

**Rocket Science Review 101:  
Ballistic Equations of Motion**

**Newton's Laws as  
Applied to  
"Rocket Science"**

**... its not just a job ... its an  
adventure**



# Real World Launch Analysis

Orbital designs a *unique mission trajectory for each Pegasus flight to maximize payload* performance while complying with the satellite and launch vehicle constraints.

Using the *3-Degree of Freedom Program* for Optimization of Simulation Trajectories(POST), a desired orbit is specified and a set of optimization parameters and constraints are designated.

Appropriate data for mass properties, aerodynamics, and motor ballistics are input. POST then selects values for the optimization parameters that target the desired orbit with specified constraints on key parameters such as angle of attack, dynamic loading, payload thermal, and ground track.

*After POST calculates optimum launch trajectory, a Pegasus-specific six degree of Freedom simulation program* used to verify Trajectory acceptability with realistic attitude dynamics, including separation analysis on all stages.

## Pegasus User's Guide



ORBITAL SCIENCES CORPORATION

- 6-DOF simulations costs **A LOT!** To run and are typically Not used for Trajectory design!
- We are going to start with A simple **2<sup>+</sup>-D** Ballistic code that works well for sounding-rocket mission profile development

# Newtonian Dynamics

- General 3-DOF Equations of Motion

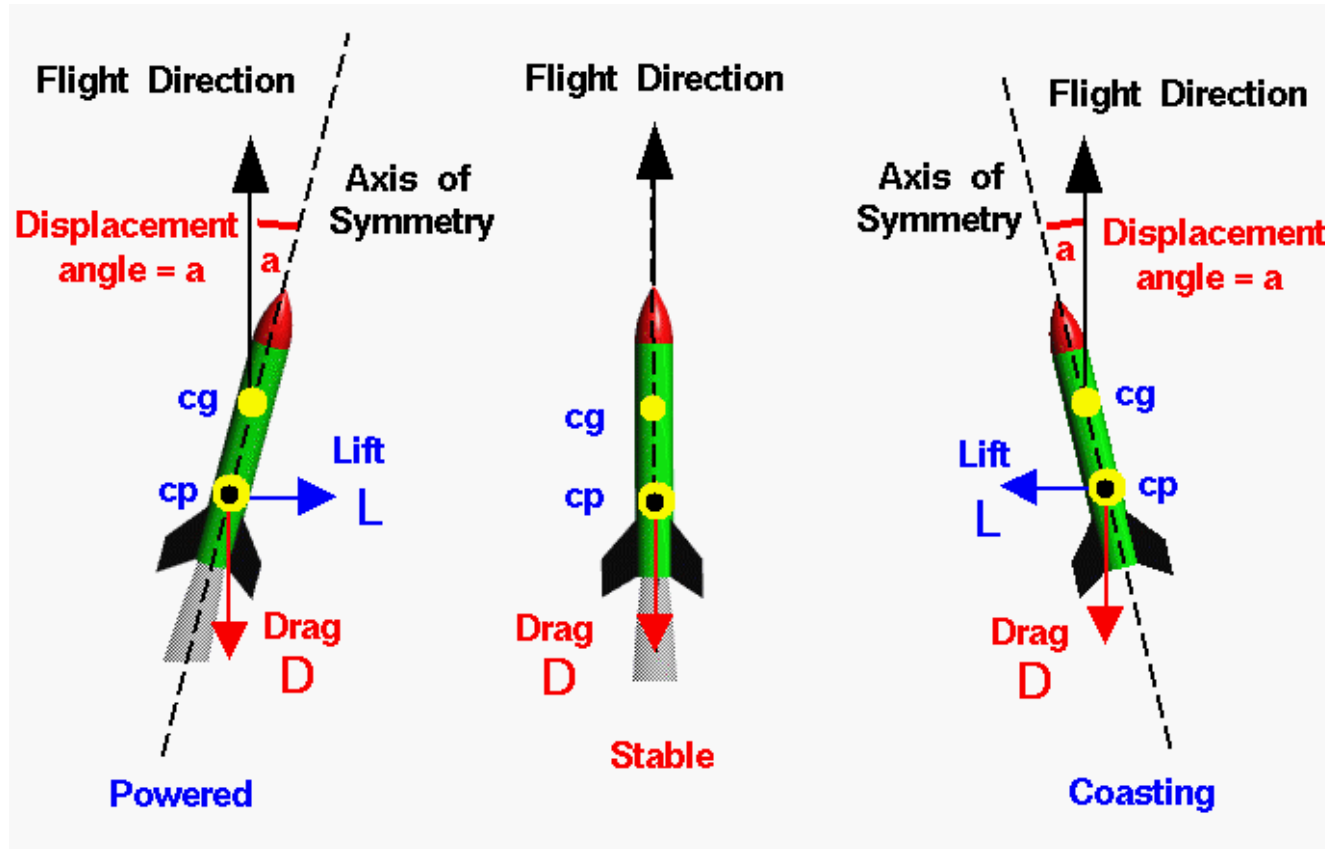
$$\bar{\mathbf{V}} = \frac{\partial \bar{\mathbf{R}}}{\partial t} + \bar{\boldsymbol{\omega}} \times \bar{\mathbf{R}}$$

$$\frac{\sum \mathbf{F}_{\text{external}}}{M} = \frac{\partial \bar{\mathbf{V}}}{\partial t} + \bar{\boldsymbol{\omega}} \times \bar{\mathbf{V}}$$

$$\dot{M}_{\text{vehicle}} = - \frac{F_{\text{thrust}}}{g_0 I_{\text{sp}}}$$

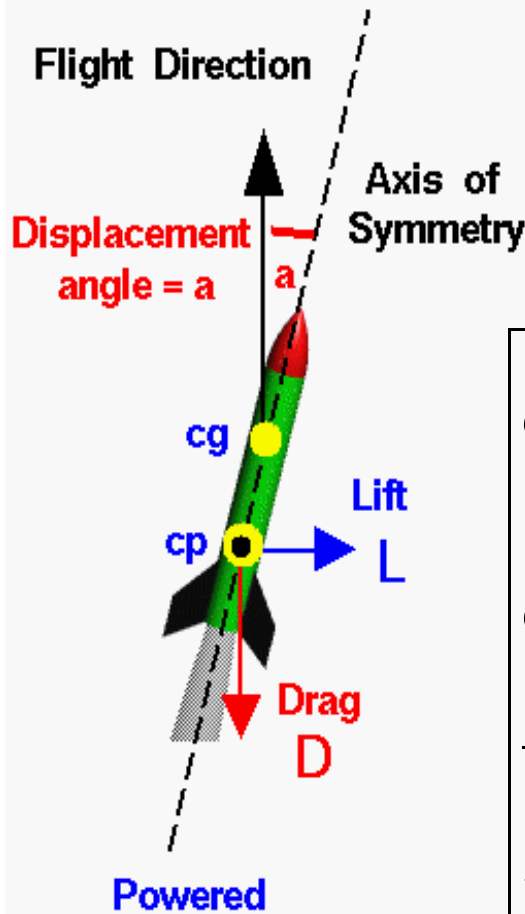
# RS 101: Summary

## External Forces Acting on Rocket



- Lift – acts perpendicular to flight path (*non-conservative*)
- Drag – acts along flight path (*non-conservative*)
- Thrust – acts along longitudinal axis of rocket (*non-conservative*)
- Gravity – acts downward (*conservative*)

## External Forces Acting on Rocket (2)



- Lift – acts perpendicular to flight path (*non-conservative*)
- Drag – acts along flight path (*non-conservative*)

### Define

$$C_L = \frac{L_{ift}}{\frac{1}{2} \cdot \rho \cdot V^2 \cdot A_{ref}}, \quad C_D = \frac{D_{rag}}{\frac{1}{2} \cdot \rho \cdot V^2 \cdot A_{ref}}$$

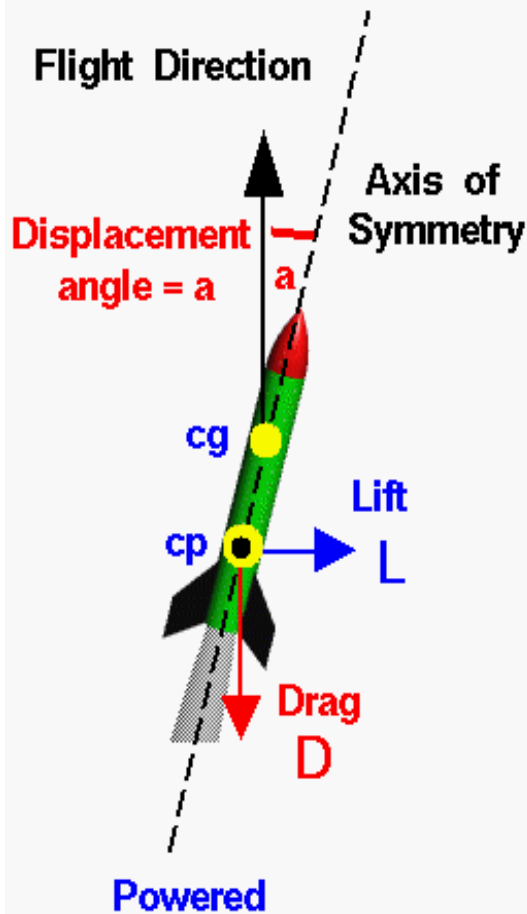
$C_L$  = lift coefficient,  $C_D$  = drag coefficient

$\frac{1}{2} \cdot \rho \cdot V^2$  = "dynamic pressure" ( $\bar{q}$ )

$A_{ref}$  = "reference area" – > typically maximum frontal area

$\rho$  = "local air density" – > function of altitude

# External Forces Acting on Rocket (3)



## •Gravitational Force

$$\vec{F}_{grav} = -\frac{G \cdot M_{earth} \cdot m_{rocket}}{r^2} \bar{i}_r \rightarrow G \cdot M_{earth} \equiv \mu_{earth} = 3.9860044 \times 10^5 \frac{km^3}{sec^3}$$

$$\frac{\vec{F}_{grav}}{m_{rocket}} = A_{grav} = -\frac{\mu}{r^2} \bar{i}_r = -g \cdot \bar{i}_r$$

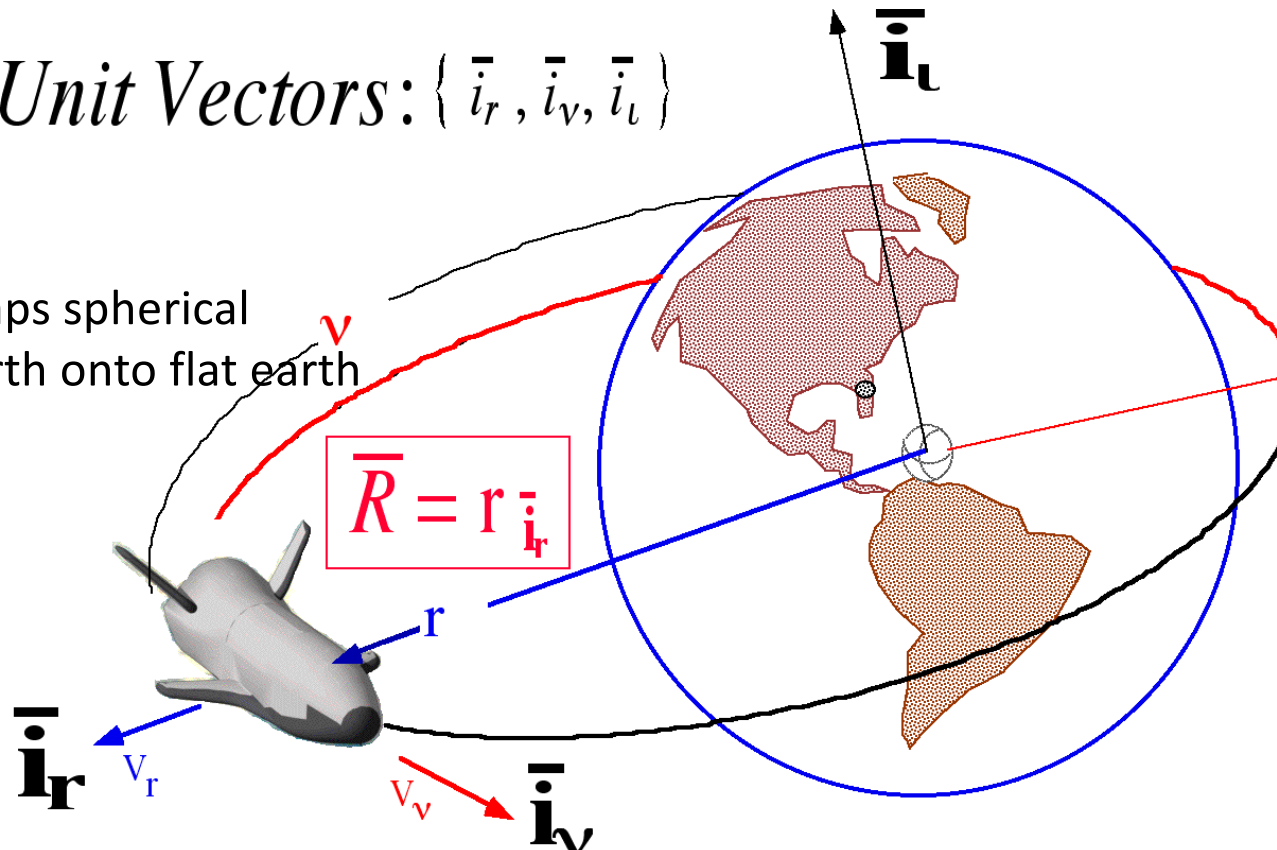
$$g_0 = \frac{3.9860044 \times 10^5 \frac{km^3}{sec^3}}{6375.4_{km}^2} \times 1000 \frac{m}{km} = 9.8067 \frac{m}{sec^2}$$

*Sea Level Acceleration of gravity at 21 deg. latitude*

# Perifocal Coordinate System

Unit Vectors:  $\{ \bar{i}_r, \bar{i}_v, \bar{i}_l \}$

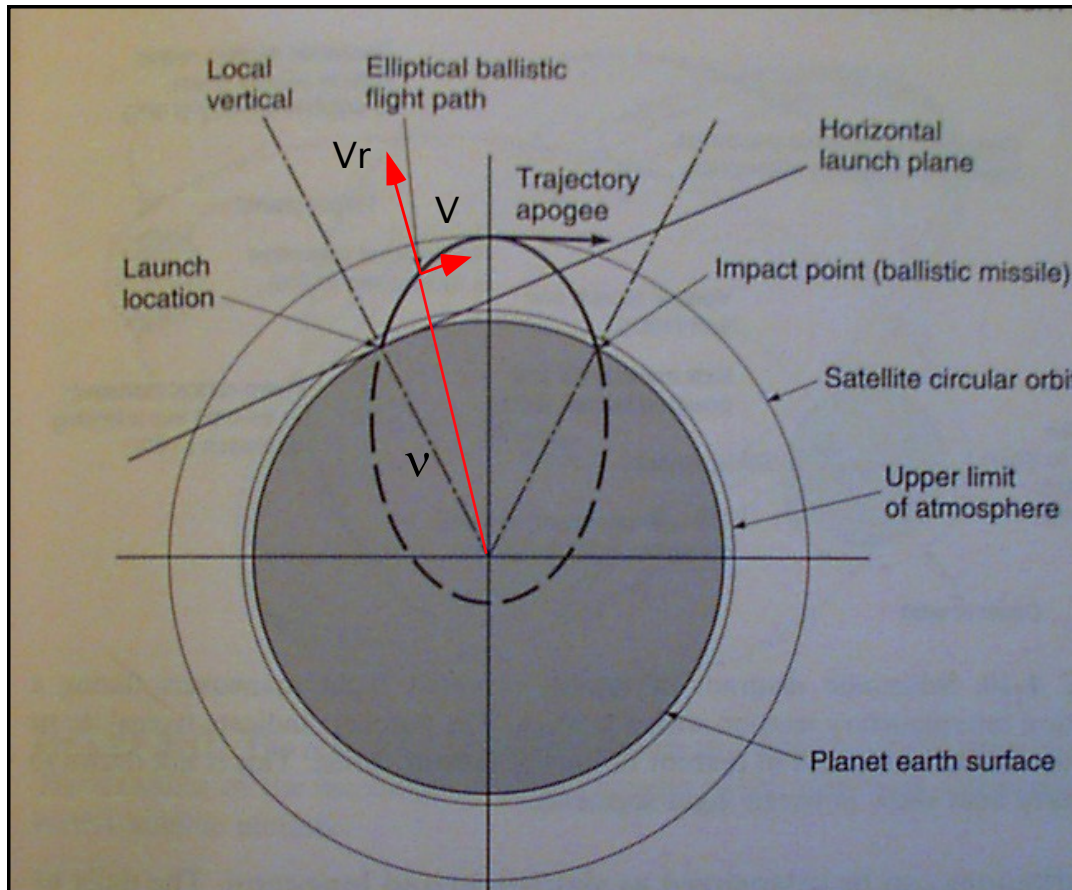
Maps spherical  
earth onto flat earth



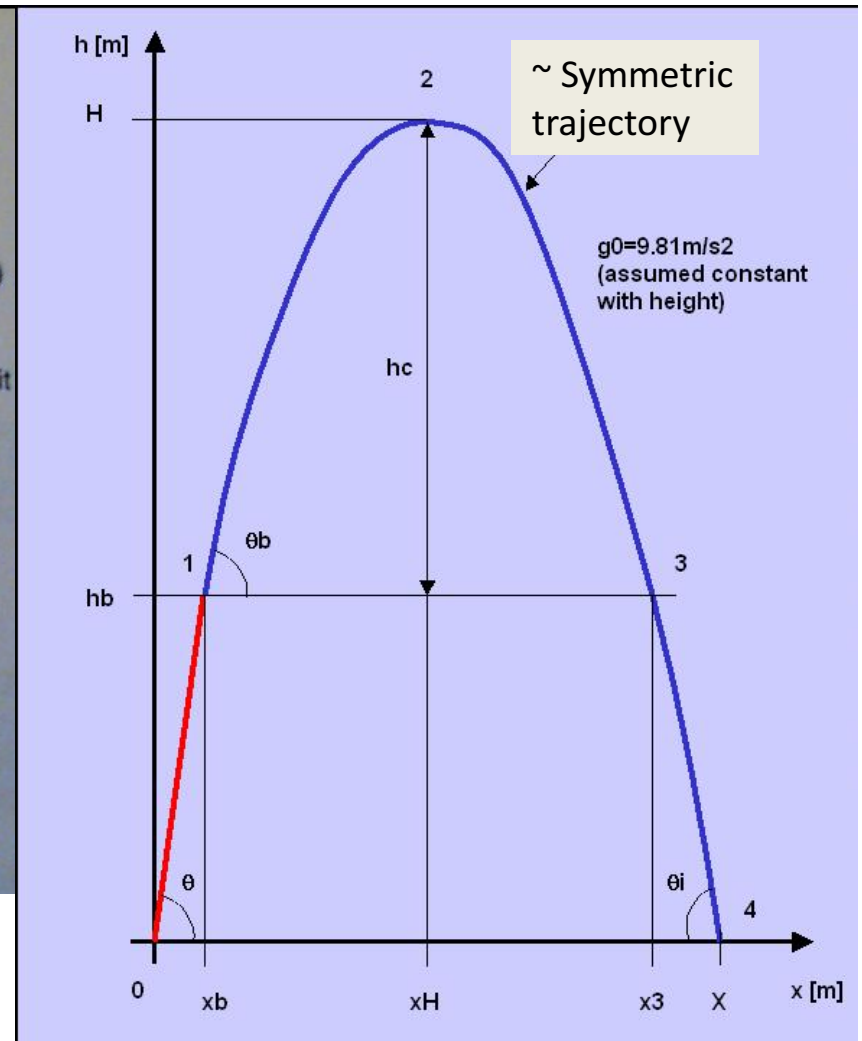
$$\begin{bmatrix} \vec{i}_r \\ \vec{i}_v \\ \vec{i}_l \end{bmatrix} = \begin{bmatrix} \text{local vertical} \\ \text{local horizontal}_{(\text{along flight path})} \\ \text{local horizontal}_{(\text{perpendicular to flight path})} \end{bmatrix}$$

# Perifocal Coordinate System Sub-orbital Image

*Sub-Orbital Launch: Spherical Earth*

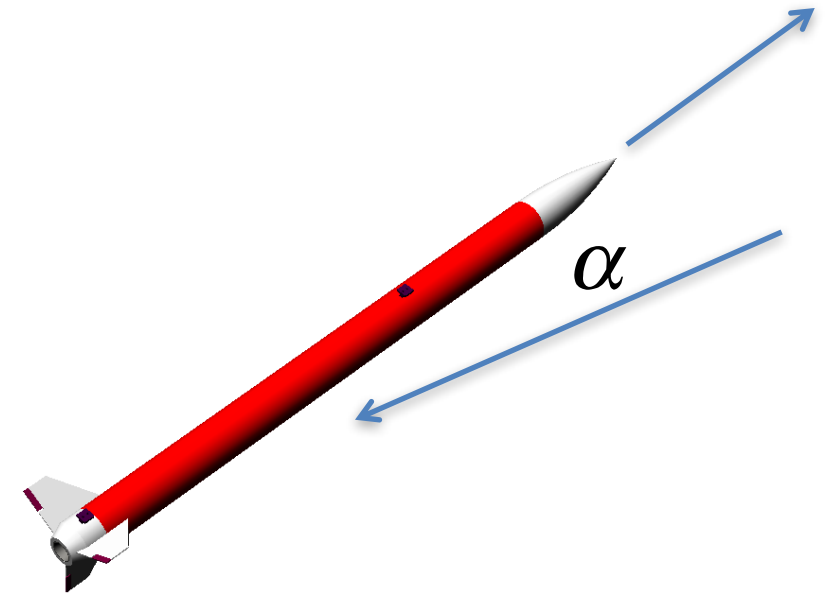
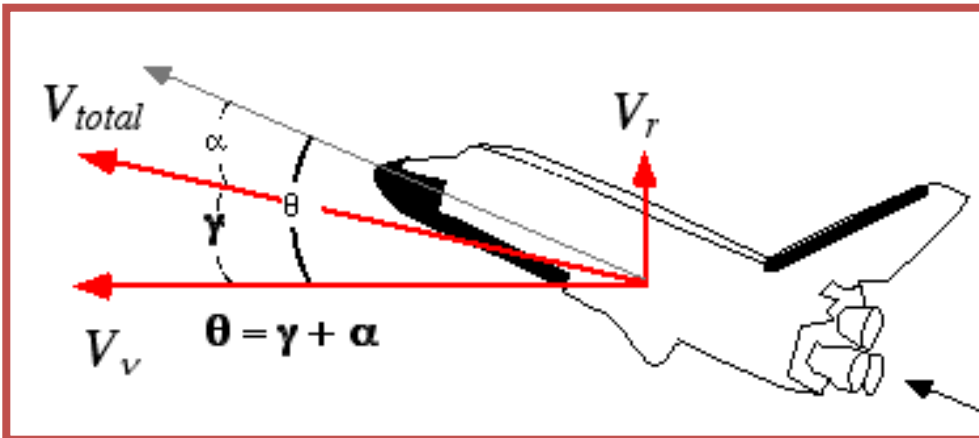


*Sub-Orbital Launch: Flat Earth*





# Ballistic versus Non-Ballistic Trajectories



- Non-ballistic trajectories sustain significantly non-zero angles of attack

*... lift is a factor in resulting trajectory*

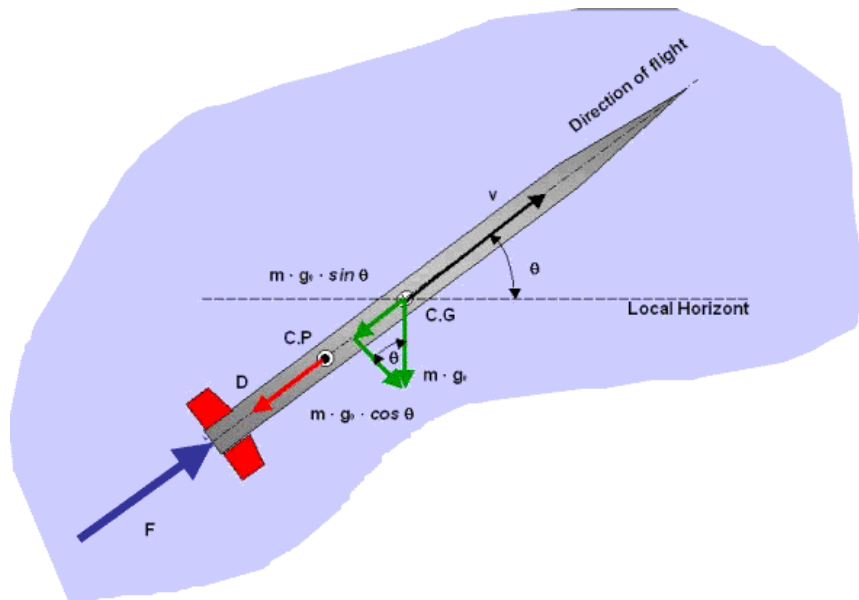
*... so is induced drag*

- *Ballistic rocket trims trajectory at  $\alpha \approx 0$*   
*... lift is a negligible factor*

$$\alpha \approx 0$$

$$\rightarrow \theta_{ballistic} = \gamma = \tan^{-1} \left[ \frac{V_r}{V_v} \right]$$

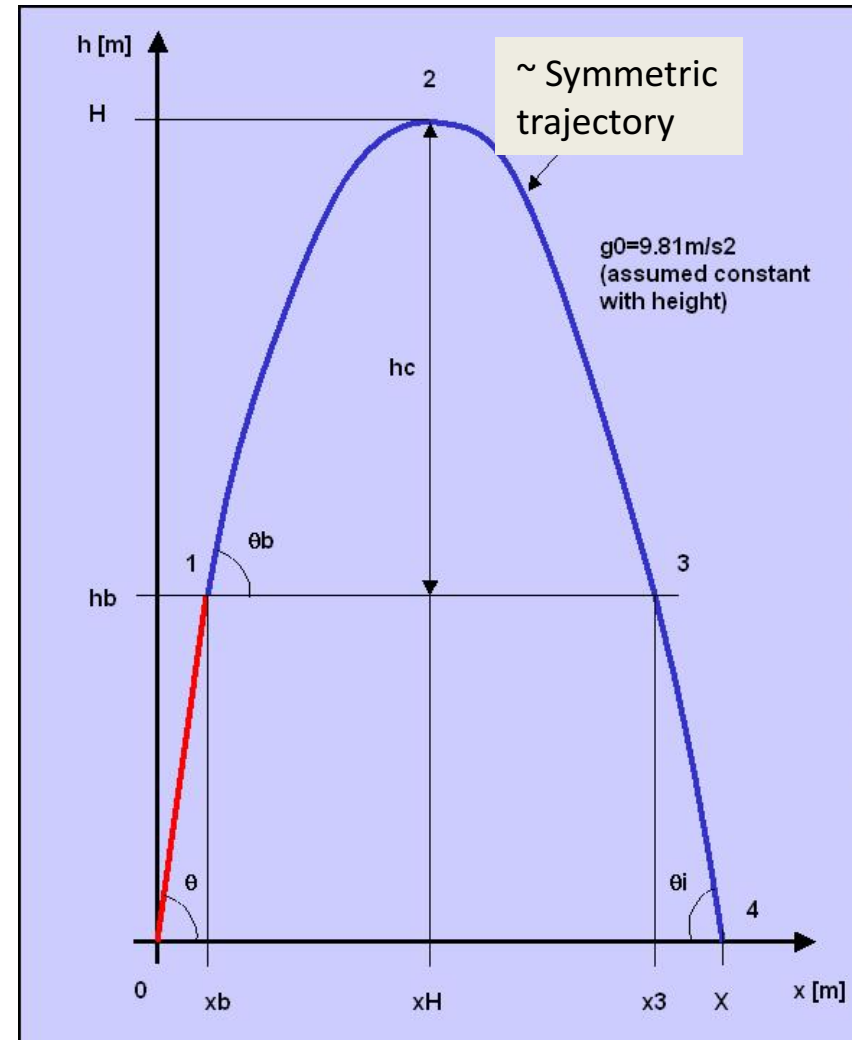
# Example of Ballistic Trajectory



- Ballistic Trajectories  
Offer minimum drag profiles

$$\alpha \approx 0$$

( Induced drag primarily due to  
Alpha-dither, assume small)



# Collected Ballistic Equations

$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{v} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \frac{V_v^2}{r} - \frac{\mu}{r^2} + \left[ \frac{F_{thrust}}{m} - \frac{\rho V_\infty^2}{2\beta} \right] \sin(\gamma) \\ -\frac{V_r V_v}{r} + \left[ \frac{F_{thrust}}{m} - \frac{\rho V_\infty^2}{2\beta} \right] \cos(\gamma) \\ \frac{V_r}{r} \\ \frac{V_v}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix} \quad \alpha=0$$

$$\begin{aligned}
 \gamma &= \tan^{-1} \left[ \frac{V_r}{V_v} \right] \\
 \beta &= \frac{M}{C_D \cdot A_{ref}} \\
 \dot{X} &= f[X, F_{thrust}]
 \end{aligned}$$

$$\vec{X} = \begin{bmatrix} V_r \\ V_v \\ r \\ v \\ M \end{bmatrix}$$

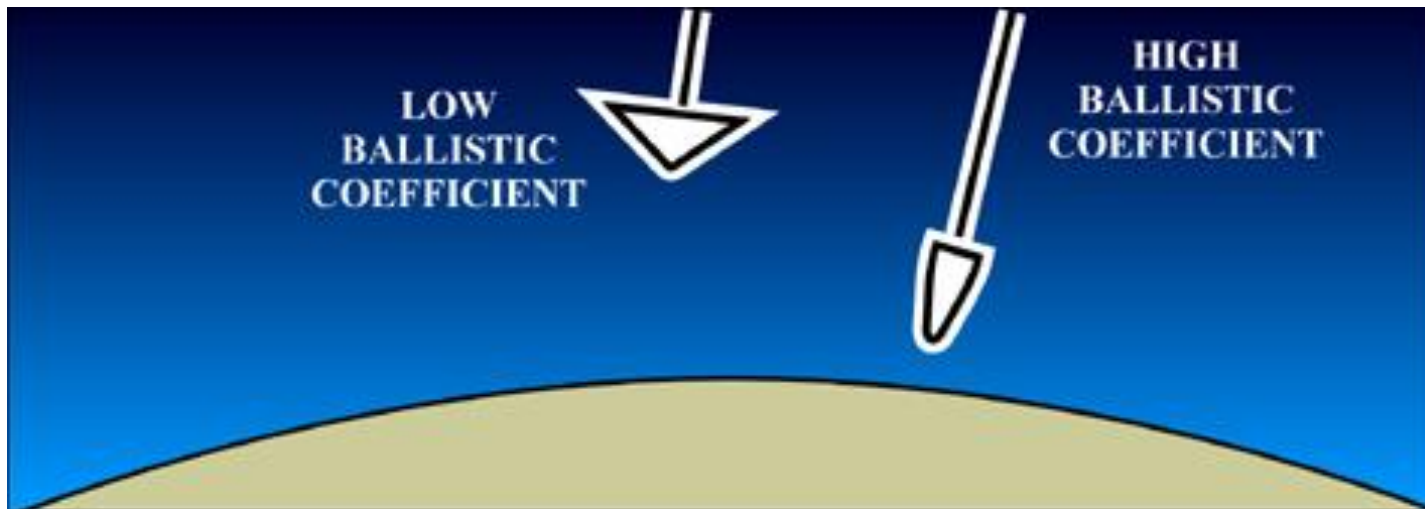
$$\left( \dot{\vec{X}} \right) = \begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{v} \\ -\dot{m}_{motor} \end{bmatrix}$$

# Ballistic Coefficient

- When effects of lift are negligible aerodynamic effects can be incorporated into a single parameter

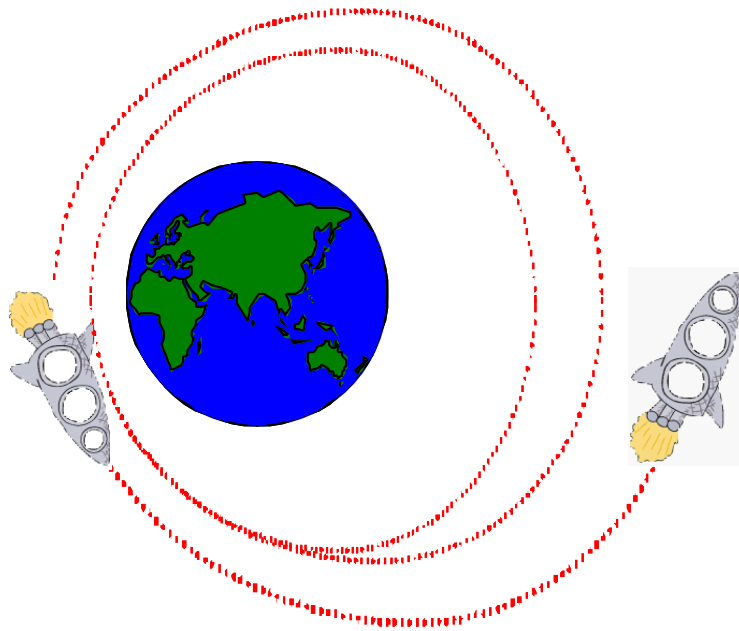
.... *Ballistic Coefficient* ....

- $\beta$  is a measure of a projectile's ability to coast. ...  $\beta = m / (C_D \cdot A_{ref})$   
...  $M$  is the projectile's mass and ...  $C_D A_{ref}$  is the drag *form factor*.
- At any given velocity and air density, the deceleration of a rocket from drag is inversely proportional to  $\beta$



**Low Ballistic  
Coefficients  
Dissipate  
More Energy  
Due to drag**

# Numerical Integration of the 2-D Launch Equations of Motion



- Simple Second Order predictor/corrector works well for Small-to-moderate step sizes ... but at larger step sizes can be come unstable
- Good to have a higher order integration scheme in our *bag of tools*
  - 4th Order Runge-Kutta method is one most commonly used
- we'll only use the second order integrators for this simulation

# Trapezoidal Rule Predictor/Corrector Algorithm

Given  $\{\widehat{\mathbf{X}}_k, \Delta t, \mathbf{F}_{\text{thrust}}\}$

Prediction Step:

$$\widetilde{\mathbf{X}}_{k+1} \equiv \widehat{\mathbf{X}}_k + \Delta t \mathbf{f}(\widehat{\mathbf{X}}_k, \mathbf{F}_{\text{thrust}})$$

Correction Step:

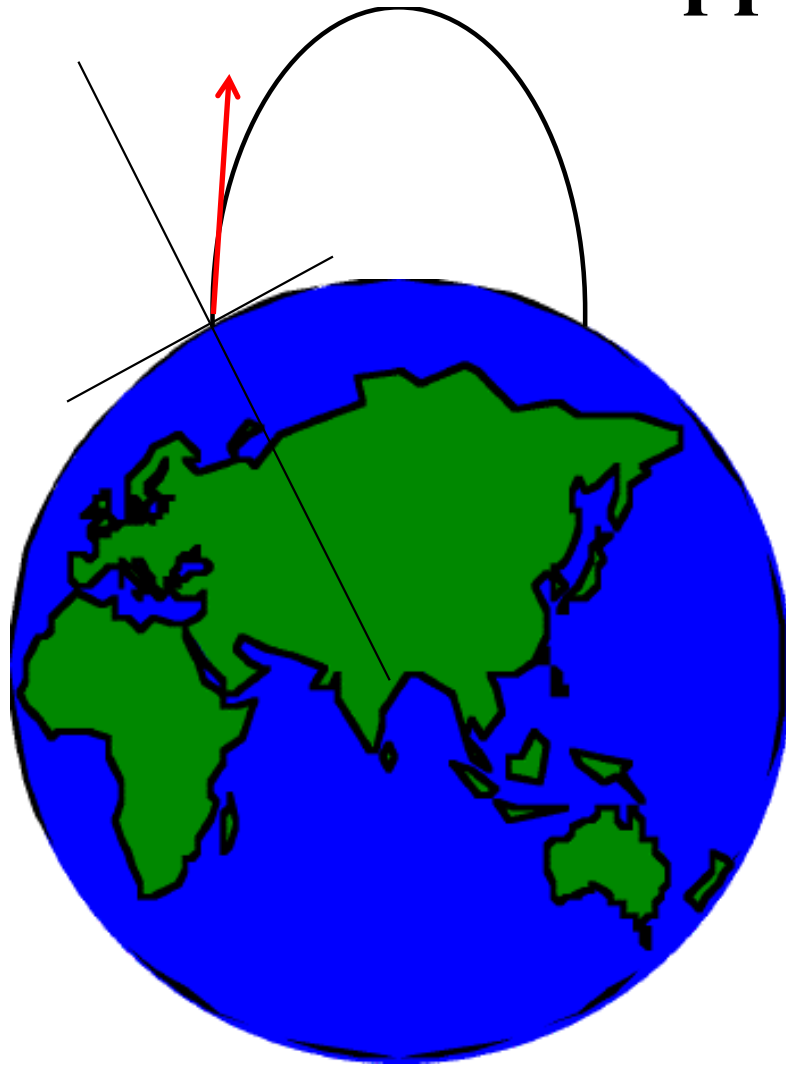
$$\widehat{\mathbf{X}}_{k+1} = \widehat{\mathbf{X}}_k + \frac{\Delta t}{2} \left[ \mathbf{f}(\widehat{\mathbf{X}}_k, \mathbf{F}_{\text{thrust}}) + \mathbf{f}(\widetilde{\mathbf{X}}_{k+1}, \mathbf{F}_{\text{thrust}}) \right]$$

Slide Indices and Repeat:

$$\widetilde{\mathbf{X}}_{k+1} \Rightarrow \widehat{\mathbf{X}}_k$$

*“trapezoidal rule”*

# Fixed Earth for Sounding Rocket Launch Approximation



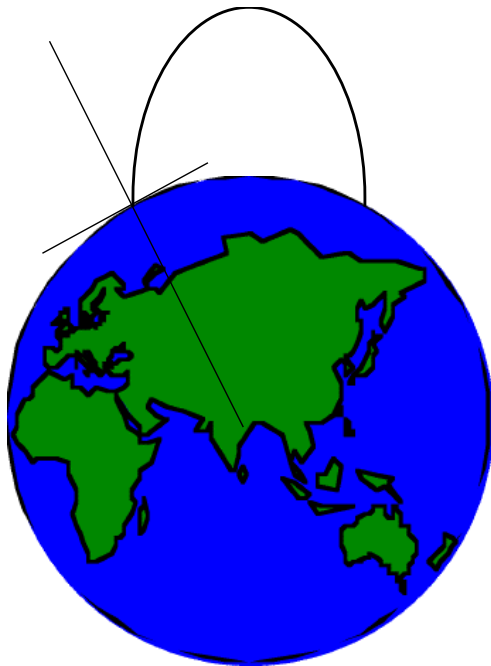
- Ignore effects of Earth's rotation
- $V_{inertial} = V_{ground}$
- $\gamma_{inertial} = \gamma_{ground}$
- Accurate for Short Duration,  
Non-orbital flights

# Ground Launch: Down Range Calculation

- Integrated trajectory gives

$$\mathbf{r}, \mathbf{V}$$

- Inertial Downrange

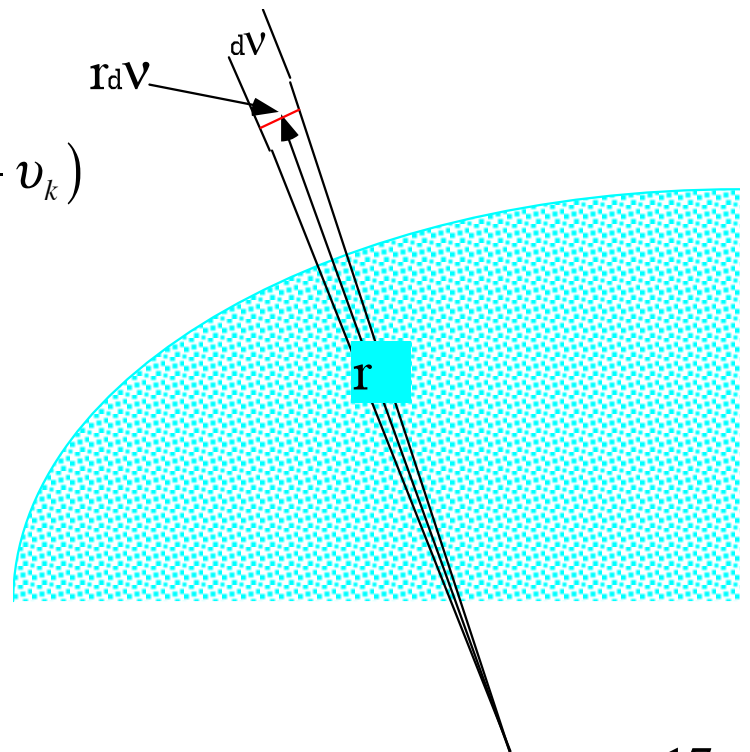


## *Horizontal Motion*

$$\Delta x = \int_{v_1}^{v_2} r \cdot dv \approx \left( \frac{r_{k+1} + r_k}{2} \right) \cdot (v_{k+1} - v_k)$$

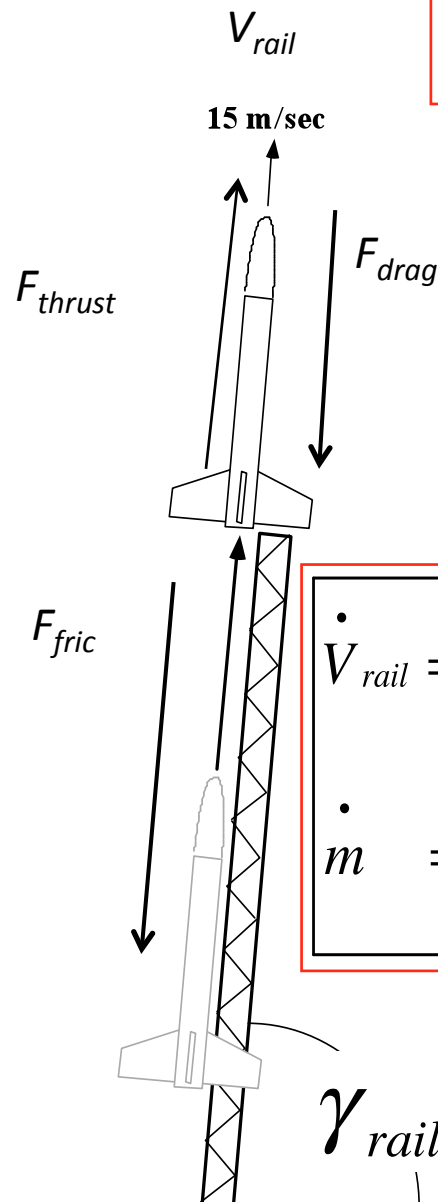
## *Recursive Formula for Horizontal Motion*

$$x_{k+1} = x_k + \left( \frac{r_{k+1} + r_k}{2} \right) \cdot (v_{k+1} - v_k)$$





# Velocity Off of the Rail



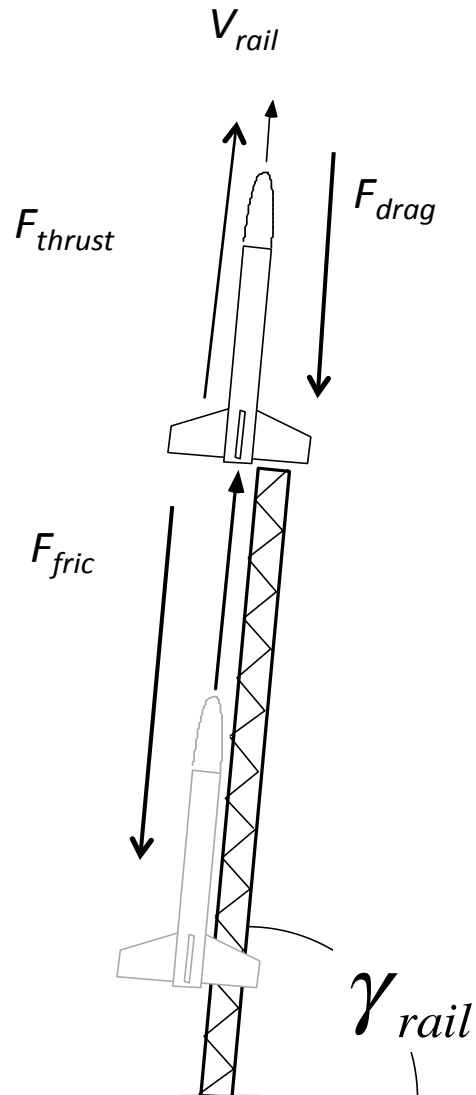
$$m \cdot A_{rail} = F_{thrust} - F_{grav} - F_{fric} - F_{drag}$$

$$\left[ \begin{array}{l} \beta = \frac{m}{C_D A_{ref}} \\ g = \frac{\mu}{(R_e + h)^2} \end{array} \right] \rightarrow \text{careful! with units}$$

$$\dot{V}_{rail} = -\frac{\rho V_{rail}^2}{2\beta} - \frac{\mu}{(R_e + h)^2} [\sin(\gamma_{rail}) + C_f \cdot \cos(\gamma_{rail})] + \frac{F_{thrust}}{m}$$

$$\dot{m} = -\frac{F_{thrust}}{g_0 I_{sp}}$$

# Exit Conditions off of Rail = Initial Conditions for Free Flight Simulation

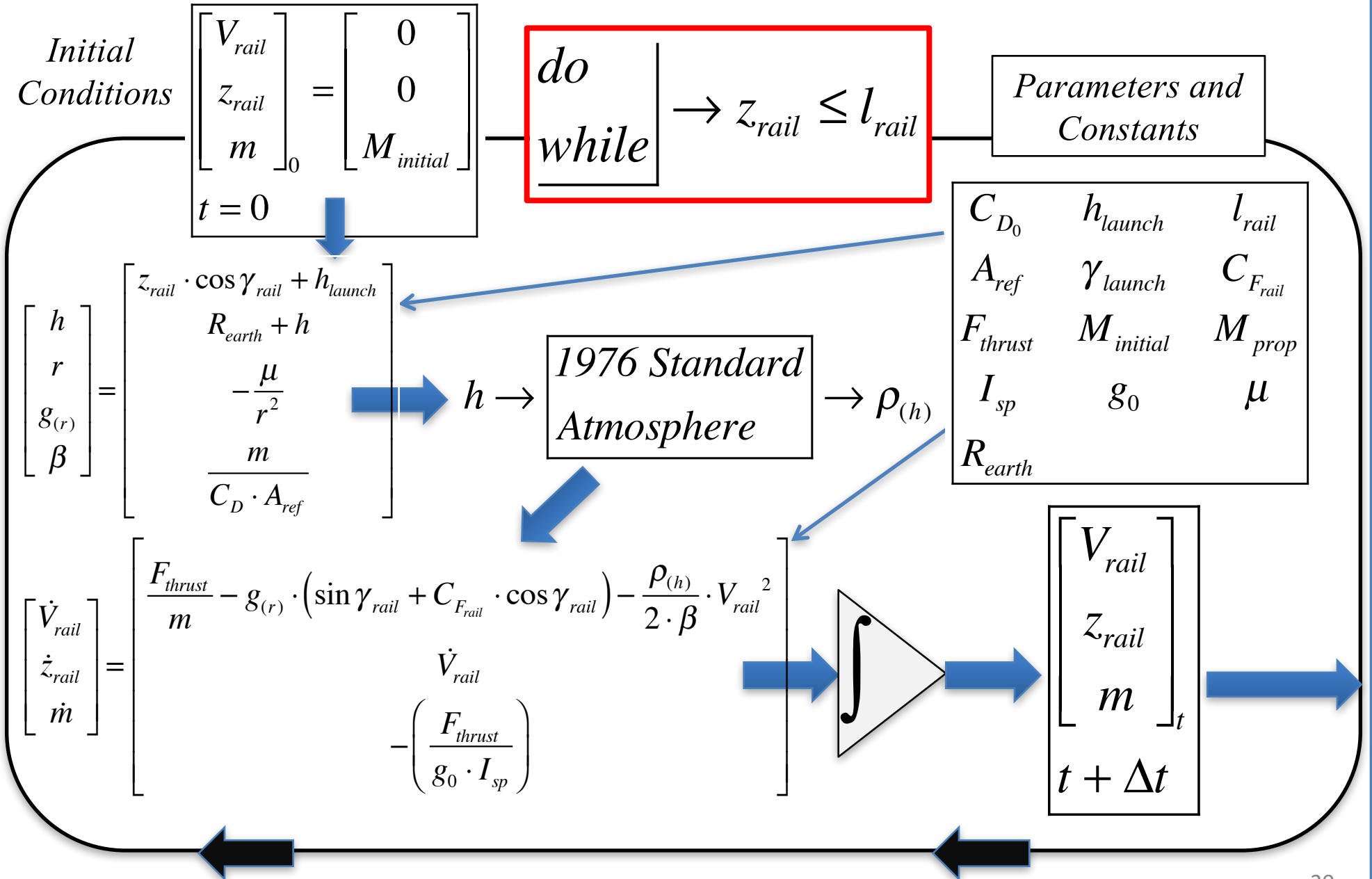


Altitude :  $h = r - R_{earth}$

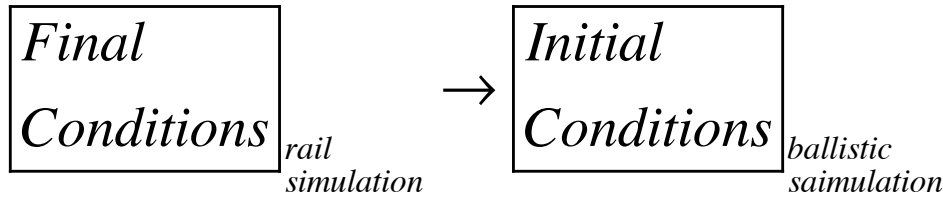
$$\begin{bmatrix} V_r \\ V_v \\ r \\ v \\ m \end{bmatrix} \text{ initial conditions}$$

$$= \begin{bmatrix} V_{rail} \cdot \sin \gamma_{rail} \\ V_v \cdot \cos \gamma_{rail} \\ R_{earth} + h_{launch} + l_{rail} \cdot \sin \gamma_{rail} \\ \left( \frac{l_{rail} \cdot \cos \gamma_{rail}}{R_{earth} + h_{launch} + l_{rail} \cdot \sin \gamma_{rail}} \right) \\ m_{rail} \end{bmatrix}$$

# Rail Simulation Block Diagram



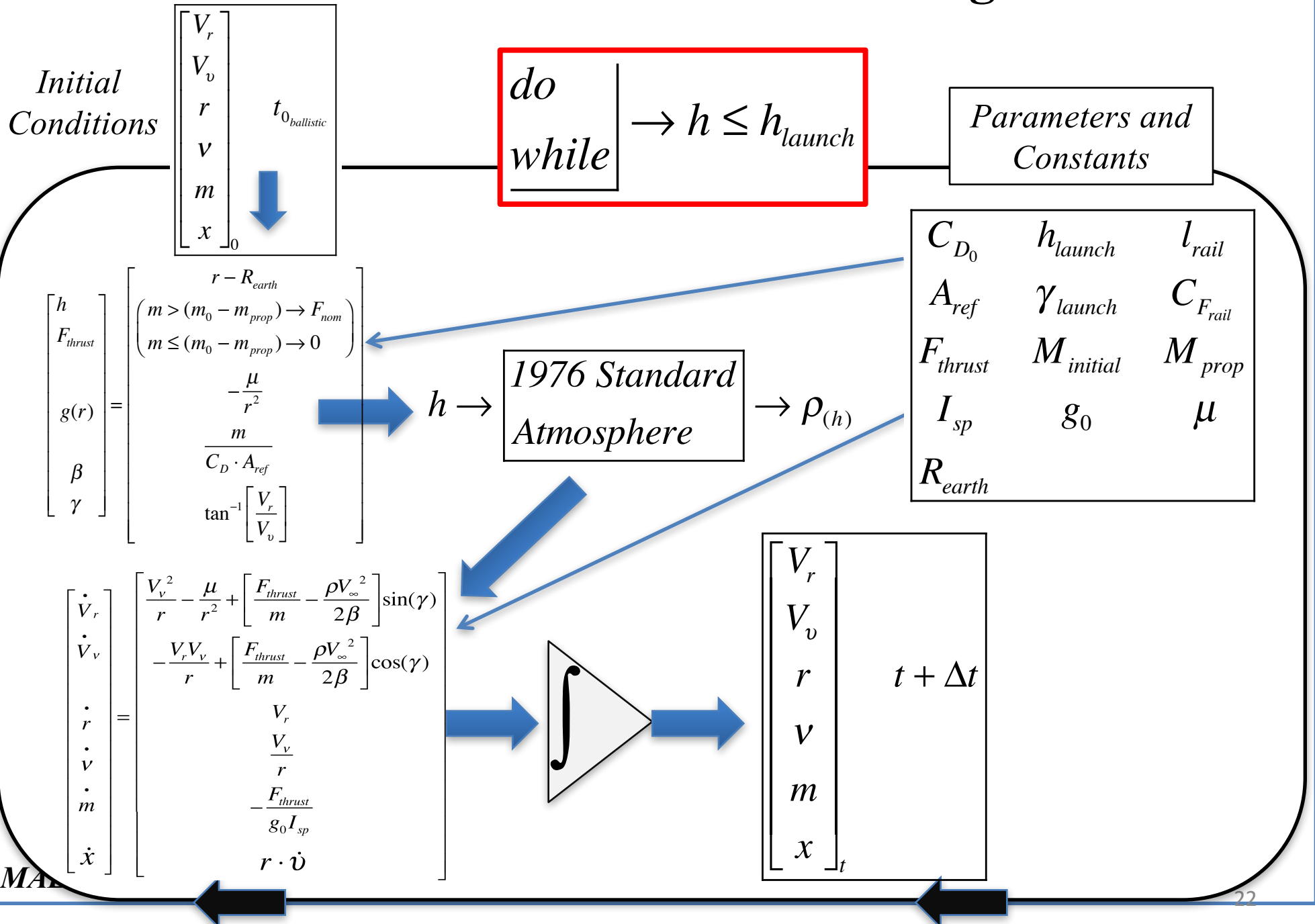
# Ballistic Simulation Initial Conditions



$$\begin{bmatrix} V_r \\ V_v \\ r \\ v \\ m \end{bmatrix}_{\text{initial conditions}} = \begin{bmatrix} V_{rail} \cdot \sin \gamma_{rail} \\ V_v \cdot \cos \gamma_{rail} \\ R_{earth} + h_{launch} + l_{rail} \cdot \sin \gamma_{rail} \\ \left( \frac{l_{rail} \cdot \cos \gamma_{rail}}{R_{earth} + h_{launch} + l_{rail} \cdot \sin \gamma_{rail}} \right) \\ m_{rail} \end{bmatrix}_{\text{Final conditions}}$$

$t_{0 \text{ ballistic}} = t_{\text{final rail}}$   
 $x_{0 \text{ ballistic}} = l_{rail} \cdot \cos \gamma_{rail} \approx 0$

# Ballistic Simulation Block Diagram



# Questions??

