Section 3
Rocket Science Review 102: Launch Energy Management

Newton's Laws as Applied to "Rocket Science"

... its not just a job ... its an adventure

RS 101: Summary and Terminology

- ***Rocket Thrust Equation***

$$F_{thrust} = \dot{m}V_{exit} + A_{exit}(P_{exit} - P_{atmosphere})$$

- ***Specific Impulse .. Total …. Instantaneous)***

$$I_{sp} = \frac{\mathrm{I}_{mpulse}}{g_0 M_{propellant}} = \frac{\int_0^t F_{thrust}\,dt}{g_0 \int_0^t \dot{m}_{propellant}\,dt} = \frac{F_{thrust}}{g_0\, \dot{m}_{propellant}}$$

- ***Rocket Equation***

$$\Delta V_{Burn} = g_0 I_{sp} \ln\left[1 + \frac{m_{propellant}}{M_{final}}\right]$$

- ***Propellant Mass Budget Equation***

$$M_{fuel + oxidizer} = \left[M_{dry} + M_{payload}\right]\left[e^{\left[\frac{\Delta V_{burn}}{g_0 I_{sp}}\right]} - 1\right]$$

- ***Load Mass Fraction***

$$\boxed{L_{mf} \equiv \frac{M_{propellant}}{M_{initial}} = \frac{M_{propellant}}{M_{final} + M_{initial}} 1 - e^{-\left(\frac{\Delta V}{g_0 \cdot I_{sp}}\right)}}$$

Summary and Terminology (2)

Available ΔV

$$\Delta V_{t_{burn}} = g_0 \cdot I_{sp}\left[\ln\left(1+P_{mf}\right)\right] -$$

$$\text{"combustion } \Delta V\text{"}$$

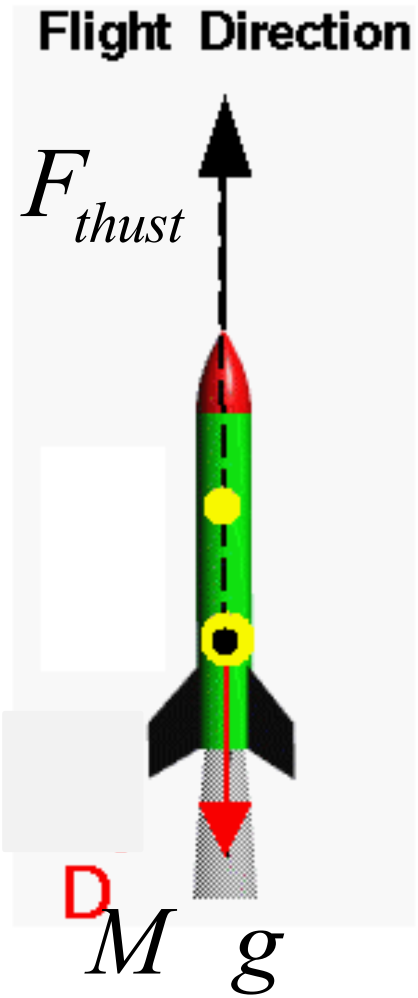
$$\int^{t_{burn}} g_{(t)} \cdot t \cdot \sin\theta_{(t)}\, dt - \sqrt{\int_0^{t_{burn}} \frac{\rho V^3}{\beta}\, dt}$$

"*gravity loss*" "*drag loss*"

$$\beta = \frac{M}{C_D \cdot A_{ref}}$$

Path dependent losses

ΔV for a Vertically Accelerating Vehicle



$$\boxed{\boxed{\Delta V_{t_{burn}} = g_0 \cdot I_{sp}\left[\ln\left(1 + P_{mf}\right)\right] - g_0 \cdot t_{burn}}}$$

$$\rightarrow \boxed{\begin{array}{l} P_{mf} = \dfrac{M_{propellant}}{M_{final}} \\ \\ t_{burn} = \dfrac{g_0 \cdot I_{sp} \cdot m_{propellamt}}{F_{thrust}} \end{array}}$$

ΔV for a Vertically Accelerating Vehicle (2)

- *Calculate burnout altitude*

$$Instantaneously:$$

$$\frac{dh}{dt}=\left(g_0\cdot I_{sp}\ln\left(\frac{M_{initial}}{M_{initial}-\dot{m}\cdot t}\right)-g_0\cdot t\right)$$

$$at\ \ Burnout:(Above\ \ ground\ \ level-AGL)$$

$$h_{t_{burn}}=\int_0^{t_{burn}}\left(g_0\cdot I_{sp}\ln\left(\frac{M_{initial}}{M_{initial}-\dot{m}\cdot\tau}\right)-g_0\cdot\tau\right)\cdot d\tau$$

- *After a lot of arithmetic!*

$$\boxed{h_{t_{burn}}=-\frac{g_0\cdot t_{burn}{}^2}{2}+\left(g_0\cdot I_{sp}\right)\cdot\left(\frac{M_{initial}}{\dot{m}}\cdot\ln\left(\frac{M_{final}}{M_{initial}}\right)+t_{burn}\cdot\left(1+\ln\left(\frac{M_{initial}}{M_{final}}\right)\right)\right)}$$

… ignoring aerodynamic drag

ΔV for a Vertically Accelerating Vehicle (3)

- *Collecting terms and simplifying*

$$h_{t_{burn}} = -\frac{g_0 \cdot t_{burn}^{\;2}}{2} + \left(g_0 \cdot I_{sp}\right) \cdot \left(\frac{M_{initial}}{\dot{m}} \cdot \ln\left(\frac{M_{final}}{M_{initial}}\right) + t_{burn} \cdot \left(1 + \ln\left(\frac{M_{initial}}{M_{final}}\right)\right)\right)$$

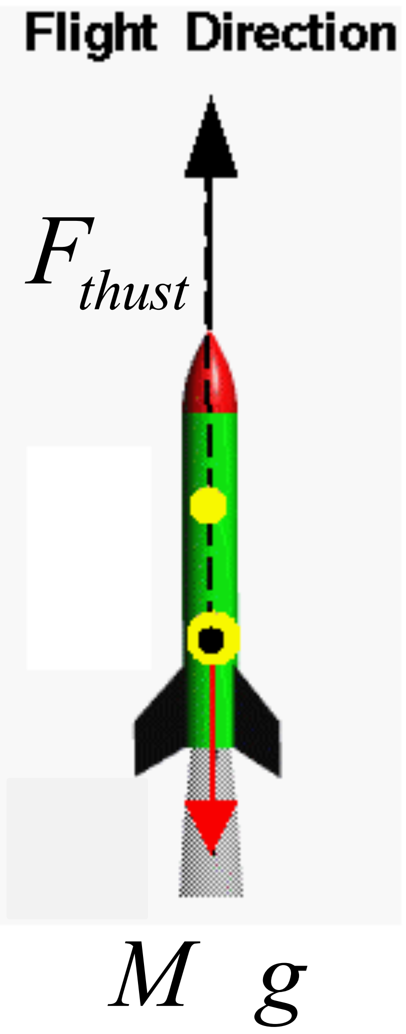
$$\frac{M_{initial}}{\dot{m}} = \frac{M_{final} + m_{propellant}}{m_{propellant} / t_{burn}}$$

$$\rightarrow \boxed{h_{t_{burn}} = g_0 \cdot t_{burn} \left\{ I_{sp} \cdot \left(1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}}\right) - \frac{t_{burn}}{2} \right\}}$$

… ignoring aerodynamic drag

ΔV for a Vertically Accelerating Vehicle (4)

•*Summary*



$$\boxed{\boxed{h_{t_{burn}} = g_0 \cdot t_{burn}\left\{I_{sp} \cdot \left(1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}}\right) - \frac{t_{burn}}{2}\right\}}}$$

$$\boxed{\boxed{V_{t_{burn}} = g_0 \cdot I_{sp}\left[\ln\left(1 + P_{mf}\right)\right] - g_0 \cdot t_{burn}}}$$

$$\rightarrow \boxed{\begin{array}{l} P_{mf} = \dfrac{M_{propellant}}{M_{final}} \\ t_{burn} = \dfrac{g_0 \cdot I_{sp} \cdot m_{propellamt}}{F_{thrust}} \end{array}}$$

… ignoring aerodynamic drag

How High will my Rocket go? (2)

Solving for **h_{apogee}**

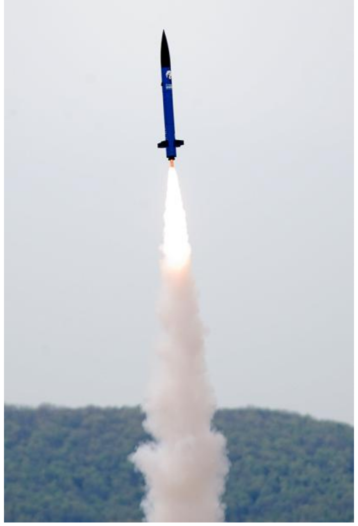
$$E_{mech} = M_{final}\left(\frac{\left(V_{apogee}\right)^2}{2} + g_0 \cdot h_{apogee}\right) \approx M_{final} \cdot g_0 \cdot h_{apogee}$$

$$h_{apogee} \approx \frac{E_{mech}}{M_{final} \cdot g_0} = \frac{\left(V_{burnout}\right)^2}{2 \cdot g_0} + h_{burnout}$$

$$\boxed{\begin{gathered} h_{apogee} \approx \\ g_0\left[\frac{1}{2}\left(I_{sp}\left[\ln\left(1+P_{mf}\right)\right] - g_0 \cdot t_{burn}\right)^2 + t_{burn}\left\{I_{sp} \cdot \left(1 - \frac{\ln\left(1+P_{mf}\right)}{P_{mf}}\right) - \frac{t_{burn}}{2}\right\}\right] \end{gathered}}$$

… ignoring aerodynamic drag

Example Energy Calculation



2009 USLI Rocket

AMW L777 Motor

"Dry" vehicle mass : *11.2451 kg*, Propellant mass: 1.7623 kg

Propellant I_{sp}: 181.49sec, Mean Motor Thrust: 774.475 Newtons

$$P_{mf} = \frac{m_{propellant}}{M_{final}} = \frac{1.7623}{11.2451} = 0.156717$$

$$t_{burn} = \frac{g_0 \; I_{sp} \; M_{propellamt}}{F_{thrust}} = \frac{9.8067 \cdot 181.49 \cdot 1.7623}{774.475} = 4.04993 \text{ sec}$$

Example Energy Calculation (2)

$$P_{mf} = \frac{m_{propellant}}{M_{final}} = 0.156717 \qquad t_{burn} = \frac{g_0 \ I_{sp} \ M_{propellamt}}{F_{thrust}} = 4.04993 \text{ sec}$$

$$h_{t_{burn}} = g_0 \cdot t_{burn} \left\{ I_{sp} \cdot \left(1 - \frac{\ln\left(1 + P_{mf}\right)}{P_{mf}} \right) - \frac{t_{burn}}{2} \right\} =$$

$$9.8067 \cdot 4.04933 \left(181.49 \left(1 - \frac{\ln\ (1 + 0.156717)}{0.156717} \right) - \frac{4.04993}{2} \right)$$ **= 431.5 meters**

$$V_{t_{burn}} = g_0 \cdot I_{sp} \left[\ln\left(1 + P_{mf}\right) \right] - g_0 \cdot t_{burn} =$$

= 219. 5 m/sec

$$9.8067 \cdot 181.49\ (\ \ln\ (1 + 0.156717)\ )\ - 9.8067 \cdot 4.004993$$

$$h_{apogee} \quad \frac{E_{mech}}{M_{final} \ \ g_0} = \frac{\left(V_{burnout}\right)^2}{2 \ \ g_0} + h_{burnout} = \frac{219.5^2}{2 \cdot 9.8067} + 431.5$$

= 2888 meters

Compare to Simulation Results

We will build this simulation later

$$h_{apogee} = 2888.71\ m$$

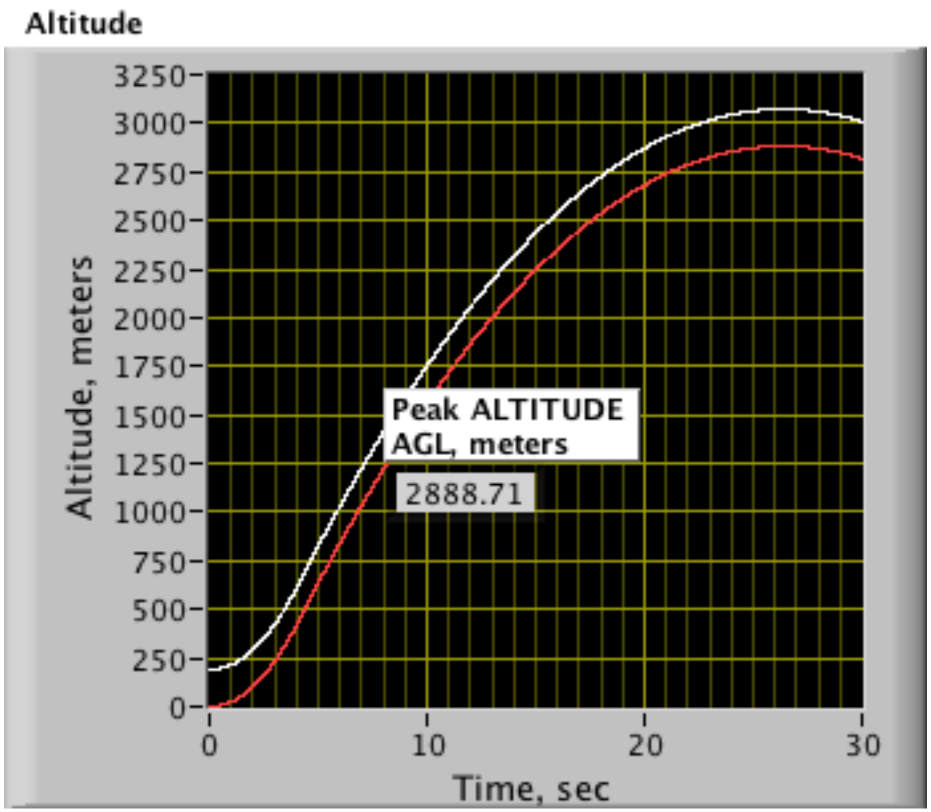
$$V_{burnout} = 219.34\ m / \sec$$

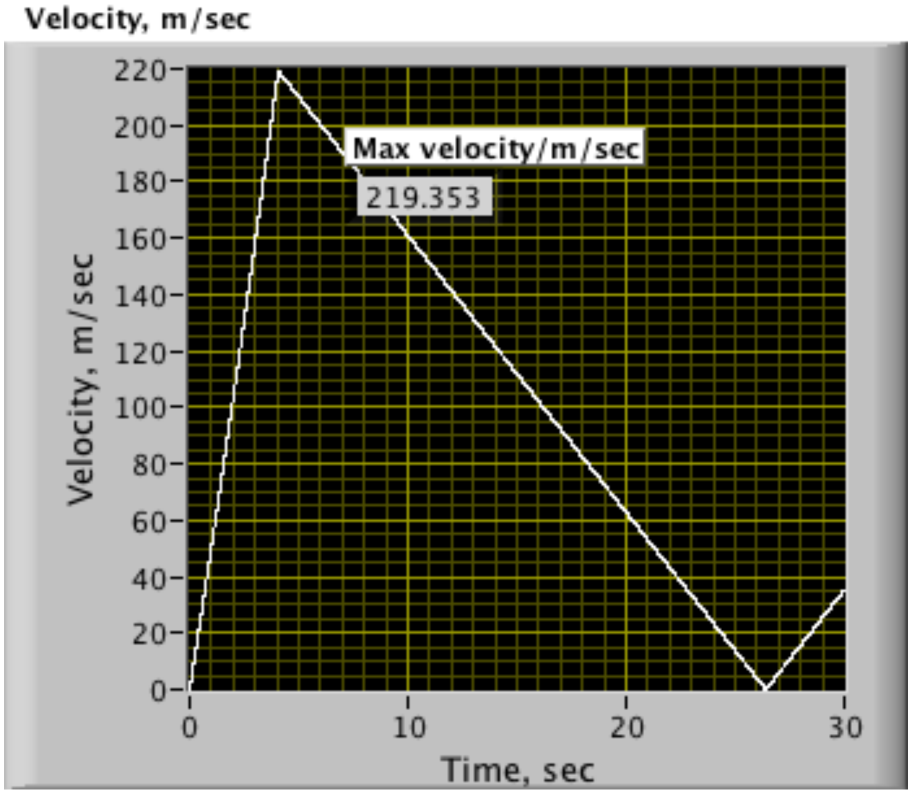
Analytical Solution

=> 2888 meters

= 219. 5 m/sec

Ignoring drag for now!



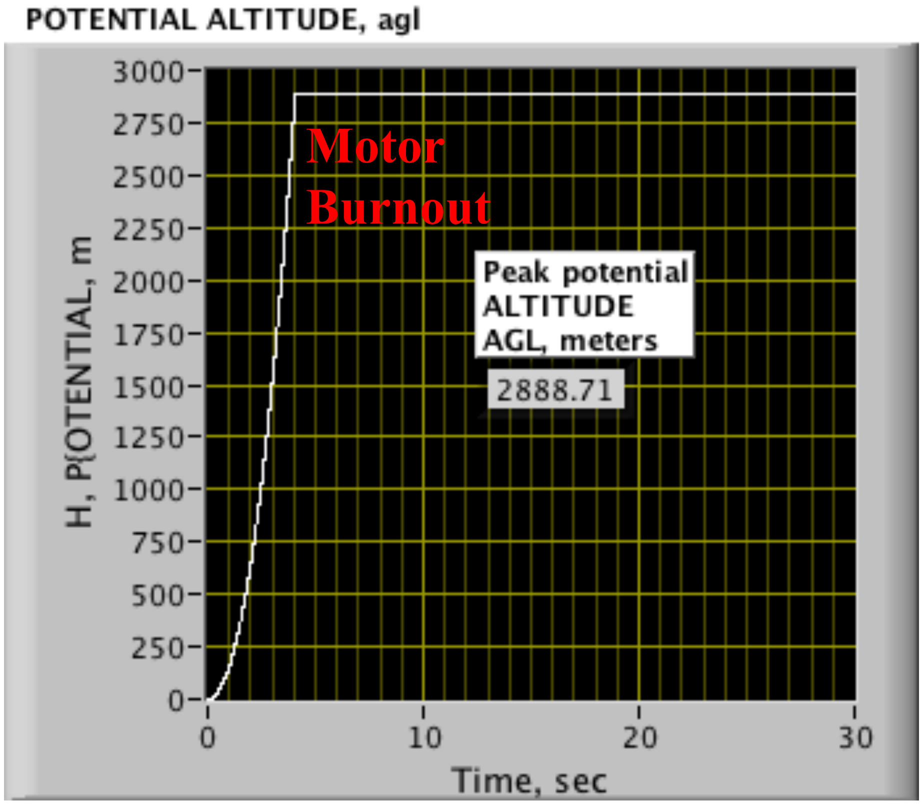


Better than 0.056%

Compare to Simulation Results (2)

$$\boxed{\begin{aligned} h_{apogee} &= 2888.71\ m \\ V_{burnout} &= 219.34\ m/\sec \end{aligned}}$$

Ignoring drag for now!



$$h_{apogee} \leftarrow h_{potential} =$$

$$\frac{E_{mech}}{M_{final}\ g_0} = \frac{\left(V_{burnout}\right)^2}{2\ g_0} + h_{burnout}$$

Compare to Fight Data

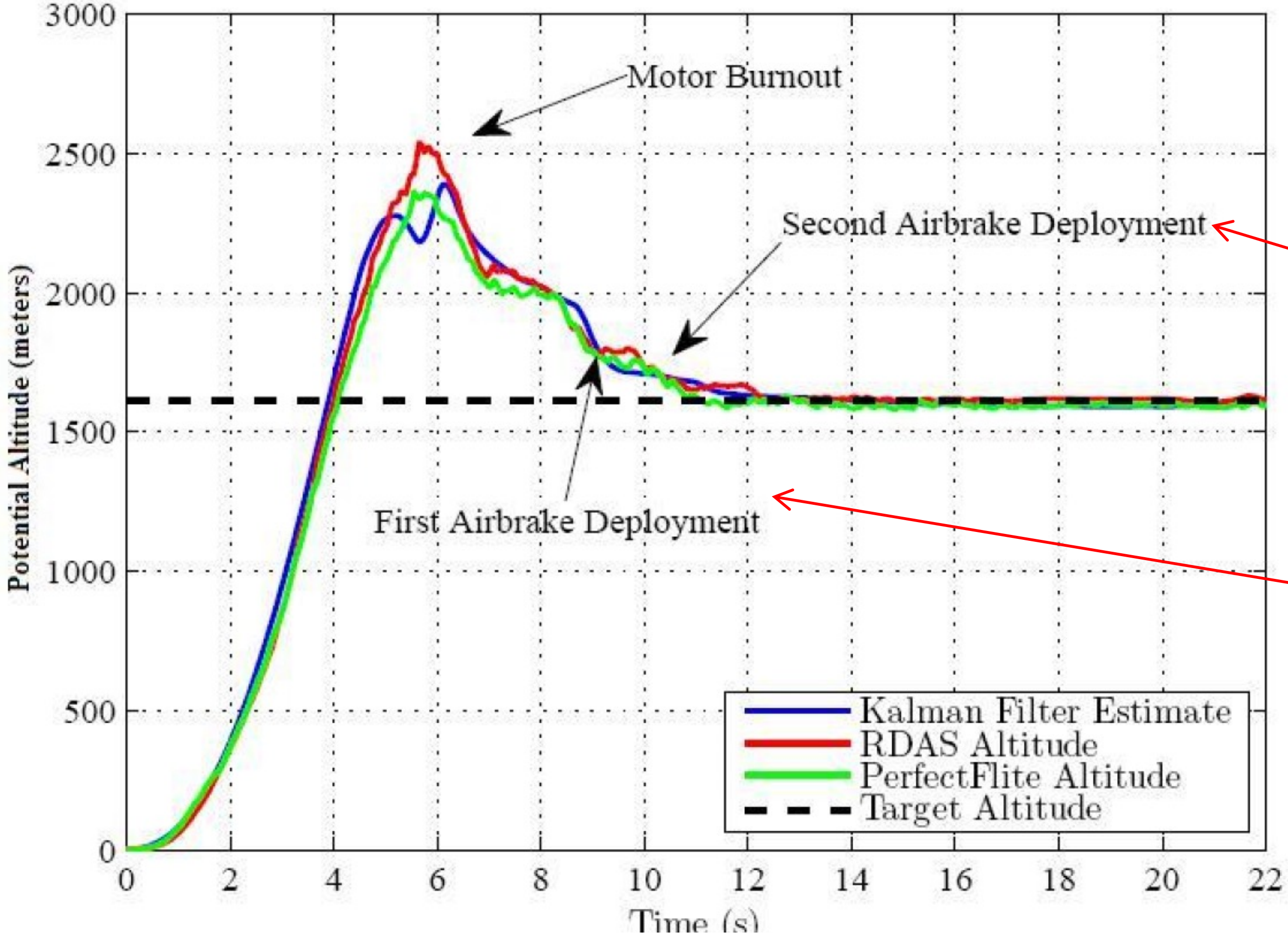
$$h_{apogee} = 2888.71\ m$$

$$V_{burnout} = 219.34\ m / \sec$$

Why the difference?

Primarily …. Drag!

~ 15% error in peak altitude



$$h_{potential} = h + \frac{V^2}{2\ \ g_0}$$

Compare to Fight Data (2)

Why the difference? We have ignored Drag!

$$\frac{\left(V_{apogee}\right)^2}{2} = \frac{E_{mech}}{M_{final}} \qquad V_{apogee} = \sqrt{\frac{2\ E_{mech}}{M_{final}}} = \sqrt{2\ g_0\ h_{potential}}$$

$$\frac{\left(V_{apogee}\right)_{calc} \quad \left(V_{apogee}\right)_{flight}}{\frac{\left(V_{apogee}\right)_{calc} + \left(V_{apogee}\right)_{flight}}{2}} \cdot 100_{\%} =$$

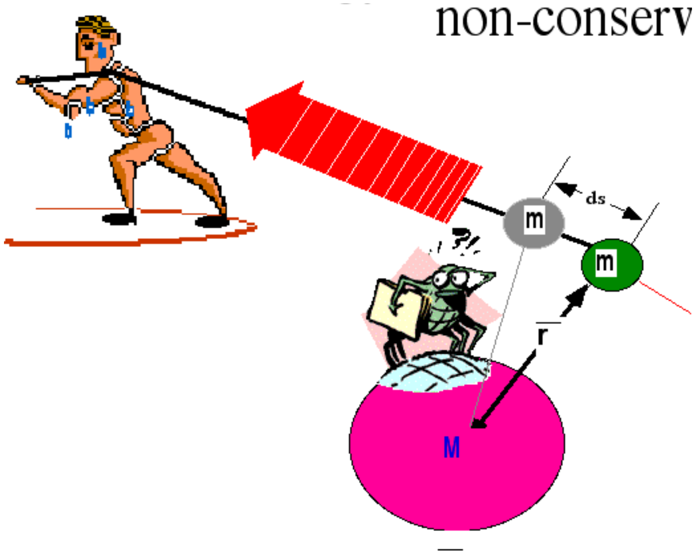
$$\frac{(2888.71 \cdot 2 \cdot 9.8067)^{0.5} - (2500 \cdot 2 \cdot 9.8067)^{0.5}}{\frac{(2888.71 \cdot 2 \cdot 9.8067)^{0.5} + (2500 \cdot 2 \cdot 9.8067)^{0.5}}{2}} \qquad = 7.22\%$$

~ 7.2 % error in delivered apogee Δ V due to aerodynamic drag

Drag Losses

$$\Delta E_{\substack{non-\\conservative}} = \int_{path} \vec{F}_{\substack{non\\-conservative}} \cdot d\vec{s} = \int_{path} \vec{F}_{\substack{non\\-conservative}} \cdot \frac{d\vec{s}}{dt} \cdot dt$$

$$\int_{t} \vec{F}_{\substack{non\\-conservative}} \cdot \vec{V} \cdot dt \approx \frac{\Delta V_{loss}^{\;2}}{2} \cdot M$$



- **Lift – acts perpendicular to flight path**
  .. *Cannot effect energy level of rocket*

- **Gravity – acts downward *(conservative)***
  … cannot effect energy level of rocket, kinetic energy potential energy as rocket climbs

- ***Apogee primarily influenced by burnout energy and atmospheric aerodynamc drag***

How Drag Losses Effect Peak Altitude

$$\Delta V_{drag} = \sqrt{2\int_0^t \frac{D_{rag} V}{M} dt} = \sqrt{2\int_0^t \frac{C_D A_{ref} \frac{1}{2}\rho V^2 \cdot V}{M} dt} = \sqrt{\int_0^t \frac{\rho \cdot V^3}{\beta} dt}$$

$$\Delta h_{drag} = \frac{{\Delta V_{drag}}^2}{2 \cdot g_0} = \frac{1}{2 \cdot g_0 \cdot \beta}\left[\int_0^t \rho \cdot V^3 \, dt\right]$$

Check units!

$$: \frac{1}{\frac{m}{\sec^2} \; \frac{kg}{m^2}} \; \frac{kg}{m^3} \; {\frac{m}{\sec}}^{3} \sec = \frac{\sec^2 \; m^5}{m \; kg} \; \frac{kg}{m^3} \; \frac{1}{\sec^2} = m$$

How Drag Losses Effect Peak Altitude (2)

$$\Delta h_{drag} = \frac{1}{2 \cdot g_0 \cdot \beta}\left[\int_0^t \rho \cdot V^3\, dt\right]$$

Correct peak altitude estimate

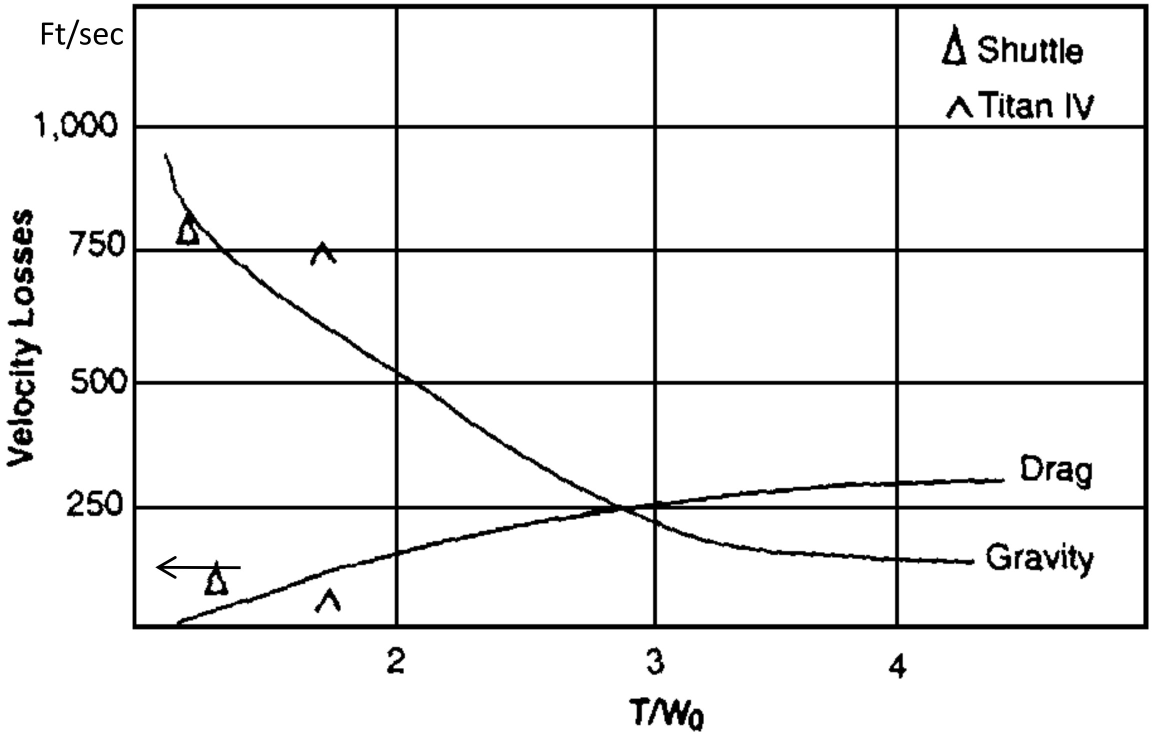
$$h_c = \frac{g_0}{4} \cdot \left(2 + P_{mf}\right) \cdot \left(I_{sp} \cdot \ln\left(1 + P_{mf}\right)\right)^2 - \Delta h_{drag} =$$

$$\underbrace{\frac{g_0}{4} \cdot \left(2 + P_{mf}\right) \cdot \left(I_{sp} \cdot \ln\left(1 + P_{mf}\right)\right)^2}_{\textbf{\textit{Path Independent}}} - \underbrace{\frac{1}{2 \cdot g_0 \cdot \beta}\left[\int_0^t \rho \cdot V^3\, dt\right]}_{\textbf{\textit{Path Dependent}}}$$

"Rule of thumb" ~ drag loss is about 5-10% of delivered ***Δ V*** from motor

Drag Losses (3)

$$D_{rag} = C_D A_{ref} \frac{1}{2} \rho V^2 \rightarrow \Delta V_{drag} = \sqrt{A_{ref} \int_0^t \frac{C_D \rho V^3}{m} dt} = \sqrt{\int_0^t \frac{\rho V^3}{\beta} dt}$$



Depending
On thrust to-weight
Off of the pad
drag losses
can be significant
During motor burn

As much as 12-15% of
Potential altitude

… path dependent!

Must simulate trajectory

Drag Coefficient is Configuration Dependent

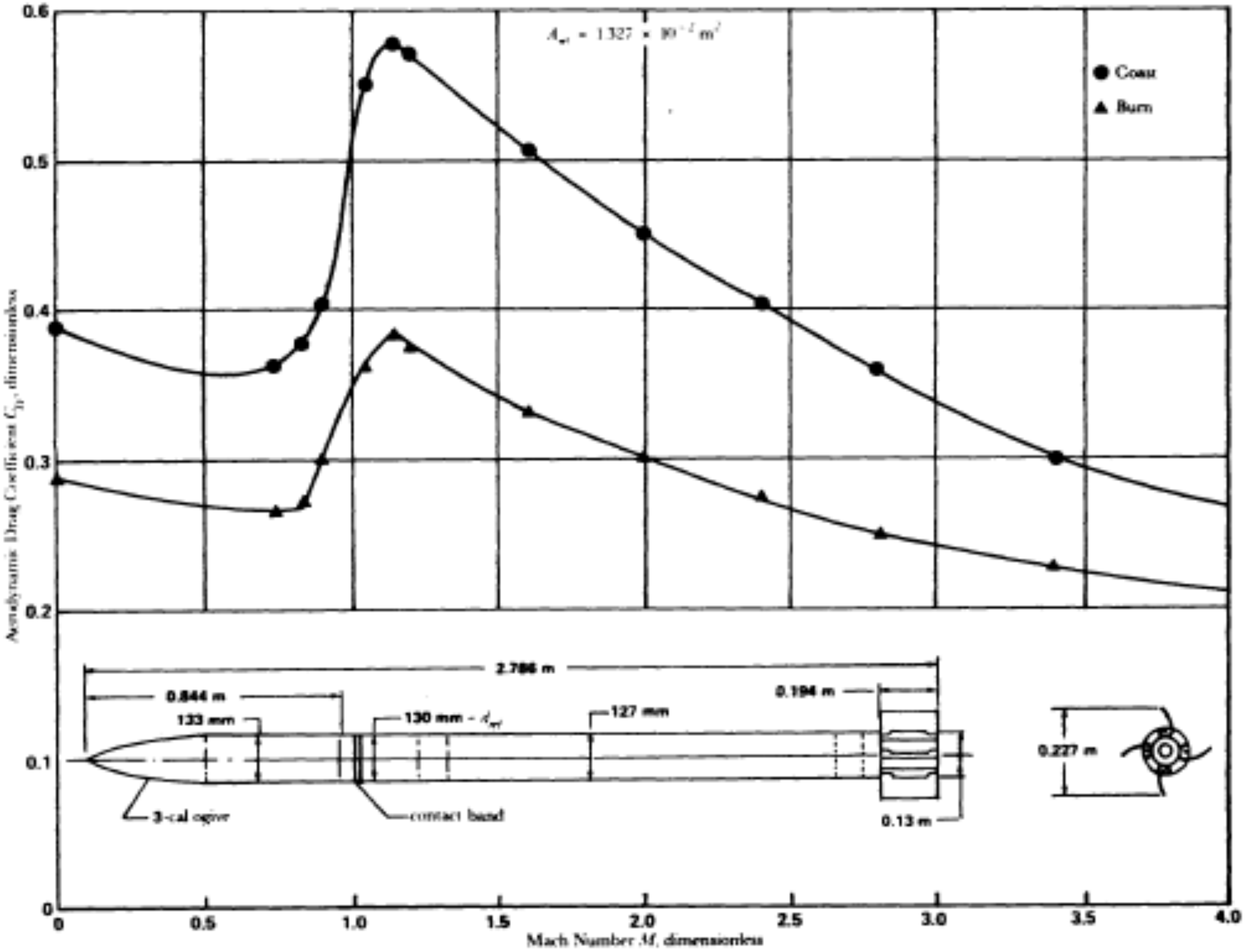


Figure 3-4. Drag Coefficient vs Mach Number-HO-mm Rocket

Drag Coefficient is Configuration Dependent (2)

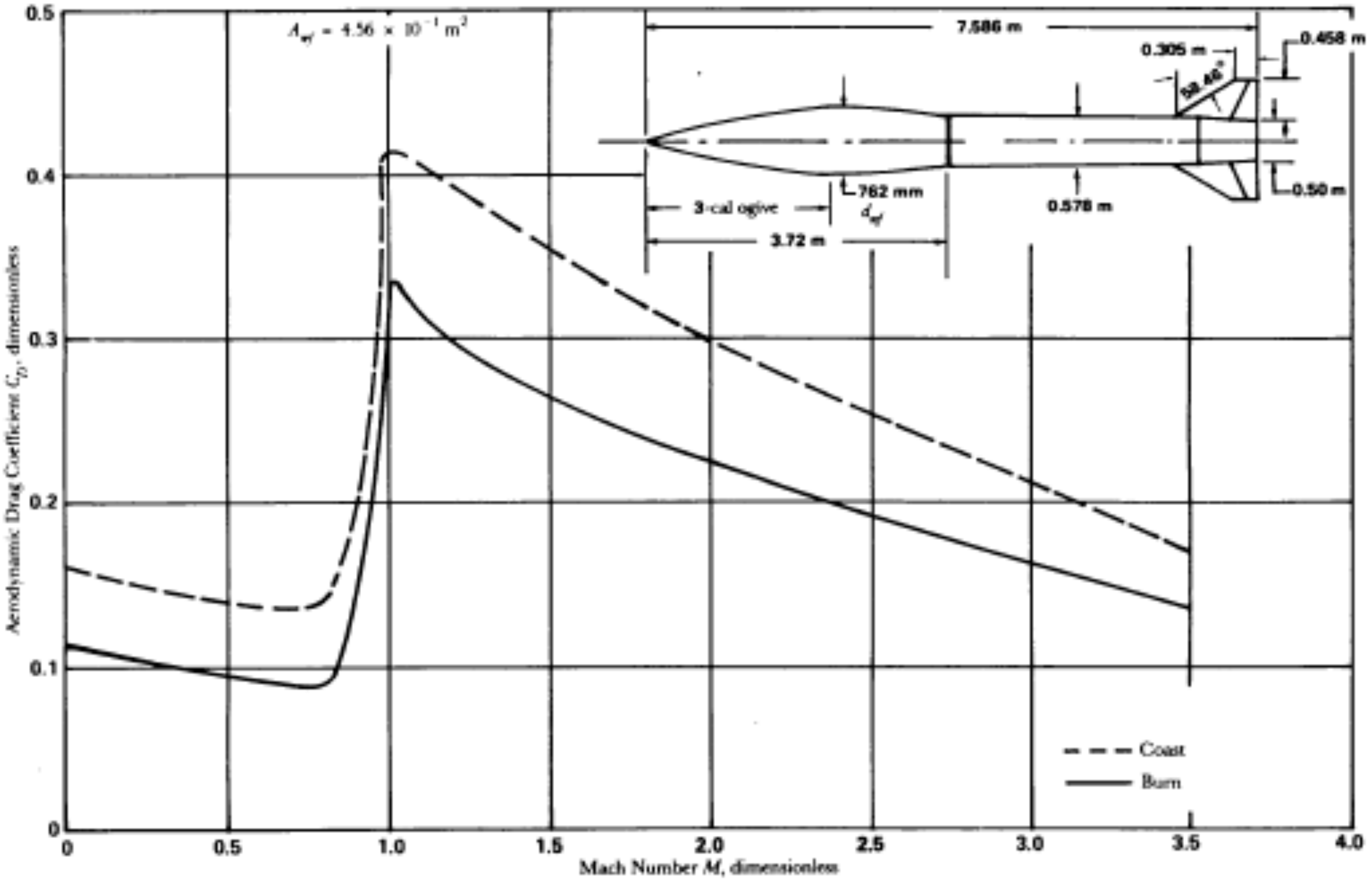


Figure 3-5. Drag Coefficient vs Mach Number—762-mm Rocket

Drag Coefficient is Configuration Dependent (3)

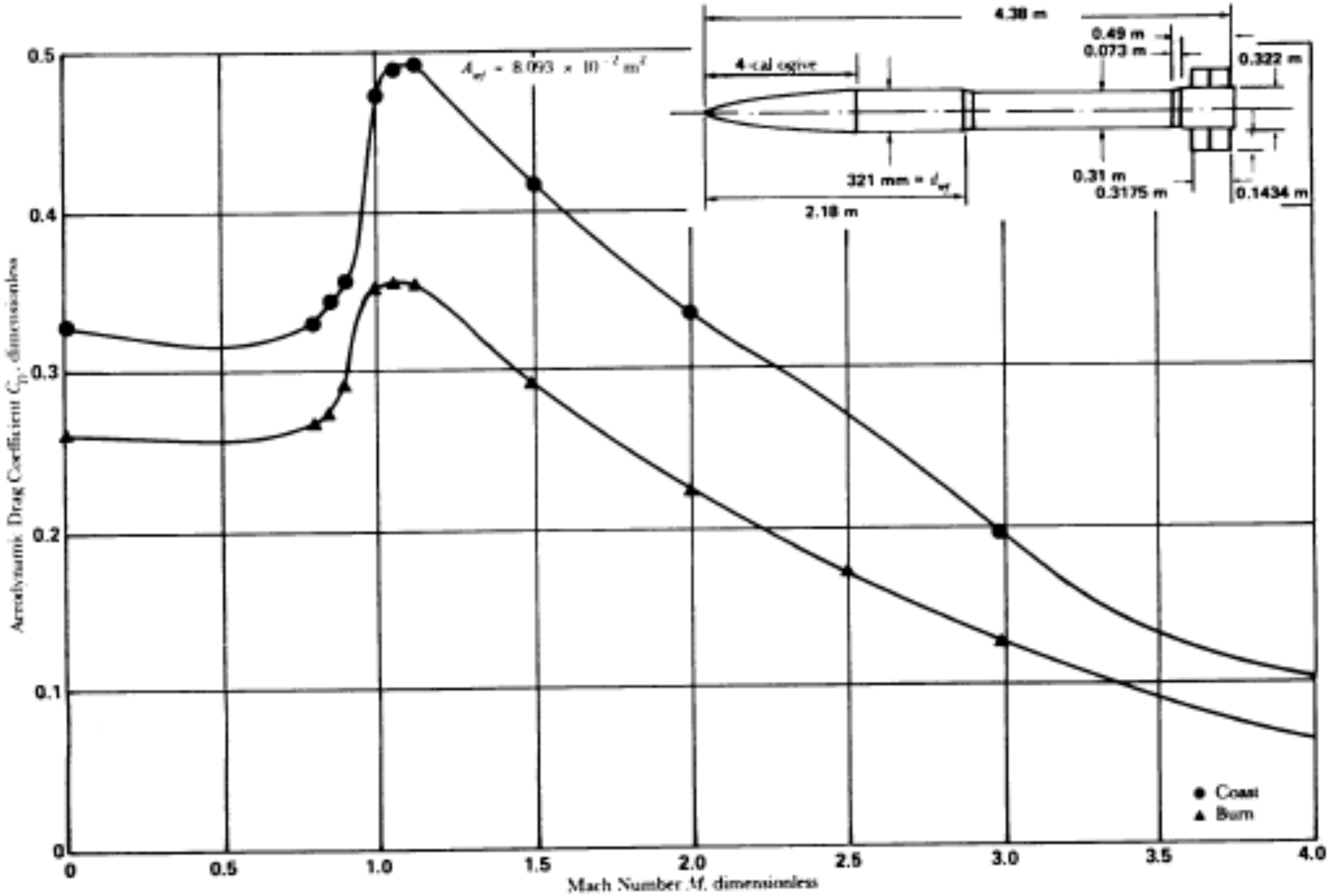


Figure 3-6. Drag Coefficient vs Mach Numbr-321-mm Rocket

A Recipe for Energy Management

Conservation of Energy:

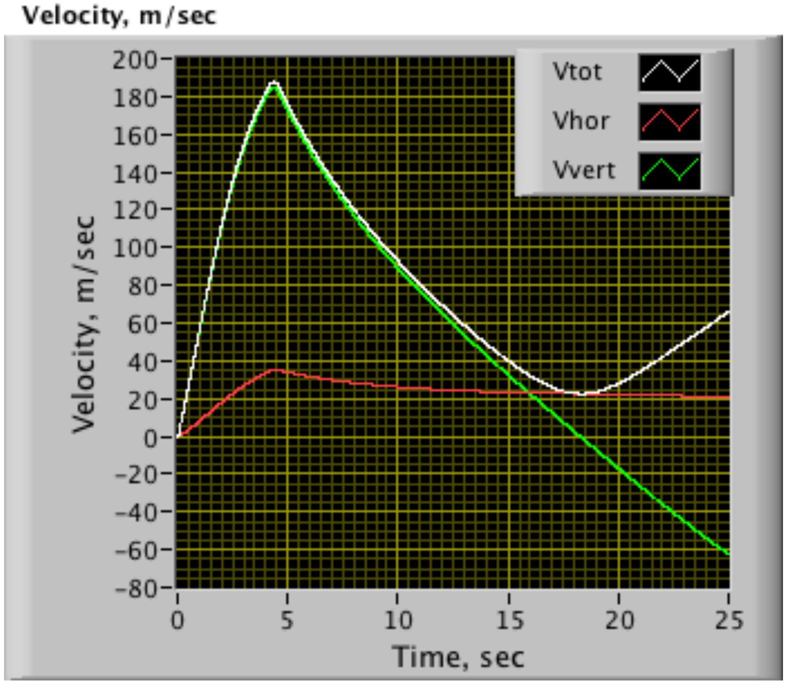
Potential + Kinetic Energy = Constant − Dissipated Energy

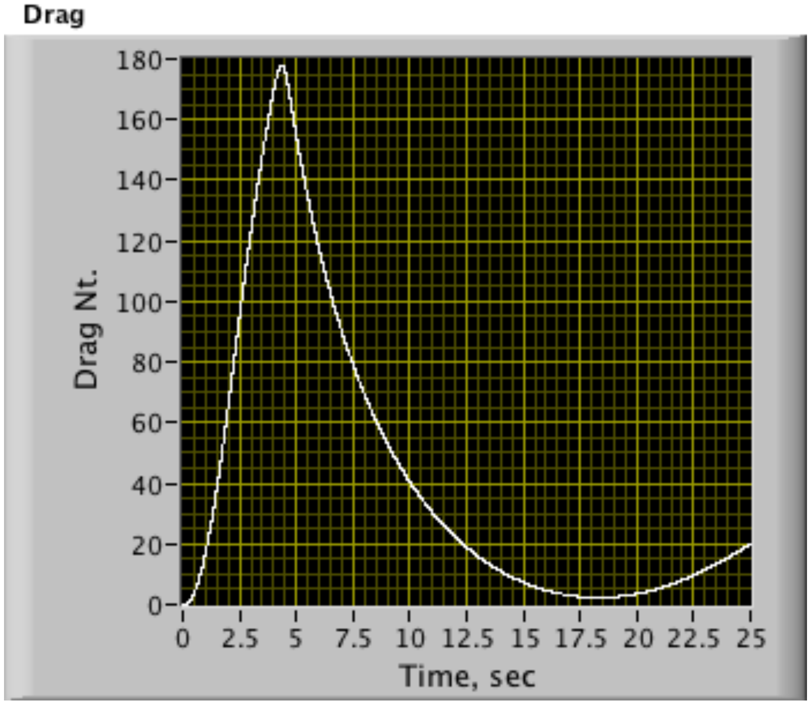
$$g \cdot h_{apogee} + \frac{V^2_{apogee}}{2} = h_{burnout} + \frac{V^2_{burnout}}{2} - \int_{t_{burnout}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

$$\rightarrow h_{apogee} = h_{burnout} + \left(\frac{V^2_{burnout}}{2 \cdot g} - \frac{V^2_{apogee}}{2 \cdot g} \right) - \frac{1}{g} \int_{t_{burnout}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

- **Velocity and drag are Very high just after motor burnout .. But diminish near apogee**

- **Specific Energy of Rocket becomes "more constant" with time**

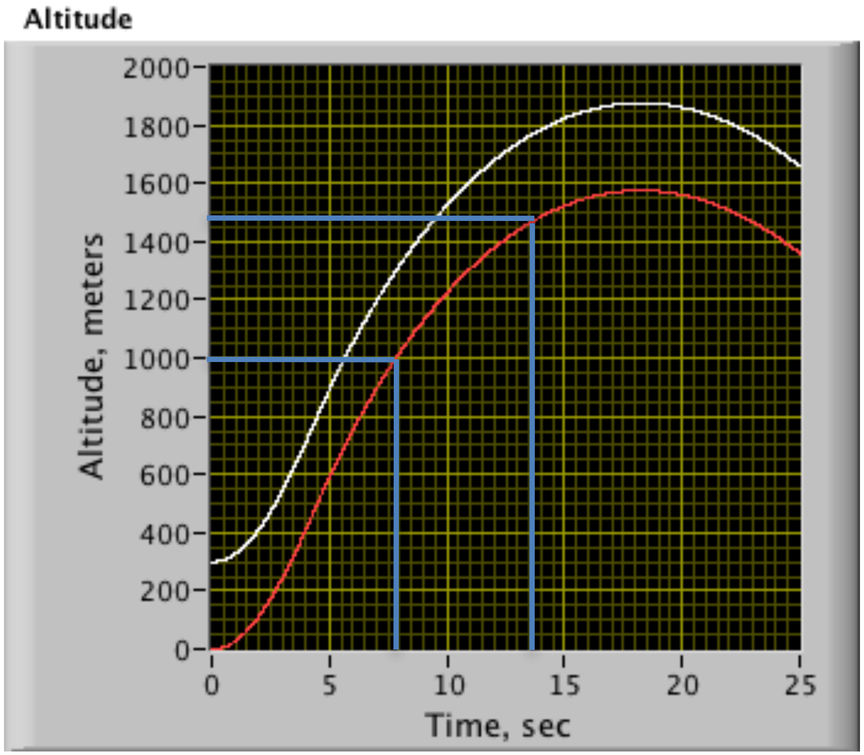


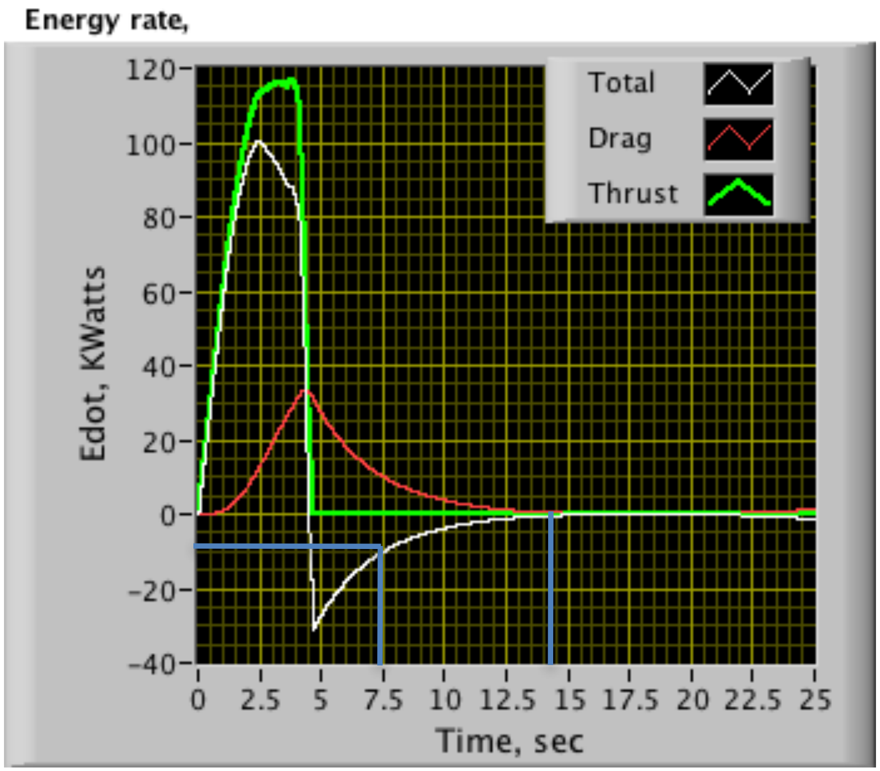


.. As drag Diminishes Near apogee

A Recipe for Energy Management (2)

- **Specific Energy of Rocket….**





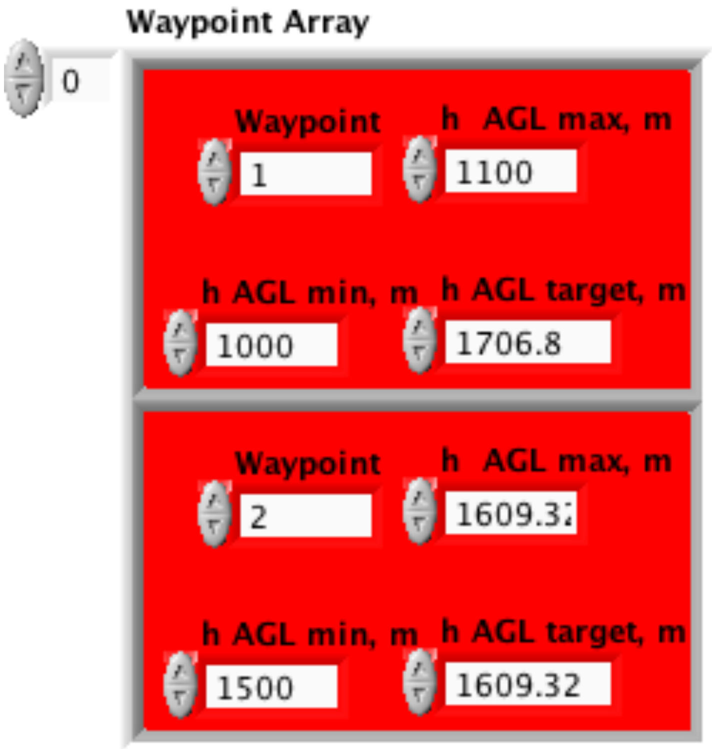
- At motor burn out Drag Energy Dissipation rate is ~3.5 times higher than at 1000 m AGL

- At 1500 m AGL Drag Energy Dissipation is essentially zero ..
  Estimated energy level~ constant

A Recipe for Energy Management (3)

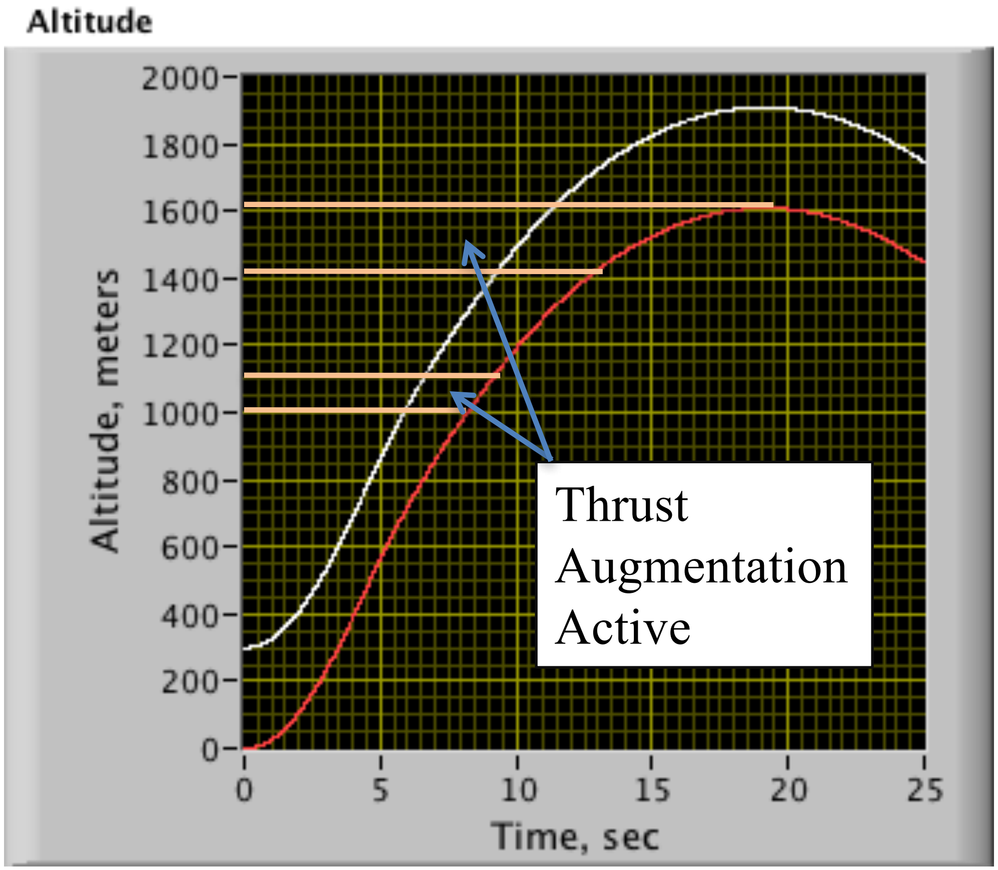
- **Potential Altitude as an Estimator of "Achievable Altitude" Becomes Increasingly More accurate as Apogee is Approached**

- **Use Augmentation Thrust to "Manage Energy" at waypoints of Increasing Altitude Along Probably trajectory**


Waypoint Array
0

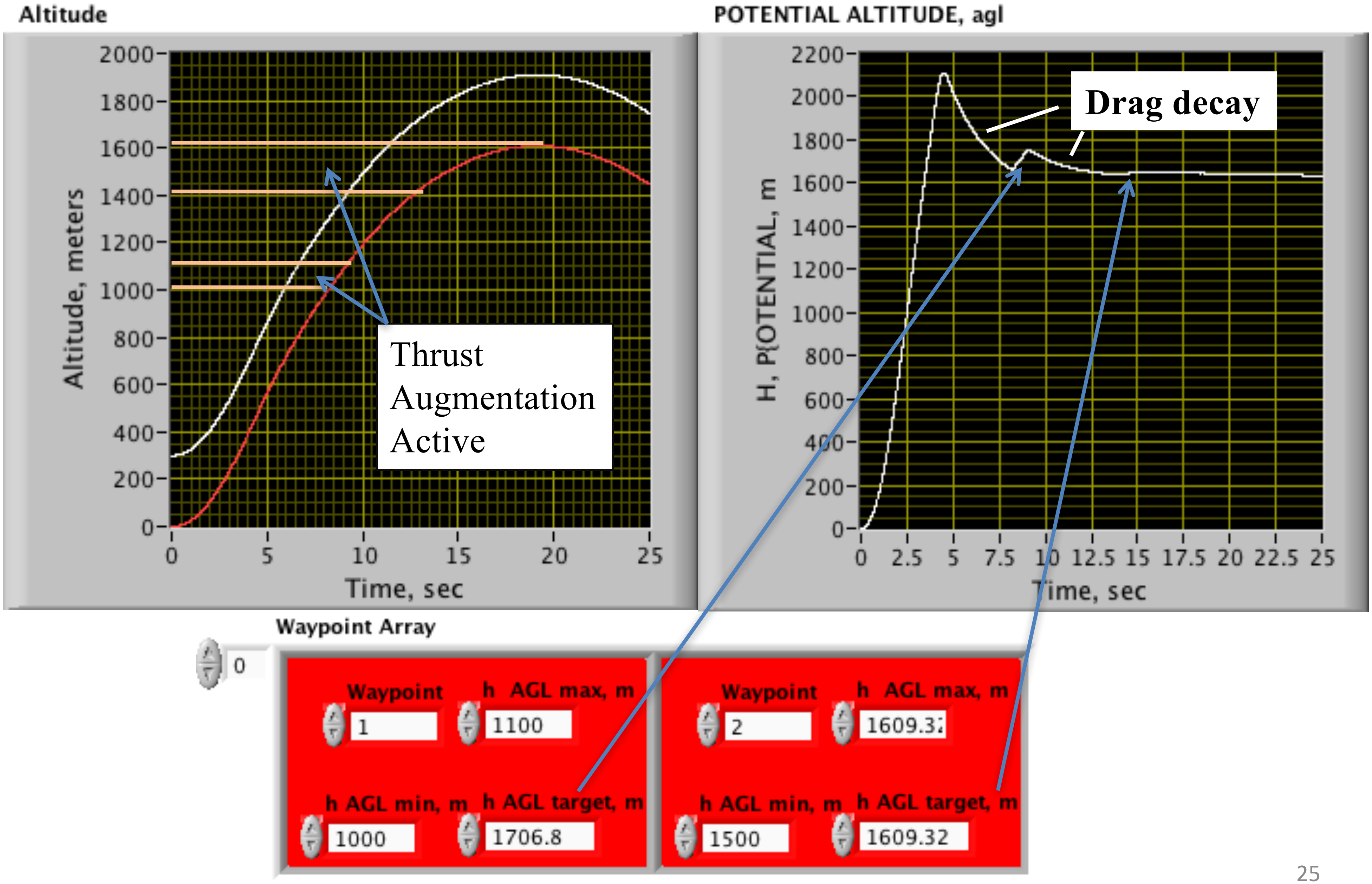
Waypoint: 1
h AGL max, m: 1100
h AGL min, m: 1000
h AGL target, m: 1706.8

Waypoint: 2
h AGL max, m: 1609.3
h AGL min, m: 1500
h AGL target, m: 1609.32



Thrust Augmentation Active

A Recipe for Energy Management (4)



Thrust Augmentation Active

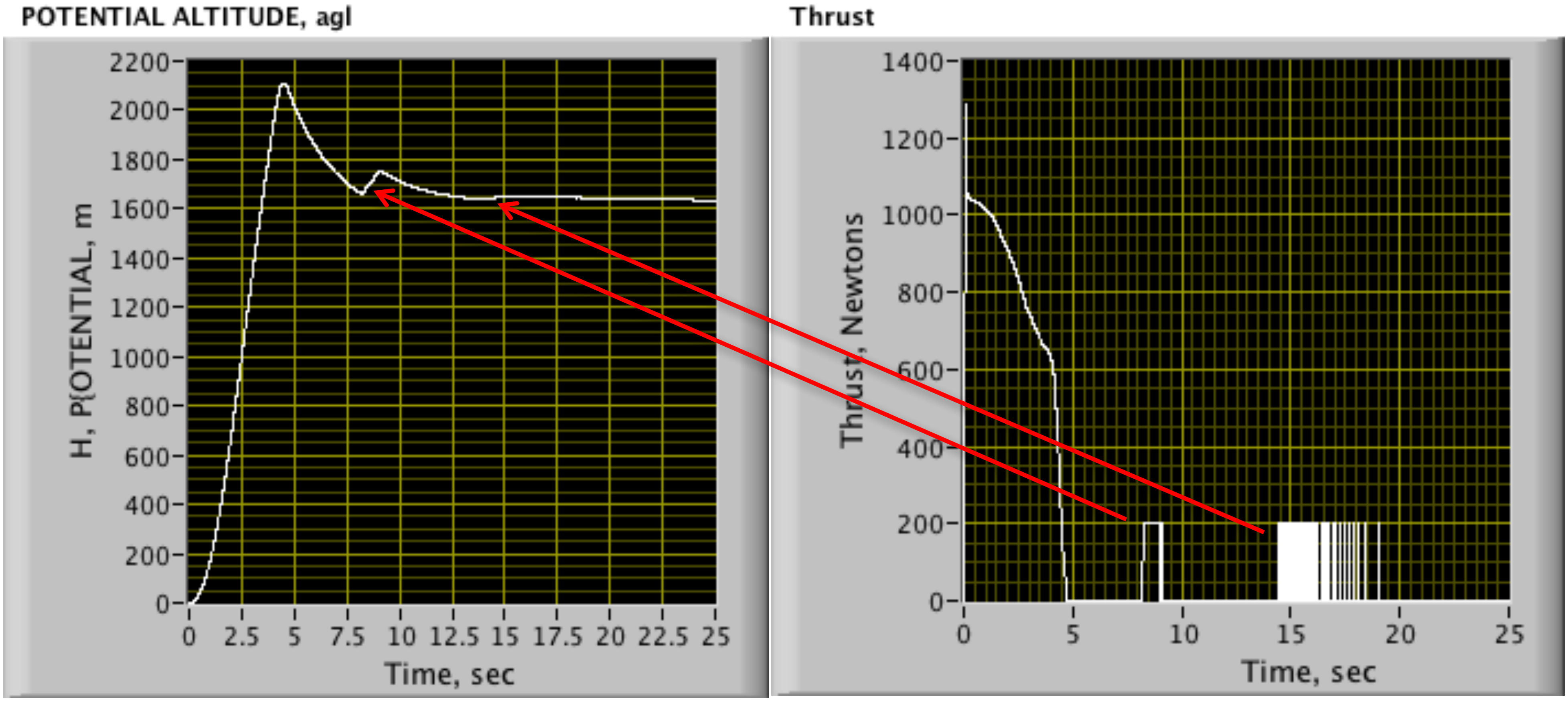
Drag decay

0

Waypoint: 1
h AGL max, m: 1100
h AGL min, m: 1000
h AGL target, m: 1706.8

Waypoint: 2
h AGL max, m: 1609.3
h AGL min, m: 1500
h AGL target, m: 1609.32

A Recipe for Energy Management (4)



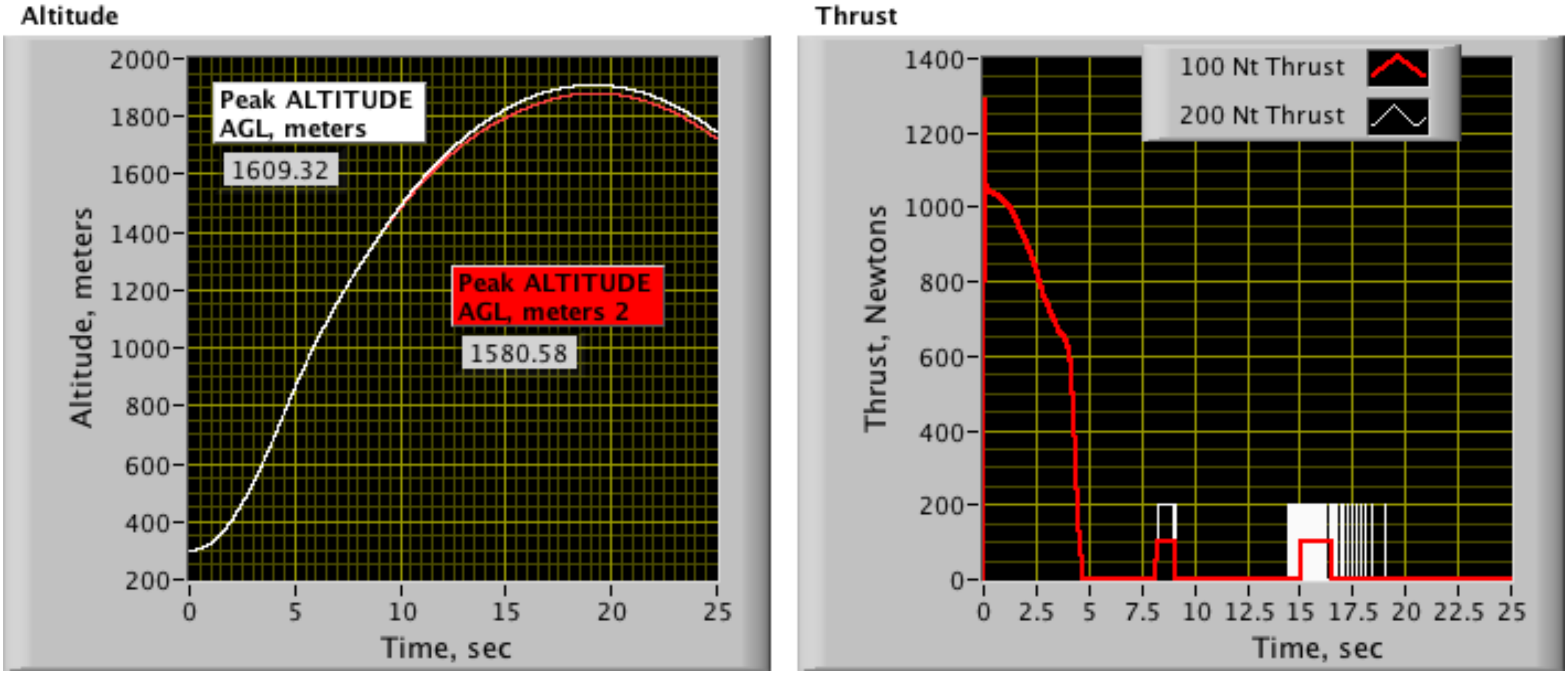
- First (mostly constant) Augmentation Impulse Boosts Energy to "achievable level"

Once we have calculated energy state (using IMU) … *1706.8 m = 5600 ft*

- Second Augmentation Impulse Boost and Maintains Energy level at Desired (Target Level)

Eneregy Level using Pulsed-modulation … *1609.32 m = 5280 ft*

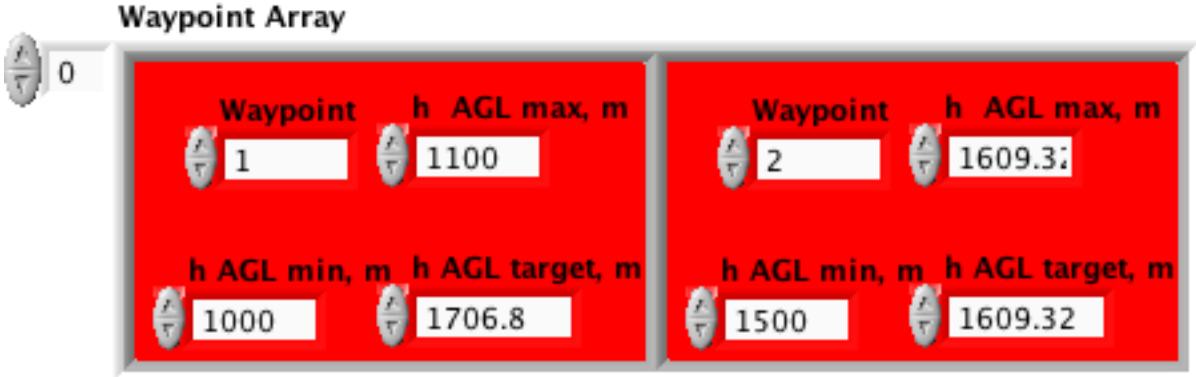
A Recipe for Energy Management (5)



Peak ALTITUDE AGL, meters 1609.32

Peak ALTITUDE AGL, meters 2 1580.58

- Lower Augmentation Thrust levels allow for more precise modulation, but are less efficient and must "burn longer"



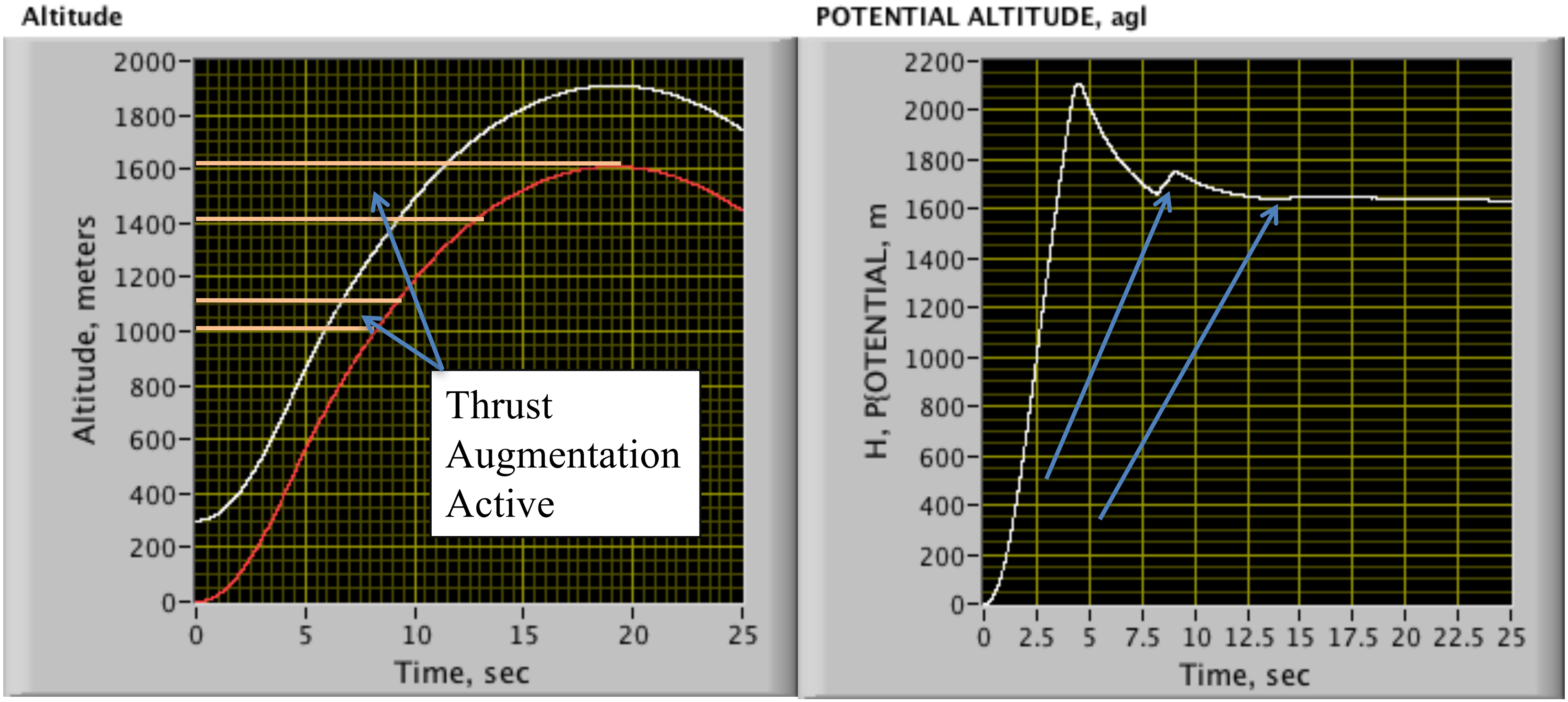
Waypoint Array 0

Waypoint 1
h AGL max, m 1100
h AGL min, m 1000
h AGL target, m 1706.8

Waypoint 2
h AGL max, m 1609.3
h AGL min, m 1500
h AGL target, m 1609.32

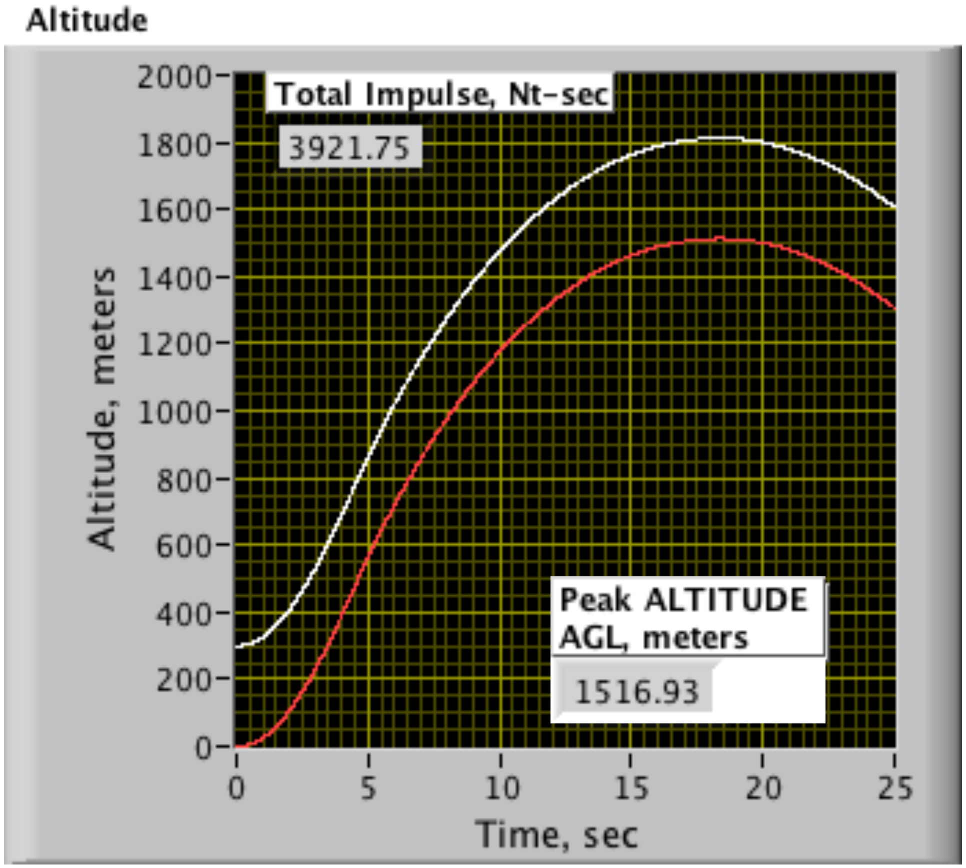
MAE 6530, Propulsion Systems II

A Recipe for Energy Management (6)



- Early Energy Management is More Effective, But less Precise

A Recipe for Energy Management (7)

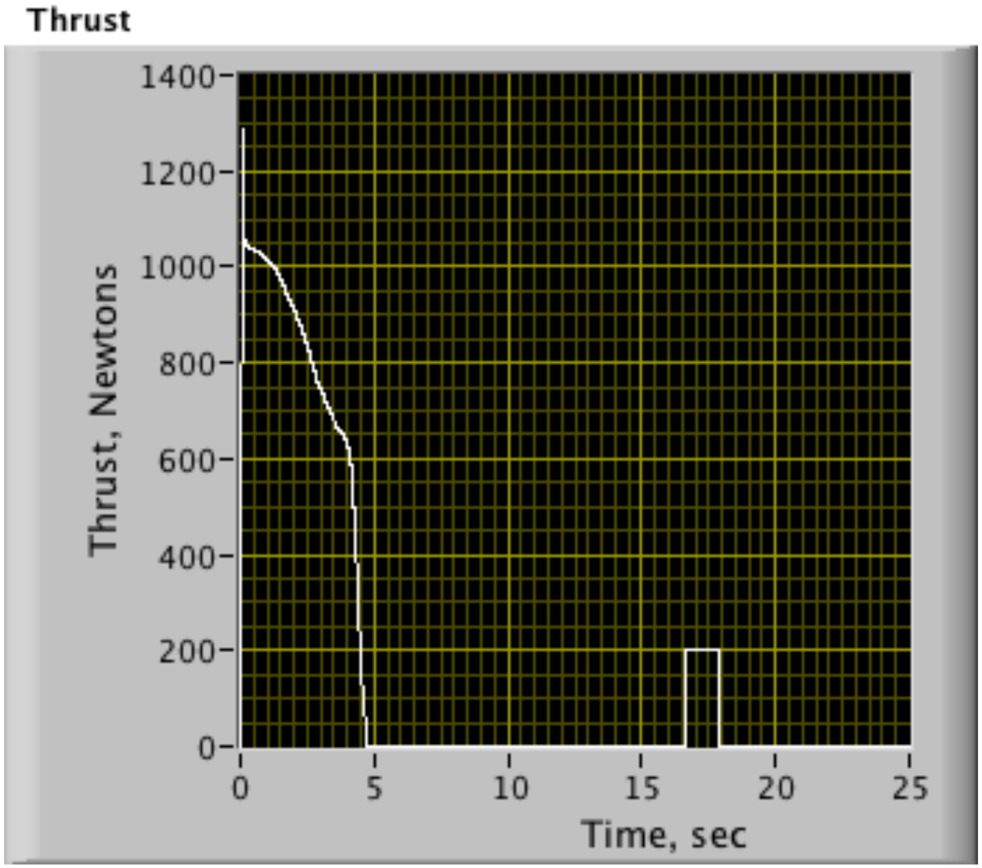


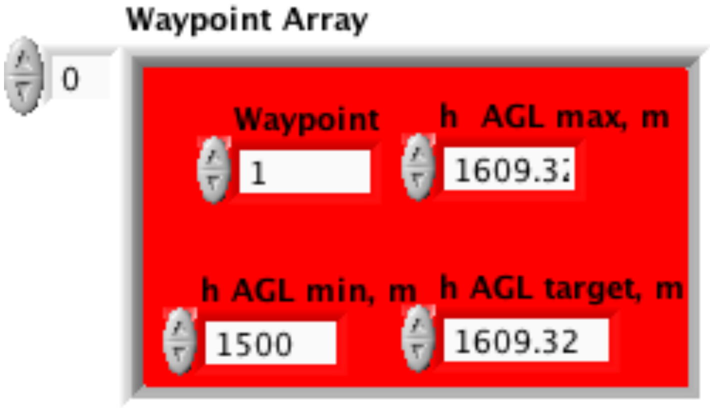
Total Impulse, Nt-sec

3921.75

Peak ALTITUDE AGL, meters

1516.93





Waypoint: 1

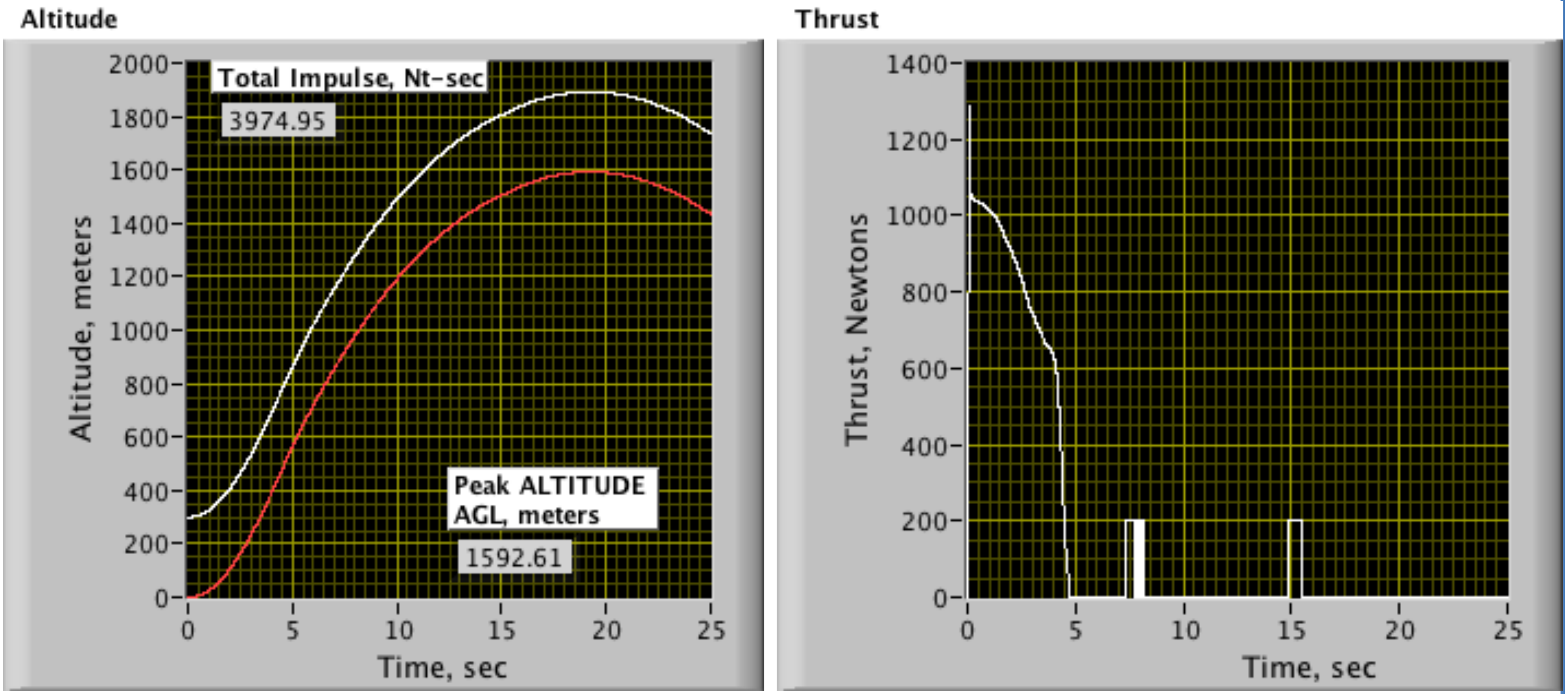
h AGL max, m: 1609.3

h AGL min, m: 1500

h AGL target, m: 1609.32

- Single Waypoint Energy Management Near Apogee
- *Insufficient Propellant to hit target*

A Recipe for Energy Management (8)



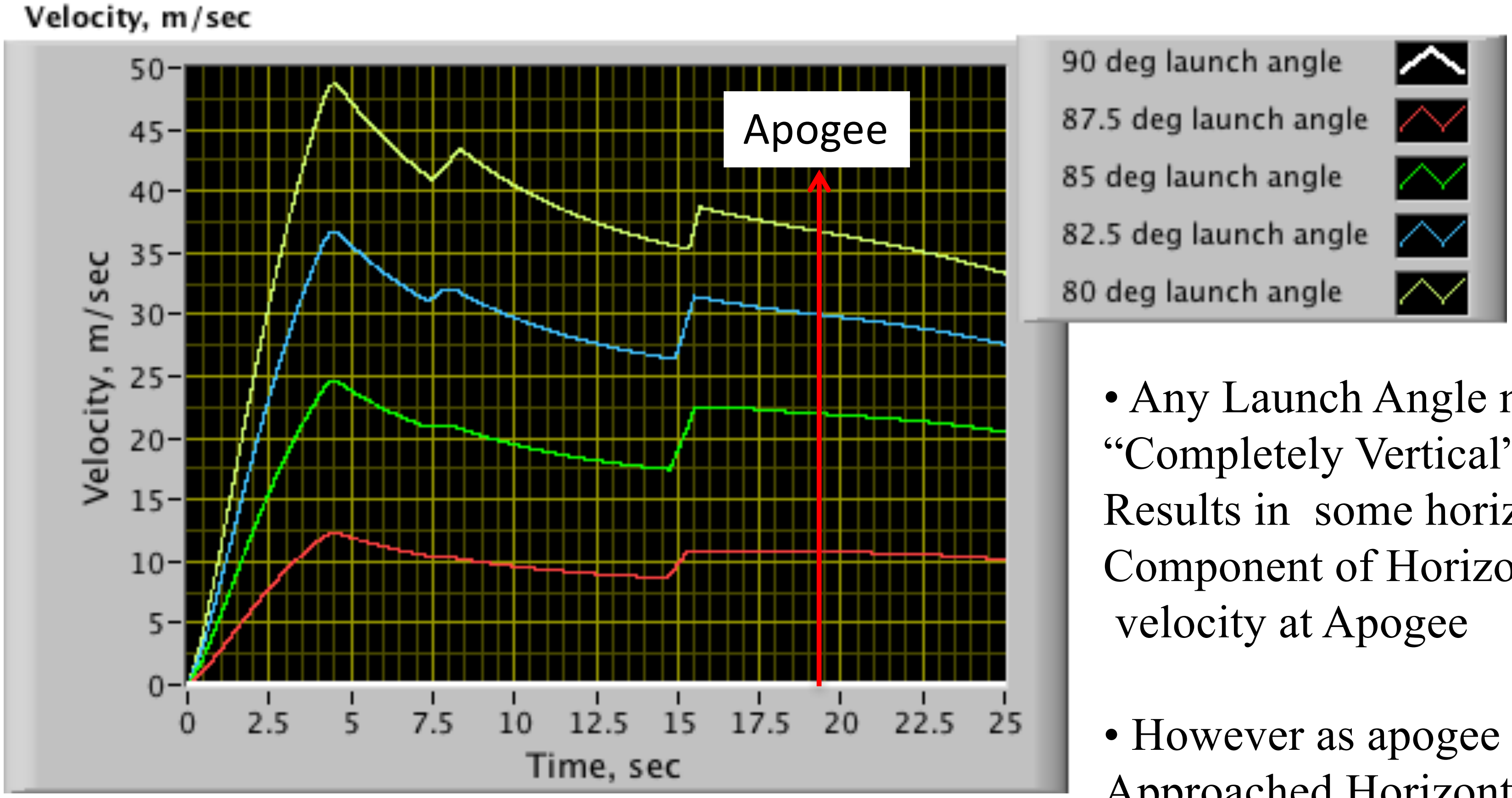
Total Impulse, Nt-sec
3974.95

Peak ALTITUDE AGL, meters
1592.61

- Earlier Implementation of First Waypoint
- Insufficient Accuracy to Hit target

- ***There will definitely be a Design "Sweet spot" .. here***

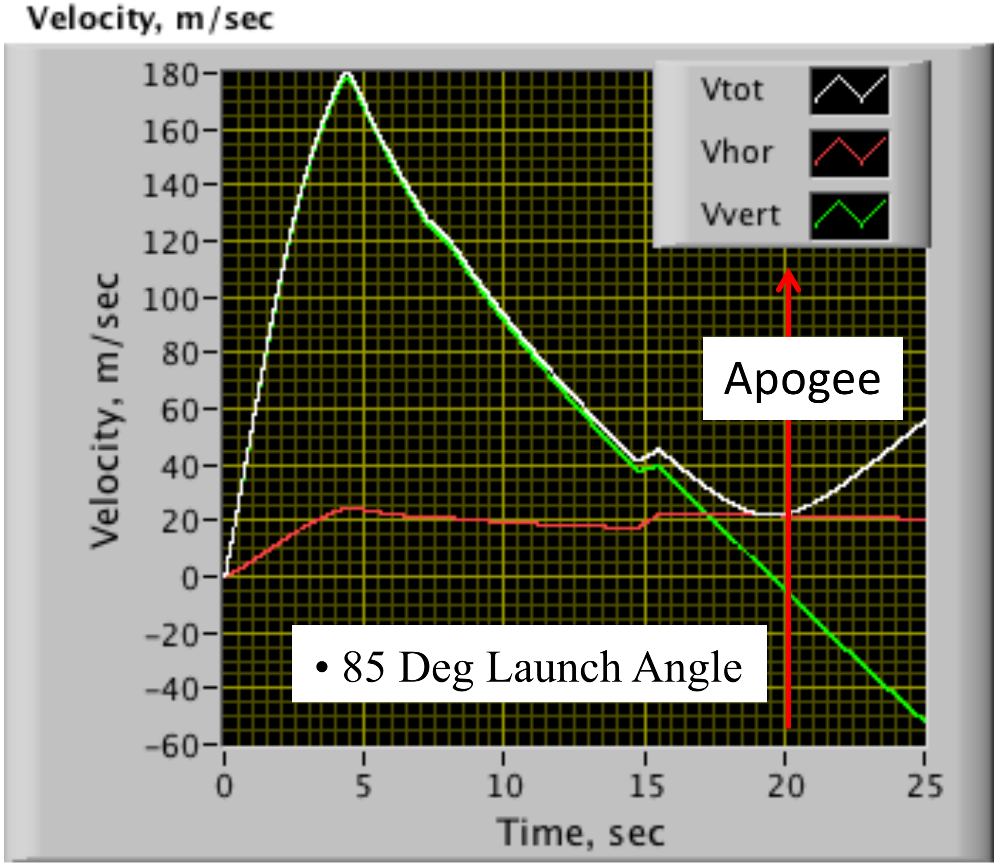
Adjusting Potential Altitude Estimate for Effects of Horizontal Velocity



- Any Launch Angle not “Completely Vertical” Results in some horizontal Component of Horizontal velocity at Apogee

- However as apogee is Approached Horizontal Velocity Component becomes ~ constant

Adjusting Potential Altitude Estimate for Effects of Horizontal Velocity (2)



- Compared to total velocity of Vehicle Horizontal component ~ constant Very soon after motor burn out

- $V_{hor_{waypoint}} \sim V_{apogee}$

$$V_{hor\ waypoint} = V_{waypoint} \cdot \cos(\gamma) \approx V_{apogee}$$

$$\rightarrow \left| \begin{array}{l} \gamma = flight\ path\ angle \\ = \tan^{-1} \dfrac{\dot{h}}{V_{hor}} \end{array} \right.$$

Adjusting Potential Altitude Estimate for Effects of Horizontal Velocity (3)

$$h_{apogee} = h_{waypoint} + \left(\frac{V^2{}_{waypoint}}{2 \cdot g} - \frac{V^2{}_{apogee}}{2 \cdot g} \right) - \frac{1}{g} \int_{t_{waypoint}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt =$$

$$= h_{waypoint} + \left(\left(\frac{V^2{}_{waypoint}}{2 \cdot g} \right)_{vertical} + \left(\frac{V^2{}_{waypoint}}{2 \cdot g} \right)_{horizontal} - \frac{V^2{}_{apogee}}{2 \cdot g} \right) - \frac{1}{g} \int_{t_{waypoint}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

$$\boxed{\rightarrow \left(\frac{V^2{}_{waypoint}}{2 \cdot g} \right)_{horizontal} \approx \frac{V^2{}_{apogee}}{2 \cdot g}}$$

$$\rightarrow h_{apogee} = h_{waypoint} + \left(\frac{V^2{}_{waypoint}}{2 \cdot g} - \frac{V^2{}_{apogee}}{2 \cdot g} \right) - \frac{1}{g} \int_{t_{waypo\,int}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt =$$

$$\rightarrow h_{apogee} = h_{waypoint} + \left(\frac{V^2{}_{waypoint}}{2 \cdot g} \right)_{vertical} - \frac{1}{g} \int_{t_{waypo\,int}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

“0” Near Apogee

Adjusting Potential Altitude Estimate for Effects of Horizontal Velocity (4)

$$\rightarrow h_{apogee} = h_{waypoint} + \left(\frac{V^2_{waypoint}}{2 \cdot g} \right)_{vertical} - \frac{1}{g} \int_{t_{waypoint}}^{t_{apogee}} \frac{\rho \cdot V^3}{\beta} dt$$

$$\rightarrow \boxed{\widehat{h}_{potential} = h_{waypoint} + \frac{V^2_{waypoint} \cdot \sin^2(\gamma)}{2 \cdot g}}$$

- Non-optimal strategy
.. But it works pretty well

- Some potential that
non-linear “bang-bang” or
Dead-band controller may
Be more propellant efficient

- But $\widehat{h}_{potential}$ is a critical
feedback parameter

continuously estimate ...

$$\left(\widehat{h}_{potential}\right)_t = h_{(t)} + \frac{V^2_{(t)} \cdot sin^2\left(\gamma_{(t)}\right)}{2 \cdot g_{(t)}}$$

at waypoint ...we have a very simple control strategy....

```
...if [ (h_min ≤ h ≤ h_max) && (h < ĥ_potential) ]
        "thrust on"
... else
        "thrust off"
```

MAE 6530, Propulsion Systems II

Accounting for Drag Losses In Potential Altitude

Ignoring drag …. At any point along the trajectory …

$$h_{potential} = h(t) + \frac{V(t) \cdot \sin(\gamma)}{2 \cdot g}$$

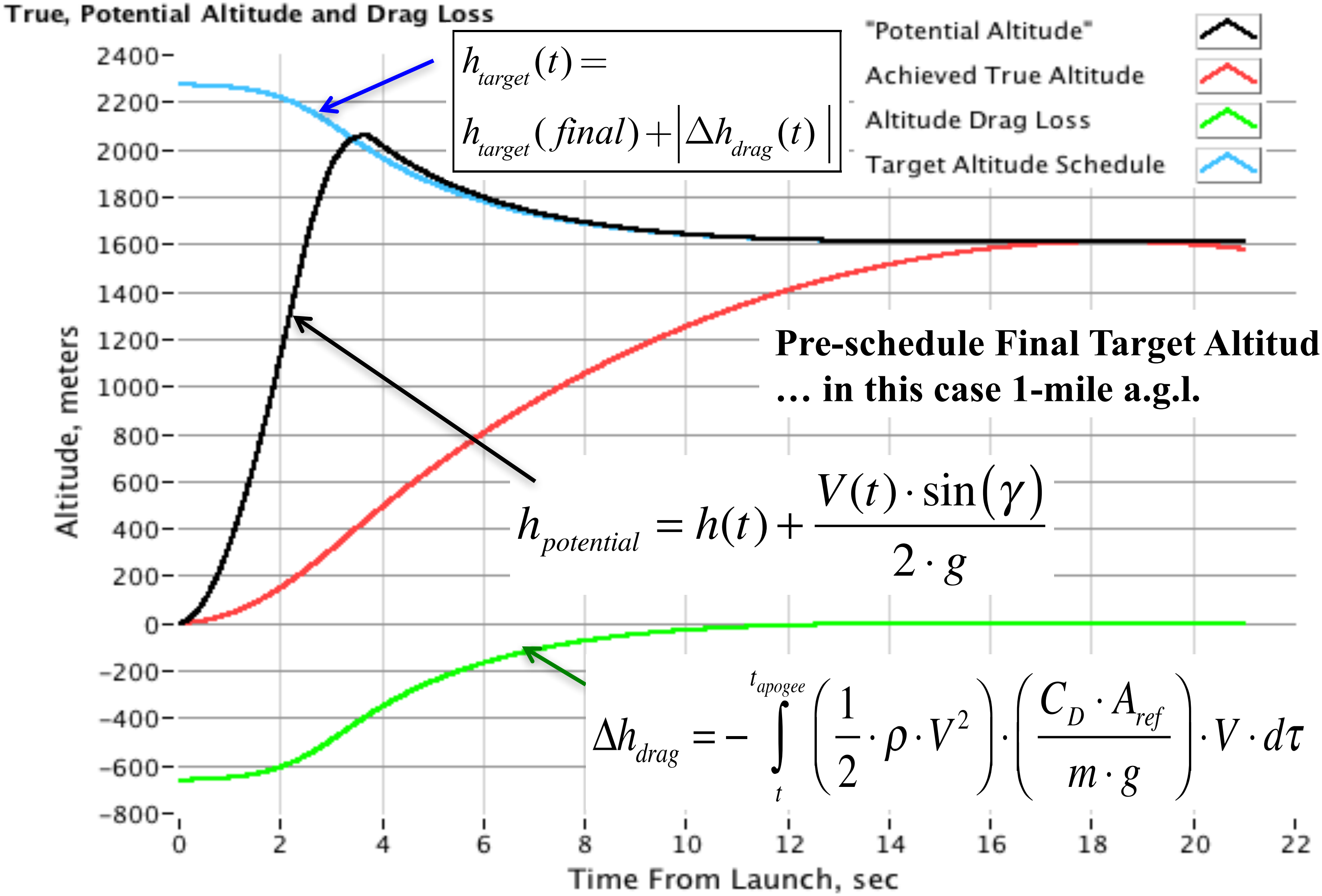
$$since \rightarrow V_{hor} = V(t) \cdot \cos(\gamma) \approx constant$$

But because of drag …. The true apogee will be…

Drag Loss Δh_{drag}

$$h_{apogee} = h_{potential} - \boxed{\int_{t}^{t_{apogee}} \left(\frac{1}{2} \cdot \rho \cdot V^2\right) \cdot \left(\frac{C_D \cdot A_{ref}}{m \cdot g}\right) \cdot V \cdot d\tau}$$

Accounting for Drag Losses In Potential Altitude 2



$$h_{target}(t) = h_{target}(final) + \left|\Delta h_{drag}(t)\right|$$

Pre-schedule Final Target Altitude … in this case 1-mile a.g.l.

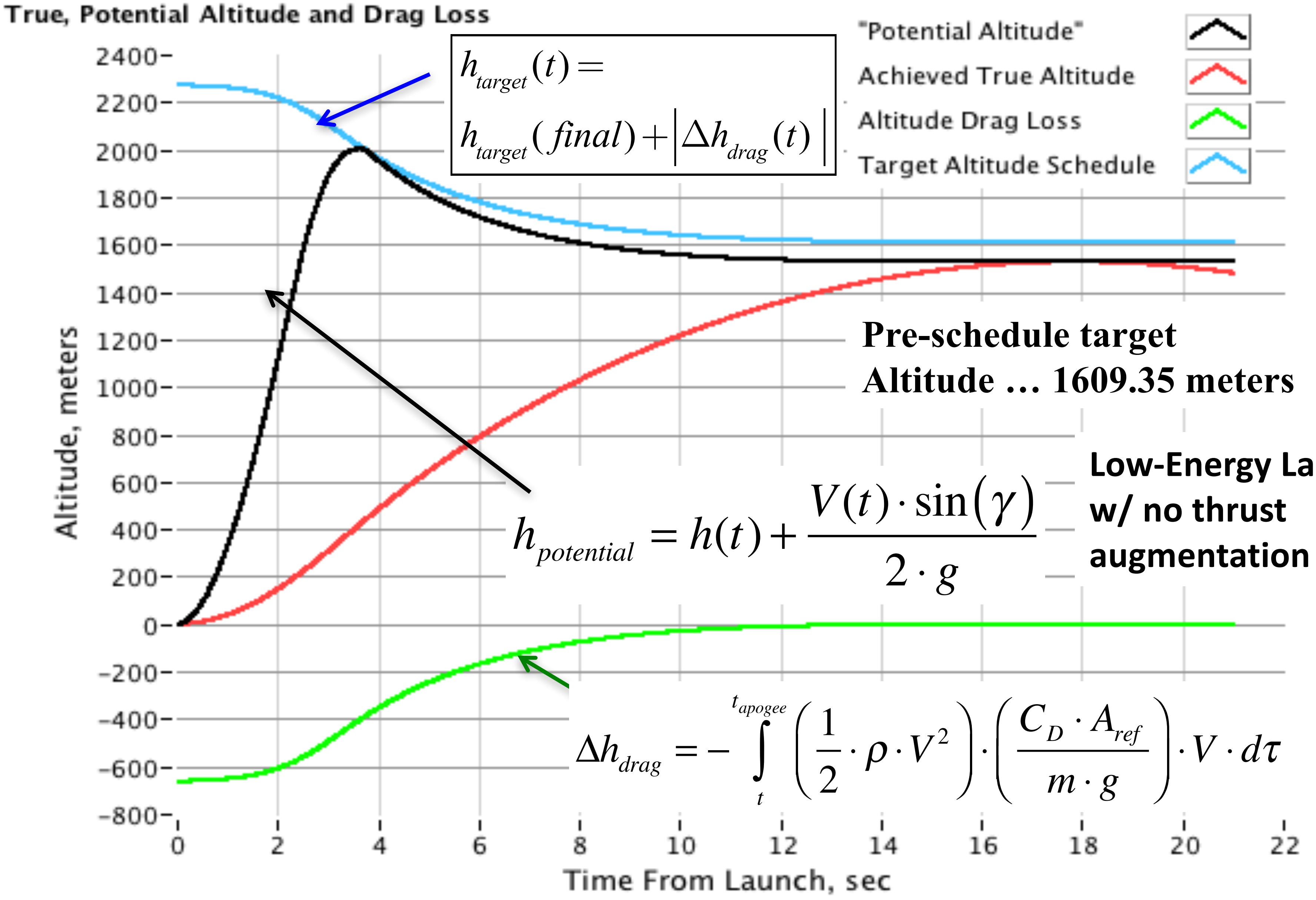
$$h_{potential} = h(t) + \frac{V(t) \cdot \sin(\gamma)}{2 \cdot g}$$

$$\Delta h_{drag} = -\int_{t}^{t_{apogee}} \left(\frac{1}{2} \cdot \rho \cdot V^2\right) \cdot \left(\frac{C_D \cdot A_{ref}}{m \cdot g}\right) \cdot V \cdot d\tau$$

UtahState UNIVERSITY

Mechanical & Aerospace Engineering

Accounting for Drag Losses In Potential Altitude 4



$$h_{target}(t) = h_{target}(final) + \left|\Delta h_{drag}(t)\right|$$

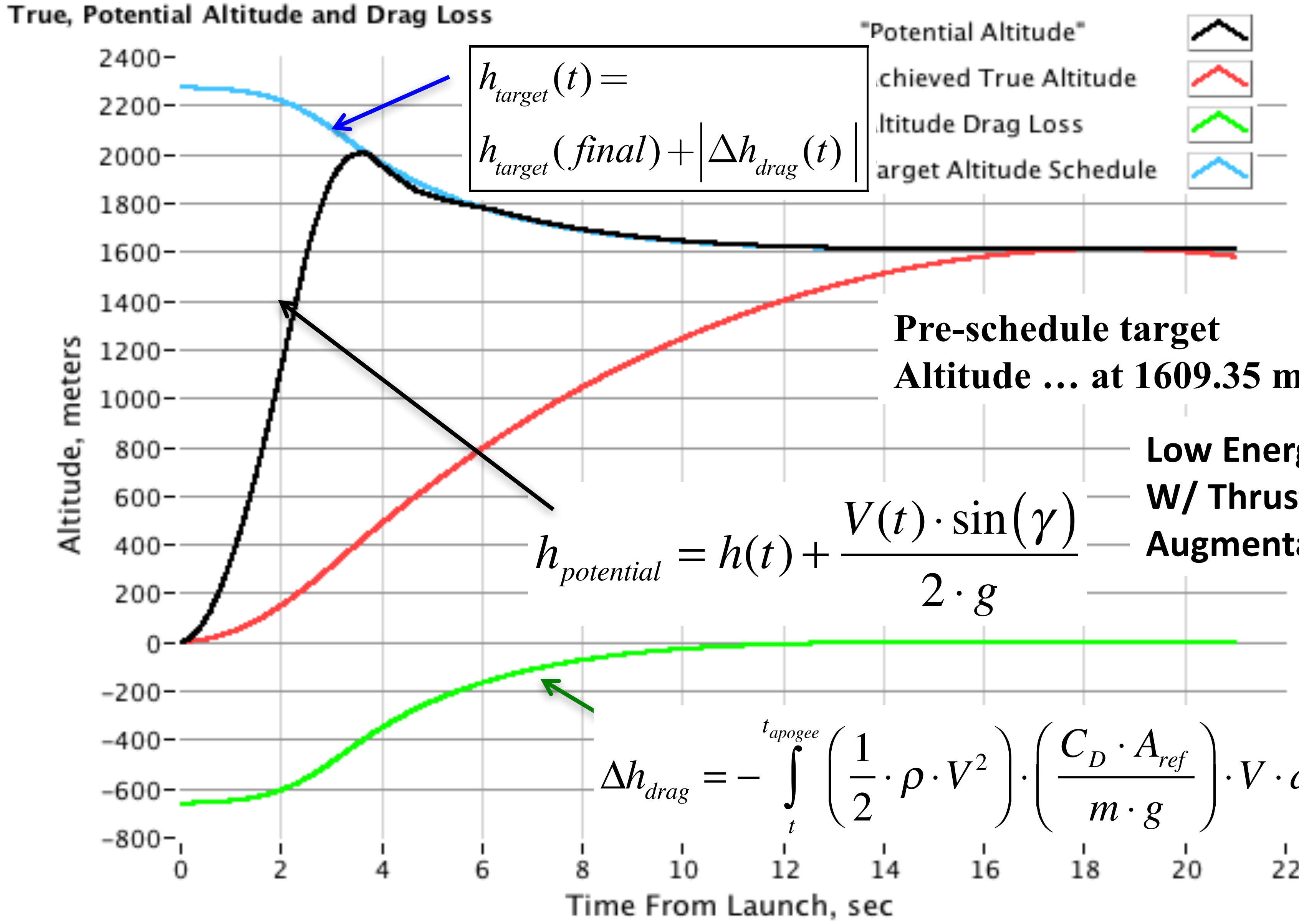
Pre-schedule target Altitude … 1609.35 meters

$$h_{potential} = h(t) + \frac{V(t)\cdot\sin(\gamma)}{2\cdot g}$$

Low-Energy Launch w/ no thrust augmentation

$$\Delta h_{drag} = -\int_{t}^{t_{apogee}} \left(\frac{1}{2}\cdot\rho\cdot V^2\right)\cdot\left(\frac{C_D\cdot A_{ref}}{m\cdot g}\right)\cdot V\cdot d\tau$$

Accounting for Drag Losses In Potential Altitude [5]



$$h_{target}(t) = h_{target}(final) + \left|\Delta h_{drag}(t)\right|$$

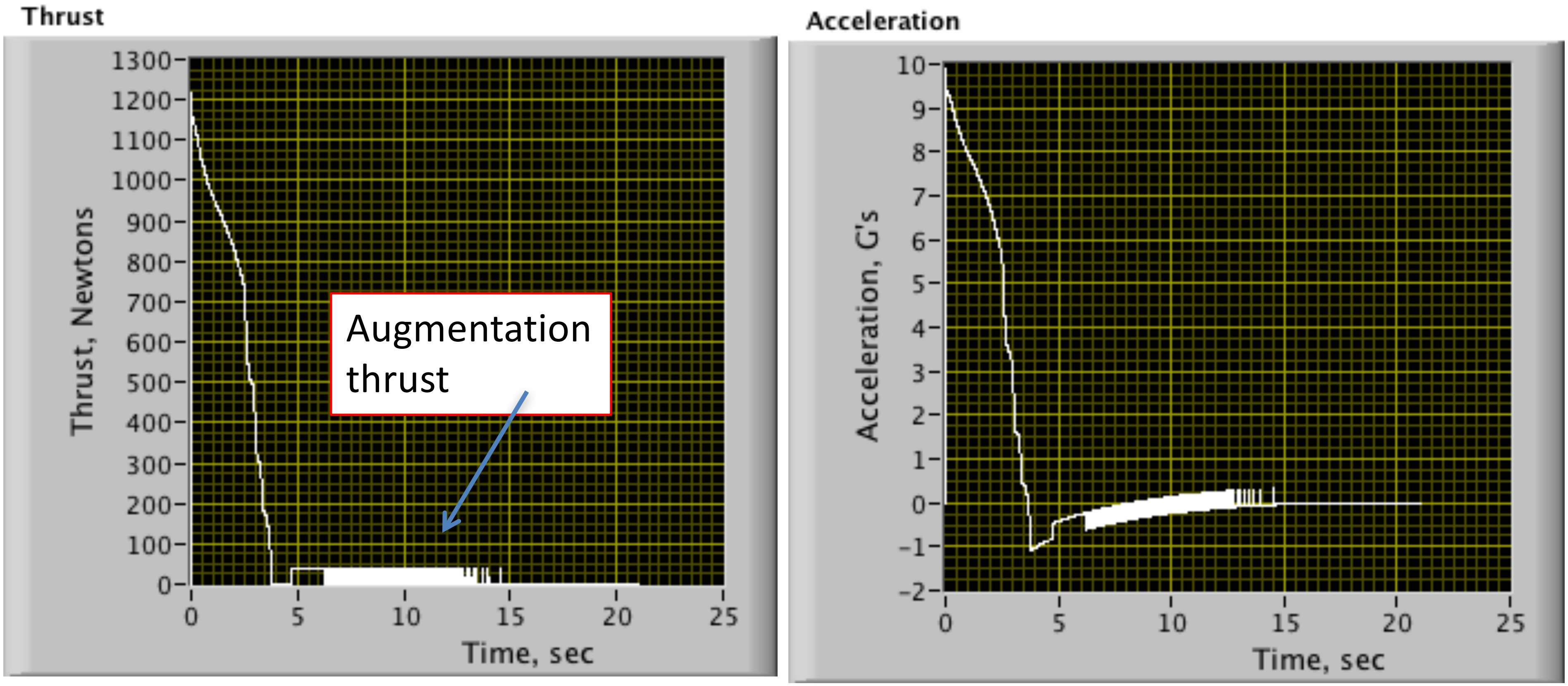
Pre-schedule target Altitude … at 1609.35 m

Low Energy Launch W/ Thrust Augmentation

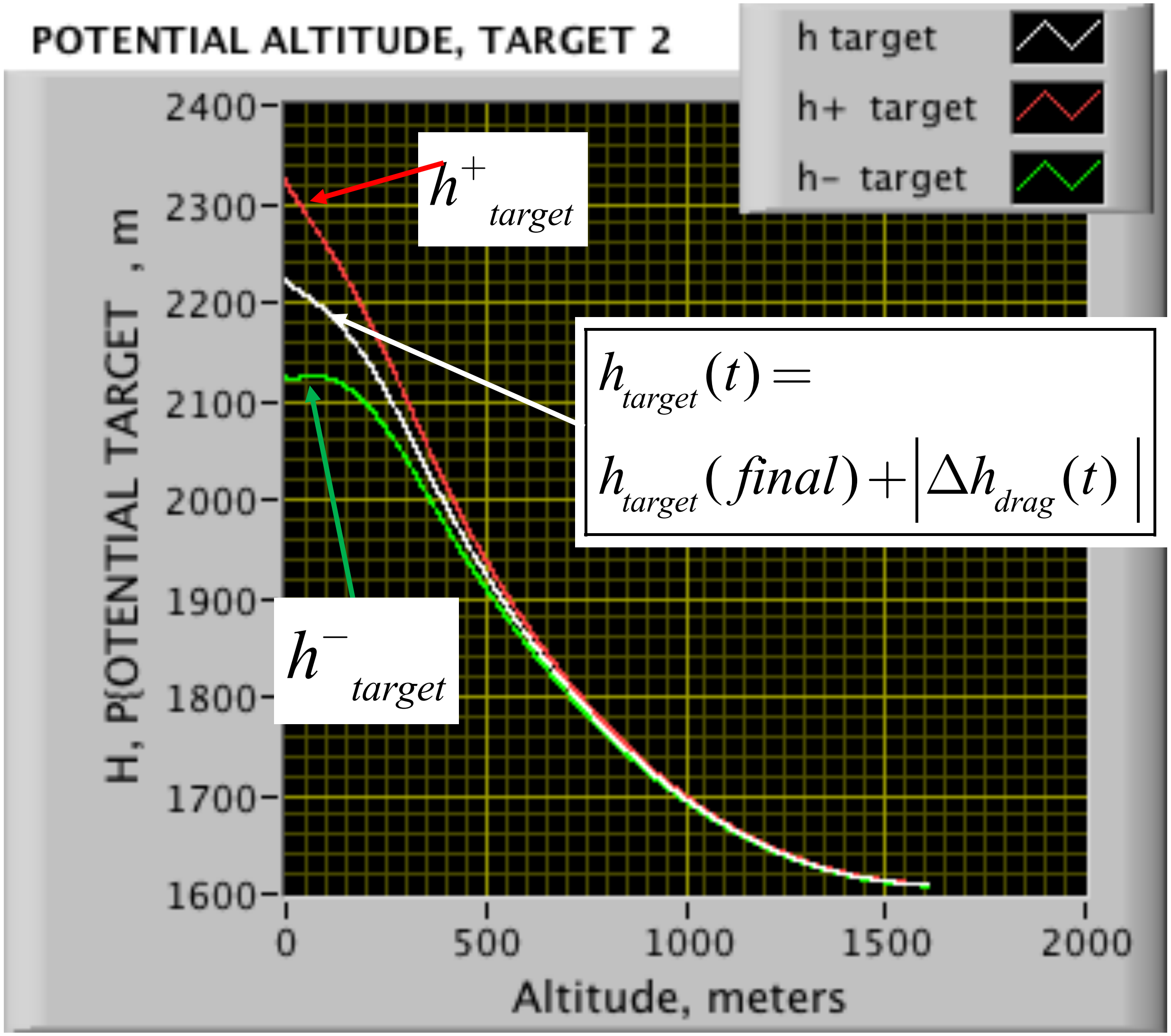
$$h_{potential} = h(t) + \frac{V(t)\cdot\sin(\gamma)}{2\cdot g}$$

$$\Delta h_{drag} = -\int_{t}^{t_{apogee}} \left(\frac{1}{2}\cdot\rho\cdot V^2\right)\cdot\left(\frac{C_D\cdot A_{ref}}{m\cdot g}\right)\cdot V\cdot d\tau$$

Accounting for Drag Losses In Potential Altitude 6



Energy Management w/ Target Envelope (1)



$$h_{target}(t) = h_{target}(final) + \left|\Delta h_{drag}(t)\right|$$

- Enveloping allows for final position error tolerance while **significantly** reducing required control activity

$$h^{+}_{target}(t) = h_{target}(t) + \delta h_{h}$$

$$h^{-}_{target}(t) = h_{target}(t) - \delta h_{h}$$

Energy Management w/ Target Envelope (2)

- Use first order decay as a function of altitude for envelope…

$$h^{\pm}_{target} = h_{\text{target}} \pm \delta h_t \rightarrow first\ order\ decay\ .... \delta\dot{h}_t + \frac{1}{\tau_h}\cdot\delta h_t = \frac{1}{\tau}\delta h_{\max}$$

$$multiply\ by \rightarrow e^{\frac{h}{\tau_h}}$$

$$\rightarrow e^{\frac{h}{\tau_h}}\cdot\left(\delta\dot{h}_t + \frac{1}{\tau_h}\cdot\delta h\right) = \frac{e^{\frac{h}{\tau_h}}}{\tau_h}\cdot\delta h_{\max} \rightarrow \frac{d}{dt}\left(e^{\frac{h}{\tau_h}}\cdot\delta h\right) = \frac{e^{\frac{h}{\tau_h}}}{\tau_h}\cdot\delta h_{\max}$$

- Integrate from launch to *h(t)* …

$$\int_{t(0)}^{t}\frac{d}{ds}\left(e^{\frac{h}{\tau_h}}\cdot\delta h\right)\cdot ds = \int_{0}^{h(t)}\frac{\delta h_{\max}}{\tau_h}\cdot e^{\frac{h}{\tau_h}}\,dh \rightarrow \boxed{e^{\frac{h(t)}{\tau_h}}\cdot\delta h(t) - e^{\frac{h(0)}{\tau_h}}\cdot\delta h(0) = \frac{\delta h_{\max}}{\tau_h}\cdot\left(e^{\frac{h(t)}{\tau_h}}\cdot\tau_h - e^{\frac{h(0)}{\tau_h}}\cdot\tau_h\right)}$$

Energy Management Target Envelope (3)

- Assuming $h(0) = 0$,

$$e^{\frac{h(t)}{\tau_h}} \cdot \delta h(t) - \delta h(0) = \delta h_{\max} \cdot \left(e^{\frac{h(t)}{\tau_h}} - 1 \right)$$

and Multiply thru by $e^{-\frac{h(t)}{\tau_h}}$

→

$$\delta h(t) = e^{-\frac{h(t)}{\tau_h}} \cdot \delta h(0) + \delta h_{\max} \cdot \left(1 - e^{-\frac{h(t)}{\tau_h}} \right)$$

Collected Target Altitude Envelope

$$\delta h_{(h)} = \delta h_{\max} \cdot \left(1 - e^{-\frac{h}{\tau_h}}\right) + e^{-\frac{h}{\tau_h}} \cdot \delta h_0$$

$$h^{+}_{target} = h_{\text{target}} + \left(\delta h_{\max} \cdot \left(1 - e^{-\frac{h}{\tau_h}}\right) + e^{-\frac{h}{\tau_h}} \cdot \delta h_0\right)$$

$$h^{+}_{target} = h_{\text{target}} - \left(\delta h_{\max} \cdot \left(1 - e^{-\frac{h}{\tau_h}}\right) + e^{-\frac{h}{\tau_h}} \cdot \delta h_0\right)$$

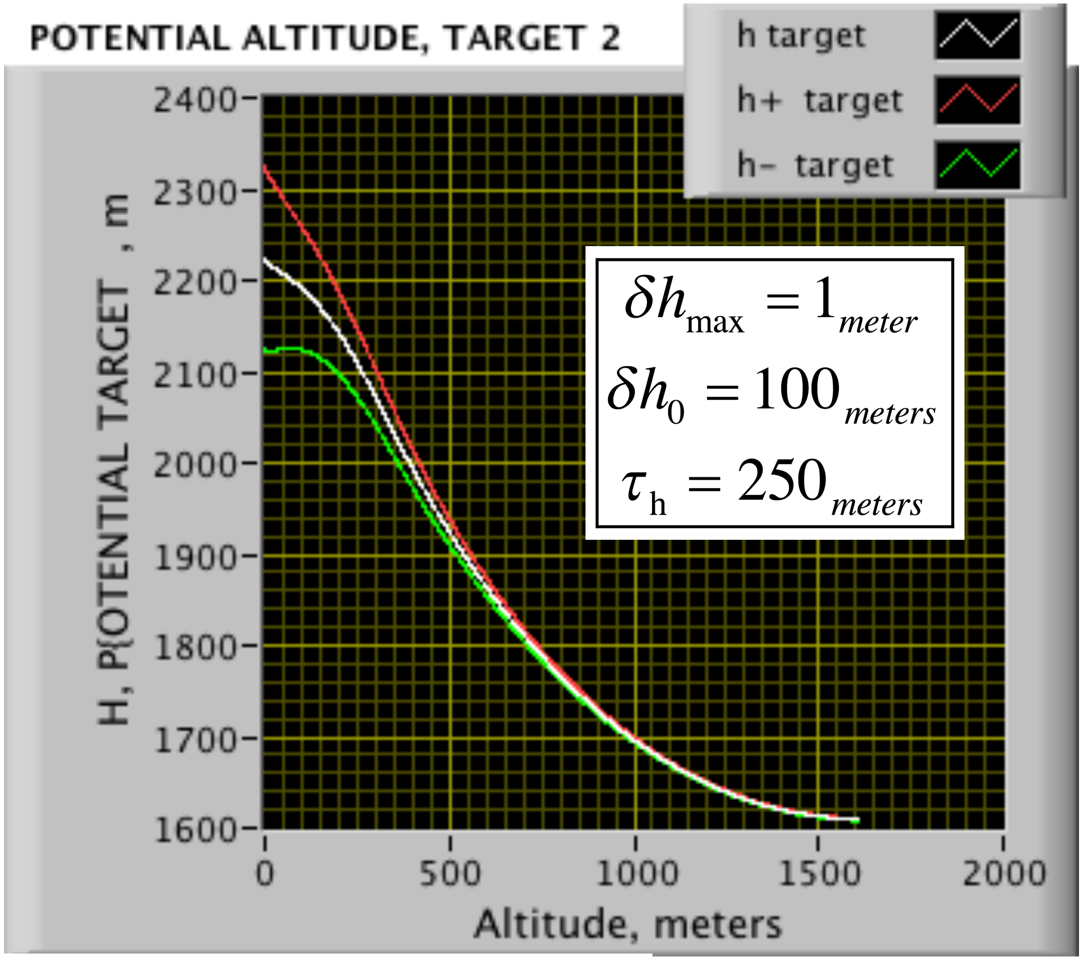
→

$$\delta h_{\max} \rightarrow \textit{sets final envelope error}$$

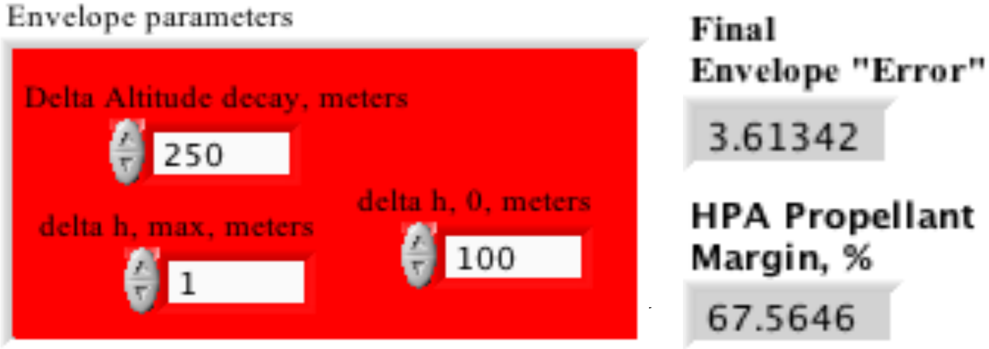
$$\delta h_0 \rightarrow \textit{sets initial envelope error}$$

$$\tau_{\text{h}} \rightarrow \textit{sets envelope decay rate}$$

Example Envelopes (1)



$$\delta h_{\max} = 1_{meter}$$
$$\delta h_0 = 100_{meters}$$
$$\tau_h = 250_{meters}$$



Envelope parameters

Delta Altitude decay, meters: 250

delta h, max, meters: 1

delta h, 0, meters: 100

Final Envelope "Error": 3.61342

HPA Propellant Margin, %: 67.5646

$$\delta h_{(h)} = \delta h_{\max} \cdot \left(1 - e^{-\frac{h}{\tau_h}}\right) + e^{-\frac{h}{\tau_h}} \cdot \delta h_0$$

$$h^+_{target} = h_{\text{target}} + \left(\delta h_{\max} \cdot \left(1 - e^{-\frac{h}{\tau_h}}\right) + e^{-\frac{h}{\tau_h}} \cdot \delta h_0\right)$$

$$h^+_{target} = h_{\text{target}} - \left(\delta h_{\max} \cdot \left(1 - e^{-\frac{h}{\tau_h}}\right) + e^{-\frac{h}{\tau_h}} \cdot \delta h_0\right)$$

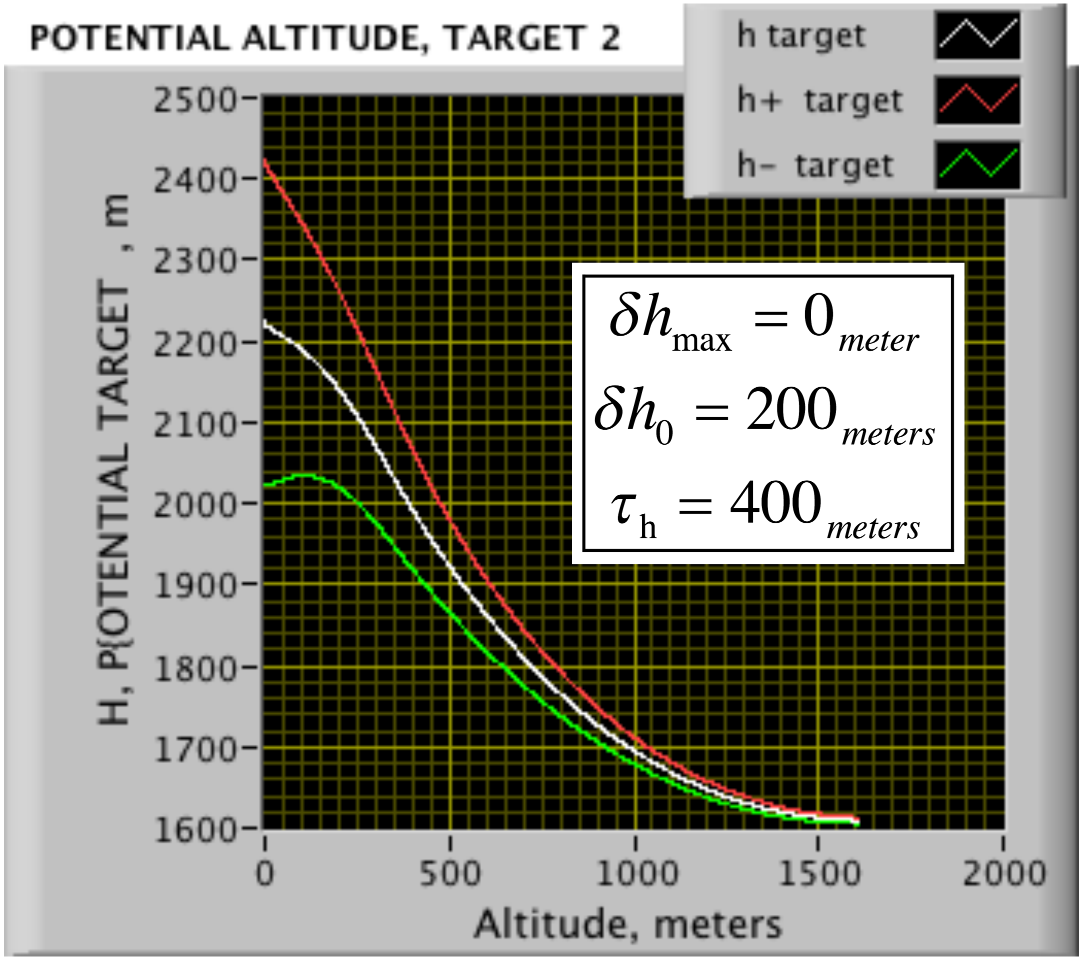
→

$\delta h_{\max} \rightarrow$ *sets final envelope error*

$\delta h_0 \rightarrow$ *sets initial envelope error*

$\tau_h \rightarrow$ *sets envelope decay rate*

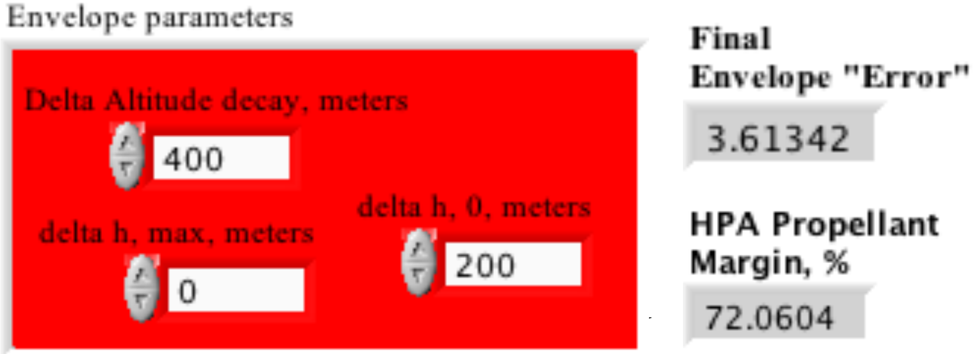
Example Envelopes (2)



$$\delta h_{\max} = 0_{meter}$$

$$\delta h_0 = 200_{meters}$$

$$\tau_\mathrm{h} = 400_{meters}$$



Envelope parameters

Delta Altitude decay, meters: 400

delta h, max, meters: 0

delta h, 0, meters: 200

Final Envelope "Error": 3.61342

HPA Propellant Margin, %: 72.0604

$$\delta h_{(h)} = \delta h_{\max} \cdot \left(1 - e^{-\frac{h}{\tau_h}}\right) + e^{-\frac{h}{\tau_h}} \cdot \delta h_0$$

$$h^+_{target} = h_{\text{target}} + \left(\delta h_{\max} \cdot \left(1 - e^{-\frac{h}{\tau_h}}\right) + e^{-\frac{h}{\tau_h}} \cdot \delta h_0\right)$$

$$h^+_{target} = h_{\text{target}} - \left(\delta h_{\max} \cdot \left(1 - e^{-\frac{h}{\tau_h}}\right) + e^{-\frac{h}{\tau_h}} \cdot \delta h_0\right)$$

→

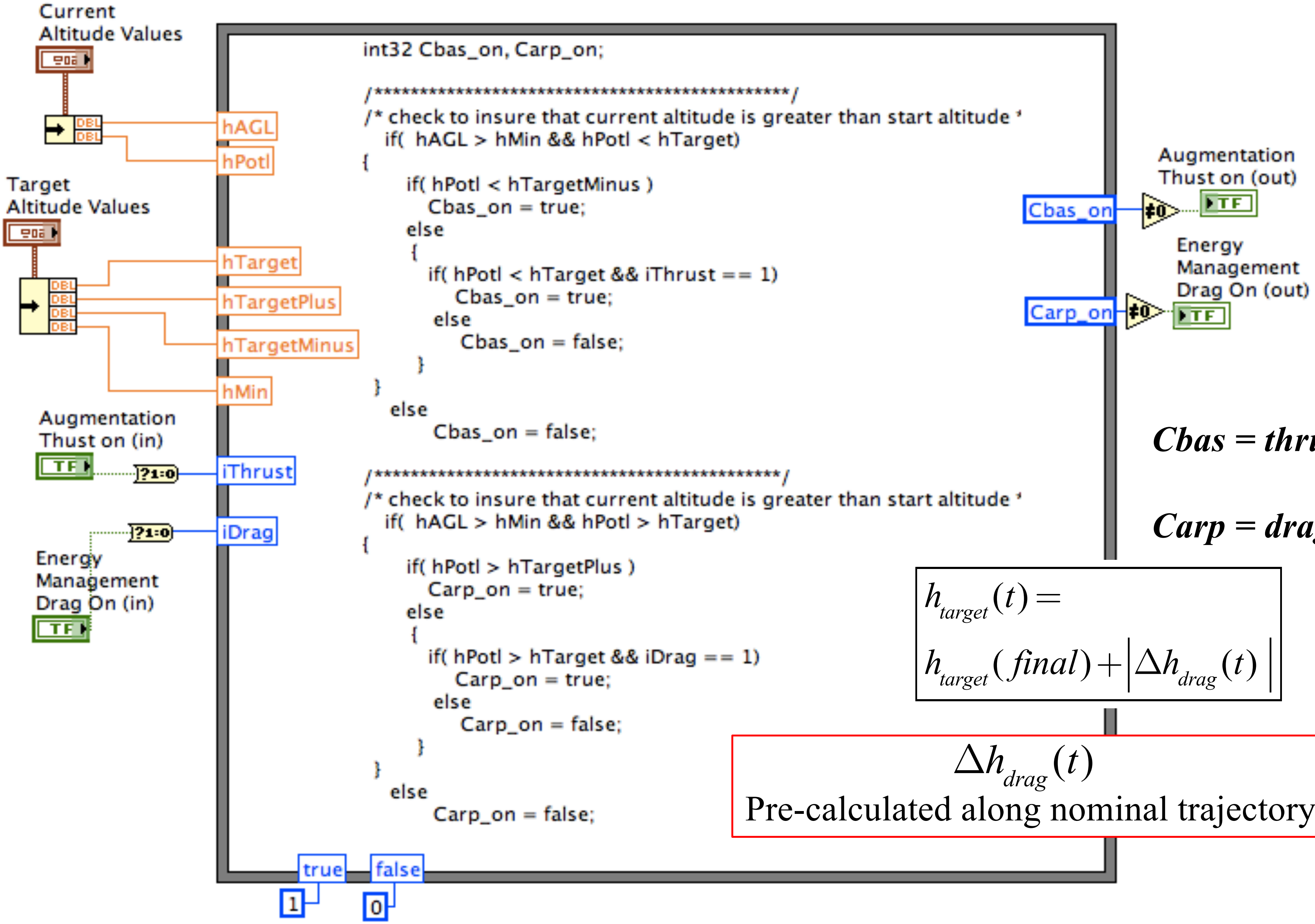
$\delta h_{\max} \rightarrow$ *sets final envelope error*

$\delta h_0 \rightarrow$ *sets initial envelope error*

$\tau_\mathrm{h} \rightarrow$ *sets envelope decay rate*

- Enveloping allows for final position error tolerance while significantly reducing required control activity

Energy Management Control Logic



```
int32 Cbas_on, Carp_on;

/************************************************/
/* check to insure that current altitude is greater than start altitude *
   if( hAGL > hMin && hPotl < hTarget)
{
     if( hPotl < hTargetMinus )
        Cbas_on = true;
     else
      {
        if( hPotl < hTarget && iThrust == 1)
            Cbas_on = true;
         else
            Cbas_on = false;
      }
 }
   else
        Cbas_on = false;

/************************************************/
/* check to insure that current altitude is greater than start altitude *
   if( hAGL > hMin && hPotl > hTarget)
{
     if( hPotl > hTargetPlus )
        Carp_on = true;
     else
      {
        if( hPotl > hTarget && iDrag == 1)
            Carp_on = true;
         else
            Carp_on = false;
      }
 }
   else
        Carp_on = false;
```

Cbas = thrust

Carp = drag

$$h_{target}(t) = h_{target}(final) + \left|\Delta h_{drag}(t)\right|$$

$$\Delta h_{drag}(t)$$

Pre-calculated along nominal trajectory

Questions??