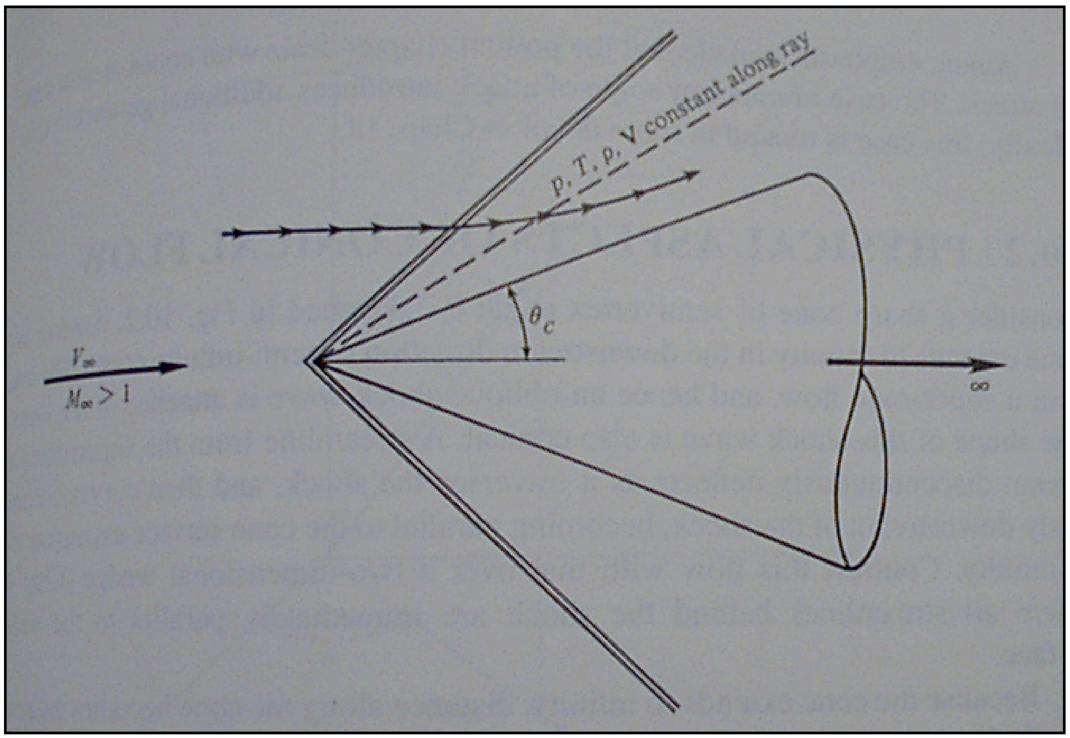
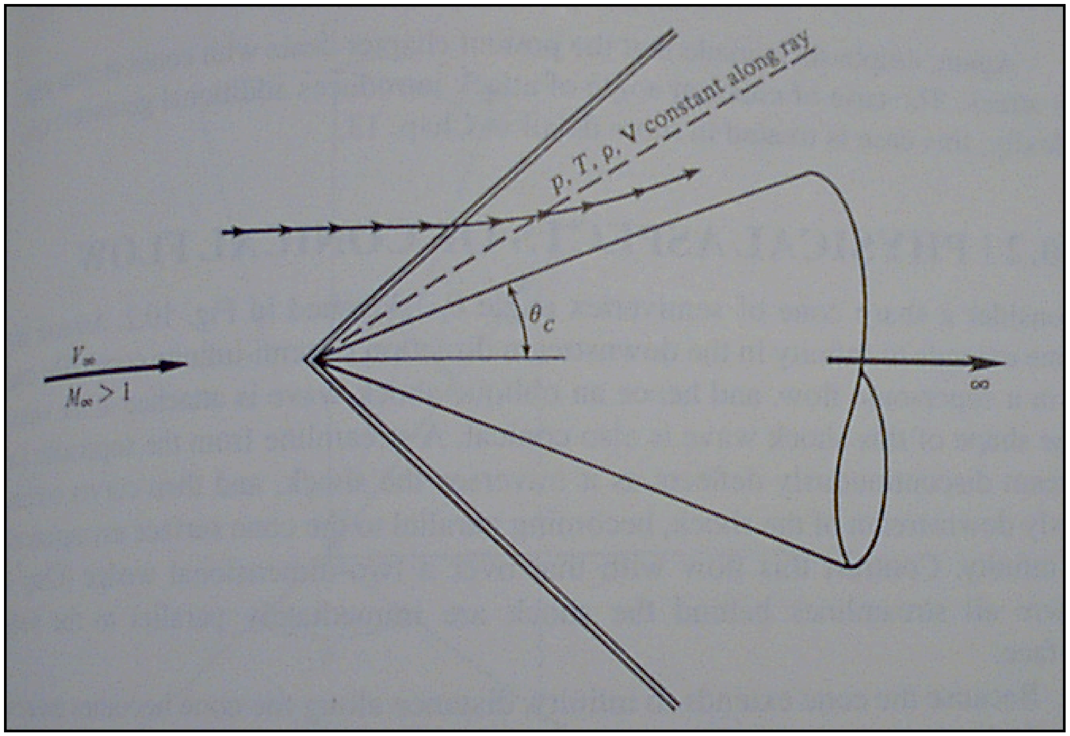
Section 2.2: Analysis of Supersonic Conical Flows

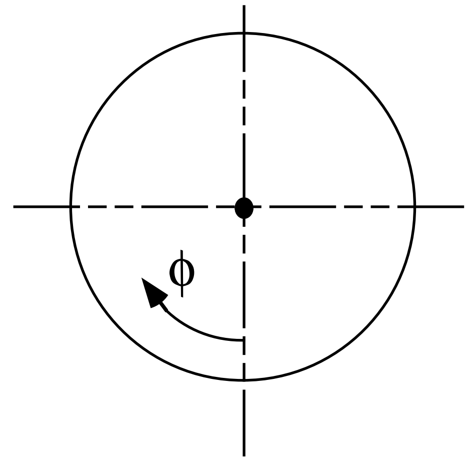
• Anderson, Chapter 10 pp. 363-375



Physical Aspects of Conical Flow

- Flow is circumferentially symmetric





$$\frac{\partial}{\partial \phi}[\quad] = 0$$

- "Axi-symmetric flow"

Physical Aspects of Conical Flow

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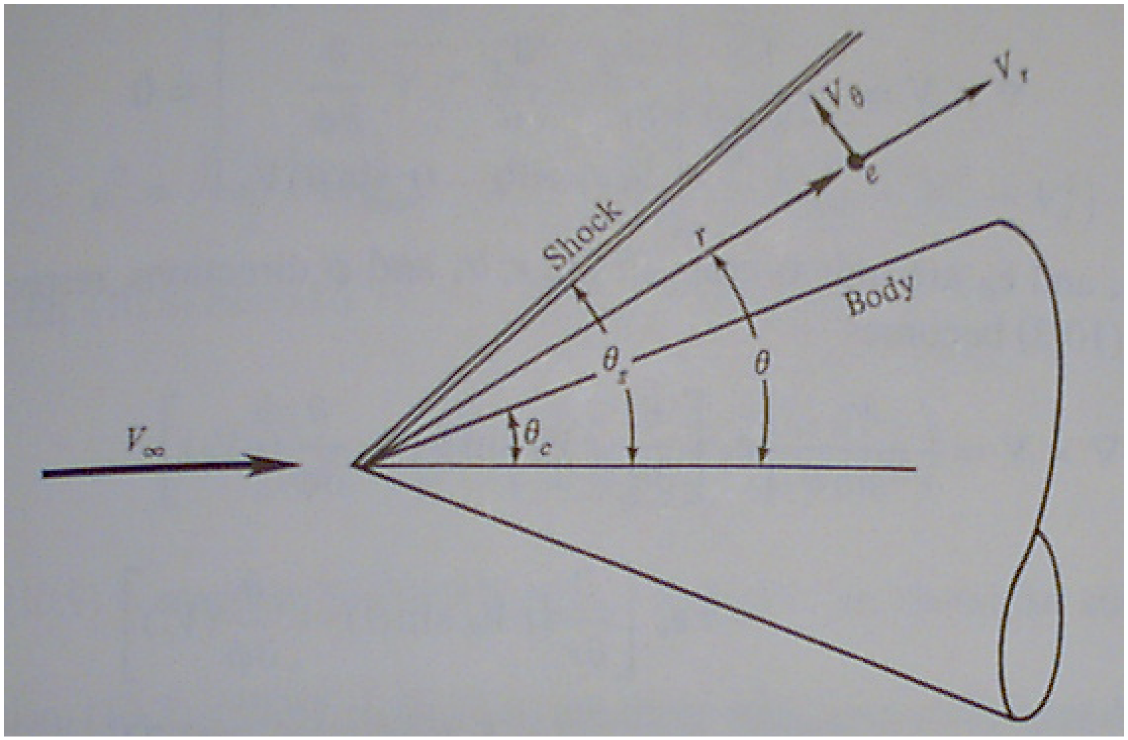
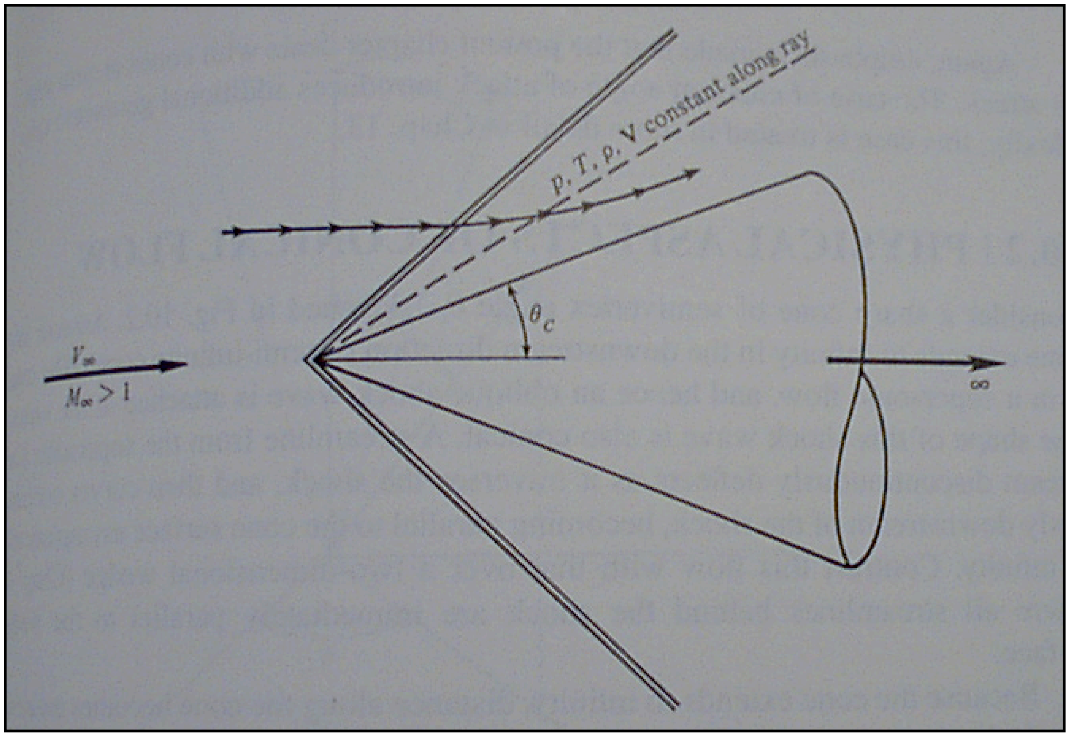


$$\frac{\partial}{\partial r}[V] = 0$$

- Flow Properties are constant along a ray from the vertex

Physical Aspects of Conical Flow

(cont'd)



- Look at shock wave "straight"

- Shock strength is the same at points 1 and 2

$$[T\nabla s]_1 = [T\nabla s]_2 \rightarrow \left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_1 = \left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_2$$ "irrotational flow"

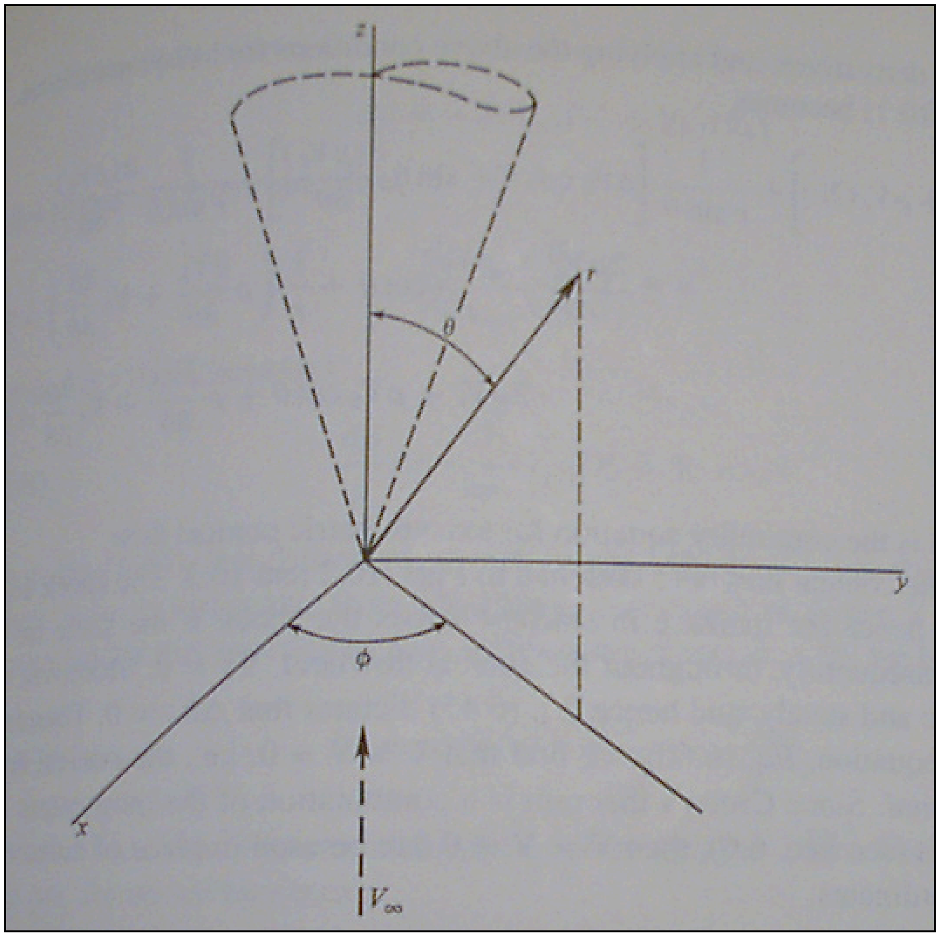
Rotational and Irrotational Flow

- Condition for Irrotational flow

$$\vec{V} \times \left(\nabla \times \vec{V} \right) = 0$$

- At every point in flow field

Spherical Coordinate System



- Axi-symmetric flow is best represented using spherical coordinate system

$$x = r\sin(\theta)\cos(\phi)$$

$$y = r\sin(\theta)\sin(\phi)$$

$$z = r\cos(\theta)$$

(Anderson conventions)

Spherical Coordinate System

(cont'd)

- **Velocity vector**

$$\vec{V} = V_r \vec{i}_r + V_\theta \vec{i}_\theta \rightarrow V_\phi = 0$$

- **Gradient vector**

$$\nabla[\;\;] = \vec{i}_r \frac{\partial}{\partial r}[\;\;] + \frac{1}{r}\vec{i}_\theta \frac{\partial}{\partial \theta}[\;\;] + \frac{1}{r\sin\theta}\vec{i}_\phi \frac{\partial}{\partial \phi}[\;\;]$$

- **Divergence**

$$\nabla \bullet \left(\vec{F}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\, F_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

Continuity Equation for Conical Flow

- Steady flow

$$\cancelto{0}{\frac{\partial}{\partial t}\rho} + \nabla \bullet \left(\rho \vec{V} \right) = 0$$

- Axi-symmetric flow -- use spherical coordinates

$$\nabla \bullet \left(\rho \vec{V} \right) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \rho V_r \right) + \frac{1}{r \sin\theta} \frac{\partial \left(\sin\theta \; \rho V_\theta \right)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \left(\rho V_\phi \right)}{\partial \phi} = 0$$

- Evaluating derivatives and letting $\frac{\partial}{\partial \phi}[\ \ ] = 0$ $\frac{\partial}{\partial r}[V] = 0$

Continuity Equation for Conical Flow (concluded)

$$\frac{2\rho V_r}{r} + \frac{\rho V_\theta}{r\sin(\theta)}\cos(\theta) + \frac{1}{r}\left[\rho\frac{\partial V_\theta}{\partial\theta} + V_\theta\frac{\partial\rho}{\partial\theta}\right] = 0 \rightarrow$$

$$\boxed{2\rho V_r + \rho V_\theta\cot(\theta) + \rho\frac{\partial V_\theta}{\partial\theta} + V_\theta\frac{\partial\rho}{\partial\theta} = 0}$$

- **Continuity Equation for Axi-symmetric Conical Flow**

Crocco's Relation for Axi-symmetric Conical Flow

- Irrotational flow

- Crocco's relationship solves both momentum and energy equation

… thus for conical flow all we need to satisfy is:

$$\vec{V} \times \left(\nabla \times \vec{V} \right) = 0$$

Crocco's Relation for Axi-symmetric Conical Flow (cont'd)

- For Arbitrary V

$$\nabla \times \vec{V} = 0$$

- Curl Operation in Spherical Coordinates

$$\nabla \times \vec{V} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{i}_r & r\ \vec{i}_\theta & r\sin\theta\ \vec{i}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & r\ V_\theta & r\sin\theta\ V_\phi \end{vmatrix} = 0$$

Crocco's Relation for Axi-symmetric Conical Flow (cont'd)

- For Arbitrary V

$$\nabla \times \vec{V} = 0$$

- Curl Operation in Spherical Coordinates

$$\nabla \times \vec{V} = \frac{1}{r^2 \sin\theta} \begin{bmatrix} \vec{i}_r \left(\frac{\partial}{\partial \theta}\left[r \sin\theta \, V_\phi \right] - \frac{\partial}{\partial \phi}\left[r \, V_\theta \right] \right) - \\ r \, \vec{i}_\theta \left[\left(\frac{\partial}{\partial r}\left[r \sin\theta \, V_\phi \right] \right) - \frac{\partial}{\partial \phi} V_r \right] + \\ r \sin\theta \, \vec{i}_\phi \left(\frac{\partial}{\partial r} r \, V_\theta - \frac{\partial}{\partial \theta} V_r \right) \end{bmatrix}$$

Crocco's Relation for Axi-symmetric Conical Flow (cont'd)

- Apply $$\frac{\partial}{\partial \phi}[\ \ ] = 0$$

- Crocco Relation reduces to

$$\nabla \times \vec{V} = \vec{i}_\phi \frac{1}{r^2 \sin\theta}\left[r\sin\theta \left(V_\theta + \frac{\partial V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta} \right)\right] = 0$$

- Now apply $$\frac{\partial}{\partial r}[V] = 0$$

Crocco's Relation for Axi-symmetric Conical Flow (concluded)

- Crocco Relation reduces to (wow!)

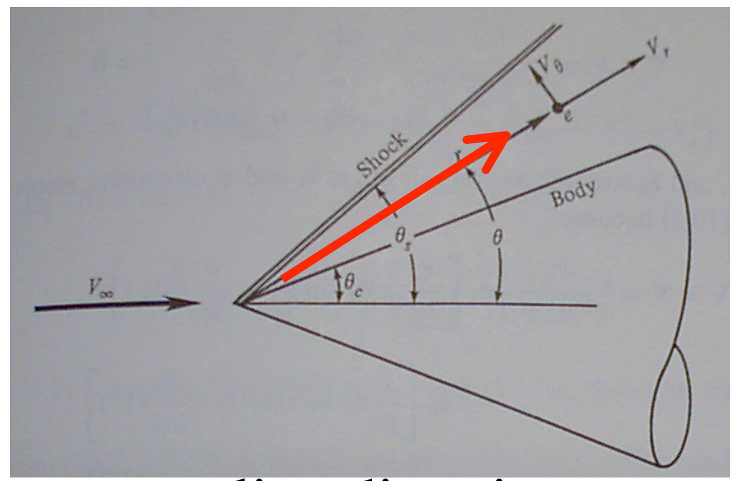
$$V_\theta = \frac{\partial V_r}{\partial \theta}$$

• Irrotationality Condition for Axi-symmetric conical flow

Euler's Equation for Axi-symmetric Conical Flow

- **Steady Axi-symmetric conical flow**

$$\rho \cancelto{0}{\frac{\partial \vec{V}}{\partial t}} + \rho\left(\vec{V} \bullet \nabla\right)\vec{V} = -\nabla P$$



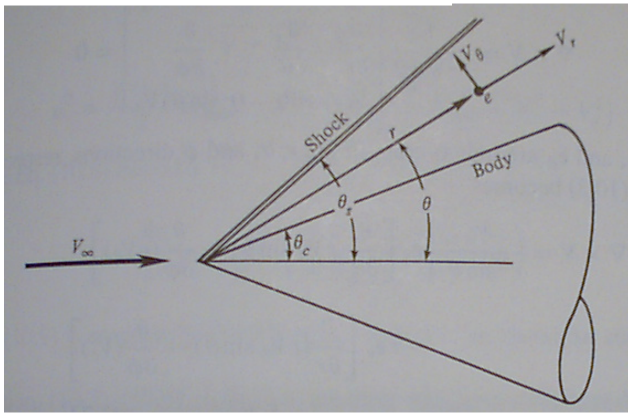
- Flow is irrotational … Apply along stream line direction

$$dP = -\rho V dV \rightarrow V^2 = V_\theta^{\ 2} + V_r^{\ 2}$$

Euler's Equation for Axi-symmetric Conical Flow (cont'd)

- **Steady Axi-symmetric conical flow**

$$dV^2 = d[V_\theta{}^2 + V_r{}^2] = 2VdV = 2V_\theta dV_\theta + 2V_r dV_r$$



$$\boxed{dP = -\rho\left[V_\theta dV_\theta + V_r dV_r\right]}$$

Divide by dρ

$$\frac{dP}{d\rho} = -\frac{\rho}{d\rho}\left[V_\theta dV_\theta + V_r dV_r\right] \rightarrow$$

$$for\ isentropic\ flow \rightarrow \frac{dP}{d\rho} = c^2$$

- **Thus behind the shock wave**

$$\frac{d\rho}{\rho} = -\frac{1}{c^2}\left[V_\theta dV_\theta + V_r dV_r\right]$$

Euler's Equation for Axi-symmetric Conical Flow (cont'd)

- But from enthalpy equation

$$h_0 = h + \frac{V^2}{2} = c_p T + \frac{V^2}{2} = \gamma R_g T \frac{c_p}{\gamma R_g} + \frac{V^2}{2} =$$

$$\gamma R_g T \frac{c_p}{\gamma\left[c_p - c_v\right]} + \frac{V^2}{2} = \frac{c^2}{\gamma - 1} + \frac{V^2}{2}$$

$$\boxed{c^2 = \frac{\gamma - 1}{2}\left(2h_0 - V^2\right)}$$

$$\boxed{h_0 = \frac{V_{\max}^2}{2} \text{....}Why\text{?...}no\ idea\text{!}}$$

Euler's Equation for Axi-symmetric Conical Flow (concluded)

- Sub into Euler's equation

$$\frac{d\rho}{\rho} = -\frac{2}{\gamma-1}\frac{\left[V_\theta dV_\theta + V_r dV_r\right]}{\left(2h_0 - V^2\right)} = \frac{2}{\gamma-1}\frac{\left[V_\theta dV_\theta + V_r dV_r\right]}{\left(\left(V_\theta^{\,2} + V_r^{\,2}\right)\text{-}2h_0\right)}$$

$$\boxed{h_0 = \frac{V_{\max}^2}{2} \;....Why?...no\ idea!}$$

Collected Equations for for Axi-symmetric Conical Flow

Continuity

$$\frac{2\rho V_r}{r} + \frac{\rho V_\theta}{r\sin(\theta)}\cos(\theta) + \frac{1}{r}\left[\rho\frac{dV_\theta}{d\theta} + V_\theta\frac{d\rho}{d\theta}\right] =$$

$$\boxed{2V + V_\theta\cot(\theta) + \left[\frac{dV_\theta}{d\theta} + \frac{V_\theta}{\rho}\frac{d\rho}{d\theta}\right] = 0}$$

- Only θ is independent variable
Can re-write partials at total derivatives

$$\boxed{\begin{aligned}&Crocco:\\ &V_\theta = \frac{dV_r}{d\theta}\end{aligned}}$$

$$\boxed{\begin{aligned}&Euler:\\ &\frac{d\rho}{d\theta} = \frac{2}{\gamma-1}\frac{\left[V_\theta\frac{dV_\theta}{d\theta} + V_r\frac{dV_r}{d\theta}\right]}{\left(\left(V_\theta^2 + V_r^2\right)\text{-}2h_0\right)}\end{aligned}}$$

Collected Equations for for Axi-symmetric Conical Flow (cont'd)

Continuity

$$\frac{2\rho V_r}{r} + \frac{\rho V_\theta}{r\sin(\theta)}\cos(\theta) + \frac{1}{r}\left[\rho\frac{dV_\theta}{d\theta} + V_\theta\frac{d\rho}{d\theta}\right] =$$

$$\boxed{2V + V_\theta\cot(\theta) + \left[\frac{dV_\theta}{d\theta} + \frac{V_\theta}{\rho}\frac{d\rho}{d\theta}\right] = 0}$$

Crocco :

$$V_\theta = \frac{dV_r}{d\theta}$$

Euler :

$$\frac{d\rho}{d\theta} = \frac{2}{\gamma - 1}\frac{\left[V_\theta\frac{dV_\theta}{d\theta} + V_r\frac{dV_r}{d\theta}\right]}{\left(\left(V_\theta{}^2 + V_r{}^2\right)\text{-}2h_0\right)}$$

Collected Equations for for Axi-symmetric Conical Flow (cont'd)

$$2V_r + V_\theta \cot(\theta) + \left[\frac{dV_\theta}{d\theta} + V_\theta \frac{2}{\gamma - 1} \frac{\left[V_\theta \frac{dV_\theta}{d\theta} + V_r \frac{dV_r}{d\theta}\right]}{\left(V_\theta{}^2 + V_r{}^2 \text{-} 2h_0\right)}\right] = 0$$

$$But\ from\ Crocco:$$

$$V_\theta = \frac{dV_r}{d\theta} \rightarrow \frac{dV_\theta}{d\theta} = \frac{d^2V_r}{d\theta^2}$$

- **Substituting in**

Collected Equations for for Axi-symmetric Conical Flow (cont'd)

$$V_r + \frac{dV_r}{d\theta}\cot(\theta) + \left[\frac{d^2V_r}{d\theta^2} + \frac{dV_r}{d\theta}\frac{2}{\gamma-1}\frac{\left[\frac{dV_r}{d\theta}\frac{d^2V_r}{d\theta^2} + V_r\frac{dV_r}{d\theta}\right]}{\left(\frac{dV_r}{d\theta}^2 + V_r^{\,2}\text{-}2h_0\right)}\right] =$$

$$\frac{\gamma-1}{2}\left(2h_0 - V_r^{\,2} - \left(\frac{dV_r}{d\theta}\right)^2\right)\left(2V_r + \frac{dV_r}{d\theta}\cot(\theta) + \frac{d^2V_r}{d\theta^2}\right) - \frac{dV_r}{d\theta}\left[V_r\frac{dV_r}{d\theta} + \frac{dV_r}{d\theta}\frac{d^2V_r}{d\theta^2}\right] = 0$$

- CLASSICAL FORM OF TAYLOR-MACCOLL EQUATION

… BUT DIFFICULT TO SOLVE NUMERICALLY IN THIS FORM

$$\boxed{h_0 = \frac{V_{\max}^2}{2} \;....Why\,?...no\;idea\,!} \rightarrow \boxed{V_{\max} = \sqrt{2h_0}}$$

Taylor-Maccoll Equation

• O.D.E for V_r in terms of θ

$$\frac{\gamma-1}{2}\left(2h_0 - V_r^{\,2} - \left(\frac{dV_r}{d\theta}\right)^2\right)\left(2V_r + \frac{dV_r}{d\theta}\cot(\theta) + \frac{d^2V_r}{d\theta^2}\right)$$

$$-\frac{dV_r}{d\theta}\left[V_r\frac{dV_r}{d\theta} + \frac{dV_r}{d\theta}\frac{d^2V_r}{d\theta^2}\right] = 0$$

V_r --> dependent variable,
θ --> independent variable

• Solve for V_r ---> Then $V_\theta = \frac{\partial V_r}{\partial \theta}$

Taylor-Maccoll Equation (cont'd)

- **BETTER SOLUTION …Nondimensionalize by** $\sqrt{2h_0}$

$$\frac{\gamma-1}{2}\left(1-\left(\frac{V_r}{\sqrt{2h_0}}\right)^2-\left(\frac{d\left(V_r/\sqrt{2h_0}\right)}{d\theta}\right)^2\right)\left(2\frac{V_r}{\sqrt{2h_0}}_r+\frac{d\left(V_r/\sqrt{2h_0}\right)}{d\theta}\cot(\theta)+\frac{d^2\left(V_r/\sqrt{2h_0}\right)}{d\theta^2}\right)$$

$$-\frac{d\left(V_r/\sqrt{2h_0}\right)}{d\theta}\left[\frac{V_r}{\sqrt{2h_0}}\frac{d\left(V_r/\sqrt{2h_0}\right)_r}{d\theta}+\frac{d\left(V_r/\sqrt{2h_0}\right)}{d\theta}\frac{d^2\left(V_r/\sqrt{2h_0}\right)}{d\theta^2}\right]=0$$

- let
- and the Taylor-Maccol equation reduces to

$$\boxed{\frac{V_r}{\sqrt{2h_0}}=\vartheta_r\rightarrow Anderson\rightarrow\vartheta=\frac{V_r}{V_{\max}}}$$

Taylor-Maccoll Equation (cont'd)

$$\frac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\left(\frac{d\vartheta_r}{d\theta}\right)^2\right)\left(2\vartheta_r+\frac{d\vartheta_r}{d\theta}\cot(\theta)+\frac{d^2\vartheta_r}{d\theta^2}\right)$$

$$-\frac{d\vartheta_r}{d\theta}\left[\vartheta_r\frac{d\vartheta_r}{d\theta}+\frac{d\vartheta_r}{d\theta}\frac{d^2\vartheta_r}{d\theta^2}\right]=0$$

$$\boxed{\frac{V_r}{\sqrt{2h_0}}=\vartheta_r \qquad \vartheta_\theta=\frac{\partial\vartheta_r}{\partial\theta}\rightarrow\vartheta=\sqrt{\vartheta_\theta^{\,2}+\vartheta_r^{\,2}}}$$

Taylor-Maccoll Equation (cont'd)

- Take a closer look at: $\frac{V}{\sqrt{2h_0}} = \vartheta$

$$\frac{V}{\sqrt{2h_0}} = \vartheta \rightarrow \vartheta^2 h_0 = \frac{V^2}{2} \rightarrow \vartheta^2\left[h + \frac{V^2}{2}\right] = \frac{V^2}{2}$$

$$\vartheta^2\left[\frac{c^2}{\gamma-1} + \frac{V^2}{2}\right] = \frac{V^2}{2} \rightarrow \vartheta^2\left[1 + \frac{\gamma-1}{2}\frac{V^2}{c^2}\right] = \frac{\gamma-1}{2}\frac{V^2}{c^2}$$

$$\rightarrow \boxed{\vartheta = \sqrt{\frac{\frac{\gamma-1}{2}M^2}{\left[1+\frac{\gamma-1}{2}M^2\right]}}}$$

- Once we find ϑ we can calculate M
  - So how do we find ϑ ?

Numerical Procedure for Solving Axi-symmetric Conical Flow Field

- Re-write

$$\frac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\left(\frac{d\vartheta_r}{d\theta}\right)^2\right)\left(2\vartheta_r+\frac{d\vartheta_r}{d\theta}\cot(\theta)+\frac{d^2\vartheta_r}{d\theta^2}\right)-\frac{d\vartheta_r}{d\theta}\left[\vartheta_r\frac{d\vartheta_r}{d\theta}+\frac{d\vartheta_r}{d\theta}\frac{d^2\vartheta_r}{d\theta^2}\right]=0$$

- As

$$\frac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\left(\frac{d\vartheta_r}{d\theta}\right)^2\right)\left(2\vartheta_r+\frac{d\vartheta_r}{d\theta}\cot(\theta)+\frac{d}{d\theta}\left(\frac{d\vartheta_r}{d\theta}\right)\right)-\frac{d\vartheta_r}{d\theta}\left[\vartheta_r\frac{d\vartheta_r}{d\theta}+\frac{d\vartheta_r}{d\theta}\frac{d}{d\theta}\left(\frac{d\vartheta_r}{d\theta}\right)\right]=0$$

- But $\vartheta_\theta = \dfrac{\partial\vartheta_r}{\partial\theta}$ • Substitute in

Numerical Procedure for Solving Axi-symmetric Conical Flow Field

- Re-write

$$\frac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\left(\frac{d\vartheta_r}{d\theta}\right)^2\right)\left(2\vartheta_r+\frac{d\vartheta_r}{d\theta}\cot(\theta)+\frac{d^2\vartheta_r}{d\theta^2}\right)-\frac{d\vartheta_r}{d\theta}\left[\vartheta_r\frac{d\vartheta_r}{d\theta}+\frac{d\vartheta_r}{d\theta}\frac{d^2\vartheta_r}{d\theta^2}\right]=0$$

- As

$$\frac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\left(\frac{d\vartheta_r}{d\theta}\right)^2\right)\left(2\vartheta_r+\frac{d\vartheta_r}{d\theta}\cot(\theta)+\frac{d}{d\theta}\left(\frac{d\vartheta_r}{d\theta}\right)\right)-\frac{d\vartheta_r}{d\theta}\left[\vartheta_r\frac{d\vartheta_r}{d\theta}+\frac{d\vartheta_r}{d\theta}\frac{d}{d\theta}\left(\frac{d\vartheta_r}{d\theta}\right)\right]=0$$

- But $\vartheta_\theta=\dfrac{\partial\vartheta_r}{\partial\theta}$ • Substitute in

Numerical Procedure for Solving Axi-symmetric Conical Flow Field (cont'd)

$$\frac{\gamma-1}{2}\left(1-\vartheta_r{}^2-\vartheta_\theta{}^2\right)\left(2\vartheta_r+\vartheta_\theta\cot(\theta)+\frac{d\vartheta_\theta}{d\theta}\right)-\vartheta_\theta\left[\vartheta_r\vartheta_\theta+\vartheta_\theta\frac{d\vartheta_\theta}{d\theta}\right]=0$$

- Solve for $\frac{\partial\vartheta_r}{\partial\theta}$

$$\left[\frac{\gamma-1}{2}\left(1-\vartheta_r{}^2-\vartheta_\theta{}^2\right)-\vartheta_\theta{}^2\right]\left(\frac{d\vartheta_\theta}{d\theta}\right)+\frac{\gamma-1}{2}\left(1-\vartheta_r{}^2-\vartheta_\theta{}^2\right)\left(2\vartheta_r+\vartheta_\theta\cot(\theta)\right)-\vartheta_\theta{}^2\vartheta_r$$

$$\rightarrow\left(\frac{d\vartheta_\theta}{d\theta}\right)=\frac{\vartheta_\theta{}^2\vartheta_r-\frac{\gamma-1}{2}\left(1-\vartheta_r{}^2-\left(\vartheta_\theta\right)^2\right)\left(2\vartheta_r+\vartheta_\theta\cot(\theta)\right)}{\left[\frac{\gamma-1}{2}\left(1-\vartheta_r{}^2-\vartheta_\theta{}^2\right)-\vartheta_\theta{}^2\right]}$$

Numerical Procedure for Solving Axi-symmetric Conical Flow Field (cont'd)

- System of first order ordinary differential equations

$$\frac{\partial \vartheta_r}{\partial \theta} = \vartheta_\theta$$

$$\frac{d\vartheta_\theta}{d\theta} = \frac{\vartheta_\theta^{\,2}\vartheta_r - \frac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\vartheta_\theta^{\,2}\right)\left(2\vartheta_r + \vartheta_\theta \cot(\theta)\right)}{\left[\frac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\vartheta_\theta^{\,2}\right)-\vartheta_\theta^{\,2}\right]}$$

Numerical Procedure for Solving Axi-symmetric Conical Flow Field (cont'd)

- Can be written in vector form as

$$\begin{bmatrix} \dfrac{\partial \vartheta_r}{\partial \theta} \\ \\ \dfrac{d\vartheta_\theta}{d\theta} \end{bmatrix} = \begin{bmatrix} \vartheta_\theta \\ \\ \dfrac{\vartheta_\theta^{\,2}\vartheta_r - \dfrac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\vartheta_\theta^{\,2}\right)\left(2\vartheta_r + \vartheta_\theta \cot(\theta)\right)}{\left[\dfrac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\vartheta_\theta^{\,2}\right)-\vartheta_\theta^{\,2}\right]} \end{bmatrix}$$

Numerical Procedure for Solving Axi-symmetric Conical Flow Field (cont'd)

- Or

$$\frac{d\vartheta}{d\theta} = F\left[\vartheta\right] \qquad \vartheta = \begin{bmatrix} \vartheta_r \\ \vartheta_\theta \end{bmatrix}$$

$$F\left[\vartheta\right] = \begin{bmatrix} \vartheta_\theta \\ \dfrac{\vartheta_\theta^{\,2}\vartheta_r - \dfrac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\vartheta_\theta^{\,2}\right)\left(2\vartheta_r + \vartheta_\theta \cot(\theta)\right)}{\left[\dfrac{\gamma-1}{2}\left(1-\vartheta_r^{\,2}-\vartheta_\theta^{\,2}\right)-\vartheta_\theta^{\,2}\right]} \end{bmatrix}$$

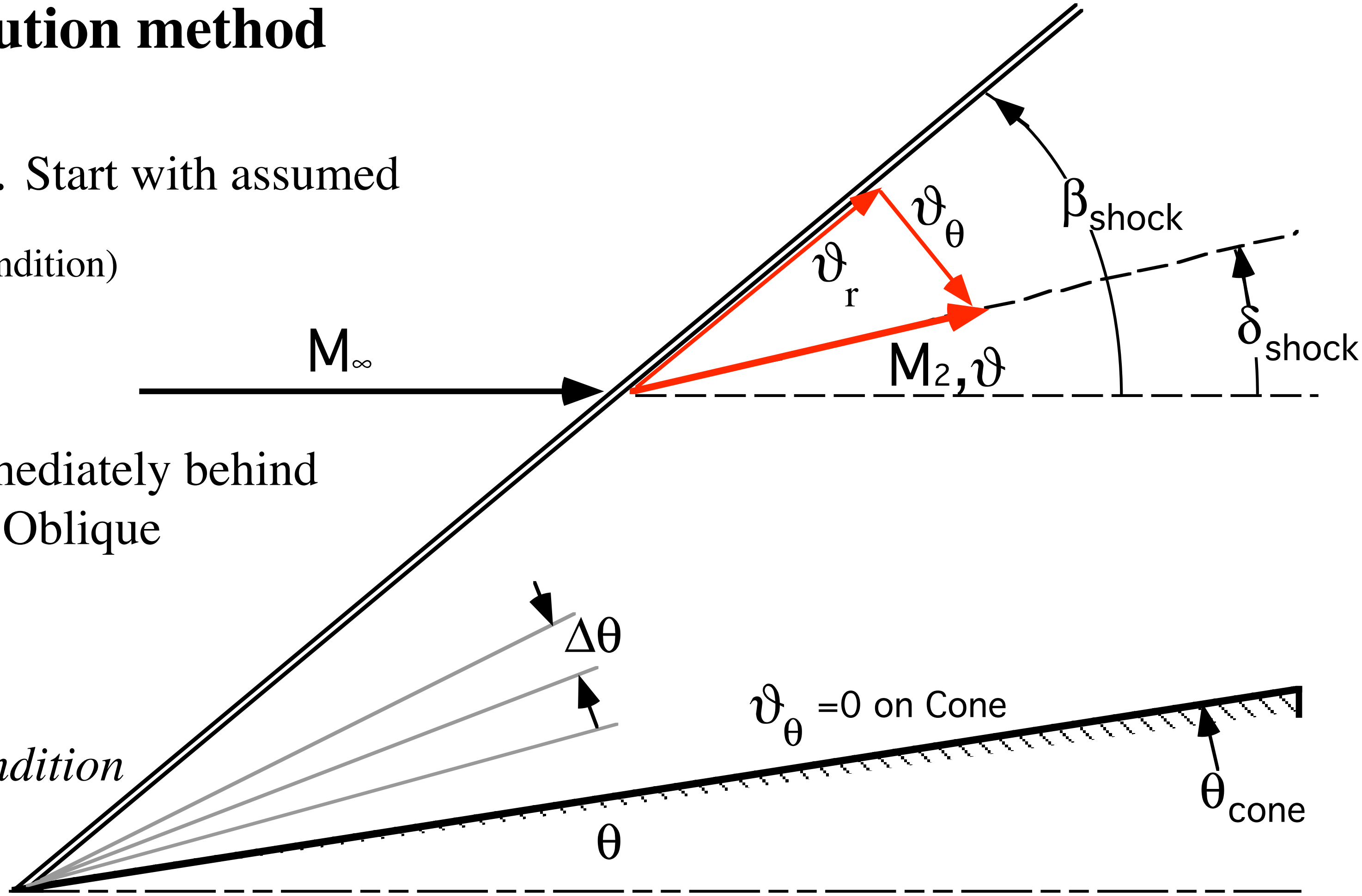
Boundary Value Problem

- **Inverse solution method**

- Given M_∞ … Start with assumed β_{shock} (initial condition)

- Properties immediately behind Shock given by Oblique Shock relations

- On cone $\vartheta_\theta=0$ *Boundary Condition*



Boundary Value Problem (cont'd)

- **Starting Conditions**

Given M_∞ … Start with assumed β_{shock} (initial condition)

Calculate M_2 (immediately behind shock)

$$Mn_\infty = M_\infty \sin\beta_{shock} \rightarrow Mn_2 = \sqrt{\frac{\left(1+\frac{(\gamma-1)}{2}Mn_\infty^{\ 2}\right)}{\left(\gamma Mn_\infty^{\ 2}-\frac{(\gamma-1)}{2}\right)}}$$

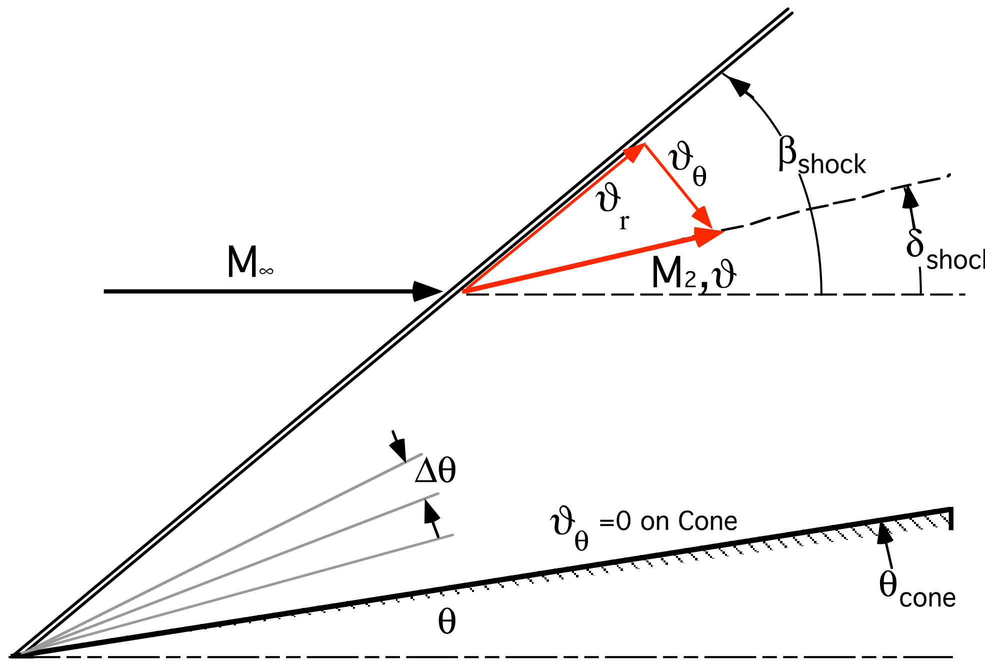
$$M_2 = \frac{Mn_2}{\sin\left(\beta_{shock}-\delta_{shock}\right)}$$

$$\tan\left(\delta_{shock}\right) = \frac{2\left\{M_\infty^{\ 2}\sin^2\left(\beta_{shock}\right)-1\right\}}{\tan\left(\beta_{shock}\right)\left[2+M_\infty^{\ 2}\left[\gamma+\cos\left(2\beta_{shock}\right)\right]\right]}$$

Boundary Value Problem (cont'd)

- **Starting Conditions**

$$\vartheta = \sqrt{\frac{\frac{\gamma-1}{2} M_2^{\,2}}{\left[1+\frac{\gamma-1}{2} M_2^{\,2}\right]}}$$



$$\vartheta_r = \vartheta \cos\left(\beta_{shock} - \delta_{shock}\right)$$

$$\vartheta_\theta = -\vartheta \sin\left(\beta_{shock} - \delta_{shock}\right)$$

Boundary Value Problem (cont'd)

- **Using Starting ϑ_r, ϑ_θ**

Integrate the equations of motion over increments of θ
Until $\vartheta_\theta = 0$ … this angle corresponds t the solid
Cone boundary corresponding to M_∞ and the assumed
β_{shock}.

$$\frac{d\vartheta}{d\theta} = F\left[\vartheta\right]$$

- So How do we integrate This?

Integration of Equations of Motion

- The Integral starts at β_{shock} and proceeds towards the cone where $\theta=\theta_{cone}$

$$\vartheta(\theta_{cone}) = \vartheta(\beta_{shock}) + \int_{\beta_{shock}}^{\theta_{cone}} F\left[\vartheta(\theta)\right] d\theta$$

- Look at small segment $d\theta$ for the integral

$$\vartheta(\theta_{j+1}) = \vartheta(\theta_j) + \int_{\theta_j}^{\theta_{j+1}} F\left[\vartheta(\theta)\right] d\theta$$

$$\theta_{j+1} = \theta_j - \Delta\theta \rightarrow \vartheta_{j+1} = \vartheta_j + \int_{\theta_j}^{\theta_j - \Delta\theta} F\left[\vartheta(\theta)\right] d\theta$$

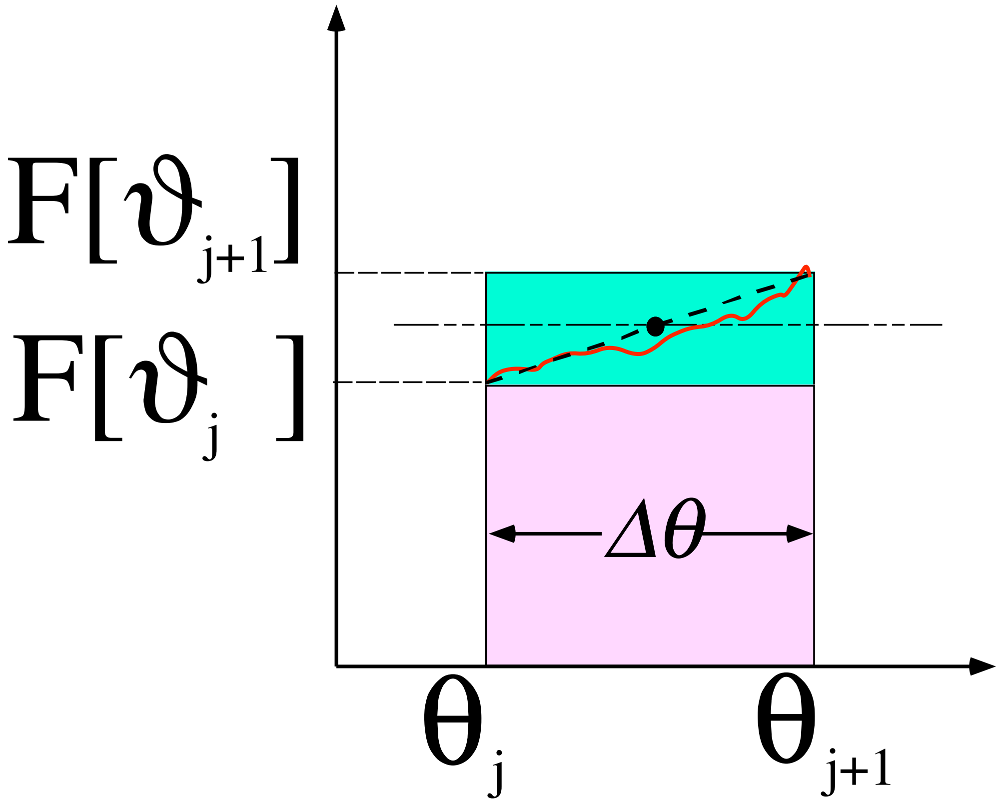
Integration of Equations of Motion

(cont'd)

- Look at $\int_{\theta_j}^{\theta_j - \Delta\theta} F\left[\vartheta(\theta)\right] d\theta$

- Approximate are under curve =

$$\left[\frac{F\left[\vartheta_j\right] + F\left[\vartheta_{j+1}\right]}{2}\right]\Delta\theta$$



Integration of Equations of Motion

(cont'd)

- The the integral is approximated by

$$\int_{\theta_j}^{\theta_j - \Delta\theta} F\left[\vartheta(\theta)\right] d\theta \approx -\left[\frac{F\left[\vartheta_j\right] + F\left[\vartheta_{j+1}\right]}{2}\right] \Delta\theta$$

- And

"trapezoidal rule"

$$\vartheta_{j+1} = \vartheta_j - \left[\frac{F\left[\vartheta_{j+1}\right] + F\left[\vartheta_j\right]}{2}\right] \Delta\theta$$

Integration of Equations of Motion

(cont'd)

- Or we perform the same argument using "finite differences"

$$\frac{d\vartheta}{d\theta} = F\left[\vartheta\right] \rightarrow\rightarrow \frac{\Delta\vartheta}{\Delta\theta} = \frac{\vartheta_{j+1} - \vartheta_j}{\theta_{j+1} - \theta_j} \approx \left[\frac{F\left[\vartheta_j\right] + F\left[\vartheta_{j+1}\right]}{2}\right]$$

$$\vartheta_{j+1} = \vartheta_j + \left(\theta_{j+1} - \theta_j\right)\left[\frac{F\left[\vartheta_j\right] + F\left[\vartheta_{j+1}\right]}{2}\right] \rightarrow \vartheta_{j+1} = \vartheta_j + \left(\theta_j - \Delta\theta - \theta_j\right)\left[\frac{F\left[\vartheta_j\right] + F\left[\vartheta_{j+1}\right]}{2}\right]$$

$$\rightarrow \vartheta_{j+1} = \vartheta_j - \Delta\theta\left[\frac{F\left[\vartheta_j\right] + F\left[\vartheta_{j+1}\right]}{2}\right]$$

- **Basic differencing scheme**

$$\vartheta_{j+1} = \vartheta_j - \left[\frac{F\left[\vartheta_{j+1}\right] + F\left[\vartheta_j\right]}{2}\right]\Delta\theta$$

Predictor / Corrector

- But at step *(j)* we don't know $F\left[\vartheta_{j+1}\right]$

- So we *predict* it …

$$\rightarrow \tilde{\tilde{\vartheta}}_{j+1} = \hat{\vartheta}_j - (\Delta\theta) F\left[\hat{\vartheta}_j\right]$$

- **Be *very careful* About units on Δθ**

- and then *correct* the prediction

$$\rightarrow \hat{\vartheta}_{j+1} = \hat{\vartheta}_j - \frac{(\Delta\theta)}{2}\left(F\left[\hat{\vartheta}_j\right] + F\left[\tilde{\tilde{\vartheta}}_{j+1}\right] \right)$$

Predictor / Corrector

- Where

$$F\left[\hat{\vartheta}_j\right]=\left[\begin{array}{c}\hat{\vartheta}_{\theta_j}\\ \\ \dfrac{\left(\hat{\vartheta}_{\theta_j}\right)^2\left(\hat{\vartheta}_{r_j}\right)-\dfrac{\gamma-1}{2}\left(1-\left(\hat{\vartheta}_{r_j}\right)^2-\left(\hat{\vartheta}_{\theta_j}\right)^2\right)\left(2\left(\hat{\vartheta}_{r_j}\right)+\left(\hat{\vartheta}_{\theta_j}\right)\cot(\theta_j)\right)}{\left[\dfrac{\gamma-1}{2}\left(1-\left(\hat{\vartheta}_{r_j}\right)^2-\left(\hat{\vartheta}_{\theta_j}\right)^2\right)-\left(\hat{\vartheta}_{\theta_j}\right)^2\right]}\end{array}\right]$$

$$F\left[\tilde{\tilde{\vartheta}}_{j+1}\right]=\left[\begin{array}{c}\tilde{\tilde{\vartheta}}_{\theta_{j+1}}\\ \\ \dfrac{\left(\tilde{\tilde{\vartheta}}_{\theta_{j+1}}\right)^2\left(\tilde{\tilde{\vartheta}}_{r_{j+1}}\right)-\dfrac{\gamma-1}{2}\left(1-\left(\tilde{\tilde{\vartheta}}_{r_{j+1}}\right)^2-\left(\tilde{\tilde{\vartheta}}_{\theta_{j+1}}\right)^2\right)\left(2\left(\tilde{\tilde{\vartheta}}_{r_{j+1}}\right)+\left(\tilde{\tilde{\vartheta}}_{\theta_{j+1}}\right)\cot(\theta_{j+1})\right)}{\left[\dfrac{\gamma-1}{2}\left(1-\left(\tilde{\tilde{\vartheta}}_{r_{j+1}}\right)^2-\left(\tilde{\tilde{\vartheta}}_{\theta_{j+1}}\right)^2\right)-\left(\tilde{\tilde{\vartheta}}_{\theta_{j+1}}\right)^2\right]}\end{array}\right]$$

Collected Algorithm

1) Compute Initial Conditions,

a) Given M_∞ … Start with assumed β_{shock}

b) Calculate M_2 (immediately behind shock)

$$Mn_\infty = M_\infty \sin \beta_{shock} \rightarrow Mn_2 = \sqrt{\frac{\left(1 + \frac{(\gamma - 1)}{2} Mn_\infty^{\ 2}\right)}{\left(\gamma Mn_\infty^{\ 2} - \frac{(\gamma - 1)}{2}\right)}}$$

$$\tan\left(\delta_{shock}\right) = \frac{2\left\{M_\infty^{\ 2} \sin^2\left(\beta_{shock}\right) - 1\right\}}{\tan\left(\beta_{shock}\right)\left[2 + M_\infty^{\ 2}\left[\gamma + \cos\left(2\beta_{shock}\right)\right]\right]}$$

$$M_2 = \frac{Mn_2}{\sin\left(\beta_{shock} - \delta_{shock}\right)}$$

Collected Algorithm

1) Compute Initial Conditions,

c) Calculate p_2,T_2 ratios (immediately behind shock)

$$\frac{p_2}{p_\infty} = 1 + \frac{2\gamma}{(\gamma+1)}\left(\left(M_\infty \sin\beta_{shock}\right)^2 - 1\right)$$

$$\frac{T_2}{T_\infty} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left(\left(M_\infty \sin\beta_{shock}\right)^2 - 1\right)\right]\left[\frac{\left(2 + (\gamma-1)\left(M_\infty \sin\beta_{shock}\right)^2\right)}{(\gamma+1)\left(M_\infty \sin\beta_{shock}\right)^2}\right]$$

Collected Algorithm (cont'd)

1) Compute Initial Conditions,

c) Compute starting non-dimensional velocities

$$\vartheta = \sqrt{\frac{\frac{\gamma-1}{2}M_2{}^2}{\left[1+\frac{\gamma-1}{2}M_2{}^2\right]}}$$

$$\vartheta_r = \vartheta\cos\left(\beta_{shock} - \delta_{shock}\right)$$

$$\vartheta_\theta = -\vartheta\sin\left(\beta_{shock} - \delta_{shock}\right)$$

Collected Algorithm (cont'd)

2) *Compute θ decrement*

$$\Delta\theta = \frac{\beta_{shock}}{N} \qquad \vartheta = \begin{bmatrix} \vartheta_r \\ \vartheta_\theta \end{bmatrix}$$

3) Integration Loop, j=1,…N

- **Be *very careful* About units on Δθ**

a) Predictor

$$\rightarrow \tilde{\tilde{\vartheta}}_{j+1} = \hat{\vartheta}_j - (\Delta\theta) F\left[\hat{\vartheta}_j\right] \quad ; \quad \theta_{j+1} = \theta_j - \Delta\theta$$

b) Corrector

$$\rightarrow \hat{\vartheta}_{j+1} = \hat{\vartheta}_j - \frac{(\Delta\theta)}{2}\left(F\left[\hat{\vartheta}_j\right] + F\left[\tilde{\tilde{\vartheta}}_{j+1}\right]\right)$$

Collected Algorithm (cont'd)

3) Integration Loop, j=1,…N (cont'd)

d) Test for convergence

$$f\left[\left|\vartheta_{\theta_j}\right| > \varepsilon\right]$$

$$\{\hat{\vartheta}_j = \hat{\vartheta}_{j+1};$$

$$\theta_j = \theta_{j+1}$$

$$return;\}$$

$$else$$

{break;}

if ϑ_θ > 0 you have a problem
Why?

Collected Algorithm (cont'd)

4) Post-process data

a) Compute total ϑ

$$\hat{\vartheta}_r = \hat{\vartheta}_N(1)$$
$$\hat{\vartheta}_\theta = \hat{\vartheta}_N(2)$$
$$\rightarrow \vartheta = \sqrt{\left(\hat{\vartheta}_r\right)^2 + \left(\hat{\vartheta}_\theta\right)^2}$$

b) Compute M_{cone}

$$M_{cone} = \sqrt{\frac{2}{(\gamma-1)}\left[\frac{(\vartheta_{cone})^2}{1-(\vartheta_{cone})^2}\right]}$$

• **Why?**

$$\vartheta = \sqrt{\frac{\frac{\gamma-1}{2}M^2}{\left[1+\frac{\gamma-1}{2}M^2\right]}}$$

Collected Algorithm (cont'd)

4) Post-process data

c) Calculate p_{cone}, T_{cone} ratios (flow behind shock is isentropic)

$$\frac{p_{cone}}{p_\infty} = \frac{p_{cone}}{p_2}\frac{p_2}{p_\infty} = \left[\frac{1+\frac{(\gamma-1)}{2}(M_2)^2}{1+\frac{(\gamma-1)}{2}(M_{cone})^2}\right]^{\frac{\gamma}{\gamma-1}} \left[1+\frac{2\gamma}{(\gamma+1)}\left((M_\infty \sin\beta_{shock})^2-1\right)\right]$$

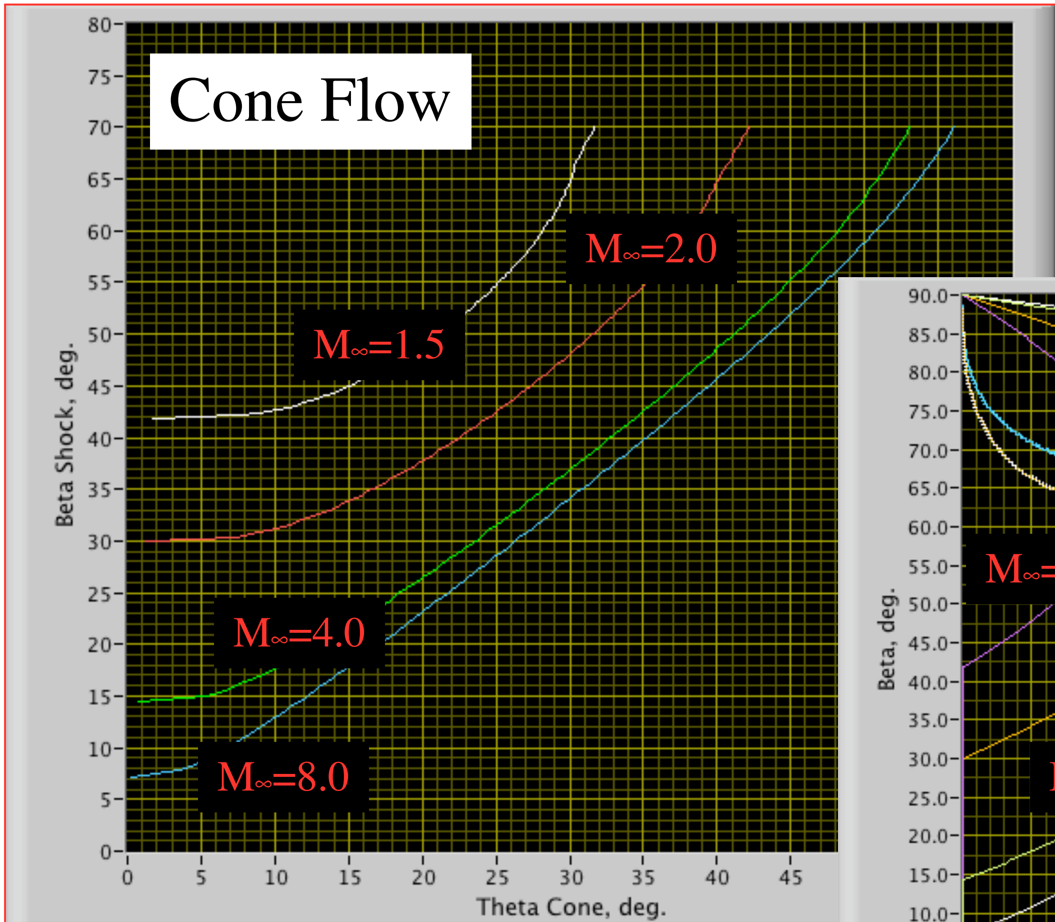
$$\frac{T_{cone}}{T_\infty} = \frac{T_{cone}}{T_2}\frac{T_2}{T_\infty} = \left[\frac{1+\frac{(\gamma-1)}{2}(M_2)^2}{1+\frac{(\gamma-1)}{2}(M_{cone})^2}\right]\left[1+\frac{2\gamma}{(\gamma+1)}\left((M_\infty \sin\beta_{shock})^2-1\right)\right]\left[\frac{\left(2+(\gamma-1)(M_\infty \sin\beta_{shock})^2\right)}{(\gamma+1)(M_\infty \sin\beta_{shock})^2}\right]$$

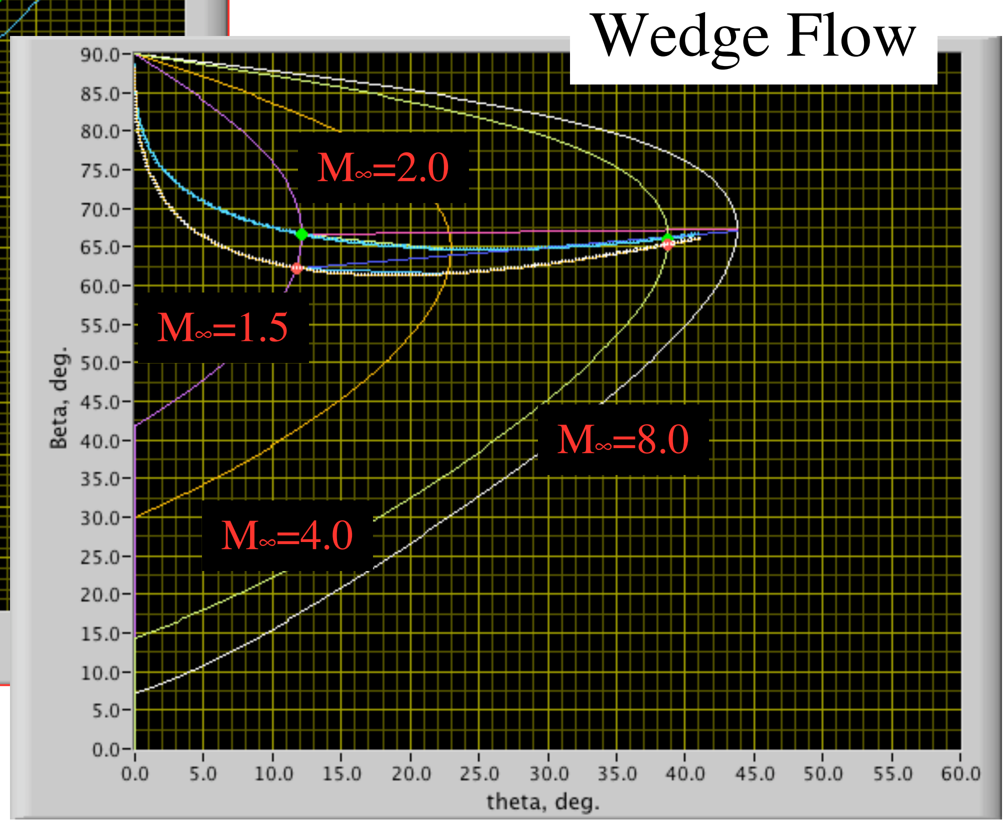
$$\frac{\rho_c}{\rho_\infty} = \frac{p_c/T_c}{p_\infty/T_\infty} = \frac{\frac{p_c}{p_\infty}}{\frac{T_c}{T_\infty}}$$

Physical Aspects of Cone Flow

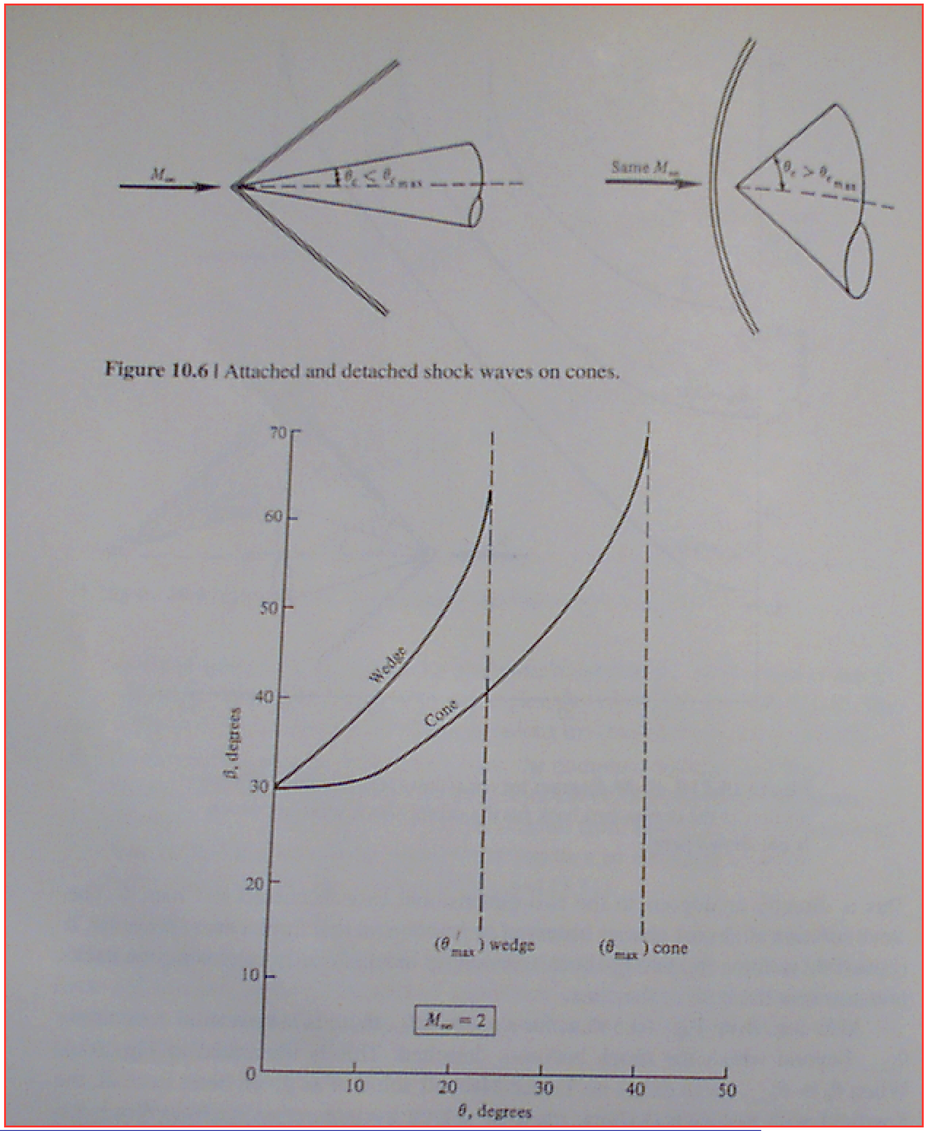
- Compare cone flow to wedge

- Cone flow supports a much larger wedge angle before shock wave detaches





Physical Aspects of Cone Flow (cont'd)

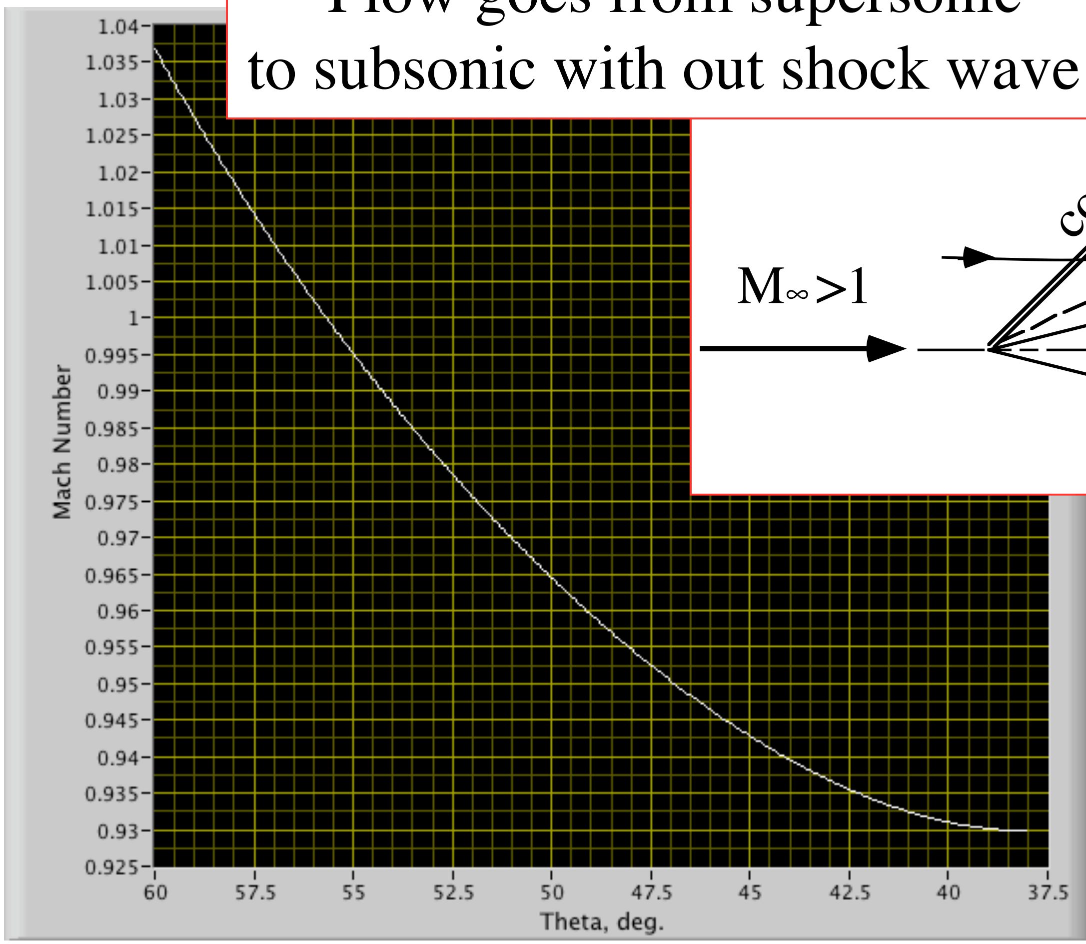


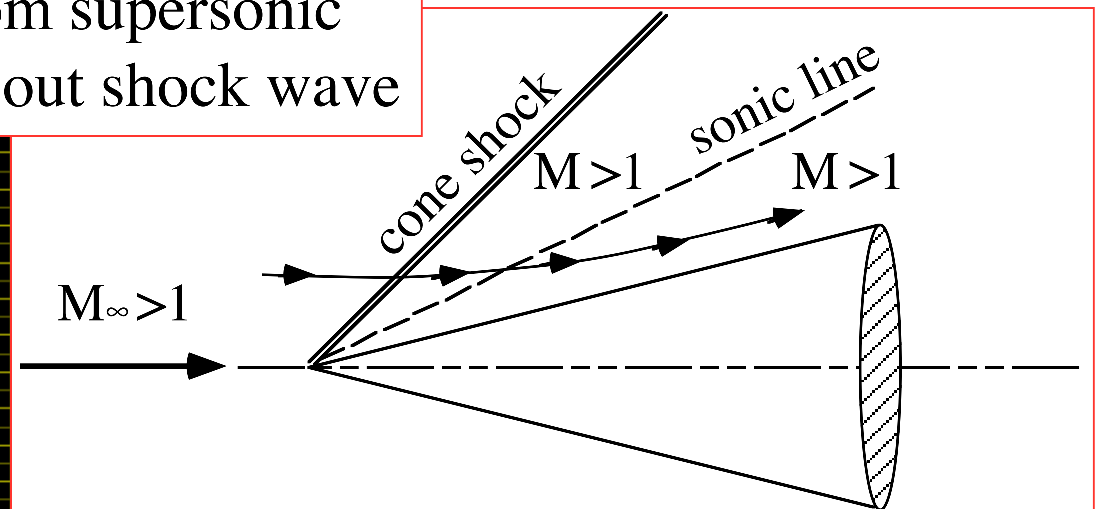
- Three-dimensional "relieving" effect

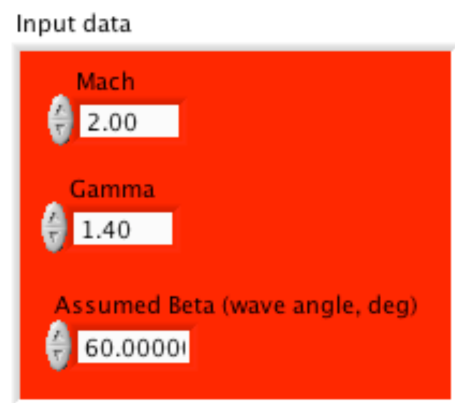
- Cone shock wave is Effectively weaker Than shock wave for Corresponding wedge angle

Isentropic Deceleration on Cone with shock wave near detachment angle

- Flow goes from supersonic to subsonic with out shock wave



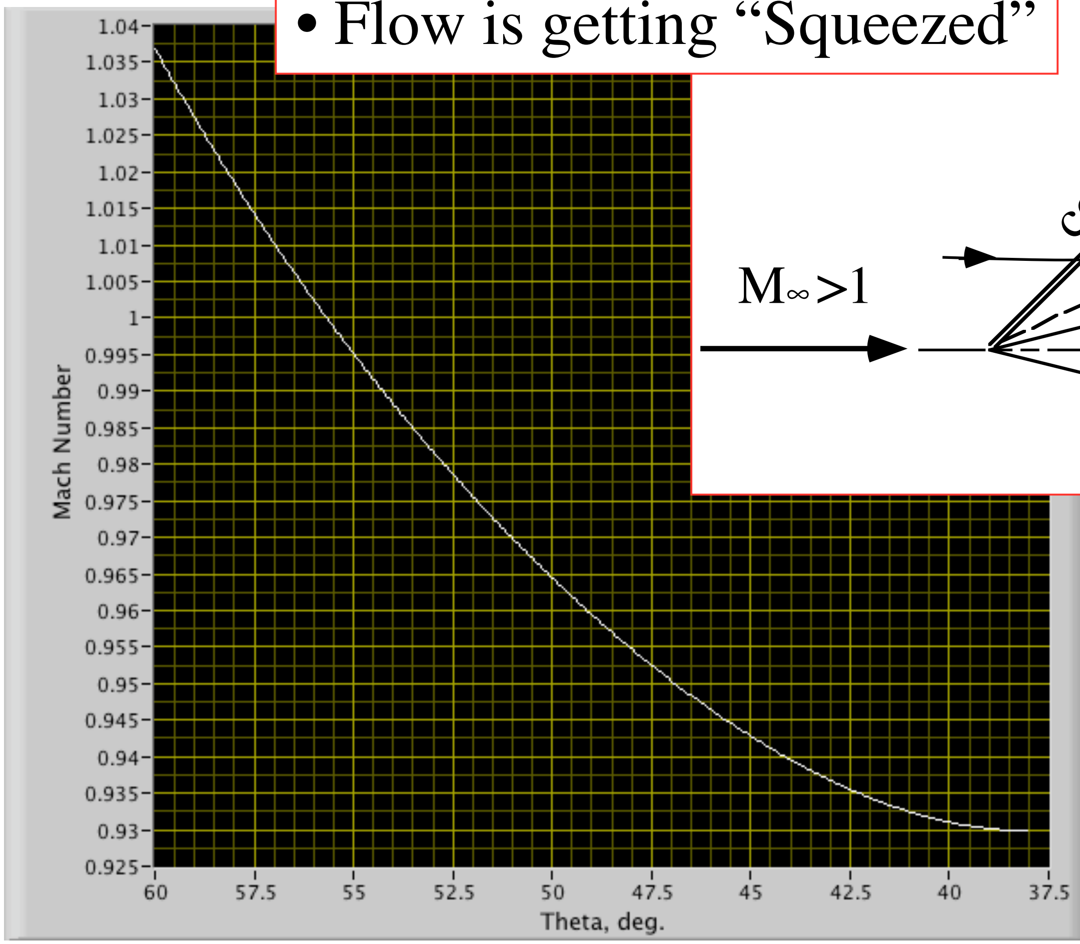


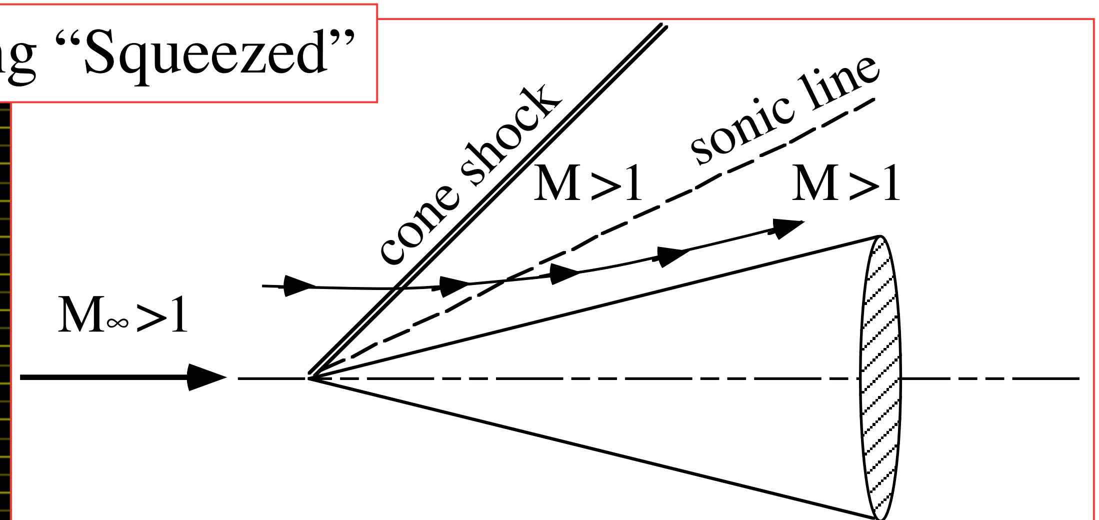


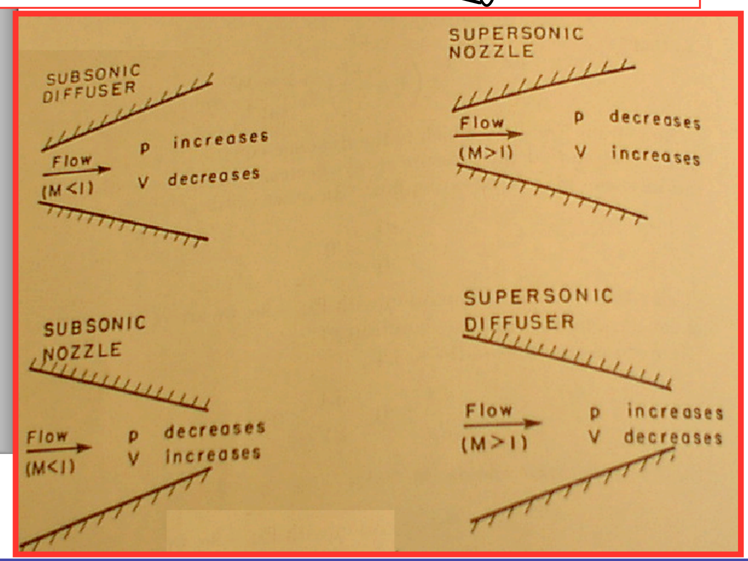
Isentropic Deceleration on Cone with shock wave near detachment angle

(cont'd)

• Flow is getting "Squeezed"

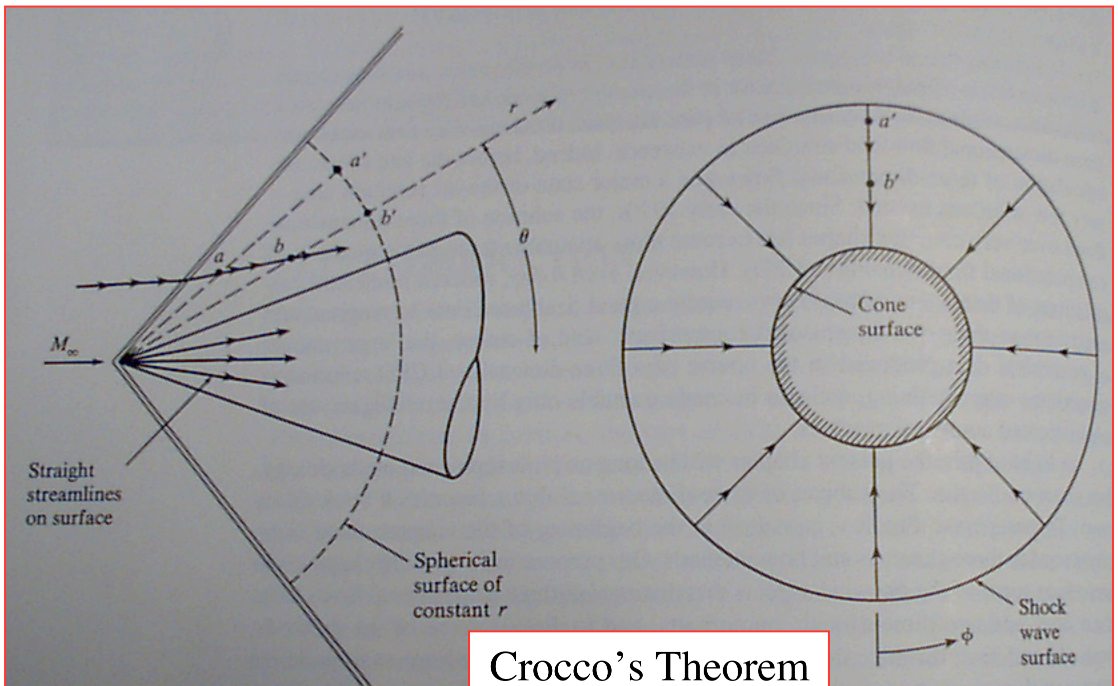






Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack

- Look at Zero-α conical flow field projected onto spherical surface



Crocco's Theorem

$$[T\nabla s]_1 = [T\nabla s]_2 \rightarrow \left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_1 = \left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]_2$$

- **Really a special case Of a 1-dimensional Flow field**

- **Shock strength is uniform … irrotational flow field**

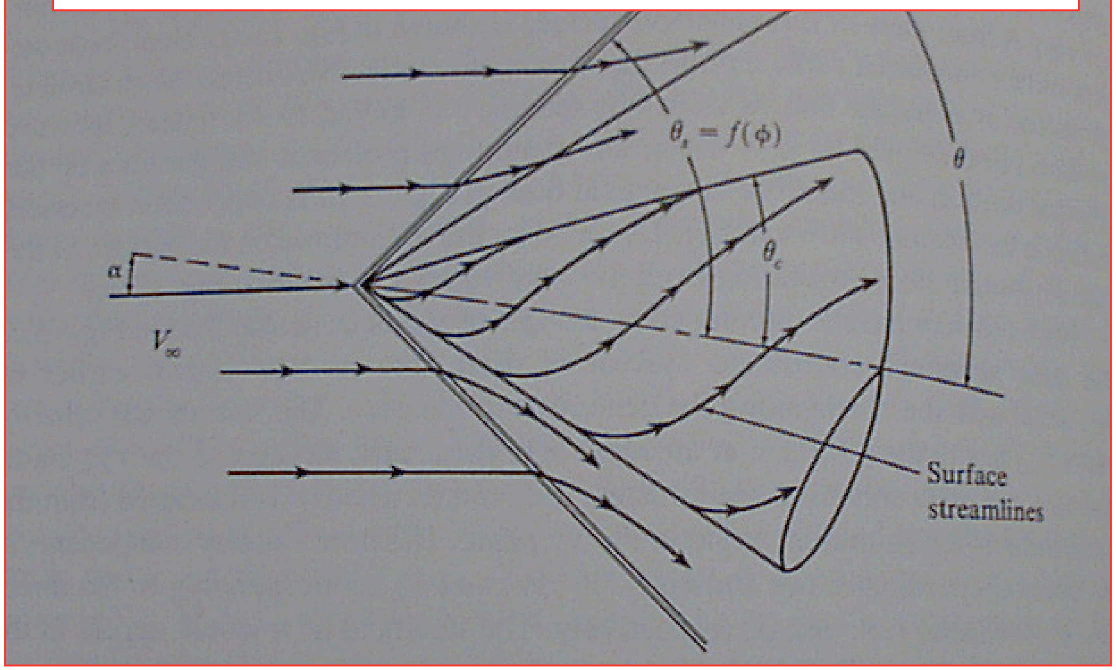
- **Simplifying conditions result in flow field That is "easy" to solve**

$$\nabla\times\vec{V} = 0$$

Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack (cont'd)

- Look at Non Zero-α conical flow field

$$\left[T\nabla s\right]_1 \neq const \rightarrow \vec{V}\times\left(\nabla\times\vec{V}\right)\neq 0$$



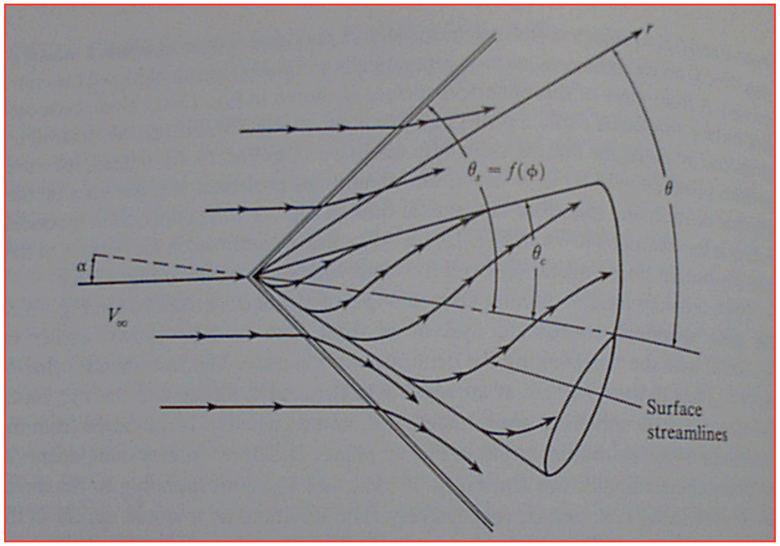
- **Shock Strength is no longer uniformly strong**
- **Tds is no longer constant**
- **Flow field is "rotational"**

$$\frac{\partial}{\partial\phi}\left[\ \ \right]\neq 0 \rightarrow \frac{\partial}{\partial r}\left[V\right]\neq 0$$

Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack (cont'd)

- Look at Non Zero-α conical flow field

1) Flow field is a function of two independent variables θ, ϕ



2) Shock wave angle β_s is different for each meridional plane (ϕ)

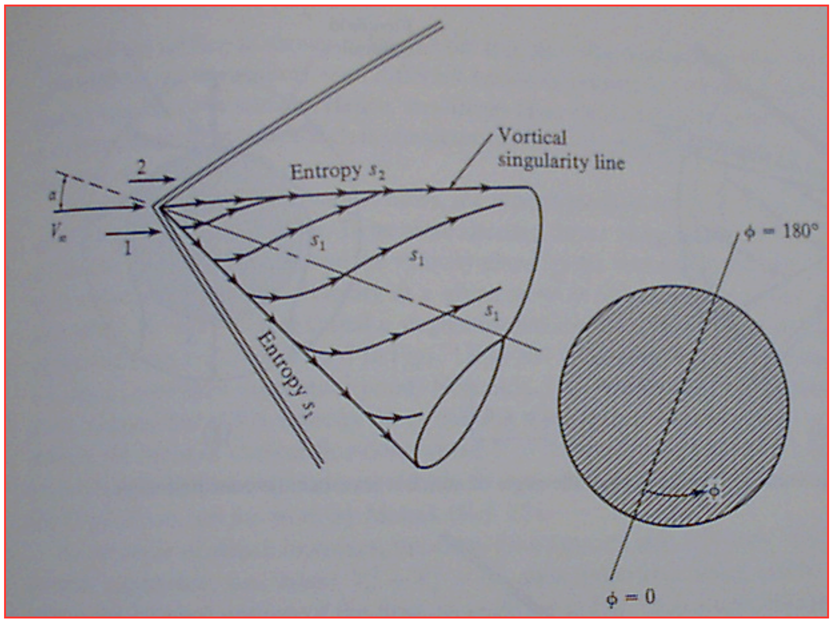
3) Stream lines about body are curved

4) From windward to leeward surface

5) Stream lines that pass thru different Point on shock wave experience different Entropy changes

Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack (cont'd)

- "Vortical Singularity"



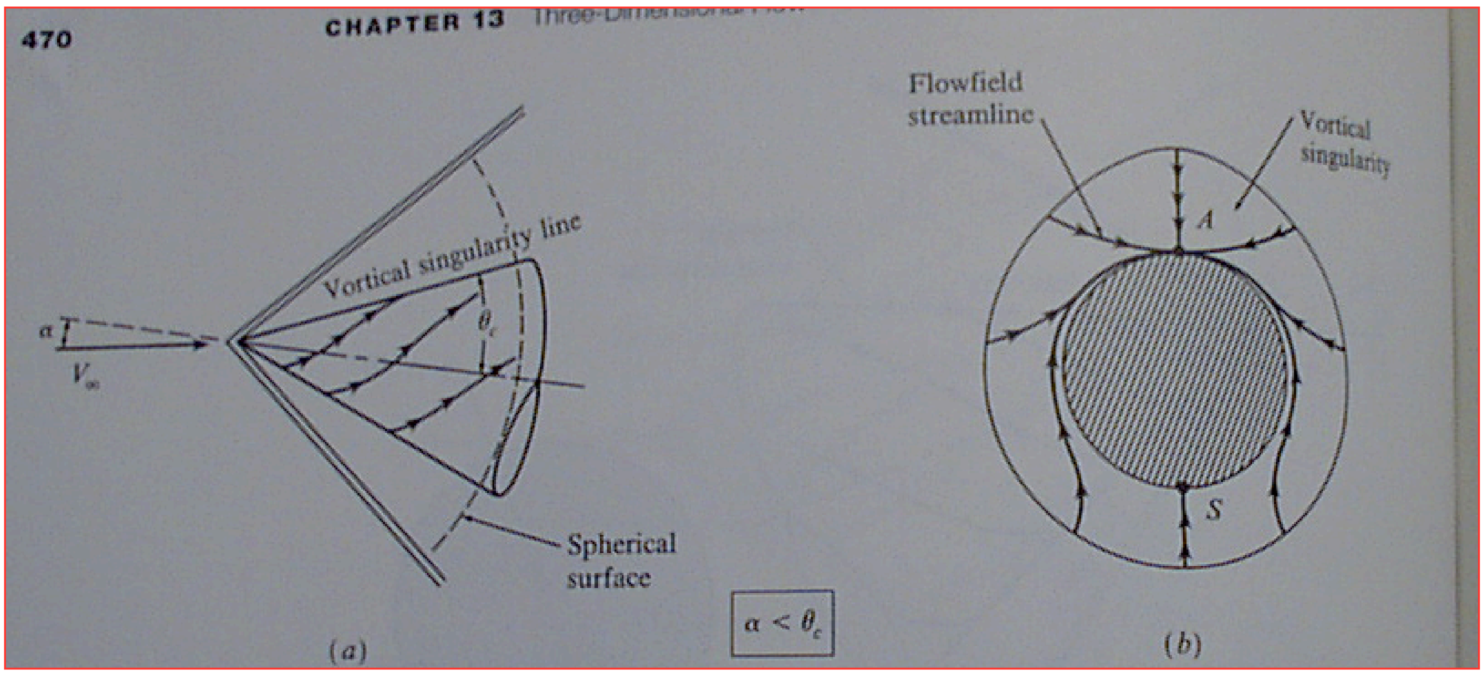
- Stream lines with different entropy levels converge on a single line on leeward side

- Ray along cone surface at 180∞ degrees to freestream wind has "multi-valued" entropy level

- Referred to as "Vortical Singularity"

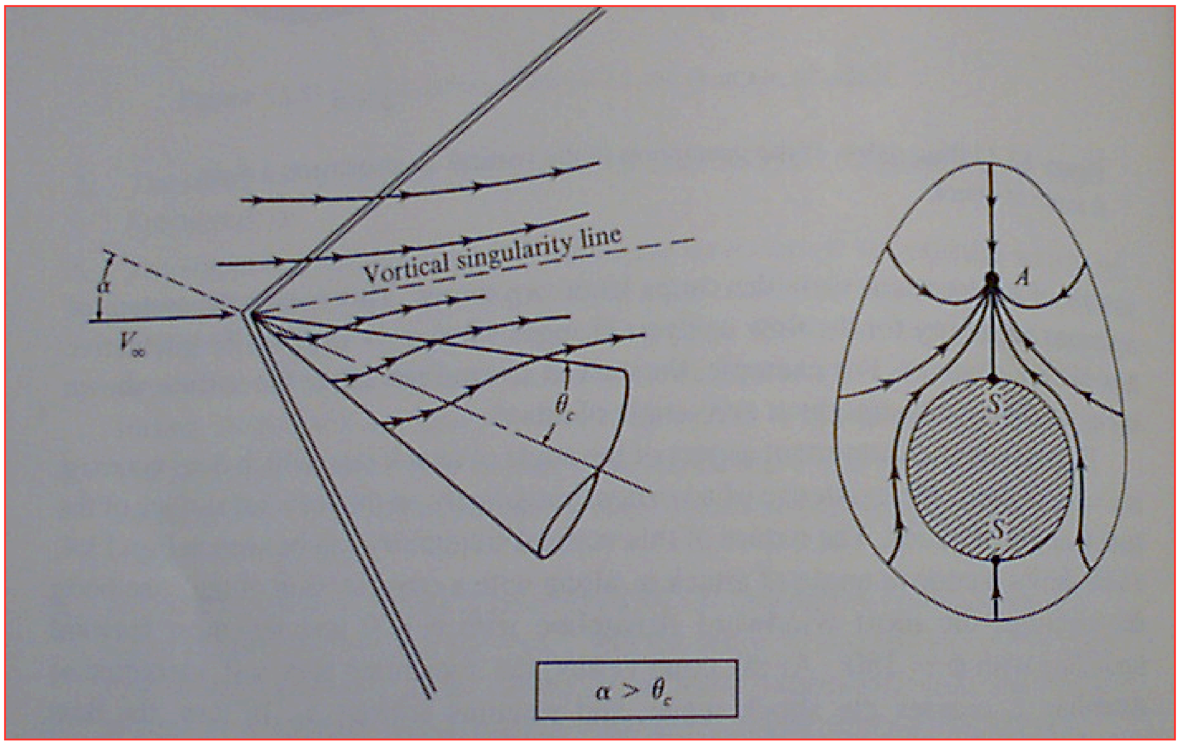
Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack (cont'd)

- When $\alpha < \theta_{cone}$ … singularity lies along cone surface



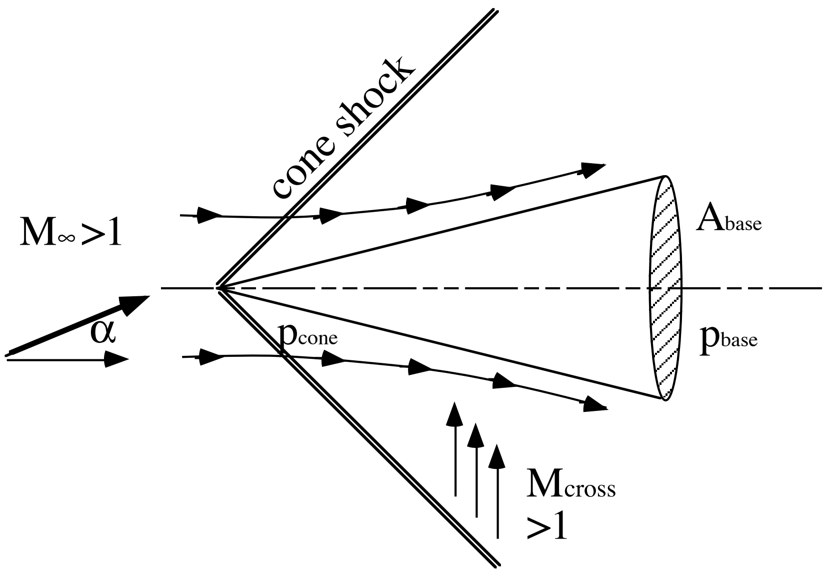
Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack (cont'd)

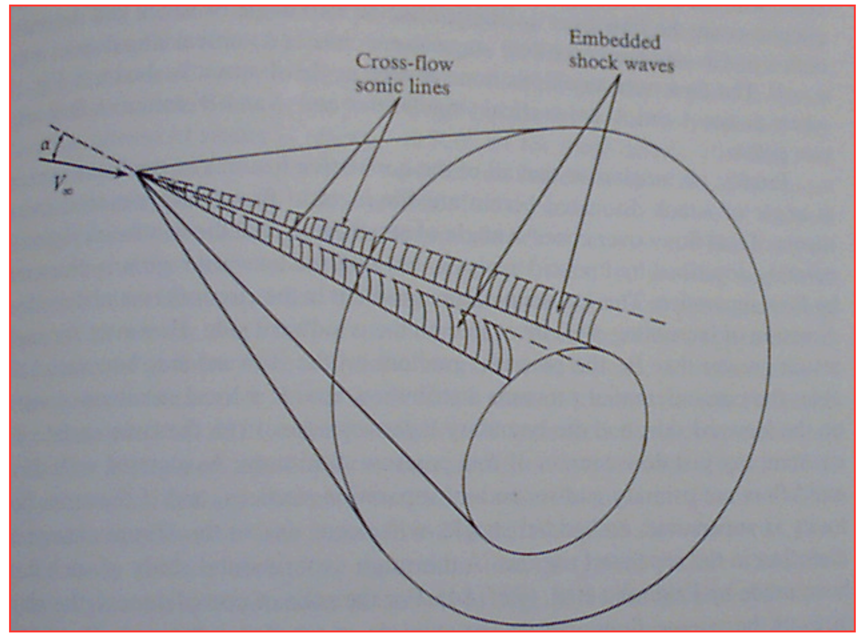
- When $\alpha > \theta_{cone}$ … singularity lies above cone surface



Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack (cont'd)

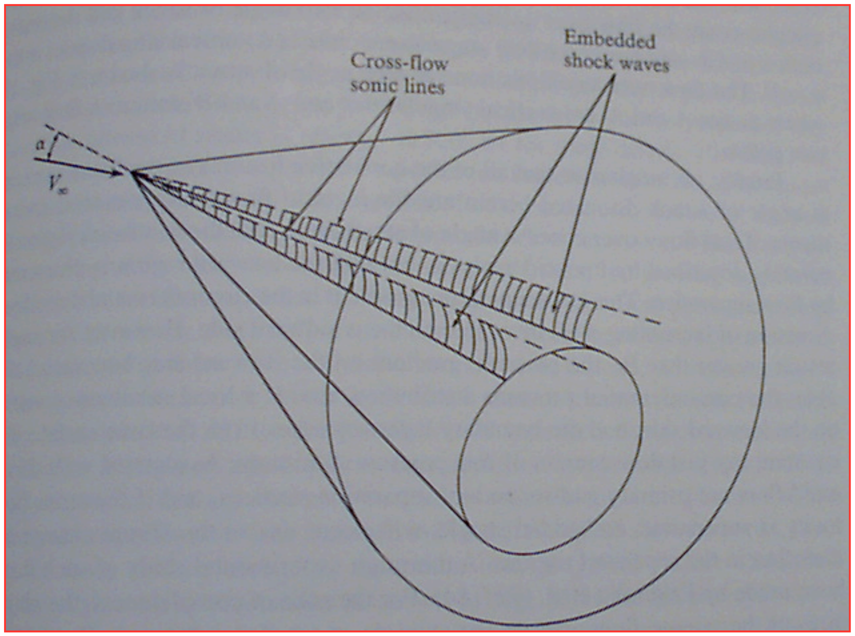
• Cross Flow Sonic Lines, cross flow Mach number > 1

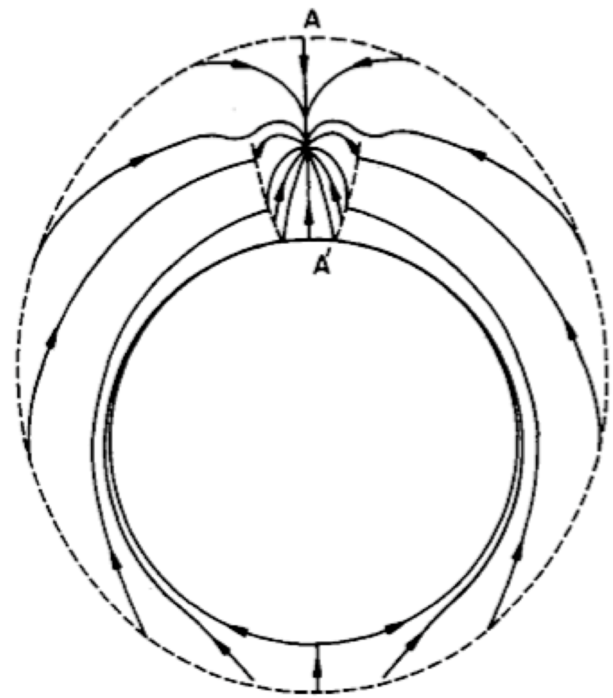




Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack (cont'd)

- Cross Flow Sonic Lines, cross flow Mach number > 1



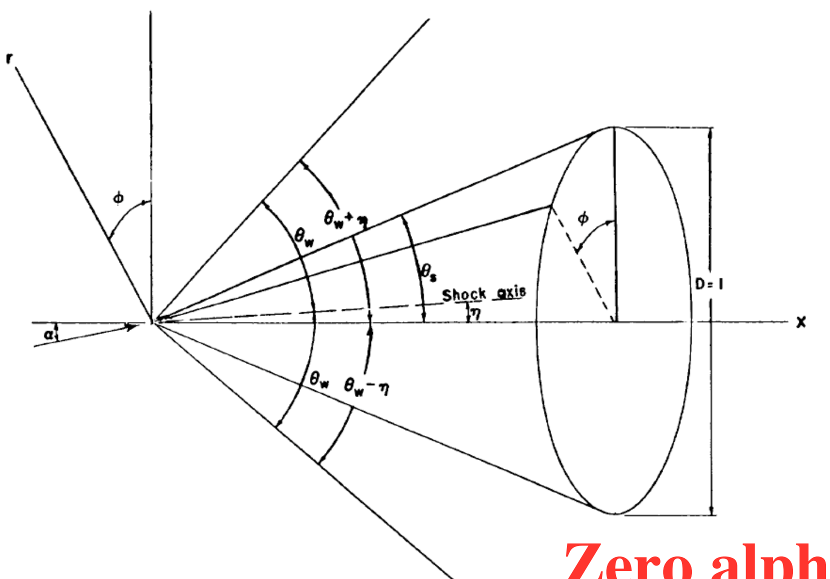


Streamline pattern at $M_\infty = 7$, incidence $= 30°$, and nose angle $= 20°$.

Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack (cont'd)

- "Small angle" approximation

- Solution formulated as "corrections" to zero-α solution



$$\vartheta_r = \vartheta_{r_{\alpha=0}} + \alpha\left[\vartheta_{r_\alpha}\cos\phi\right]$$

$$\vartheta_\theta = \vartheta_{\theta_{\alpha=0}} + \alpha\left[\vartheta_{\theta_\alpha}\cos\phi\right]$$

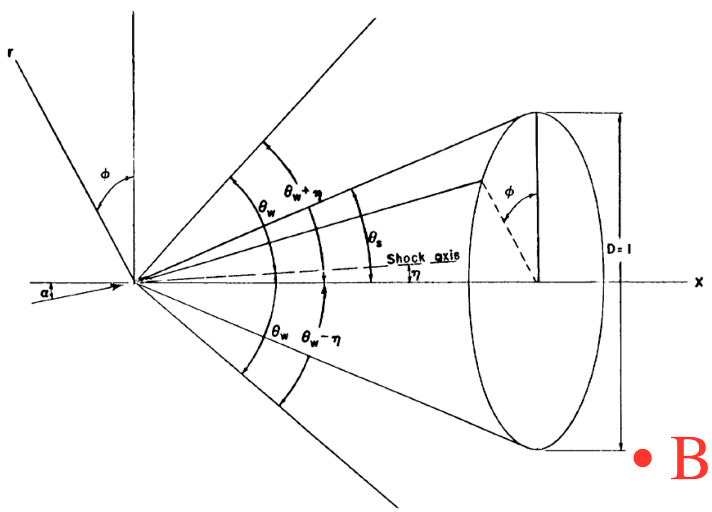
$$\vartheta_\phi = \alpha\left[\vartheta_{\phi_\alpha}\cos\phi\right]$$

Zero alpha condition

"Correction"

Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack (cont'd)

- "Taylor-Maccoll" type of equation for alpha perturbations



$$\frac{1}{f}\frac{d^2\vartheta_{r_\alpha}}{d\theta^2} + \frac{A}{f}\frac{d\vartheta_{r_\alpha}}{d\theta} + \frac{B}{f} + C = 0 \rightarrow \frac{d\vartheta_{r_\alpha}}{d\theta} = \vartheta_{\theta_\alpha}$$

$$A, B, C, D = func(zero\ alpha\ solution)$$

$$f = \frac{P_{\alpha=0} + \alpha\left[P_\alpha \cos\phi\right]}{P_{\alpha=0}} - \gamma\frac{\rho_{\alpha=0} + \alpha\left[\rho_\alpha \cos\phi\right]}{\rho_{\alpha=0}}$$

- Boundary conditions (behind shock wave)

$$\left(\vartheta_{r_\alpha}\right)_{shock} = \frac{c^2 f}{\gamma(\gamma-1)(\vartheta_{r_{\alpha=0}} + \vartheta_{\theta_{\alpha=0}}\cot\beta_{shock})}$$

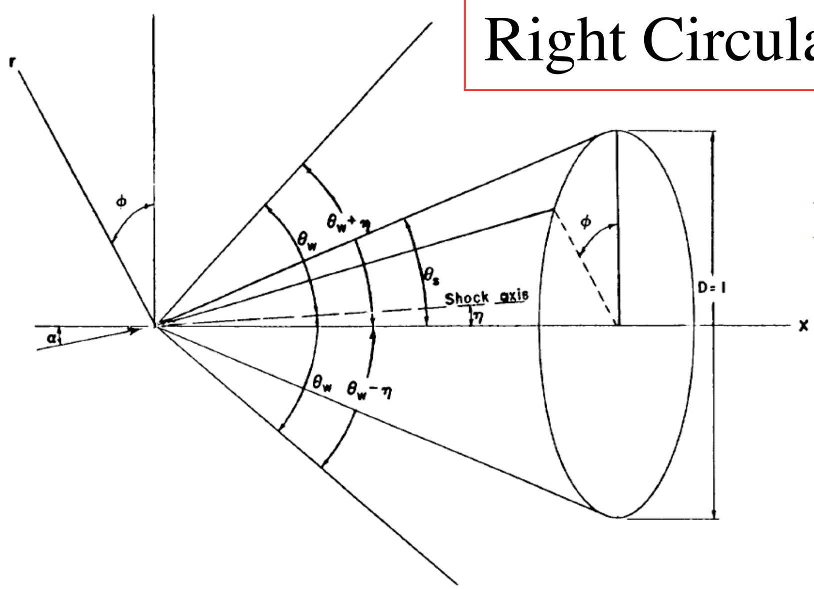
$$\left(\vartheta_{\theta_\alpha}\right)_{shock} = \frac{2c^2\cot\theta(2\vartheta_{r_{\alpha=0}} - \vartheta_{\theta_{\alpha=0}}\cot\beta_{shock})}{\gamma(\gamma^2-1)(\vartheta_{r_{\alpha=0}} + \vartheta_{\theta_{\alpha=0}}\cot\beta_{shock})^2}$$

Qualitative Aspects of Conical Supersonic Flow Fields at Angles of Attack (cont'd)

- Solution is similar (but more complex) than zero alpha case
  Starts at shock wave and works toward surface

- Solution tables for right cones

NASA SP 3007 "Tables for Flow Around Right Circular Cones at Small Angle of Attack"



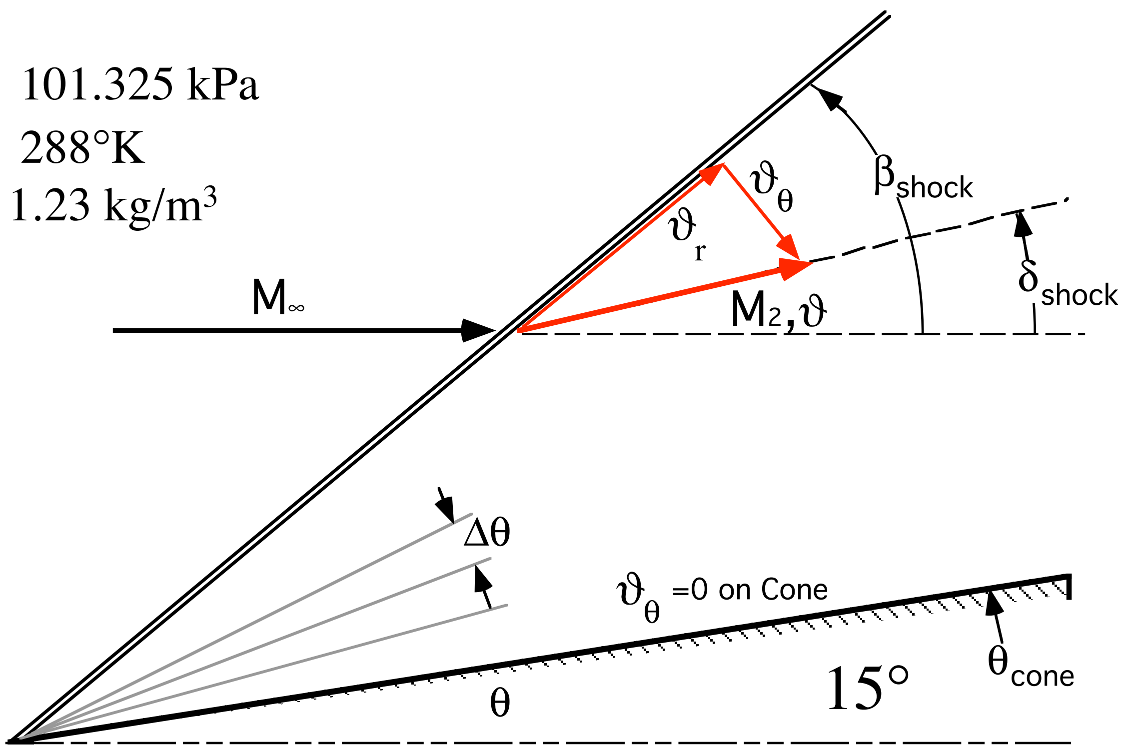
Link to paper in appendix to this section

Homework 12

- Code Taylor-Maccoll algorithm for cone flow

- Solve for flow conditions on surface of Cone at freestream Mach 2.0 with 15° half angle

p_∞	=	101.325 kPa
T_∞	=	288°K
ρ_∞	=	1.23 kg/m^3



Part 1 Solution

11.1 (a) $\theta_{shock} = 0.592 \text{ rad} = \boxed{33.9°}$

(b) $p_s/p_\infty = 1.286 \;\therefore p_s = 1.286\,(1.01 \times 10^5) = \boxed{1.3 \times 10^5 \text{ N/m}^2}$

$\rho_s/\rho_\infty = 1.196 \;\therefore \rho_s = 1.196\,(1.23) = \boxed{1.47 \text{ kg/m}^3}$

$T_s/T_\infty = 1.075 \;\therefore T_s = 1.075\,(288) = \boxed{310°\text{K}}$

$M_s = \boxed{1.835}$

(c) $p_c/p_\infty = 1.566$ $\boxed{p_c = 1.58 \times 10^5 \text{ N/m}^2}$

$\rho_c/\rho_\infty = 1.377$ $\boxed{\rho_c = 1.69 \text{ kg/m}^3}$

$T_c/T_\infty = 1.137$ $\boxed{T_c = 327°\text{K}}$

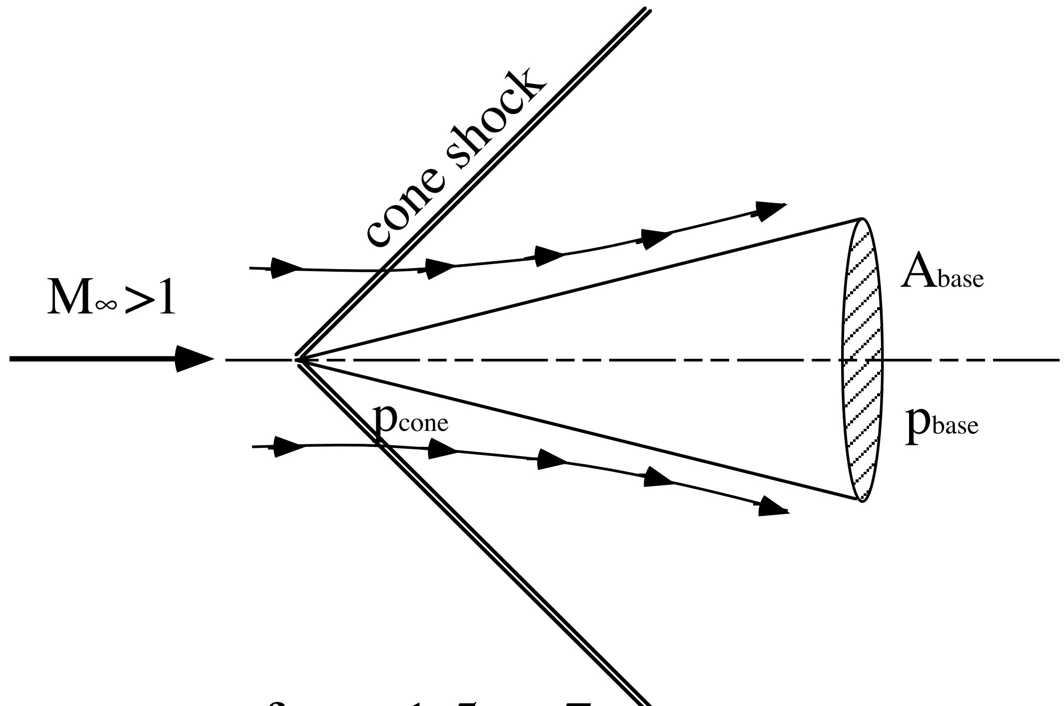
$\boxed{M_c = 1.707}$

Homework 12 (Continued)

- Define $C_{D_{cone}} = \dfrac{D_{cone}}{\bar{q}_\infty A_{base}}$

• Hint: You'll have to do trial And error for each mach number to get the Shock angle correct

- Derive an expression for the cone wave drag as a function of the cone surface pressure (p_{cone}) and the base pressure (p_{base})

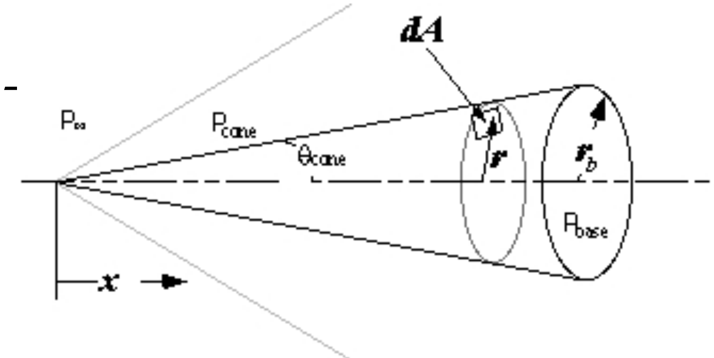


- Assume $p_{base} = p_\infty$
  plot $C_{D_{cone}}$ versus Mach over range from 1.5 to 7

Part 2 Solution

11.2

$$dA = r\,d\theta \frac{dx}{\cos\theta} \qquad dD = p_{cone}\left(r\,d\theta \frac{dx}{\cos\theta}\right)\sin\theta -$$



$$dD = p_{cone}\,(r\,d\theta\,dx)\tan\theta - p_{base}\,r\,d\theta\,dr$$

from cone geometry

$$\frac{r}{x} = \tan\theta \;\therefore x = \frac{r}{\tan\theta} \text{ and } dx = \frac{dr}{\tan\theta} \quad \text{… sub in}$$

$$dD = (p_{cone} - p_{base})\,r\,dr\,d\theta$$

$$C_{D_{cone}} = \frac{D_{cone}}{\bar{q}_\infty A_{base}} = \frac{1}{\frac{\gamma}{2} M_\infty^2}\left[\frac{p_{cone}}{p_\infty} - \frac{p_{base}}{p_\infty}\right]$$

Integrate around circumference

$$D = \int_o^{r_b}\int_0^{2\pi} dD = 2\pi\,(p_{cone} - p_{base})\frac{r_b^2}{2} = \pi(p_{cone} - p_{base})\,r_b^2$$

Normalize …

$$C_D = \frac{D}{q_\infty A_b} = \frac{D}{q_\infty \pi r_b^2} = \frac{p_{cone} - p_{base}}{q_\infty} \rightarrow \text{if } P_{base} = P_{cone} \rightarrow \boxed{C_{D_{cone}} \frac{p_{cone} - p_\infty}{q_\infty}}$$

Part 2 Solution (cont'd)

when $p_b = p_\infty$, For $\theta_c = 15°$

M_∞	$C_{D_{cone}}$
1.5	0.24
2.0	0.202
3.0	0.173
4.0	0.161
5.0	0.154
6.0	0.150
7.0	0.148

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Part 2 Solution (concluded)