UtahState UNIVERSITY

Mechanical & Aerospace Engineering

Section 3
Rocket Science Review 101: Ballistic Equations of Motion

Newton's Laws as Applied to "Rocket Science"

... its not just a job ... its an adventure

Real World Launch Analysis

Orbital designs a ***unique mission trajectory for each Pegasus flight to maximize payload*** performance while complying with the satellite and launch vehicle constraints.

Using the ***3-Degree of Freedom Program*** for Optimization of Simulation Trajectories(POST), a desired orbit is specified and a set of optimization parameters and constraints are designated.

Appropriate data for mass properties, aerodynamics, and motor ballistics are input. POST then selects values for the optimization parameters that target the desired orbit with specified constraints on key parameters such as angle of attack, dynamic loading, payload thermal, and ground track.

After POST calculates optimum launch trajectory, a Pegasus-specific six degree of Freedom simulation program used to verify Trajectory acceptability with realistic attitude dynamics, including separation analysis on all stages.

Pegasus User's Guide

Pegasus®

ORBITAL SCIENCES CORPORATION

- 6-DOF simulations costs ***A LOT!*** To run and are typically Not used for Trajectory design!

- We are going to start with A simple ***2^{+}-D*** Ballistic code that works well for sounding-rocket mission profile development

Newtonian Dynamics

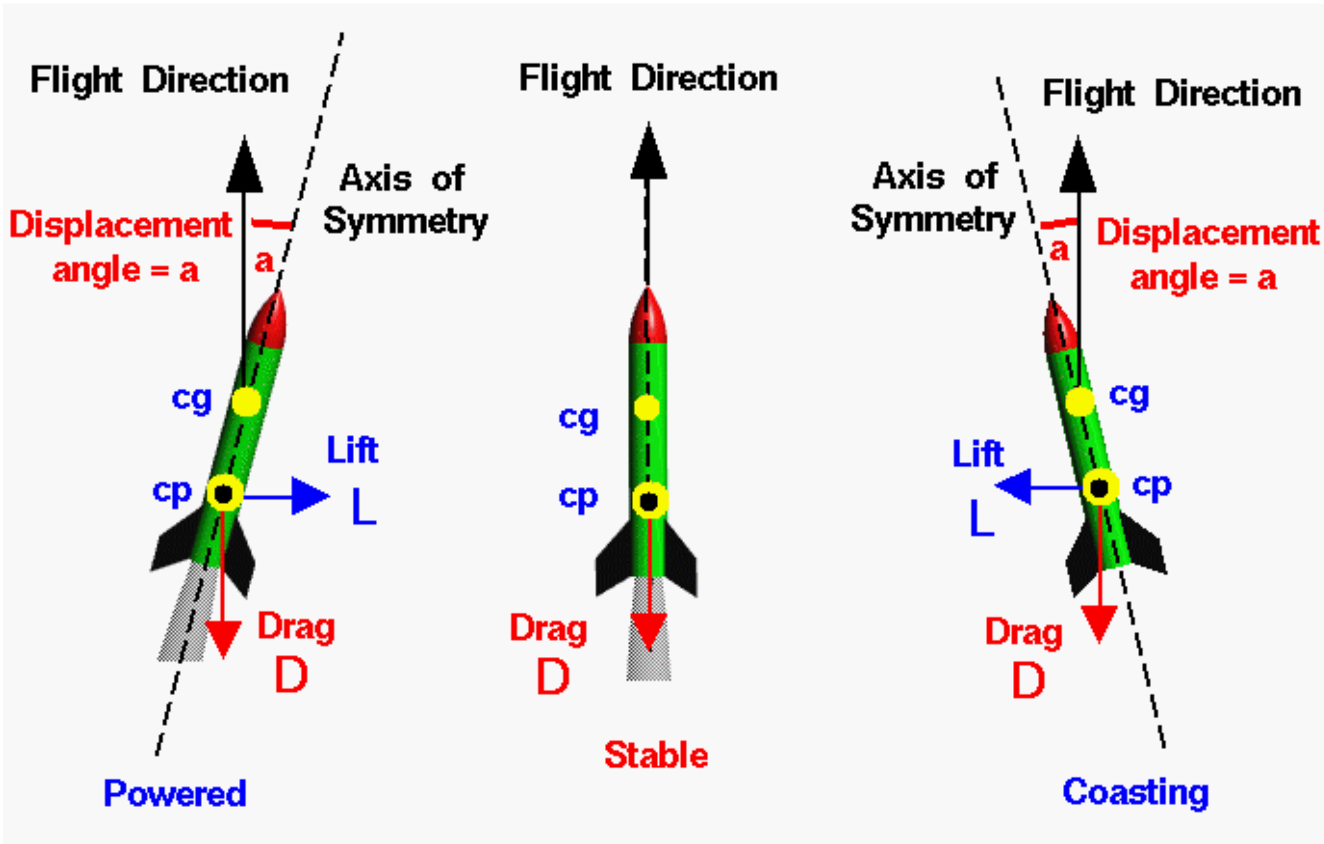
- General 3-DOF Equations of Motion

$$\overline{V} = \frac{\partial \overline{R}}{\partial t} + \overline{\omega} \times \overline{R}$$

$$\frac{\sum F_{external}}{M} = \frac{\partial \overline{V}}{\partial t} + \overline{\omega} \times \overline{V}$$

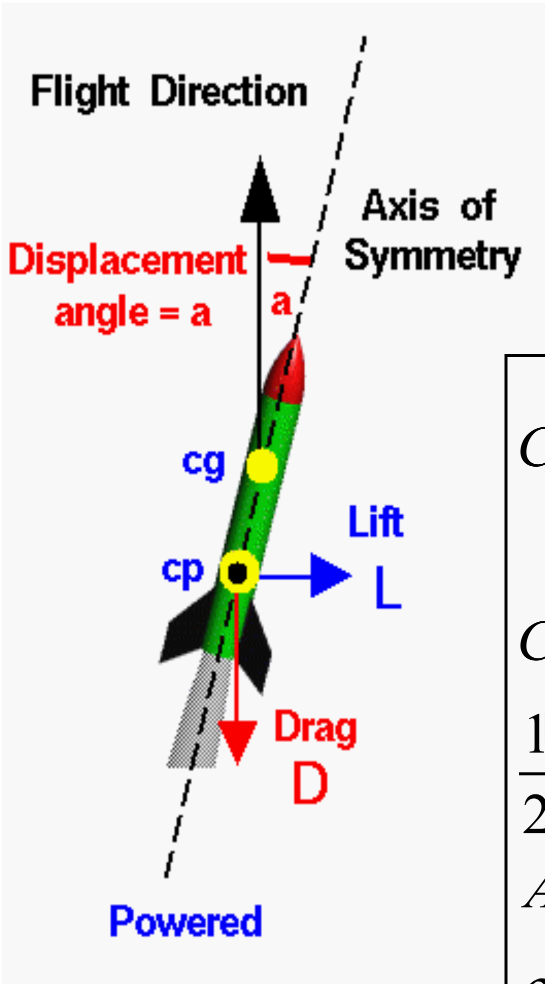
$$\dot{M}_{vehicle} = -\frac{F_{thrust}}{g_0 I_{sp}}$$

RS 101: Summary
External Forces Acting on Rocket



- **Lift – acts perpendicular to flight path *(non-conservative)***
- **Drag – acts along flight path *(non-conservative)***
- **Thrust – acts along longitudinal axis of rocket *(non-conservative)***
- **Gravity – acts downward *(conservative)***

External Forces Acting on Rocket (2)



- •**Lift – acts perpendicular to flight path *(non-conservative)***
- •**Drag – acts along flight path *(non-conservative)***

Define

$$C_L = \frac{L_{ift}}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A_{ref}}, \quad C_D = \frac{D_{rag}}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A_{ref}}$$

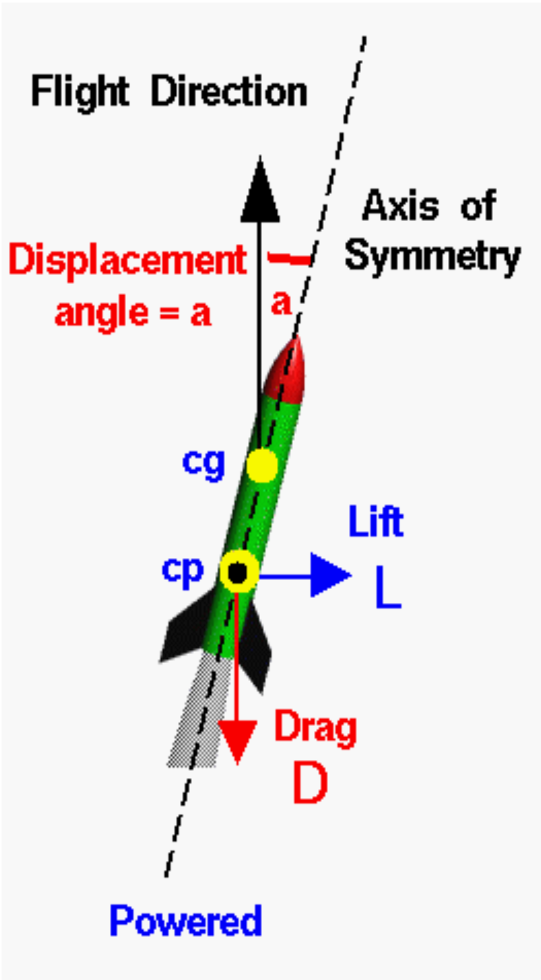
$$C_L = lift\ coefficient, \quad C_D = drag\ coefficient$$

$$\frac{1}{2}\cdot\rho\cdot V^2 = \text{"}dynamic\ pressure\text{"}(\bar{q})$$

$$A_{ref} = \text{"}reference\ area\text{"} -> typically\ maximum\ frontal\ area$$

$$\rho = \text{"}local\ air\ density\text{"} -> function\ of\ altitude$$

External Forces Acting on Rocket (3)



- **Gravitational Force**

$$\vec{F}_{grav} = -\frac{G \cdot M_{earth} \cdot m_{rocket}}{r^2}\bar{i}_r \rightarrow G \cdot M_{earth} \equiv \mu_{earth} = 3.9860044 \times 10^5 \frac{km^3}{sec^3}$$

$$\frac{\vec{F}_{grav}}{m_{rocket}} = A_{grav} = -\frac{\mu}{r^2}\bar{i}_r = -g \cdot \bar{i}_r$$

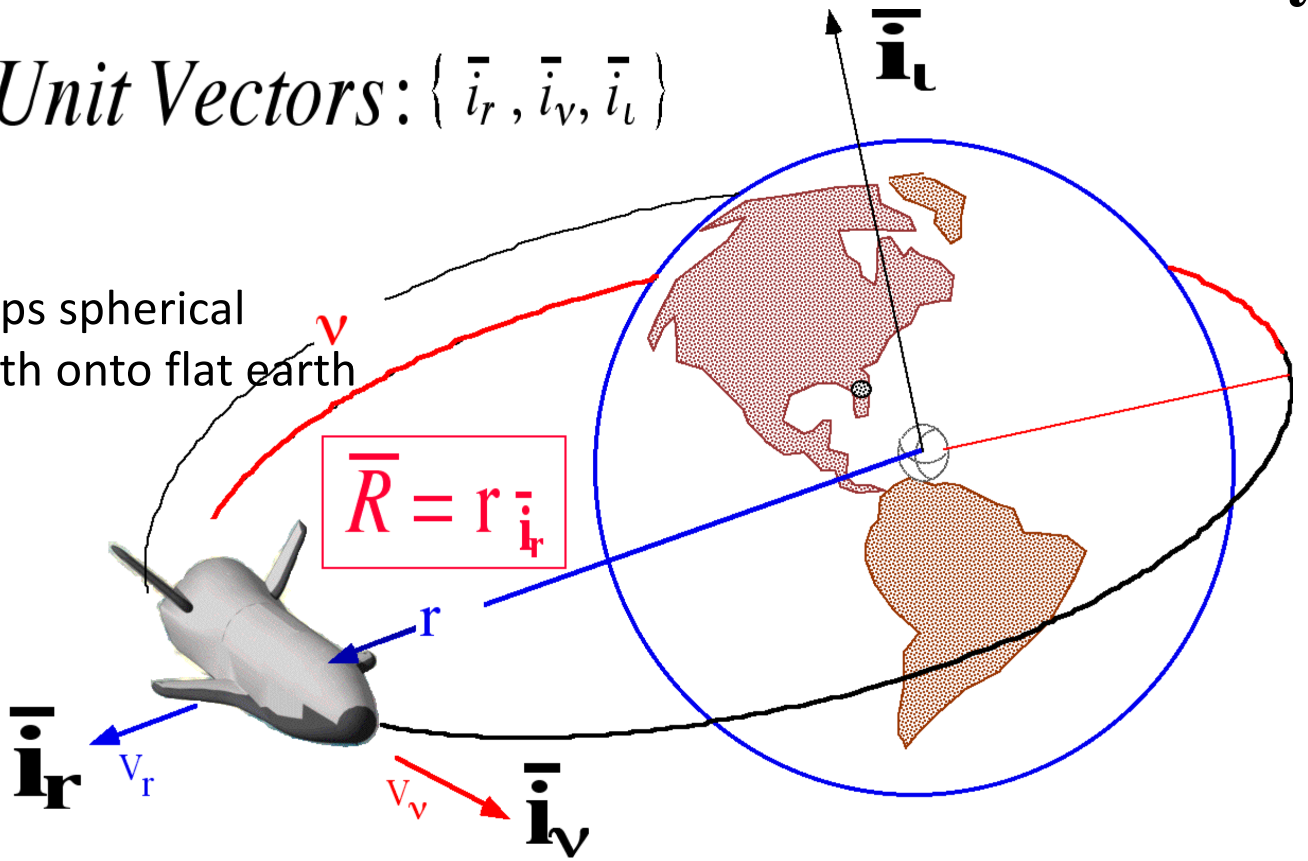
$$g_0 = \frac{3.9860044 \times 10^5 \frac{km^3}{sec^3}}{6375.4_{km}^{\;2}} \times 1000 \frac{m}{km} = 9.8067 \frac{m}{sec^2}$$

Sea Level Acceleration of gravity at 21 deg. latitude

Perifocal Coordinate System

$$Unit\ Vectors: \{\ \bar{i}_r\ ,\ \bar{i}_\nu,\ \bar{i}_\iota\ \}$$

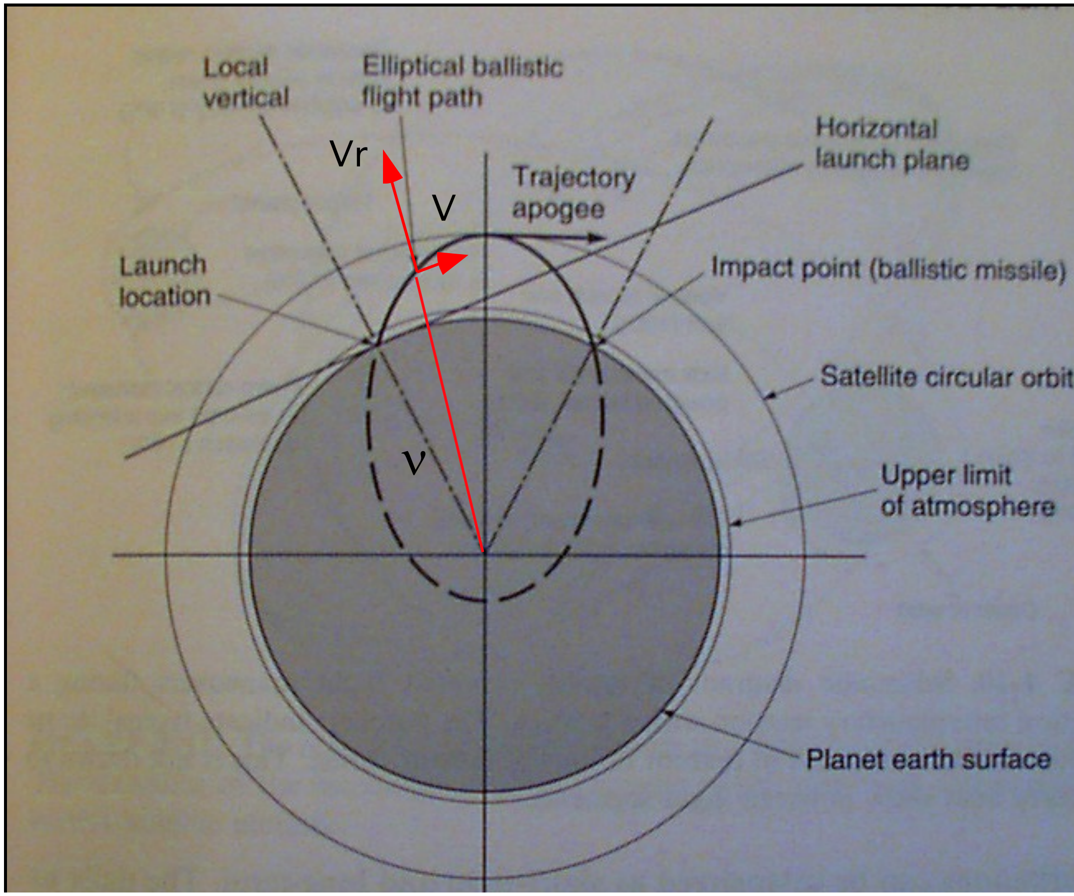
Maps spherical earth onto flat earth



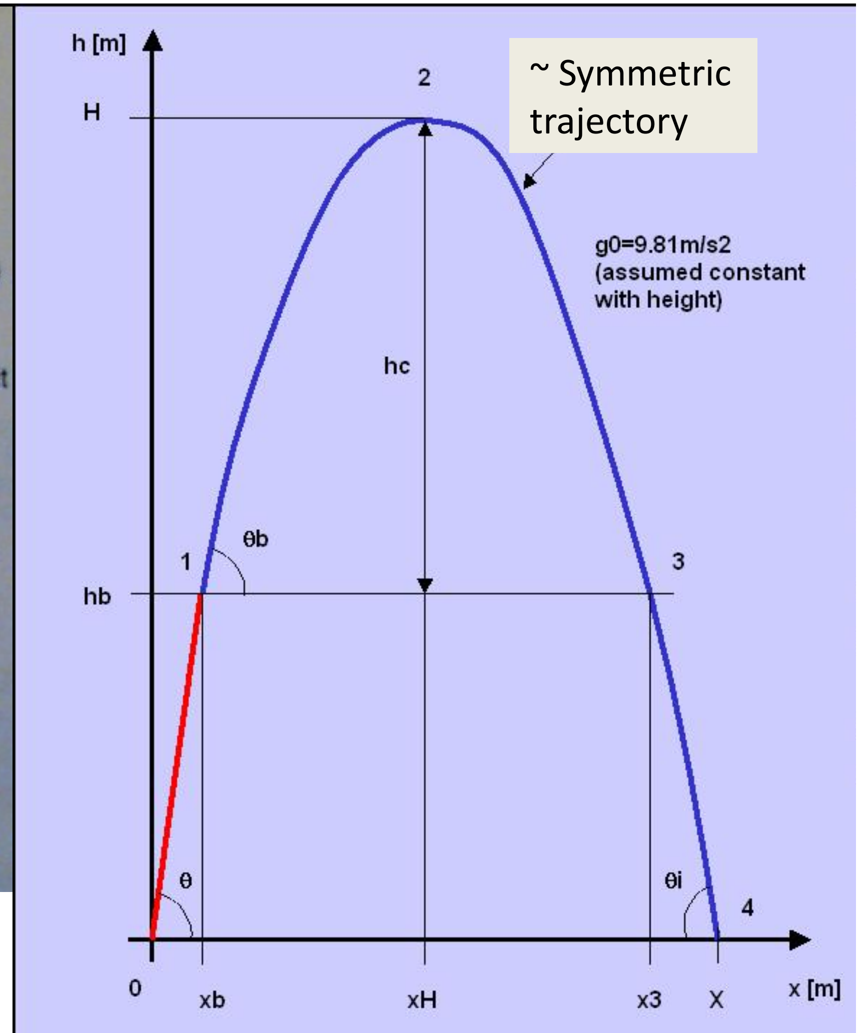
$$\begin{bmatrix} \vec{i}_r \\ \vec{i}_\nu \\ \vec{i}_\iota \end{bmatrix} = \begin{bmatrix} local\ vertical \\ local\ horizontal_{(along\ flight\ path)} \\ local\ horizontal_{(perpendicular\ to\ flight\ path)} \end{bmatrix}$$

Perifocal Coordinate System
Sub-orbital Image

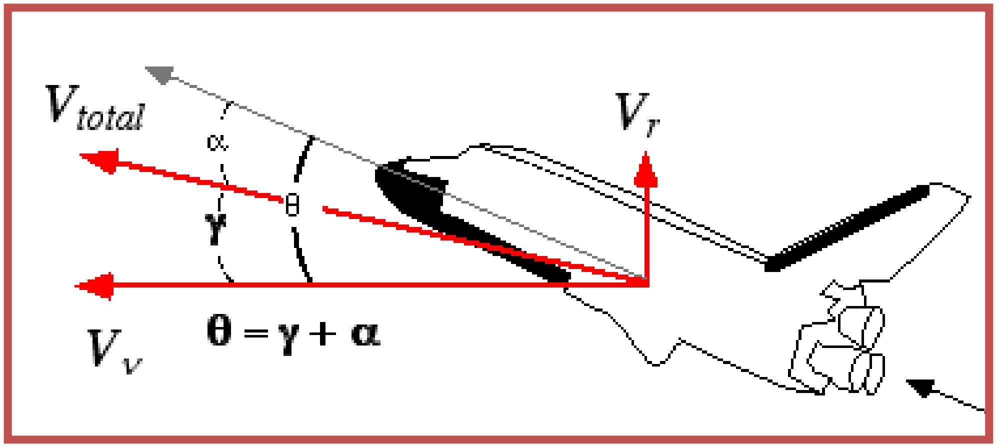
Sub-Orbital Launch: Spherical Earth

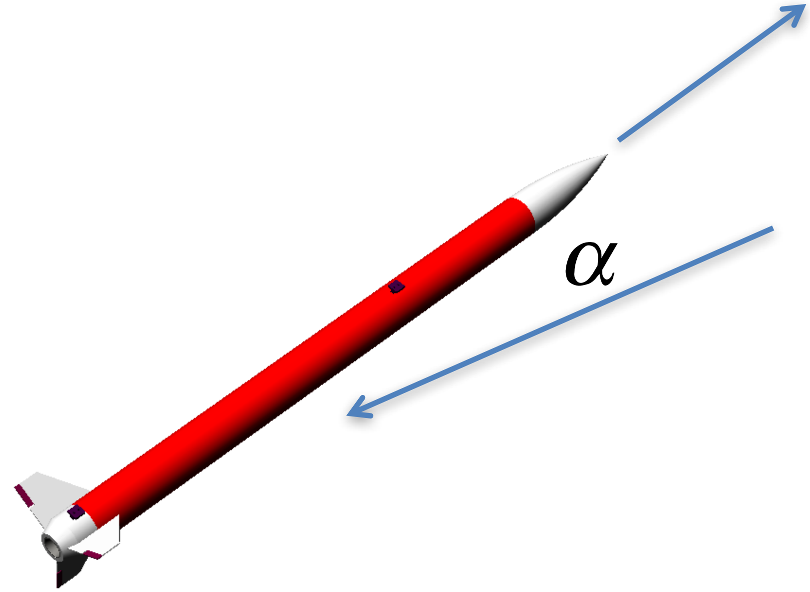


Sub-Orbital Launch: Flat Earth



Ballistic versus Non -Ballistic Trajectories





- Non-ballistic trajectories sustain significantly non -zero angles of attack

… lift is a factor in resulting trajectory
… so is induced drag

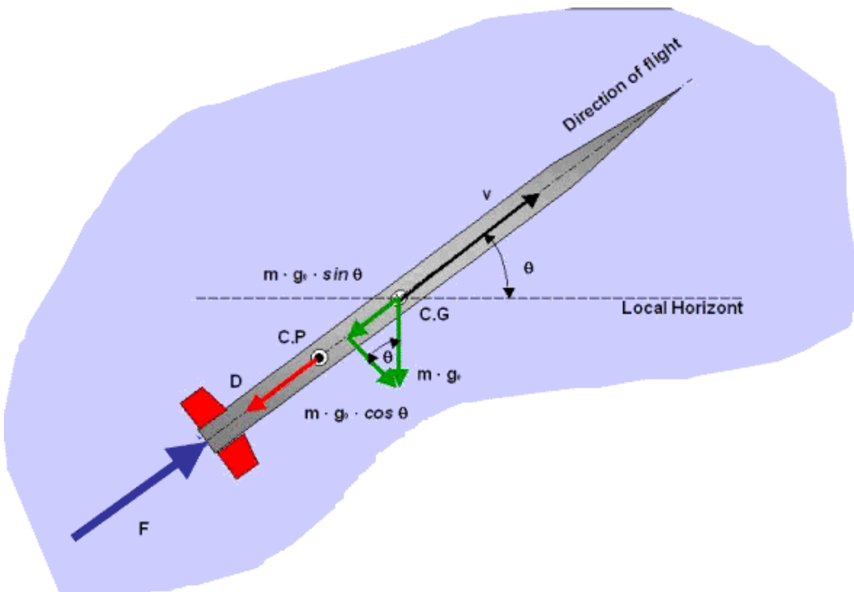
- *Ballistic rocket trims trajectory at* $\alpha \approx 0$

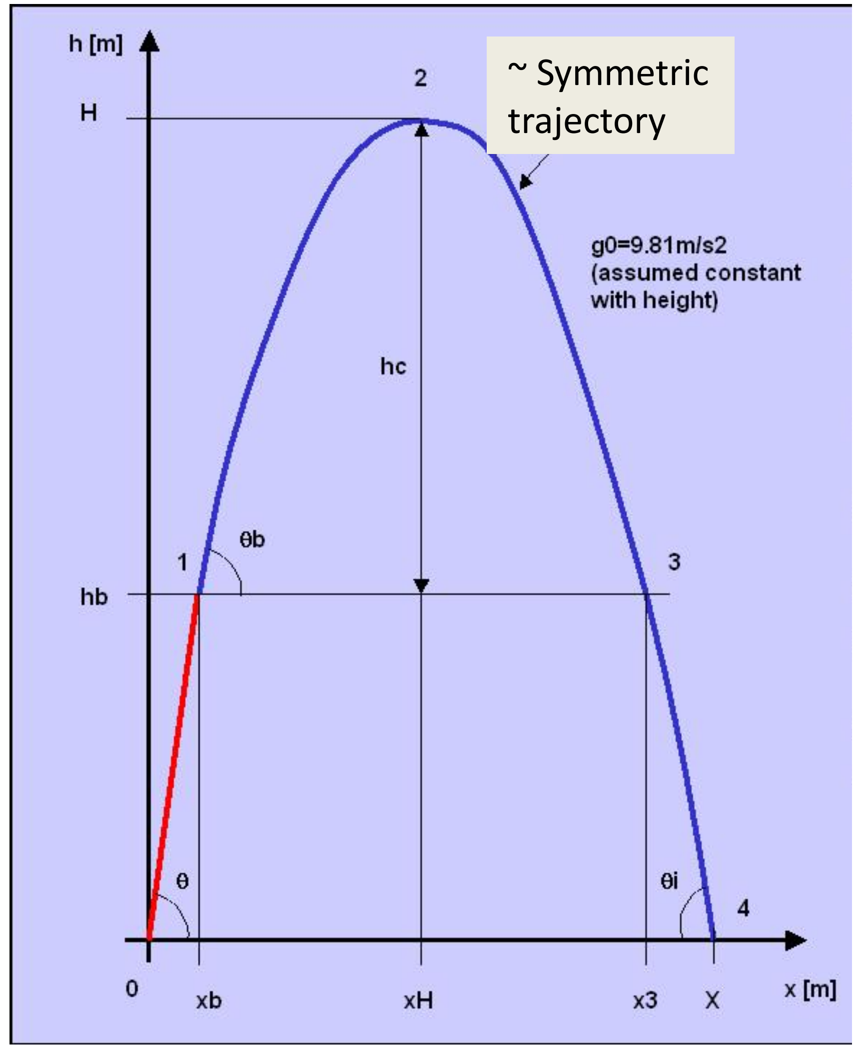
… lift is a negligible factor

$$\alpha \approx 0$$

$$\rightarrow \theta_{ballistic} = \gamma = \tan^{-1}\left[\frac{V_r}{V_\upsilon}\right]$$

Example of Ballistic Trajectory





- Ballistic Trajectories

Offer minimum drag profiles

$$\alpha \approx 0$$

(Induced drag primarily due to Alpha-dither, assume small)

Collected Ballistic Equations

$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{v} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \frac{V_v^2}{r} - \frac{\mu}{r^2} + \left[\frac{F_{thrust}}{m} - \frac{\rho V_\infty^2}{2\beta}\right]\sin(\gamma) \\ -\frac{V_r V_v}{r} + \left[\frac{F_{thrust}}{m} - \frac{\rho V_\infty^2}{2\beta}\right]\cos(\gamma) \\ V_r \\ \frac{V_v}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix}_{\alpha=0}$$

$$\gamma = \tan^{-1}\left[\frac{V_r}{V_v}\right]$$

$$\beta = \frac{M}{C_D \cdot A_{ref}}$$

$$\dot{X} = f\left[X, F_{thrust}\right]$$

$$\vec{X} = \begin{bmatrix} V_r \\ V_v \\ r \\ v \\ M \end{bmatrix}$$

$$\left(\dot{\vec{X}}\right) = \begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ v \\ -\dot{m}_{motor} \end{bmatrix}$$

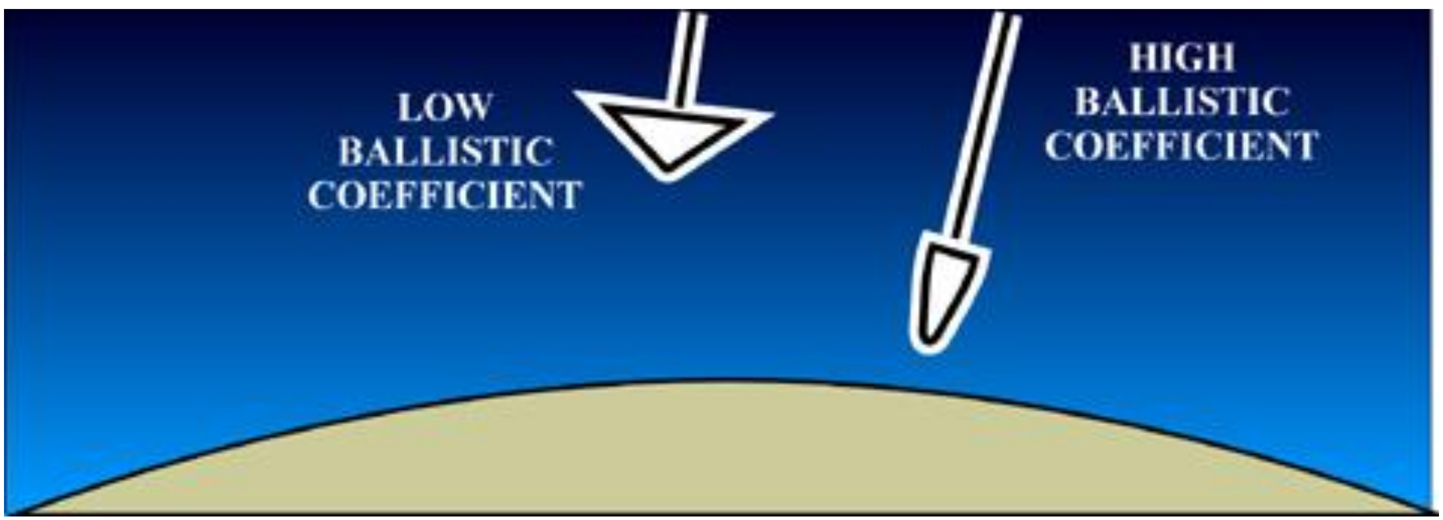
Ballistic Coefficient

- When effects of lift are negligible aerodynamic effects can be incorporated into a single parameter

…. ***Ballistic Coefficient*** **….**

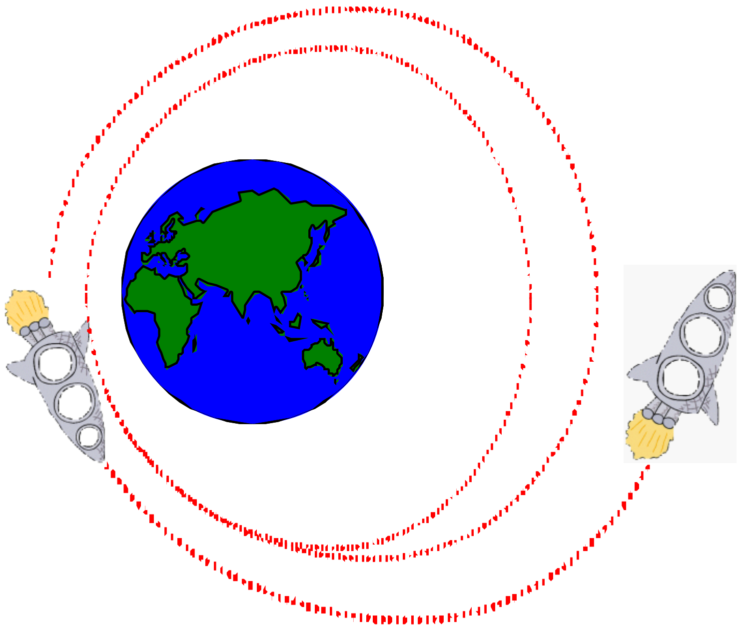
- β is a measure of a projectile's ability to coast. … $\beta = m / \left(C_D \cdot A_{ref} \right)$
… *M* is the projectile's mass and … $C_D A_{ref}$ is the drag *form factor*.

- At any given velocity and air density, the deceleration of a rocket from drag is inversely proportional to β



Low Ballistic Coefficients Dissipate More Energy Due to drag

Numerical Integration of the 2-D Launch Equations of Motion



•Simple Second Order predictor/corrector works well for
Small-to-moderate step sizes … but at larger step sizes can
be come unstable

• Good to have a higher order integration scheme in our *bag of tools*

• 4th Order Runge-Kutta method is one most commonly used

• we'll only use the second order integrators for this simulation

Trapezoidal Rule
Predictor/Corrector Algorithm

Given $\left\{\hat{\mathbf{X}}_\mathbf{k},\ \Delta\mathbf{t},\ \mathbf{F}_{\mathbf{thrust}}\right\}$

Prediction Step:

$$\tilde{\hat{\mathbf{X}}}_{\mathbf{k}+1} \equiv \hat{\mathbf{X}}_\mathbf{k} + \Delta\mathbf{t}\ \ \mathbf{f}\left(\hat{\mathbf{X}}_\mathbf{k},\ \mathbf{F}_{\mathbf{thrust}}\right)$$

Correction Step:

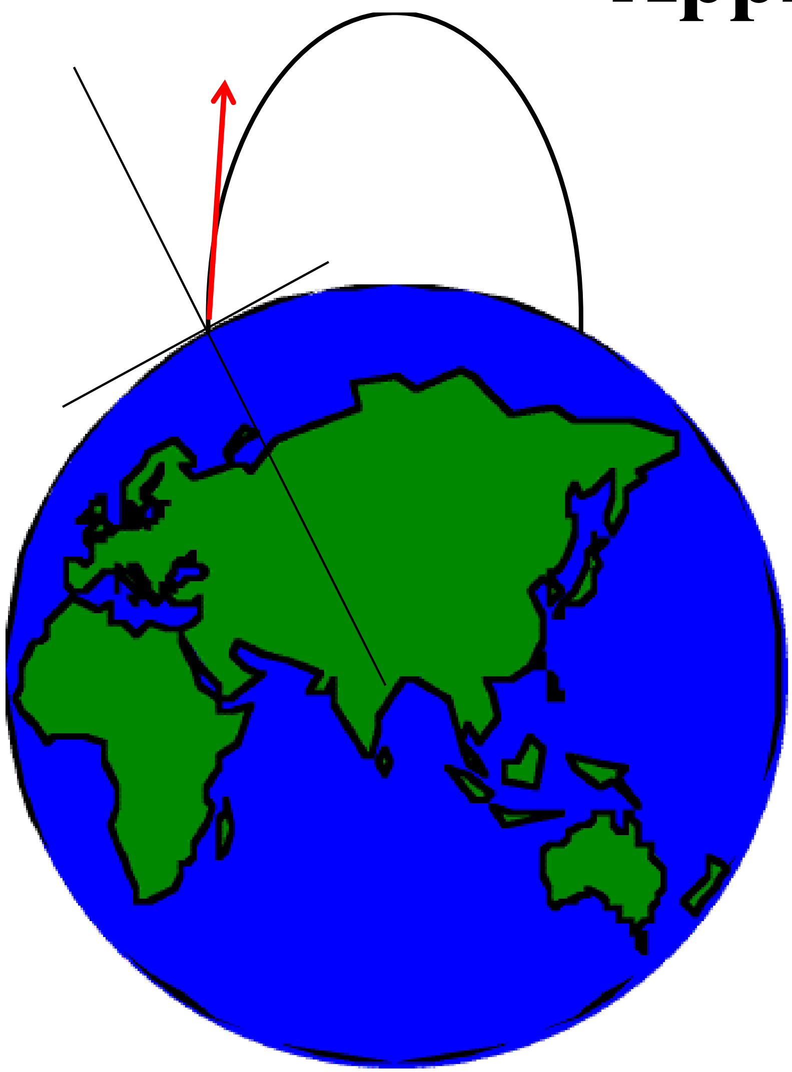
$$\hat{\mathbf{X}}_{\mathbf{k}+1} = \hat{\mathbf{X}}_\mathbf{k} + \frac{\Delta\mathbf{t}}{2}\left[\mathbf{f}\left(\hat{\mathbf{X}}_\mathbf{k},\ \mathbf{F}_{\mathbf{thrust}}\right) + \ \mathbf{f}\left(\tilde{\hat{\mathbf{X}}}_{\mathbf{k}+1},\ \mathbf{F}_{\mathbf{thrust}}\right)\right]$$

Slide Indices and Repeat:

$$\hat{\mathbf{X}}_{\mathbf{k}+1} \Rightarrow \hat{\mathbf{X}}_\mathbf{k}$$

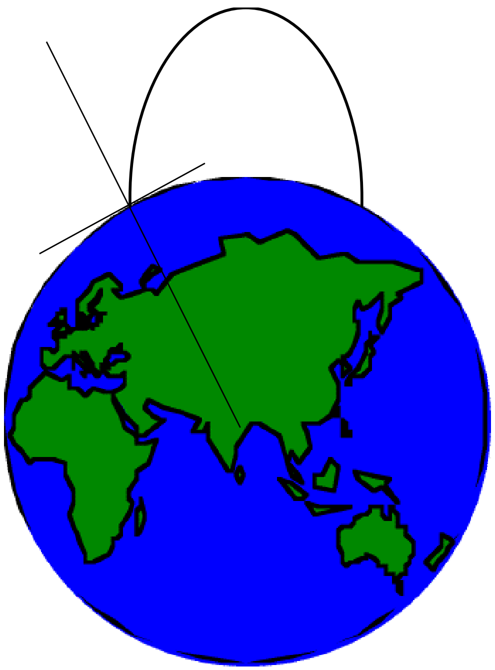
"trapezoidal rule"

Fixed Earth for Sounding Rocket Launch Approximation



- Ignore effects of Earth's rotation
- $V_{inertial} = V_{ground}$
- $\gamma_{inertial} = \gamma_{ground}$
- Accurate for Short Duration, Non-orbital flights

Ground Launch: Down Range Calculation



- Integrated trajectory gives

$$r, \nu$$

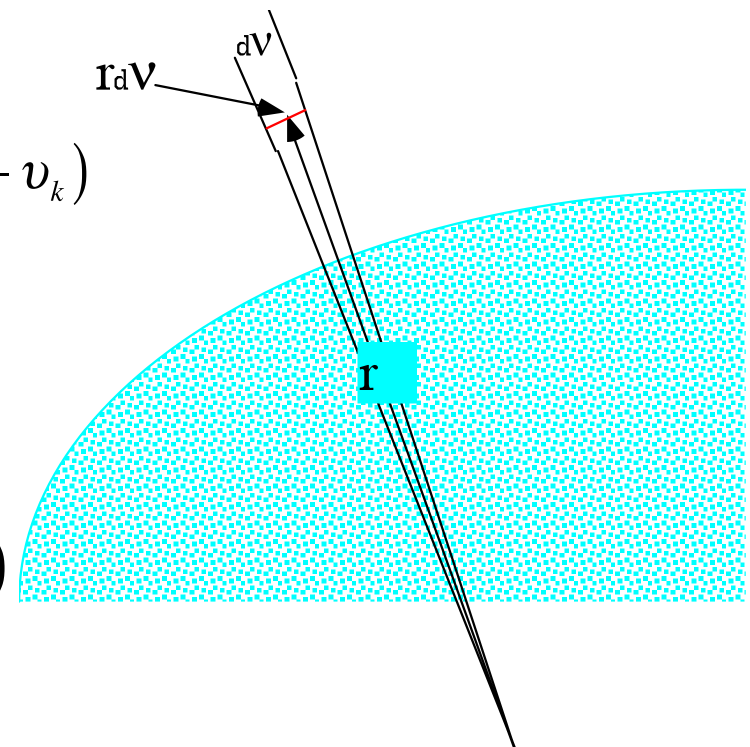
- Inertial Downrange

Horizontal Motion

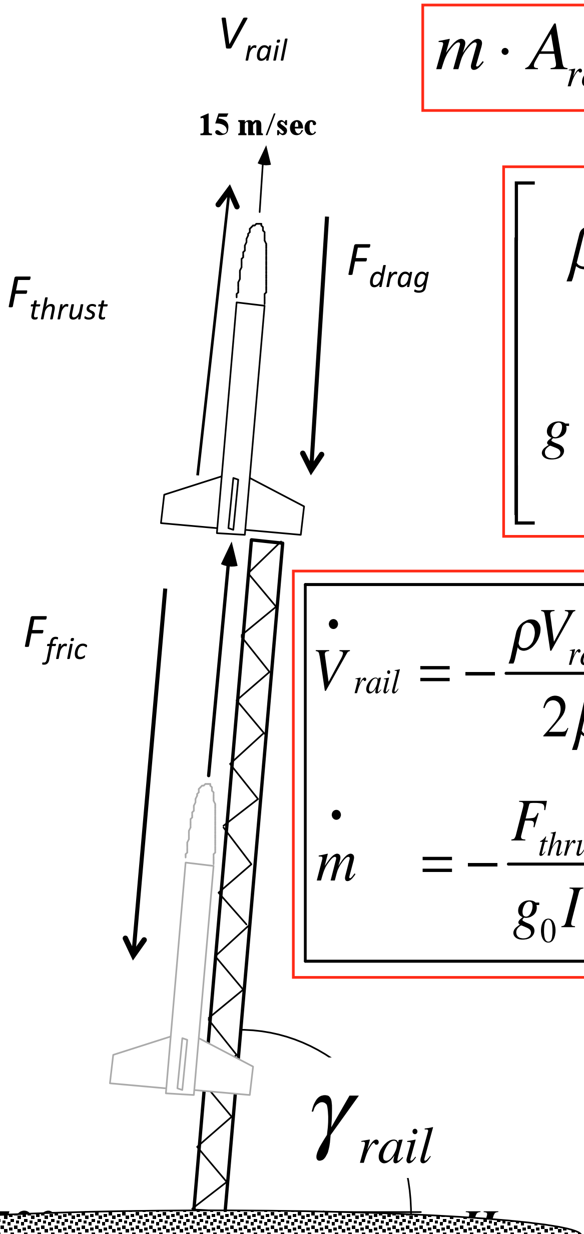
$$\Delta x = \int_{\upsilon_1}^{\upsilon_2} r \cdot d\upsilon \approx \left(\frac{r_{k+1} + r_k}{2}\right) \cdot \left(\upsilon_{k+1} - \upsilon_k\right)$$

Recursive Formula for Horizontal Motion

$$x_{k+1} = x_k + \left(\frac{r_{k+1} + r_k}{2}\right) \cdot \left(\upsilon_{k+1} - \upsilon_k\right)$$



Velocity Off of the Rail



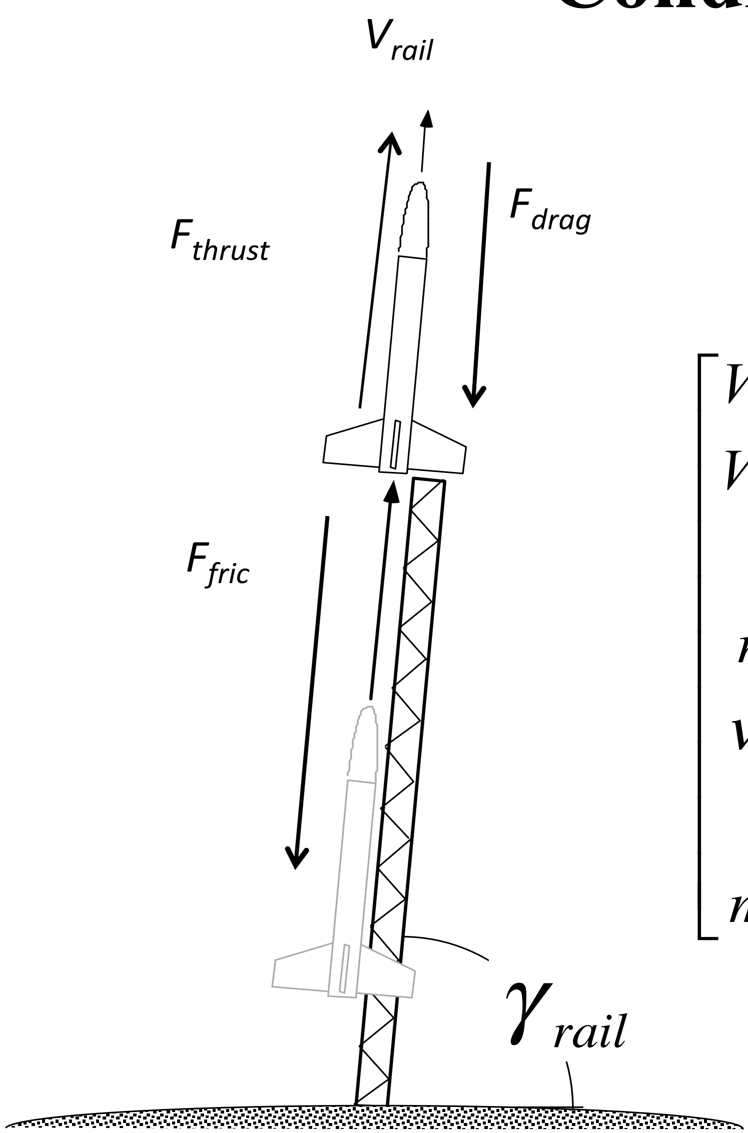
$$m \cdot A_{rail} = F_{thrust} - F_{grav} - F_{fric} - F_{drag}$$

$$\left[\begin{array}{l} \beta = \dfrac{m}{C_D A_{ref}} \\ g = \dfrac{\mu}{(R_e + h)^2} \end{array} \right] \rightarrow careful! \ with \ units$$

$$\dot{V}_{rail} = -\frac{\rho V_{rail}^2}{2\beta} - \frac{\mu}{(R_e + h)^2}\left[\sin(\gamma_{rail}) + C_f \cdot \cos(\gamma_{rail})\right] + \frac{F_{thrust}}{m}$$

$$\dot{m} = -\frac{F_{thrust}}{g_0 I_{sp}}$$

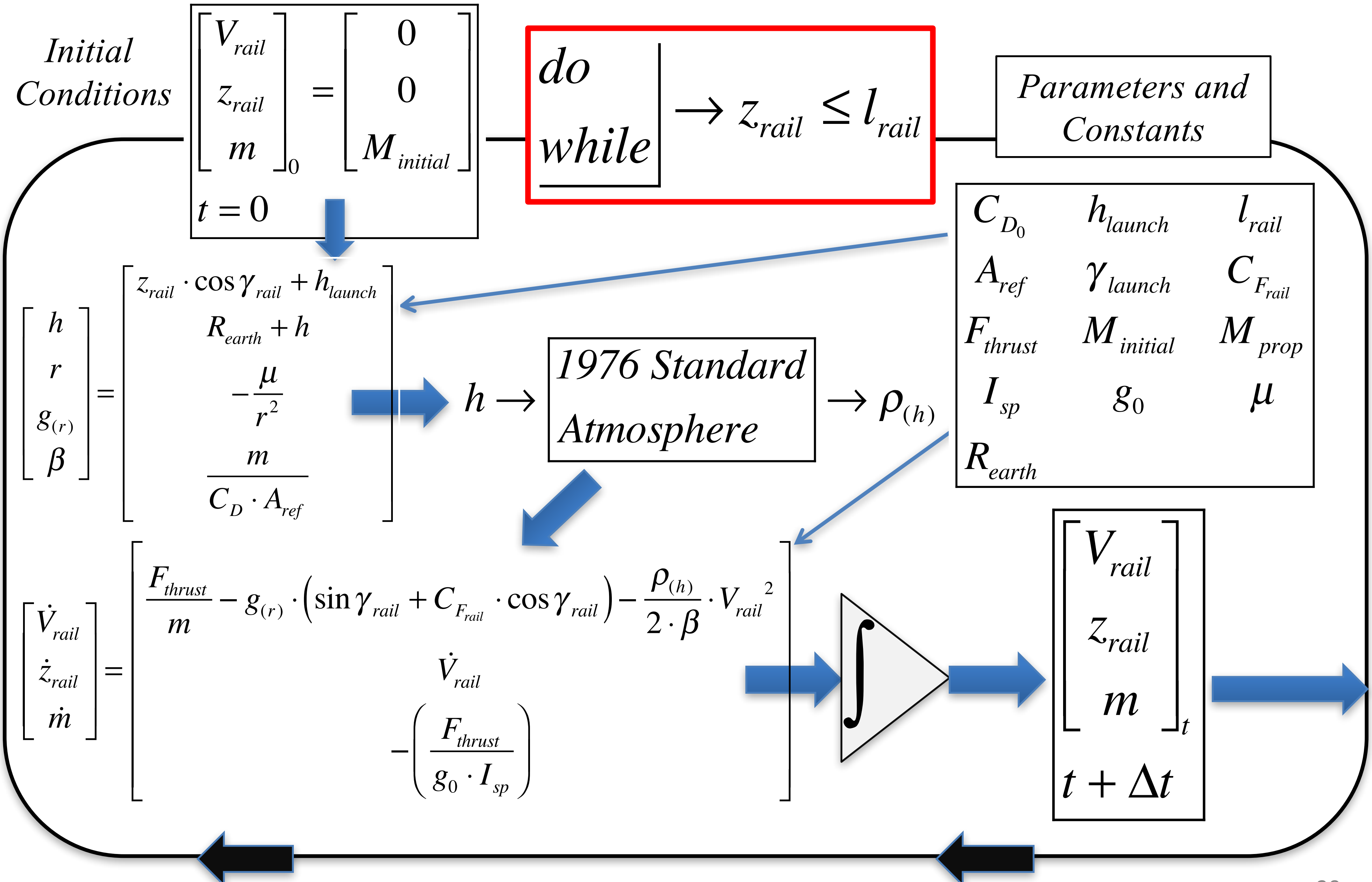
Exit Conditions off of Rail = Initial Conditions for Free Flight Simulation



$$Altitude: h = r - R_{earth}$$

$$\begin{bmatrix} V_r \\ V_\nu \\ r \\ \nu \\ m \end{bmatrix}_{initial\ conditions} = \begin{bmatrix} V_{rail} \cdot \sin\gamma_{rail} \\ V_\nu \cdot \cos\gamma_{rail} \\ R_{earth} + h_{launch} + l_{rail} \cdot \sin\gamma_{rail} \\ \left(\dfrac{l_{rail} \cdot \cos\gamma_{rail}}{R_{earth} + h_{launch} + l_{rail} \cdot \sin\gamma_{rail}} \right) \\ m_{rail} \end{bmatrix}$$

Rail Simulation Block Diagram



Ballistic Simulation Initial Conditions

$$\begin{bmatrix} Final \\ Conditions \end{bmatrix}_{\substack{rail \\ simulation}} \rightarrow \begin{bmatrix} Initial \\ Conditions \end{bmatrix}_{\substack{ballistic \\ saimulation}}$$

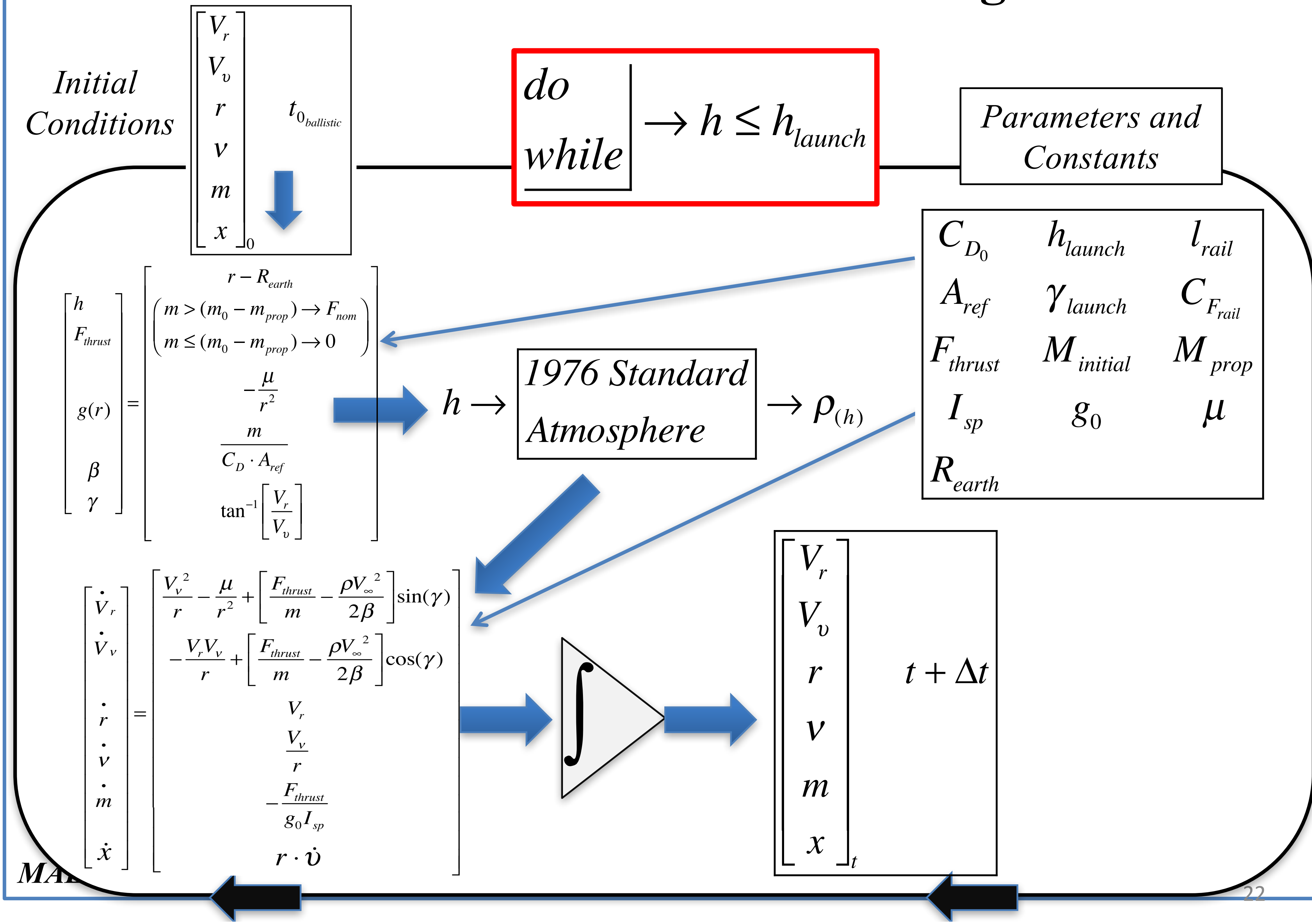
$$\begin{bmatrix} V_r \\ V_v \\ r \\ \nu \\ m \end{bmatrix}_{\substack{initial \\ conditions}} = \begin{bmatrix} V_{rail} \cdot \sin\gamma_{rail} \\ V_v \cdot \cos\gamma_{rail} \\ R_{earth} + h_{launch} + l_{rail} \cdot \sin\gamma_{rail} \\ \left(\dfrac{l_{rail} \cdot \cos\gamma_{rail}}{R_{earth} + h_{launch} + l_{rail} \cdot \sin\gamma_{rail}} \right) \\ m_{rail} \end{bmatrix}$$

Final conditions

$$t_{0_{ballistic}} = t_{final_{rail}}$$

$$x_{0_{ballistic}} = l_{rail} \cdot \cos\gamma_{rail} \approx 0$$

Ballistic Simulation Block Diagram




Questions??