

Section 7.1 : An Introduction to the Differential Conservation for Inviscid Flows

$$\text{Continuity} : \nabla \cdot \left(\rho \vec{V} \right) = 0$$

• Anderson,
Chapter 6 pp. 239-260

$$\text{Momentum} : \rho \left(\vec{V} \cdot \nabla \right) \vec{V} = -\nabla P$$

$$\text{Energy} : \nabla \cdot \left[\left(e + \frac{\|V^2\|}{2} \right) \rho \vec{V} \right] = -\nabla \cdot \left(p \vec{V} \right) + \rho \left(\dot{q} \right)$$

$$\text{Crocco(entropy)} : T \nabla s = \nabla h_o - \vec{V} \times \left(\nabla \times \vec{V} \right)$$

Differential Form of Continuity Equation

- Recall from section 2, integral form of continuity equation is:

$$-\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) = \frac{\partial}{\partial t} \left(\iiint_{C.V.} \rho dv \right)$$

“continuity equation”

- Look at first term

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right)$$

- Can we write this surface integral in terms of a volume integral?

Divergence Theorem

- From advanced calculus recall that (Gauss Theorem)

$$\iiint_v \nabla \cdot \vec{V} dv = \iint_s \left(\vec{V} \cdot \vec{ds} \right)$$

- Applying to mass flux across control surface

$$\iint_{C.S.} \left(\rho \vec{V} \cdot \vec{ds} \right) = \iiint_v \nabla \cdot \left(\rho \vec{V} \right) dv$$

- The divergence theorem is a mathematical statement of the physical fact that, in the absence of the creation or destruction of matter, the density within a region of space can change only by having it flow into or away from the region through its boundary.

Divergence Theorem

(cont'd)

- Applying divergence theorem to continuity equation

$$-\iiint_V \nabla \cdot (\rho \vec{V}) dv = \frac{\partial}{\partial t} \left(\iiint_{c.v.} \rho dv \right) \rightarrow \iiint_{c.v.} \left[\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \vec{V}) \right] dv = 0 \rightarrow$$

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \vec{V}) = 0$$

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \vec{V}) = 0$$

Differential form of continuity equation

Differential Form of Momentum Equation

- Recall from section 2, integral form of momentum equation for inviscid flow is

$$\iiint_{C.V.} \rho \vec{f}_b dv - \iint_{C.S.} (p) \vec{dS} = \iiint_{C.V.} \frac{\partial}{\partial t} (\rho \vec{V}) dv + \iint_{C.S.} (\rho \vec{V} \bullet \vec{ds}) \vec{V}$$

- Look at term $\iint_{C.S.} (p) \vec{dS}$

- Can we write the surface integral in terms of volume integral?

Analog of Divergence Theorem

- Start with Gauss theorem
$$\iiint_v \nabla \cdot \vec{V} dv = \iint_s \left(\vec{V} \cdot \vec{ds} \right)$$

Let $\vec{V} = \vec{\lambda} P \rightarrow \iiint_v \nabla \cdot \left(\vec{\lambda} P \right) dv = \iint_s \left(\left(\vec{\lambda} P \right) \cdot \vec{ds} \right)$

$$\iiint_v \vec{\lambda} \cdot \nabla P dv = \iint_s \left(\vec{\lambda} \cdot \left(P \vec{ds} \right) \right) \rightarrow \vec{\lambda} \cdot \left(\iiint_v \nabla P dv \right) = \vec{\lambda} \cdot \left(\iint_s P \vec{ds} \right) \rightarrow$$

$$\boxed{\iiint_v \nabla P dv = \iint_s P \vec{ds}}$$

Momentum Equation

(revisited)

- Applying analog of divergence theorem to momentum equation

$$\iint_{C.S.} (p) \vec{dS} = \iiint_{C.V.} \nabla P dv$$

- Momentum equation becomes

$$\iiint_{C.V.} \rho \vec{f}_b dv - \iiint_{C.V.} \nabla P dv = \iiint_{C.V.} \frac{\partial}{\partial t} \left(\rho \vec{V} \right) dv + \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V}$$

- Now look at $\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V}$

Momentum Equation

(revisited, cont'd)

- Apply Gauss Theorem

$$\iint_{C.S.} \left(\rho \vec{V} \cdot \vec{ds} \right) \vec{V} = \left(\iiint_v \nabla \cdot \left(\rho \vec{V} \right) dv \right) \vec{V}$$

- Momentum equation becomes

$$\iiint_{C.V.} \rho \vec{f}_b dv - \iiint_{C.V.} \nabla P dv = \iiint_{C.V.} \frac{\partial}{\partial t} \left(\rho \vec{V} \right) dv + \iiint_v \nabla \cdot \left(\rho \vec{V} \right) dv \vec{V}$$

Momentum Equation

(revisited, cont'd)

- Look at scalar components

-- Longitudinal (x) direction $\vec{V} \rightarrow u$

$$\iiint_{C.V.} \rho f_x dv - \iiint_{C.V.} \frac{\partial P}{\partial x} dv = \iiint_{C.V.} \frac{\partial}{\partial t} (\rho u) dv + \iiint_v \nabla \cdot \left(\rho u \vec{V} \right) dv$$

$$\iiint_{C.V.} \left(\rho f_x - \frac{\partial P}{\partial x} - \frac{\partial}{\partial t} (\rho u) - \nabla \cdot \left(\rho u \vec{V} \right) \right) dv = 0$$

Momentum Equation

(revisited, cont'd)

- Integrand must be zero for all cases

$$\rho f_x - \frac{\partial P}{\partial x} - \frac{\partial}{\partial t}(\rho u) - \nabla \cdot \left(\rho u \vec{V} \right)$$

- Rearranging

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot \left(\rho u \vec{V} \right) = -\frac{\partial P}{\partial x} + \rho f_x$$

X-COMPONENT ... differential momentum equation

Momentum Equation

(revisited, cont'd)

- Lateral component, y

$$\frac{\partial}{\partial t}(\rho v) + \nabla \cdot \left(\rho \vec{v} \vec{V} \right) = -\frac{\partial P}{\partial y} + \rho f_y$$

- Normal component, z

$$\frac{\partial}{\partial t}(\rho w) + \nabla \cdot \left(\rho \vec{w} \vec{V} \right) = -\frac{\partial P}{\partial z} + \rho f_z$$

Momentum Equation (cont'd)

- Re-visit longitudinal momentum equation

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot \left(\rho u \vec{V} \right) = -\frac{\partial P}{\partial x} + \rho f_x$$

- Expand derivative and gradient operator

$$\rho \frac{\partial}{\partial t} u + u \frac{\partial}{\partial t} \rho + u \nabla \cdot \left(\rho \vec{V} \right) + \left(\rho \vec{V} \right) \cdot \nabla u = -\frac{\partial P}{\partial x} + \rho f_x$$

- Multiply differential form of continuity equation by u

$$u \frac{\partial}{\partial t} \rho + u \nabla \cdot \left(\rho \vec{V} \right) = 0$$

Momentum Equation (cont'd)

- Subtract from expanded longitudinal momentum equation

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot \left(\rho u \vec{V} \right) = -\frac{\partial P}{\partial x} + \rho f_x$$

- Result is

$$\rho \frac{\partial}{\partial t} u + \left(\rho \vec{V} \right) \cdot \nabla u = -\frac{\partial P}{\partial x} + \rho f_x$$

Momentum Equation (cont'd)

- Performing same operations on y, and z momentum equations we get collected set

$$\left[\begin{array}{l} \rho \frac{\partial}{\partial t} u + \left(\rho \vec{V} \right) \cdot \nabla u = -\frac{\partial P}{\partial x} + \rho f_x \\ \rho \frac{\partial}{\partial t} v + \left(\rho \vec{V} \right) \cdot \nabla v = -\frac{\partial P}{\partial y} + \rho f_y \\ \rho \frac{\partial}{\partial t} w + \left(\rho \vec{V} \right) \cdot \nabla w = -\frac{\partial P}{\partial z} + \rho f_z \end{array} \right]$$

Momentum Equation (concluded)

- But

$$\left(\rho \vec{V} \cdot \nabla \right) \vec{V} = \begin{bmatrix} \left(\rho \vec{V} \right) \cdot \nabla u \\ \left(\rho \vec{V} \right) \cdot \nabla v \\ \left(\rho \vec{V} \right) \cdot \nabla w \end{bmatrix}$$

- And momentum equation can be written in vector form as

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho \left(\vec{V} \cdot \nabla \right) \vec{V} = -\nabla P + \rho \vec{f}$$

Euler's Equation

- If we ignore body forces then

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho \left(\vec{V} \cdot \nabla \right) \vec{V} = -\nabla P$$

- Look at steady 1-D form of above

$$dp = -\rho V dV \quad (\text{"euler's equation"})$$

Euler's Equation (concluded)

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho \left(\vec{V} \cdot \nabla \right) \vec{V} = -\nabla P$$

- 3-D form of Euler equation derived back in section 3

(“euler’s equation”)

Energy Equation

- From section 2 derivation, integral form of energy equation is:

$$\frac{\partial}{\partial t} \iiint_{C.V.} \left(\rho \left(e + \frac{\|V^2\|}{2} \right) dv \right) + \iint_{C.S.} \rho \vec{V} \cdot d\vec{S} \left(e + \frac{\|V^2\|}{2} \right) =$$

$$\iiint_{C.V.} (\rho \vec{f} dv) \cdot \vec{V} - \iint_{C.S.} (p d\vec{S}) \cdot \vec{V} + \iiint_{C.V.} \left(\rho \left(\dot{q} \right) dv \right)$$

- Control surface integrals are converted to control volume using the derived forms of the gauss theorem

$$\iint_{C.S.} \rho \vec{V} \cdot d\vec{S} \left(e + \frac{\|V^2\|}{2} \right) = \iint_{C.S.} \left(e + \frac{\|V^2\|}{2} \right) \rho \vec{V} \cdot d\vec{S} = \iiint_{C.V.} \nabla \cdot \left(e + \frac{\|V^2\|}{2} \right) \rho \vec{V} dv$$

Energy Equation (concluded)

$$\iint_{C.S.} (pd\vec{S}) \cdot \vec{V} = \iiint_v \nabla p dv \cdot \vec{V}$$

- Substituting into energy equation

$$\frac{\partial}{\partial t} \iiint_{C.V.} \left(\rho \left(e + \frac{\|V^2\|}{2} \right) dv \right) + \iiint_{C.V.} \nabla \cdot \left(e + \frac{\|V^2\|}{2} \right) \rho \vec{V} dv - \iiint_{C.V.} (\rho \vec{f} dv) \cdot \vec{V} + \iiint_{C.V.} \nabla p dv \cdot \vec{V} - \iiint_{C.V.} \left(\rho \left(\dot{q} \right) dv \right) = 0$$

- Integrand must be zero for all conditions

“differential form of energy equation”

$$\frac{\partial}{\partial t} \left[\rho \left(e + \frac{\|V^2\|}{2} \right) \right] + \nabla \cdot \left(e + \frac{\|V^2\|}{2} \right) \rho \vec{V} = -\nabla \cdot (p \vec{V}) + \rho \left(\dot{q} \right) + \rho \left(\vec{f} \cdot \vec{V} \right)$$

Crocco's Theorem

- Write Euler's equation

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho \left(\vec{V} \cdot \nabla \right) \vec{V} = -\nabla P$$

- Recall from section 1 that entropy equation can be written as

$$dh = Tds - pdv + pdv + vdp =$$

$$dh = Tds + vdp$$

- Extend to 3-D space by letting $d \rightarrow \nabla$

$$T \nabla s = \nabla h - v \nabla p = \nabla h - \frac{\nabla p}{\rho}$$

Crocco's Theorem: (cont'd)

- Substitute in Euler equation

$$T\nabla s = \nabla h + \frac{\rho \frac{\partial \vec{V}}{\partial t} + \rho \left(\vec{V} \cdot \nabla \right) \vec{V}}{\rho} = \nabla h + \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla \right) \vec{V}$$

- But from definition of total enthalpy

$$h = h_o - \frac{V^2}{2} \rightarrow \nabla h = \nabla h_o - \nabla \left(\frac{V^2}{2} \right)$$

Crocco's Theorem: (cont'd)

- Substitute into entropy equation

$$T \nabla s = \nabla h_o - \nabla \left(\frac{V^2}{2} \right) + \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla \right) \vec{V}$$

- But from “curl theorem” (more calculus!)

$$\vec{V} \times \left(\nabla \times \vec{V} \right) = \left(\vec{V} \cdot \nabla \right) \vec{V} - \nabla \left(\frac{V^2}{2} \right)$$

Crocco's Theorem: (cont'd)

- Substitute into entropy equation

$$T \nabla s = \nabla h_o + \frac{\partial \vec{V}}{\partial t} - \vec{V} \times (\nabla \times \vec{V})$$

“Crocco's Theorem”

Steady Flow Relationships with no Body Forces

$$\text{Continuity} : \nabla \cdot (\rho \vec{V}) = 0$$

$$\left\{ \frac{\partial}{\partial t} \rho = 0, \frac{\partial \vec{V}}{\partial t} = 0, \vec{f} = 0 \right\}$$

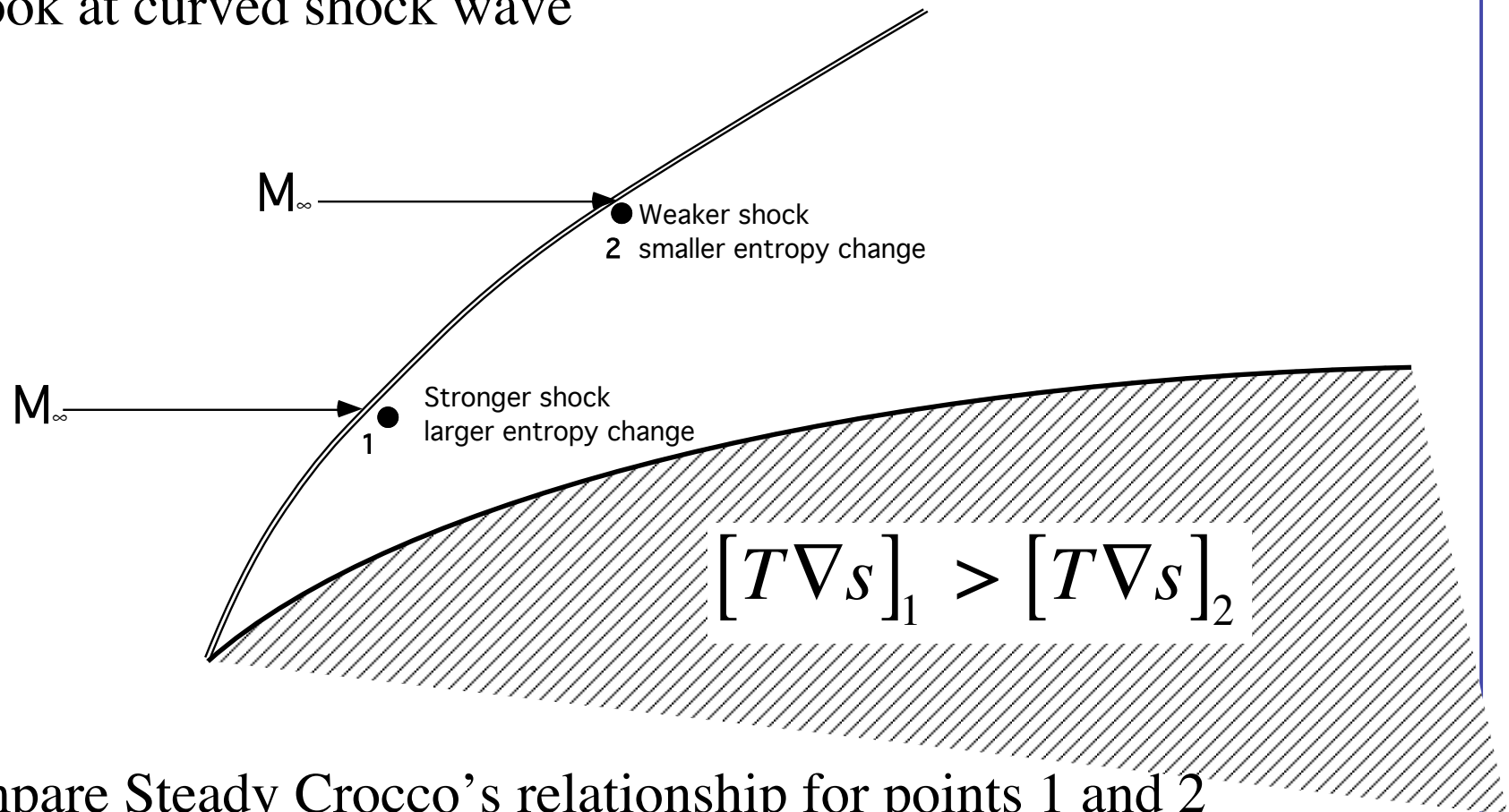
$$\text{Momentum} : \rho \left(\vec{V} \cdot \nabla \right) \vec{V} = -\nabla P$$

$$\text{Energy} : \nabla \cdot \left[\left(e + \frac{\|V^2\|}{2} \right) \rho \vec{V} \right] = -\nabla \cdot (p \vec{V}) + \rho \left(\dot{q} \right)$$

$$\text{Crocco(entropy)} : T \nabla s = \nabla h_o - \vec{V} \times (\nabla \times \vec{V})$$

Rotational and Irrotational Flow

- Look at curved shock wave



- Compare Steady Crocco's relationship for points 1 and 2

Rotational and Irrotational Flow (cont'd)

- Across shock total enthalpy is constant

$$\left[T \nabla s + \vec{V} \times \left(\nabla \times \vec{V} \right) \right]_1 = \left[T \nabla s + \vec{V} \times \left(\nabla \times \vec{V} \right) \right]_2$$

- or

$$\left[\vec{V} \times \left(\nabla \times \vec{V} \right) \right]_1 - \left[\vec{V} \times \left(\nabla \times \vec{V} \right) \right]_2 = [T \nabla s]_2 - [T \nabla s]_1$$

$$[T \nabla s]_1 > [T \nabla s]_2$$

Rotational and Irrotational Flow (cont'd)

- Thus

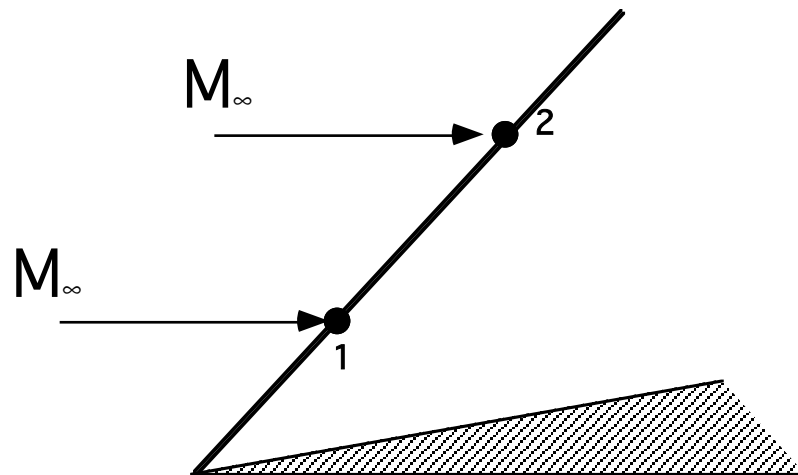
$$\left[\vec{V} \times (\nabla \times \vec{V}) \right]_1 - \left[\vec{V} \times (\nabla \times \vec{V}) \right]_2 < 0$$

$$\left[\vec{V} \times (\nabla \times \vec{V}) \right]_1 < \left[\vec{V} \times (\nabla \times \vec{V}) \right]_2$$

- $\vec{V} \times (\nabla \times \vec{V})$ Measure of “spin” in flow .. “Vorticity”
- Not constant ... “rotational flow” ... complex flow field

Rotational and Irrotational Flow (cont'd)

- Look at straight oblique shock wave



- Shock strength is the same at points 1 and 2

$$[T\nabla_s]_1 = [T\nabla_s]_2 \rightarrow \left[\vec{V} \times (\nabla \times \vec{V}) \right]_1 = \left[\vec{V} \times (\nabla \times \vec{V}) \right]_2 \quad \text{“irrotational flow”}$$

Rotational and Irrotational Flow (cont'd)

- Condition for Irrotational flow

$$\vec{V} \times \left(\nabla \times \vec{V} \right) = 0$$

- At every point in flow field
- If you put a pinwheel in and it spins ... “rotational flow”