$\iint_{C.S.} (p d\vec{S}) \bullet \vec{V} = \iiint_{v} \nabla p dv \bullet \vec{V}$$

- Substituting into energy equation

$$\frac{\partial}{\partial t}\iiint_{C.V.}\left(\rho\left(e+\frac{\left\|V^2\right\|}{2}\right)dv\right)+\iiint_{C.V.}\nabla\bullet\left(e+\frac{\left\|V^2\right\|}{2}\right)\rho\vec{V}\,dv-\iiint_{C.V.}(\rho\vec{f}\,dv)\bullet\vec{V}+\iiint_{C.V.}\nabla p dv\bullet\vec{V}--\iiint_{C.V.}\left(\rho\left(\dot{q}\right)dv\right)=0$$

- Integrand must be zero for all conditions

"differential form of energy equation"

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{\left\|V^2\right\|}{2}\right)\right]+\nabla\bullet\left(e+\frac{\left\|V^2\right\|}{2}\right)\rho\vec{V}=-\nabla\bullet\left(p\vec{V}\right)+\rho\left(\dot{q}\right)+\rho\left(\vec{f}\bullet\vec{V}\right)$$

Crocco's Theorem

- Write Euler's equation

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho\left(\vec{V} \bullet \nabla\right)\vec{V} = -\nabla P$$

- Recall from section 1 that entropy equation can be written as

$$dh = Tds - pdv + pdv + vdp =$$

$$dh = Tds + vdp$$

- Extend to 3-D space by letting $d \rightarrow \nabla$

$$T\nabla s = \nabla h - v\nabla p = \nabla h - \frac{\nabla p}{\rho}$$

Crocco's Theorem: (cont'd)

- Substitute in Euler equation

$$T\nabla s = \nabla h + \frac{\rho \frac{\partial \vec{V}}{\partial t} + \rho\left(\vec{V} \bullet \nabla\right)\vec{V}}{\rho} = \nabla h + \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \nabla\right)\vec{V}$$

- But from definition of total enthalpy

$$h = h_o - \frac{V^2}{2} \rightarrow \nabla h = \nabla h_o - \nabla\left(\frac{V^2}{2}\right)$$

Crocco's Theorem: (cont'd)

- Substitute into entropy equation

$$T\nabla s = \nabla h_o - \nabla\left(\frac{V^2}{2}\right) + \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \nabla\right)\vec{V}$$

- But from "curl theorem" (more calculus!)

$$\vec{V} \times \left(\nabla \times \vec{V}\right) = \left(\vec{V} \bullet \nabla\right)\vec{V} - \nabla\left(\frac{V^2}{2}\right)$$

Crocco's Theorem: (cont'd)

- Substitute into entropy equation

$$T\nabla s = \nabla h_o + \frac{\partial \vec{V}}{\partial t} - \vec{V} \times \left(\nabla \times \vec{V} \right)$$

"Crocco's Theorem"

Steady Flow Relationships with no Body Forces

$$Continuity : \nabla \bullet \left(\rho \vec{V} \right) = 0$$

$$\left\{ \frac{\partial}{\partial t} \rho = 0, \frac{\partial \vec{V}}{\partial t} = 0, \vec{f} = 0 \right\}$$

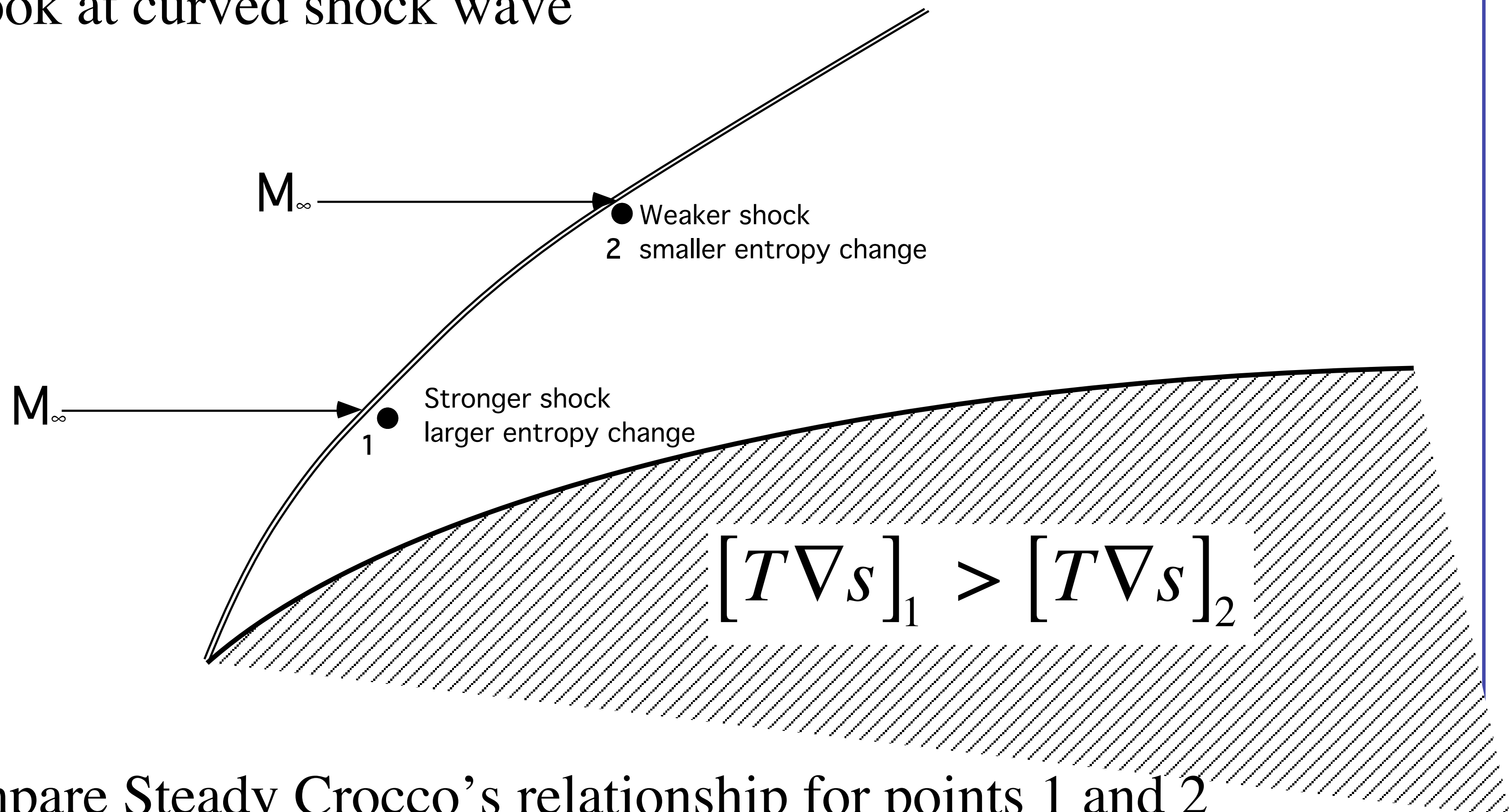
$$Momentum : \rho \left(\vec{V} \bullet \nabla \right) \vec{V} = -\nabla P$$

$$Energy : \nabla \bullet \left[\left(e + \frac{\left\| V^2 \right\|}{2} \right) \rho \vec{V} \right] = -\nabla \bullet \left(p \vec{V} \right) + \rho \left(\dot{q} \right)$$

$$Crocco(entropy) : T \nabla s = \nabla h_o - \vec{V} \times \left(\nabla \times \vec{V} \right)$$

Rotational and Irrotational Flow

- Look at curved shock wave



$$\left[T\nabla s\right]_1 > \left[T\nabla s\right]_2$$

- Compare Steady Crocco's relationship for points 1 and 2

Rotational and Irrotational Flow (cont'd)

- Across shock total enthalpy is constant

$$\left[T\nabla s+\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_1=\left[T\nabla s+\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_2$$

- or

$$\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_1-\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_2=\left[T\nabla s\right]_2-\left[T\nabla s\right]_1$$

$$\left[T\nabla s\right]_1>\left[T\nabla s\right]_2$$

Rotational and Irrotational Flow (cont'd)

- Thus

$$\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_1-\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_2 < 0$$

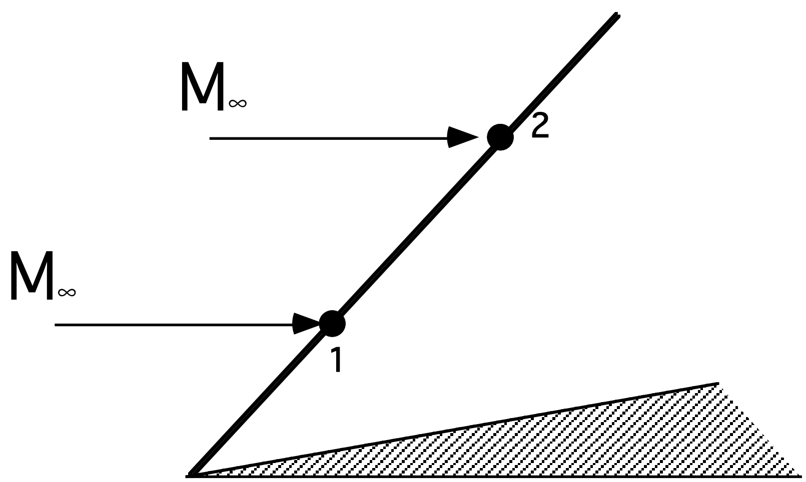
$$\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_1 < \left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_2$$

- $\vec{V}\times\left(\nabla\times\vec{V}\right)$ Measure of "spin" in flow .. "Vorticity"

- Not constant … "rotational flow" … complex flow field

Rotational and Irrotational Flow (cont'd)

- Look at straight oblique shock wave



- Shock strength is the same at points 1 and 2

$$[T\nabla s]_1 = [T\nabla s]_2 \rightarrow \left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_1 = \left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_2$$ "irrotational flow"

Rotational and Irrotational Flow (cont'd)

- Condition for Irrotational flow

$$\vec{V} \times \left(\nabla \times \vec{V} \right) = 0$$

- At every point in flow field

- If you put a pinwheel in and it spins … "rotational flow"