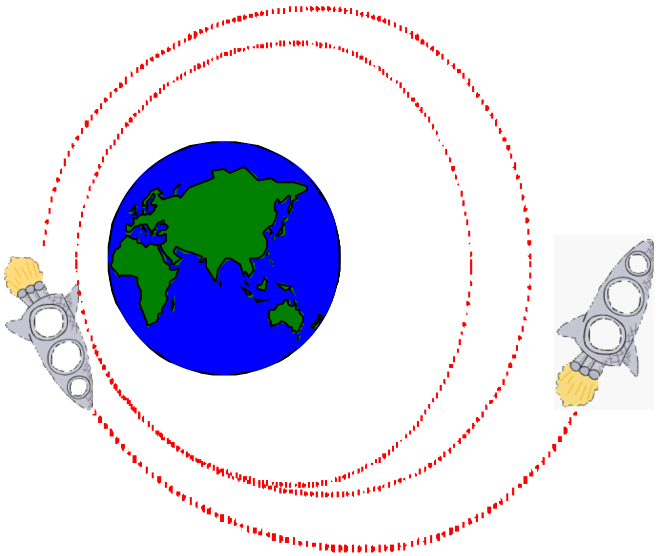
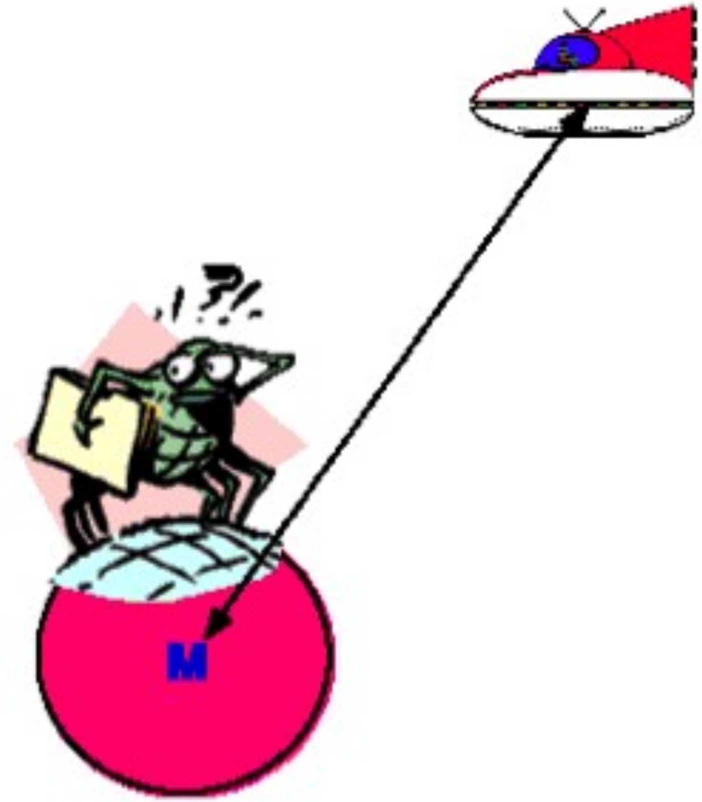
Utah State University

Mechanical & Aerospace Engineering

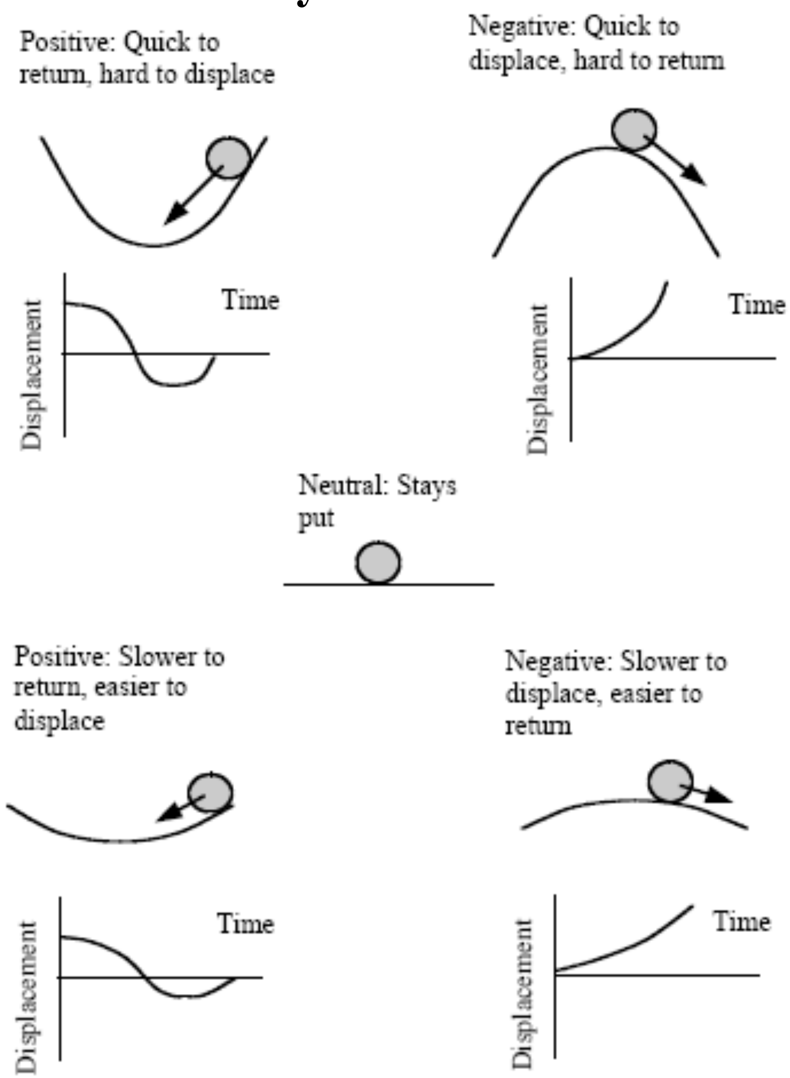
Section 7.4
Static and Dynamic Stability, Longitudinal Pitch Dynamics



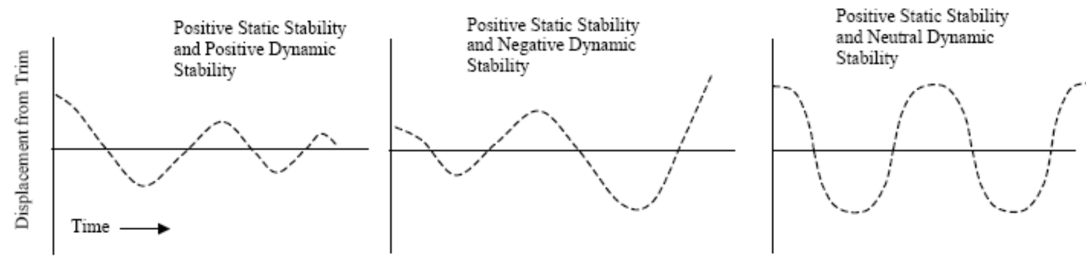


Static Vs. Dynamic Stability

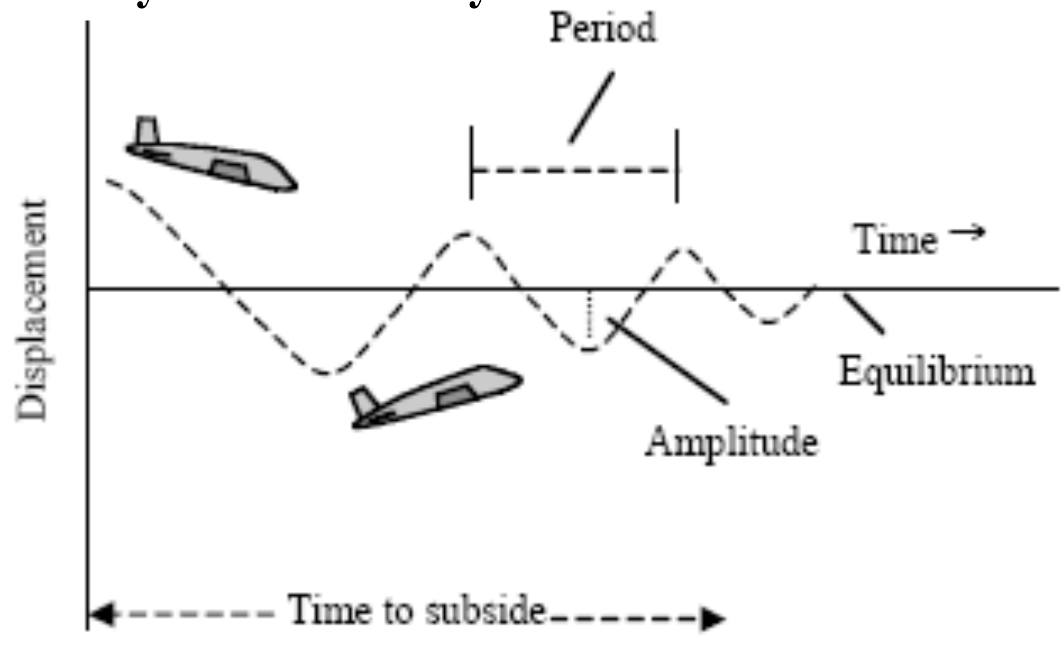
Static Stability



Dynamic Stability

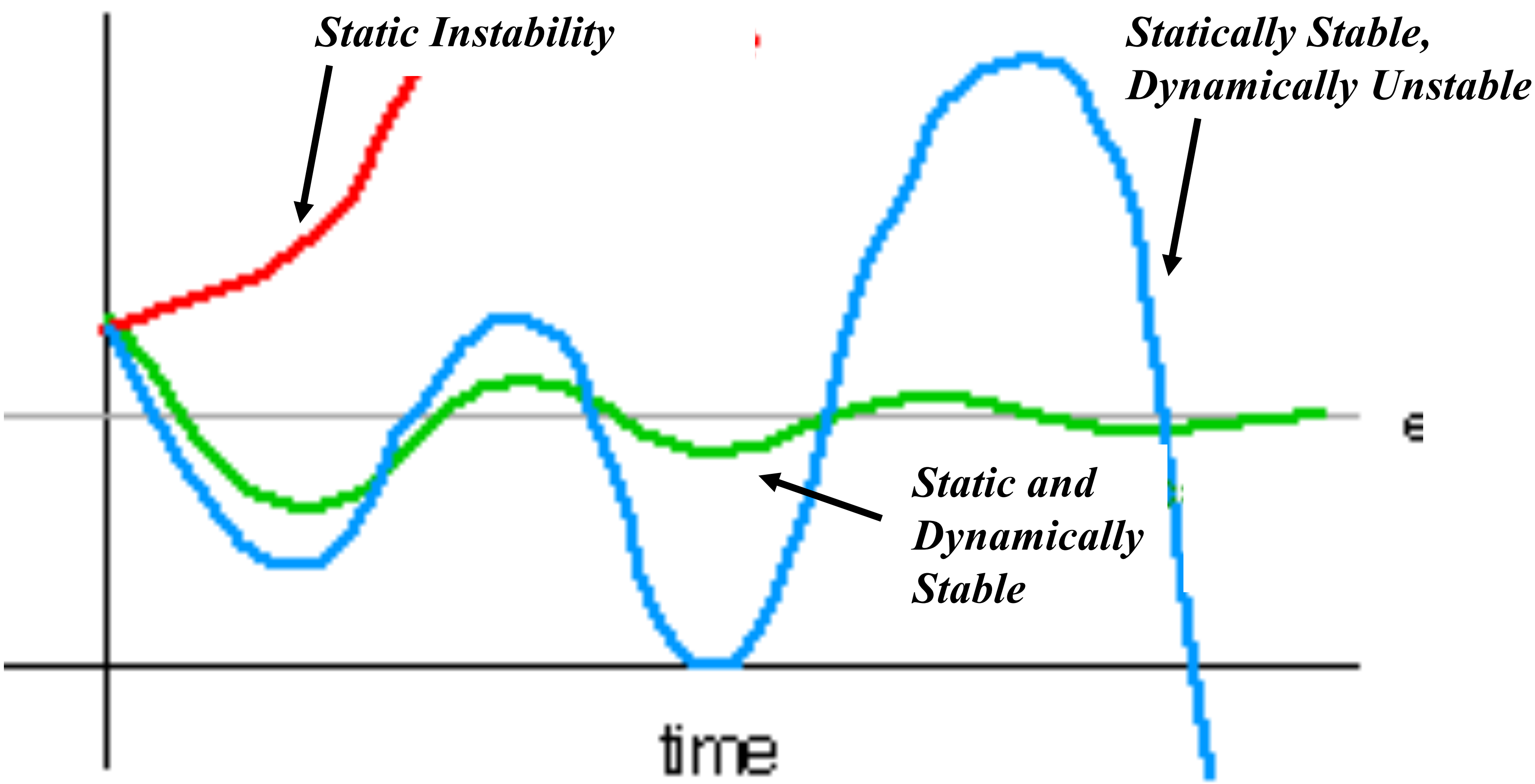


Dynamic Stability



Period/Time to subside = damping ratio, ζ

Static Versus Dynamic Stability (2)



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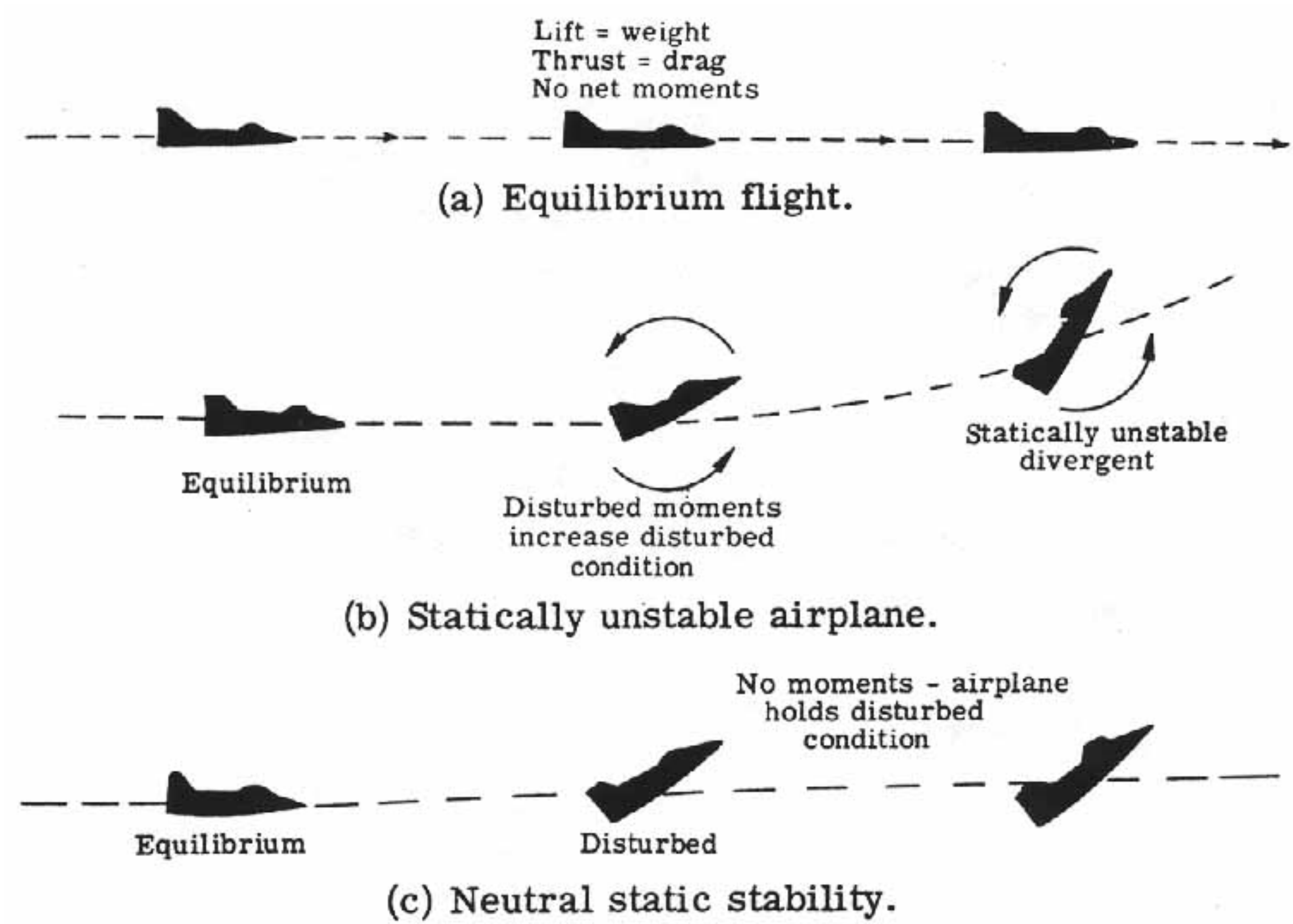
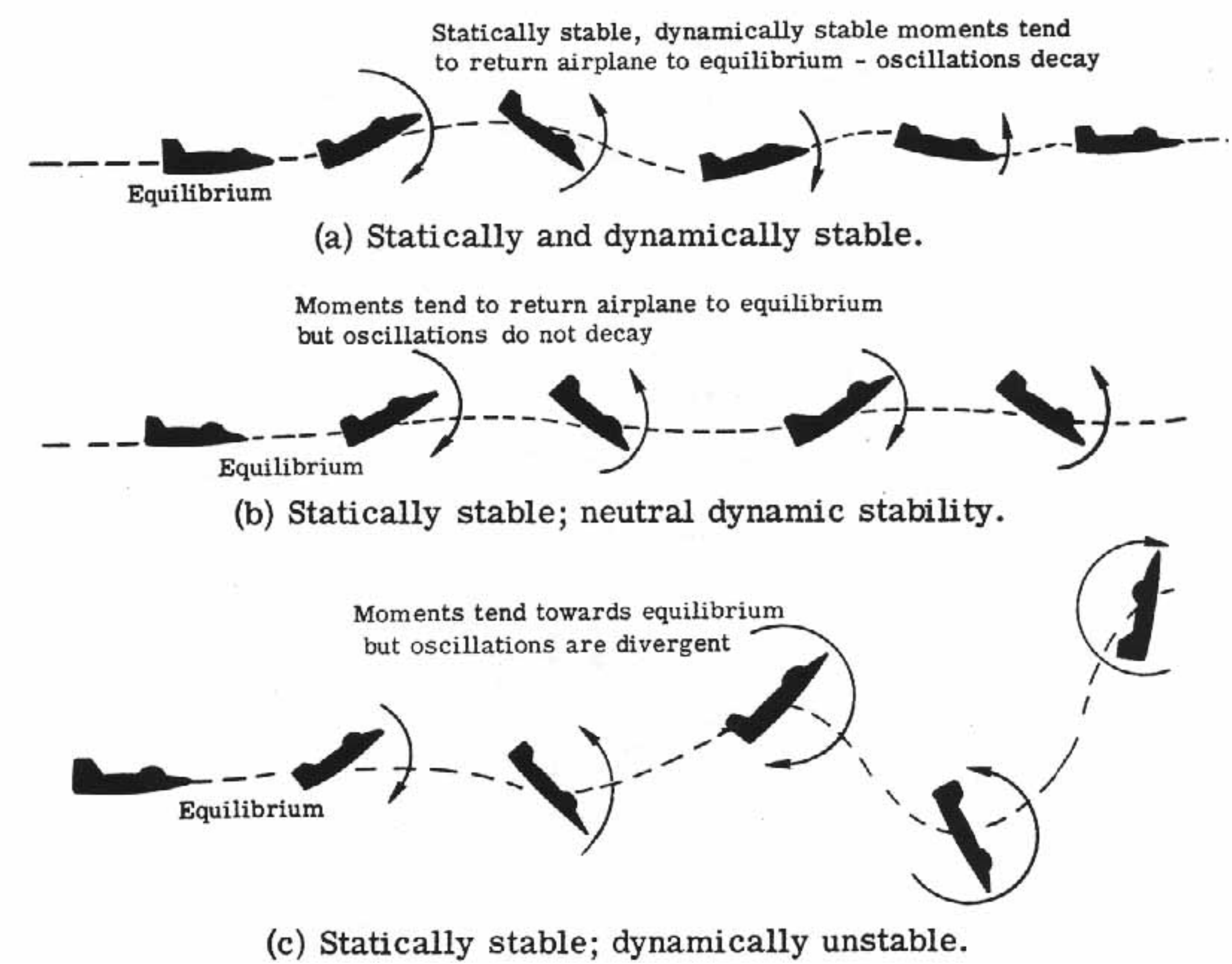
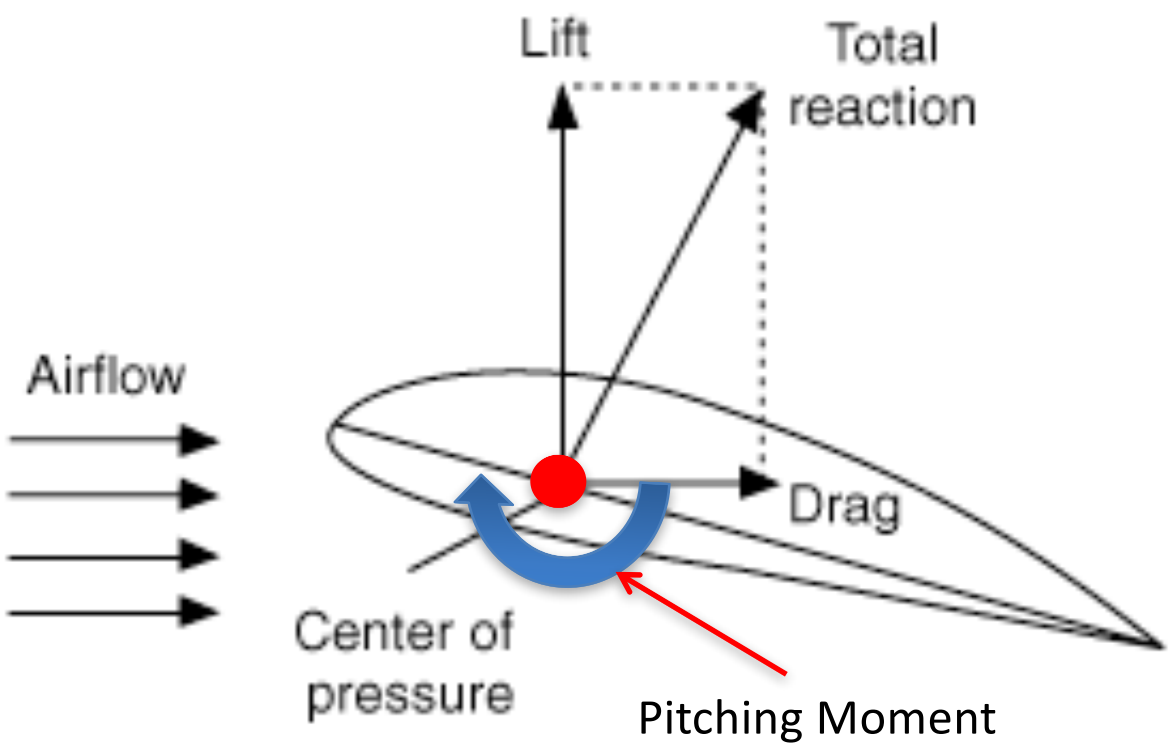
Airframe Static Stability



Airframe Dynamic Stability



Center of Pressure



•Aerodynamic *Lift, Drag, and Pitching Moment* Can be though of acting at a single point … the **Center of Pressure (*CP*)** of the vehicle

•Sometimes (*and not quite correctly*) referred to as the **Aerodynamic Center (Ac)**

•***For our purposes applied to an axisymmetric rocket configuration, Ac and CP are synonymous***

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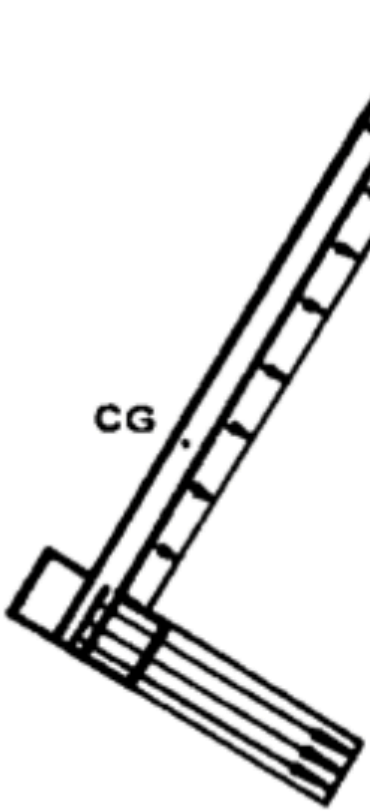
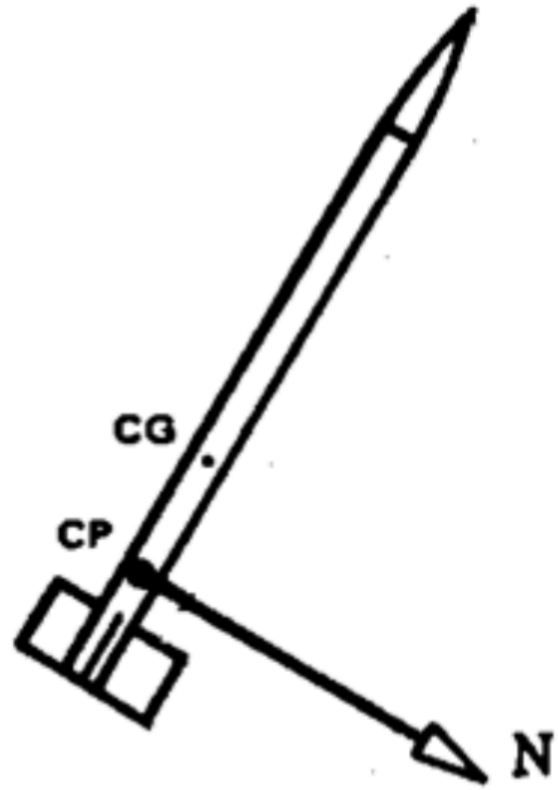
Center of Pressure



Actual Distributed Pressure Forces on Body Surface

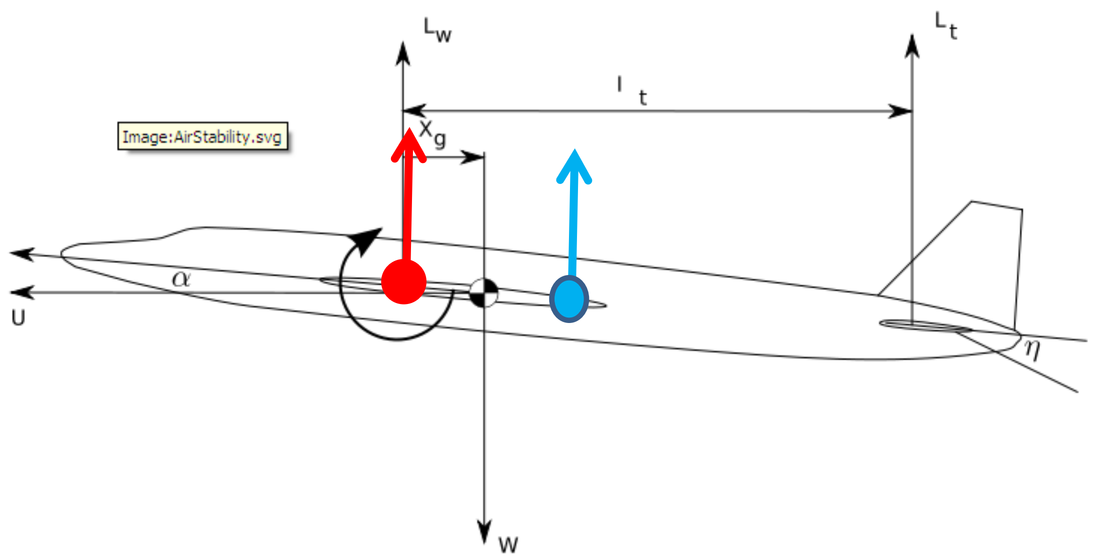


Equivalent Point Load with Identical Moment About CG

Flight Vehicle Static Stability

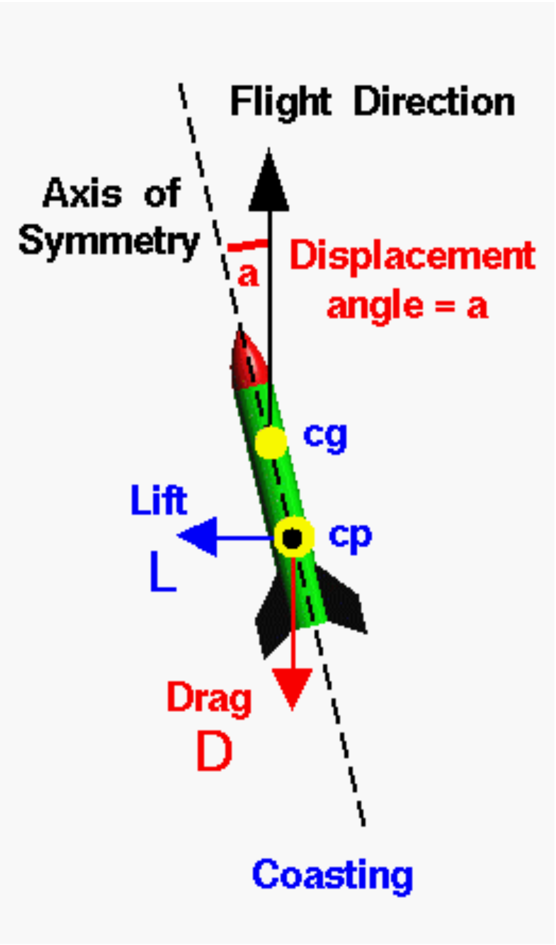
- **If center of gravity (*cg)* is forward of the *Cp*, vehicle responds to a disturbance by producing aerodynamic moment that returns Angle of attack of vehicle towards angle that existed prior to the disturbance. *(static stability)***

- **If *cg* is behind the center of pressure, vehicle will respond to a disturbance by producing an aerodynamic moment that continues to drive angle of attack further away from starting position. *(static instability)***

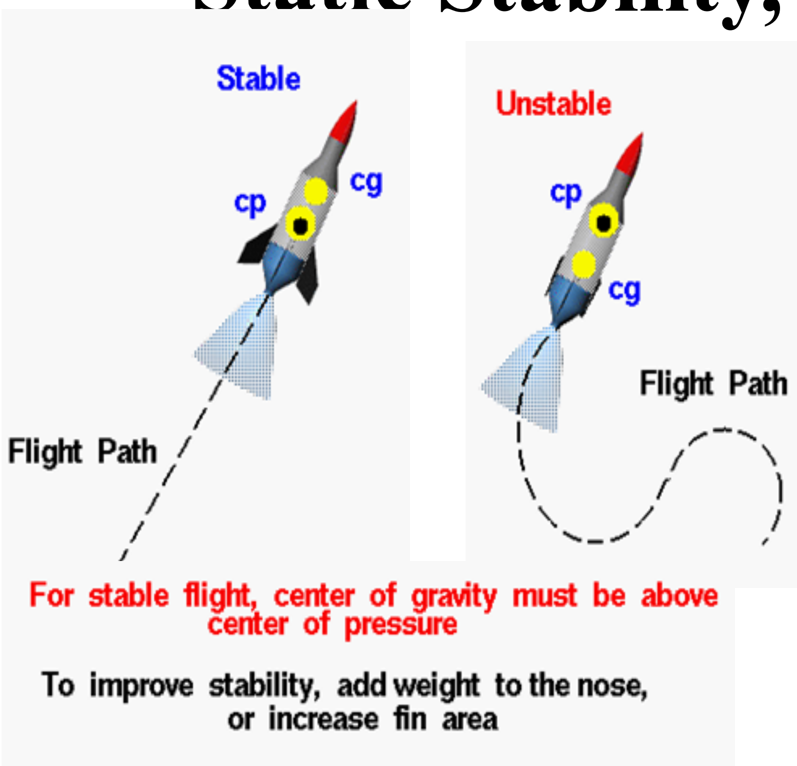


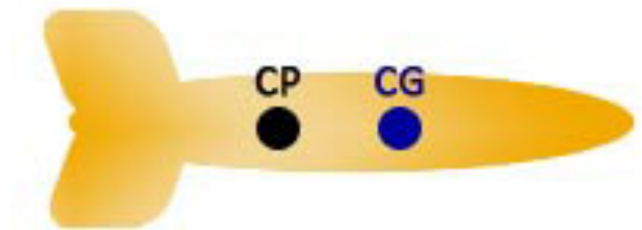
Static Stability, Rocket Flight Example

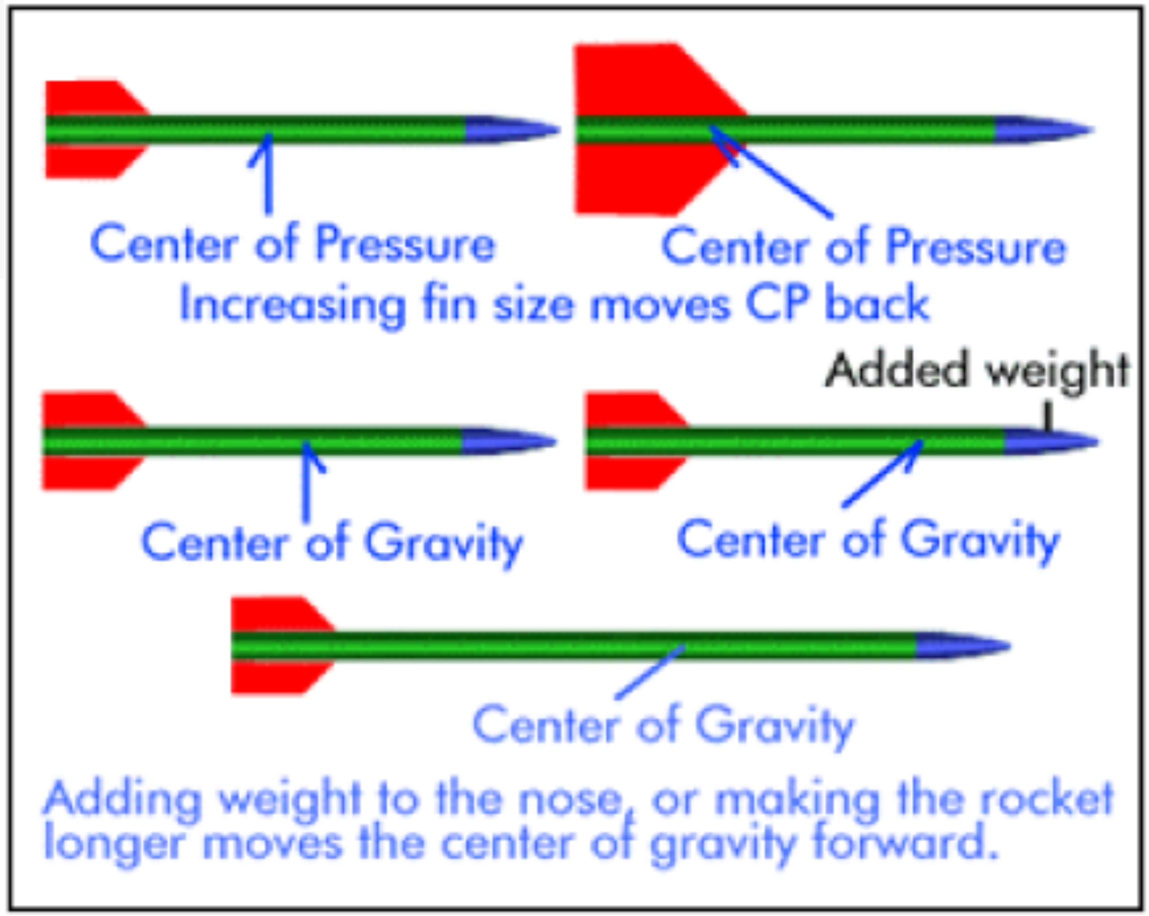
- **During flight small wind gusts or thrust offsets cause the rocket to "wobble" … change attitude**
- **Rocket rotates about center of gravity *(cg)***
- **Lift and drag both act through center of pressure *(Cp)***
- **When *cp* is behind *cg*, aerodynamic forces provide a "restoring force" … rocket is said to be "statically stable"**
- **When *Cp* ahead of *cg*, aerodynamic forces provide a "destabilizing force" … rocket is said to be "unstable"**
- ***Condition for a statically for a stable rocket is that center of pressure must be located behind longitudinal center of gravity.***



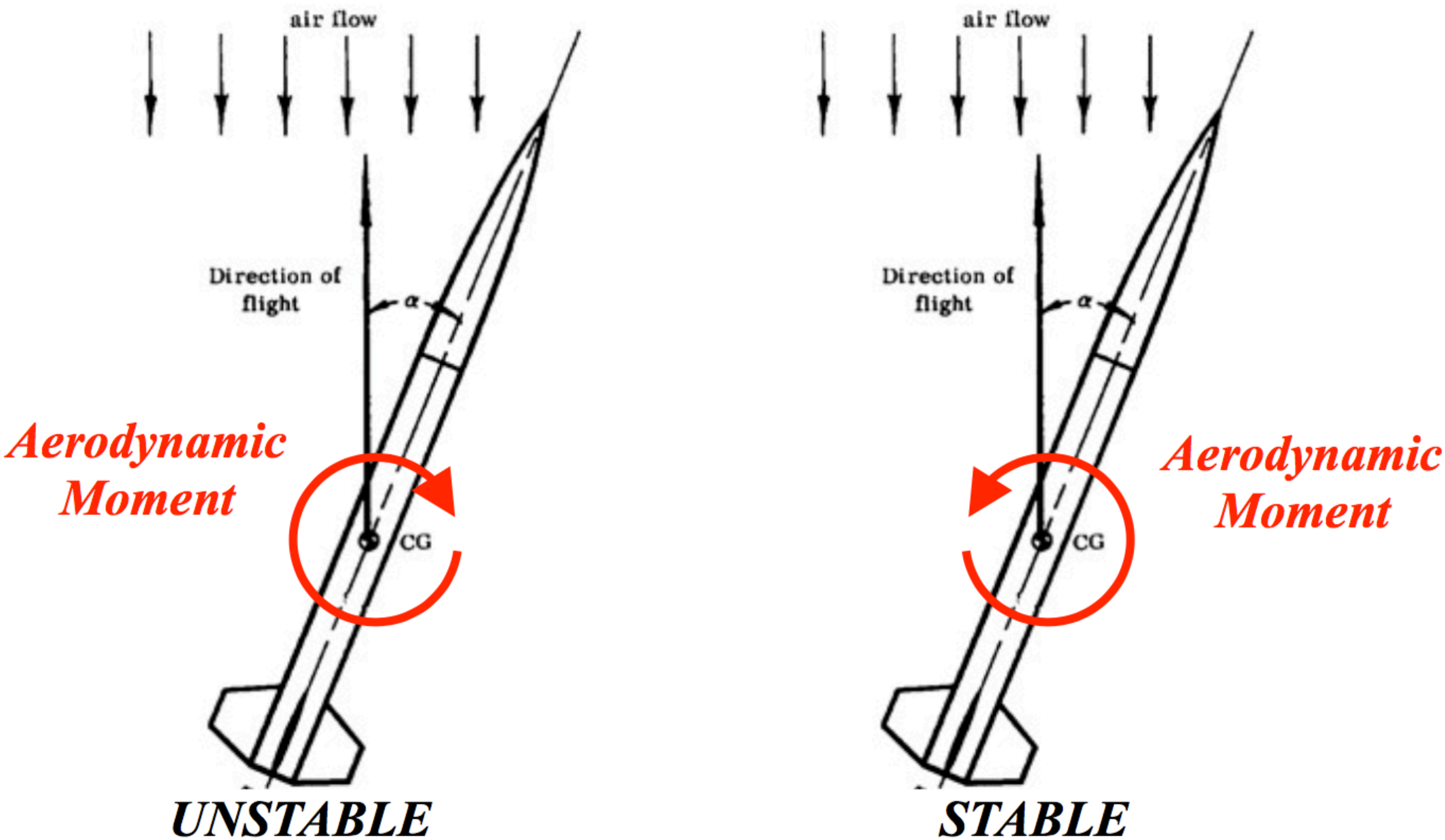
Static Stability, Rocket Flight Example (2)



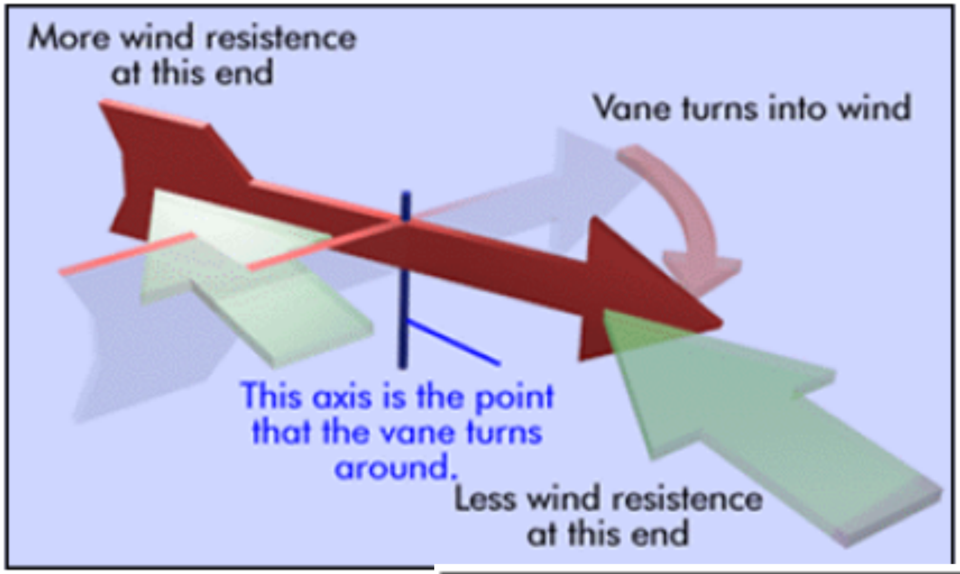


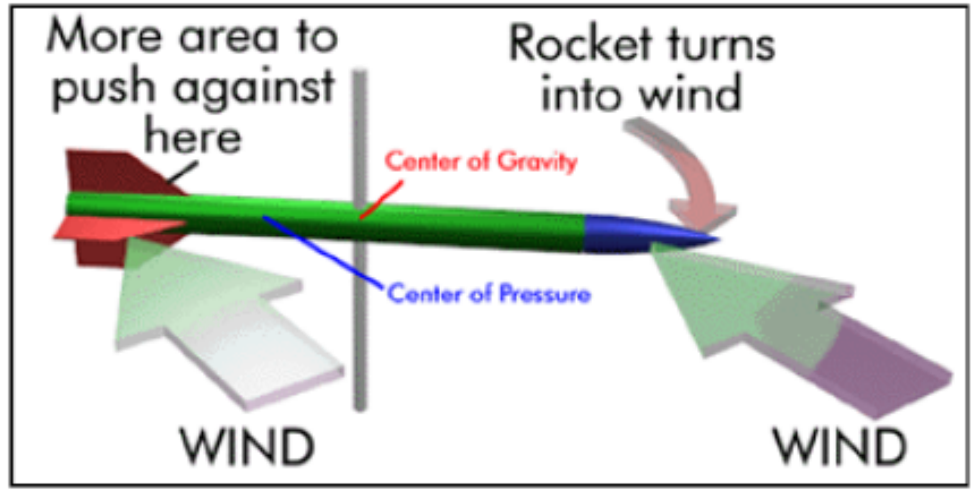


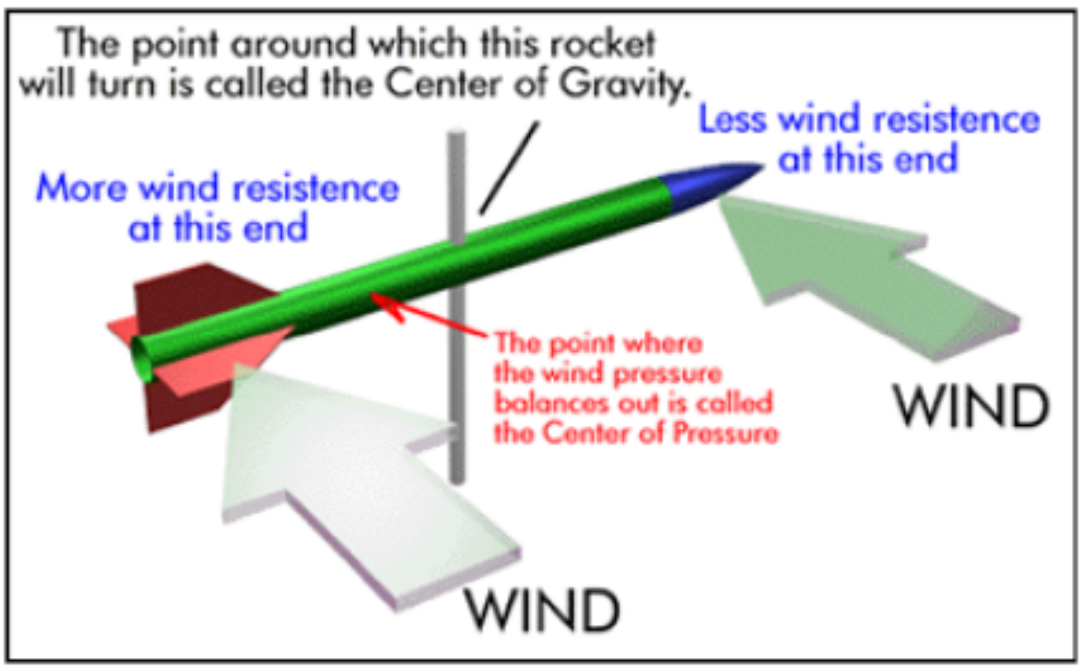
Static Stability, Rocket Flight Example (3)



Weather Vane Analogy of Static Stability





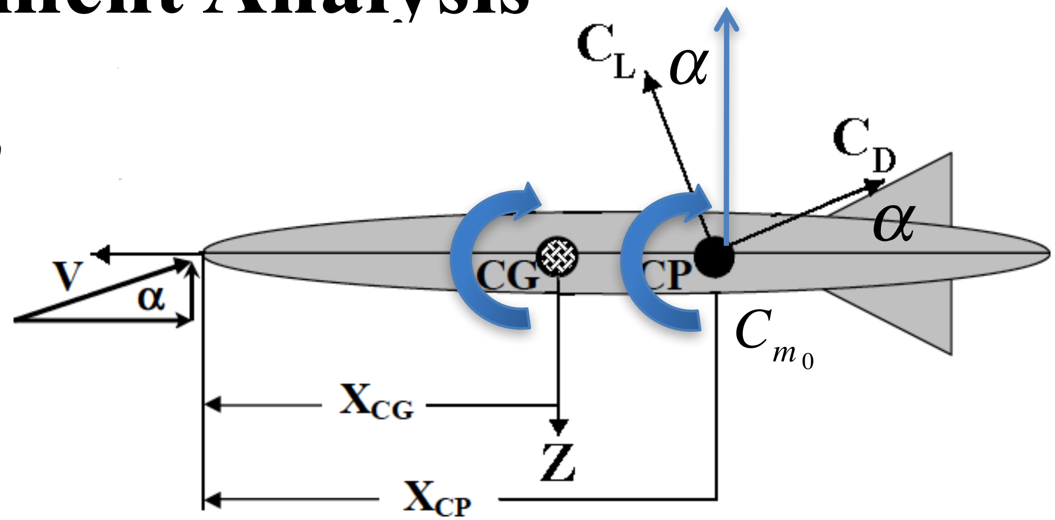


Static Margin and Pitching Moment

- ***Static margin*** used to characterize static stability and controllability of aircraft and missiles.
- ***For aircraft systems … Static margin*** defined as non-dimensional distance between center of gravity (*cg*) and aerodynamic center (*ac*) of the aircraft.
- **For missile systems … Static margin** defined as non-dimensional distance between center of gravity (*cg*) and the center of pressure (*Cp*).
- ***Static Stability*** requires that the pitching moment C_m about the rotation point, become negative as we increase C_L:

Pitching Moment Analysis

C_{m0} → *pitching moment about Cp*
$C_{m(\alpha)}$ → *pitching moment about cg*
c_{ref} = *"chord" reference length*



Sum Moments abut cg

$$C_m(\alpha) = C_{m_0} + \left(\frac{X_{cg} - X_{Cp}}{c_{ref}}\right) \cdot \left(C_L \cdot \cos\alpha + C_D \cdot \sin\alpha\right)$$

$$\rightarrow \textit{Define} \ldots \mathrm{X}_{sm} = \left(\frac{X_{Cp} - X_{cg}}{c_{ref}}\right) \quad \rightarrow C_m(\alpha) = C_{m_0} - \mathrm{X}_{sm} \cdot \left(C_L \cdot \cos\alpha + C_D \cdot \sin\alpha\right)$$

→ *Linearize Pitching* Moment *Equation*

$$\ldots C_m(\alpha) = C_{m_0} + \frac{\partial C_m}{\partial \alpha} \cdot \alpha$$

Pitching Moment Analysis (2)

$$\rightarrow \textit{Define} \ldots C_{m\alpha} = \frac{\partial C_m}{\partial \alpha} = -X_{sm} \cdot \left(\frac{\partial C_L}{\partial \alpha} \cdot \cos\alpha - C_L \cdot \sin\alpha + \frac{\partial C_D}{\partial \alpha} \cdot \sin\alpha + C_D \cdot \cos\alpha \right) =$$

$$-X_{sm} \cdot \left[\left(\frac{\partial C_L}{\partial \alpha} + C_D \right) \cdot \cos\alpha - \left(C_L - \frac{\partial C_D}{\partial \alpha} \right) \cdot \sin\alpha \right]$$

$$\rightarrow \textit{Small } \alpha \textit{ approximation} \ldots \frac{\partial C_m}{\partial \alpha} = -X_{sm} \cdot \left(\left(\frac{\partial C_L}{\partial \alpha} + C_D \right) - \left(C_L - \frac{\partial C_D}{\partial \alpha} \right) \cdot \alpha \right)$$

$$\rightarrow \textit{Neglect } \alpha^2 \text{ term} \ldots C_m(\alpha) = C_{m_0} - X_{sm} \cdot \left(\frac{\partial C_L}{\partial \alpha} + C_D \right) \cdot \alpha - \left(C_L - \frac{\partial C_D}{\partial \alpha} \right) \cdot \alpha^2$$

(the α^2 term is crossed out)

$$\rightarrow C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha} = -X_{sm} \cdot \left(\frac{\partial C_L}{\partial \alpha} + C_D \right)$$

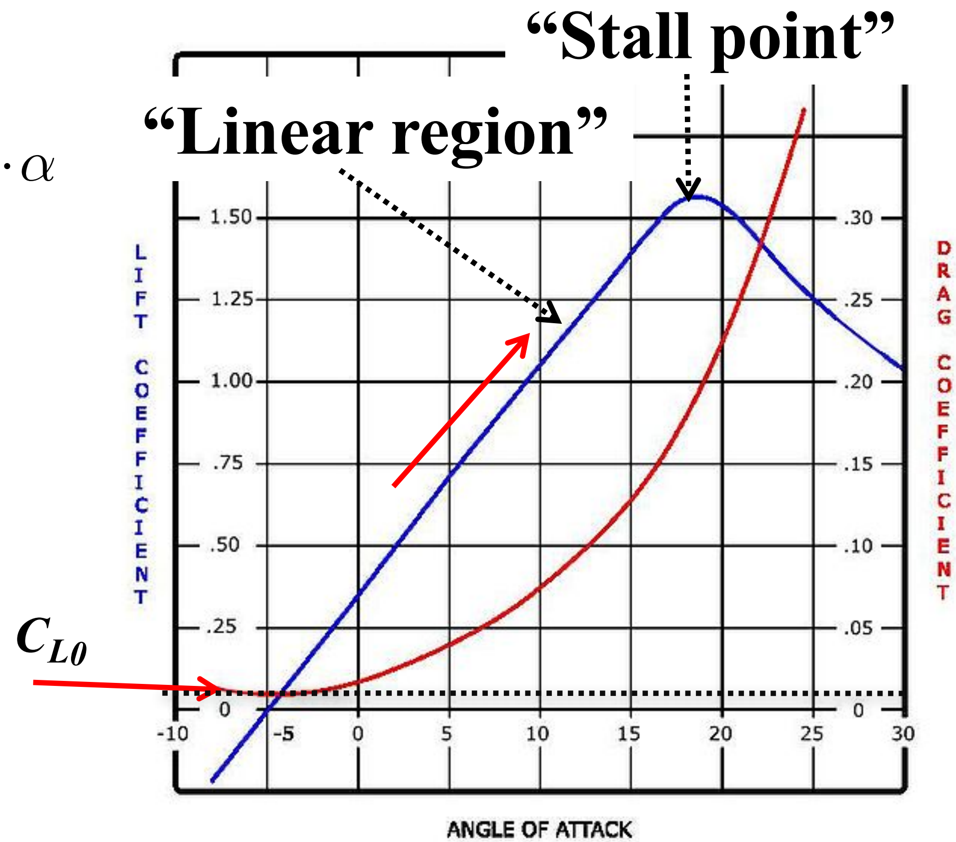
Pitching Moment Analysis (3)

Linear Airfoil Theory

$$C_L = C_{L_0} + \frac{\partial C_L}{\partial \alpha}\cdot\alpha \equiv \; C_{L_0} + C_{L_0} + \frac{\partial C_L}{\partial \alpha}\cdot\alpha$$

$$C_D = C_{D_0} + \frac{{C_L}^2}{\pi\cdot\varepsilon\cdot A_r}$$

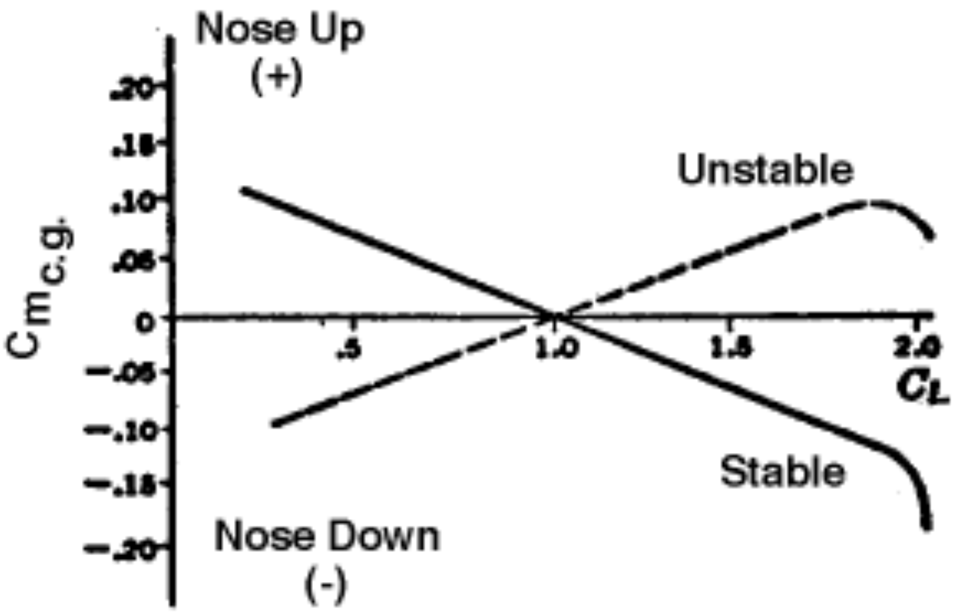
$$\rightarrow \left[\frac{\frac{\partial C_L}{\partial \alpha}}{C_D}\right] > 0$$



$$\frac{\partial C_m}{\partial \alpha} \rightarrow "C_{m_\alpha}" .....Ideally...$$

$$C_{m_\alpha} = -X_{sm}\left(\frac{\partial C_L}{\partial \alpha} + C_D\right) \rightarrow \left[\frac{\frac{\partial C_L}{\partial \alpha}}{C_D}\right] > 0 \rightarrow X_{sm} > 0...\text{static stability} \rightarrow C_{m_\alpha} < 0...\text{static stability}$$

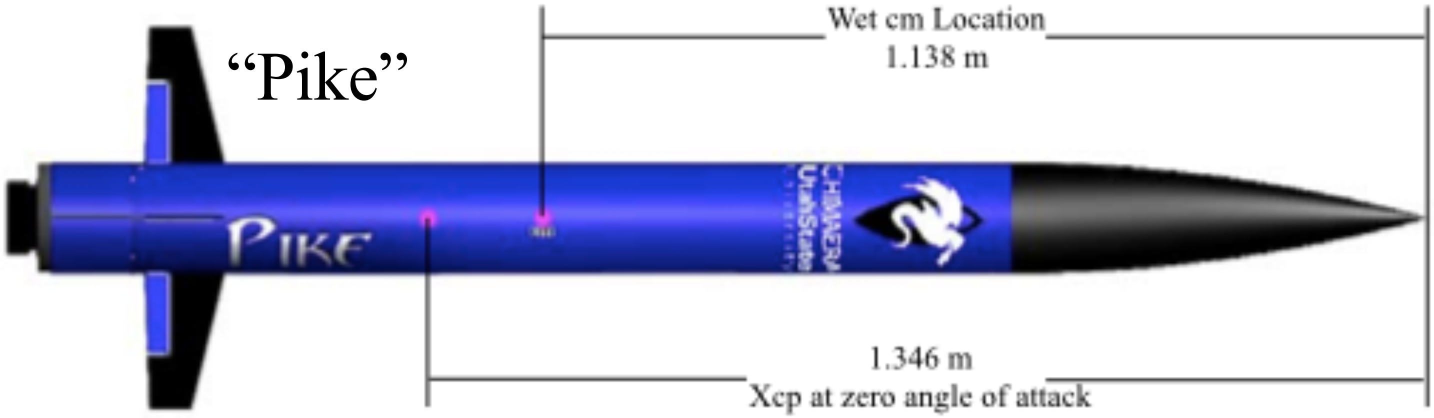
Pitching Moment Analysis (4)



For a Rocket Static margin is the distance between the CG and the CP; divided by body tube diameter.

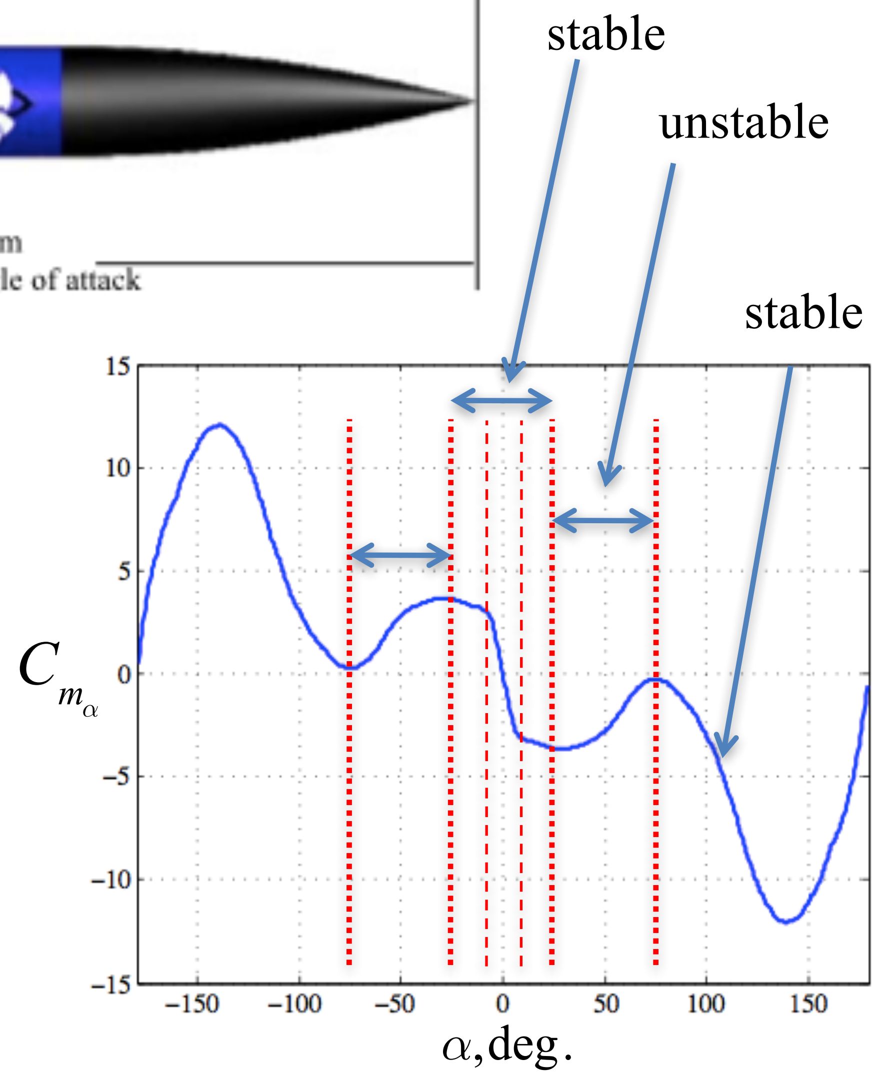
$$X_{sm} > 0 \ldots \underline{\text{static stability}} \rightarrow \boxed{C_{m_\alpha} < 0 \ldots \underline{\text{static stability}}}$$

Pitching Moment Analysis (6)

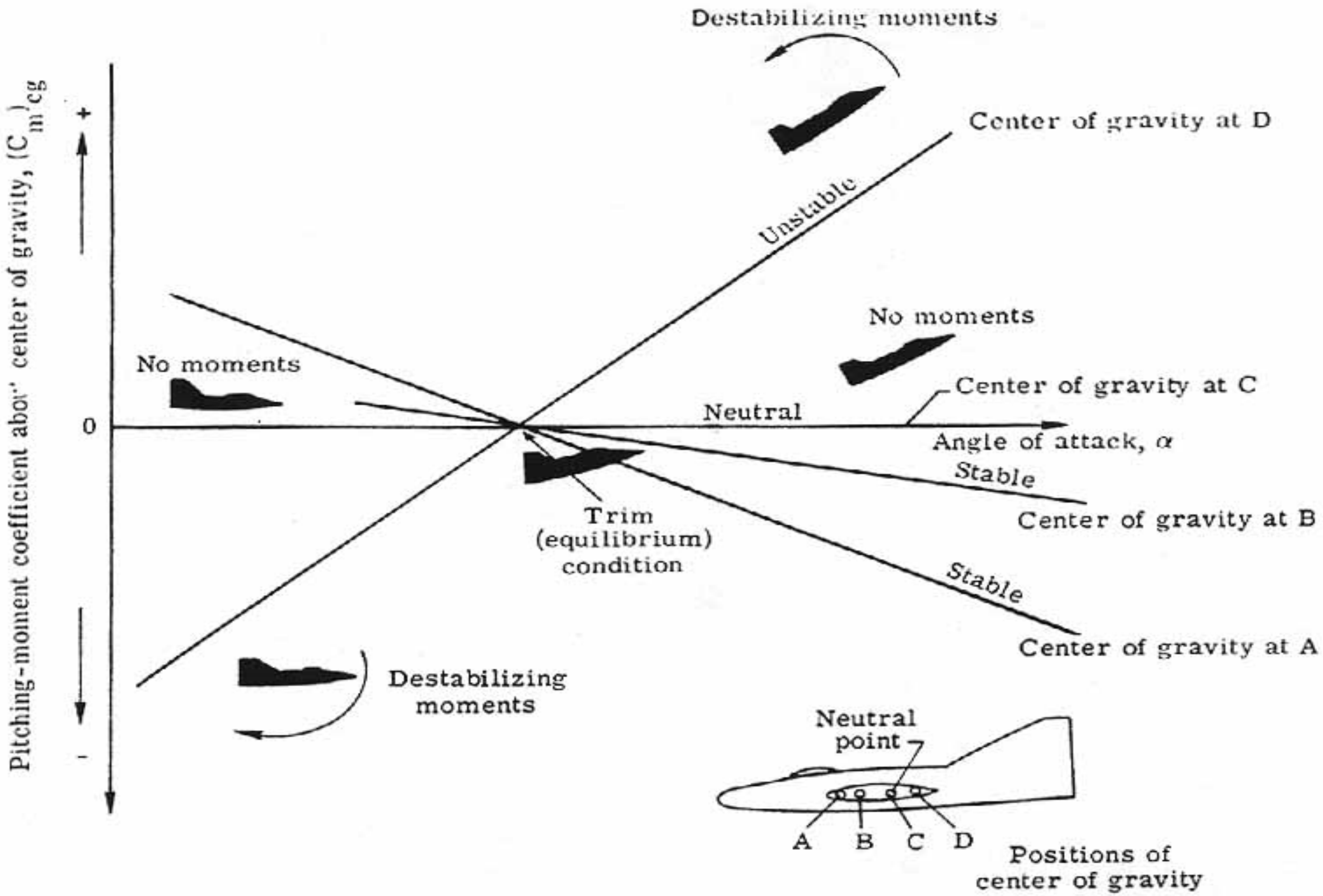


- Even $X_{sm} > 0$ (static stability) rockets become unstable at higher angles of attack

- "Strong stability" region limited to very low angle of attack range



Pitching Moment Analysis (5)



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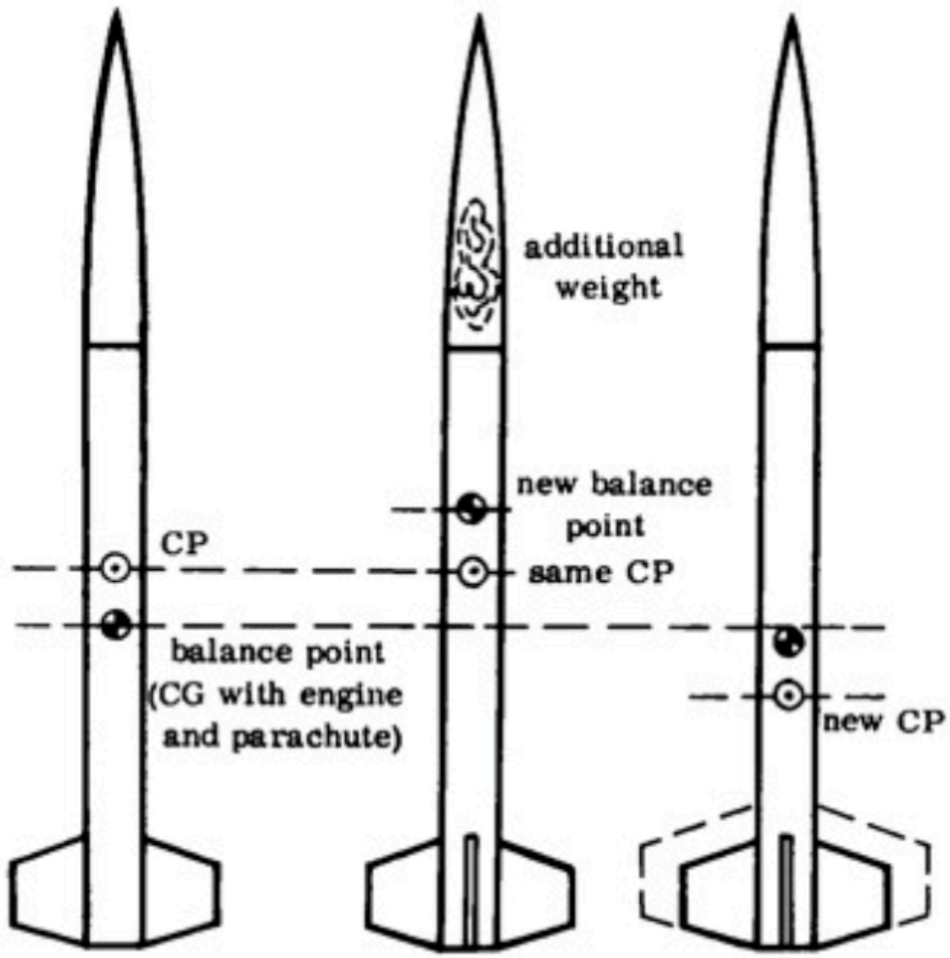
Achieving Static Stability

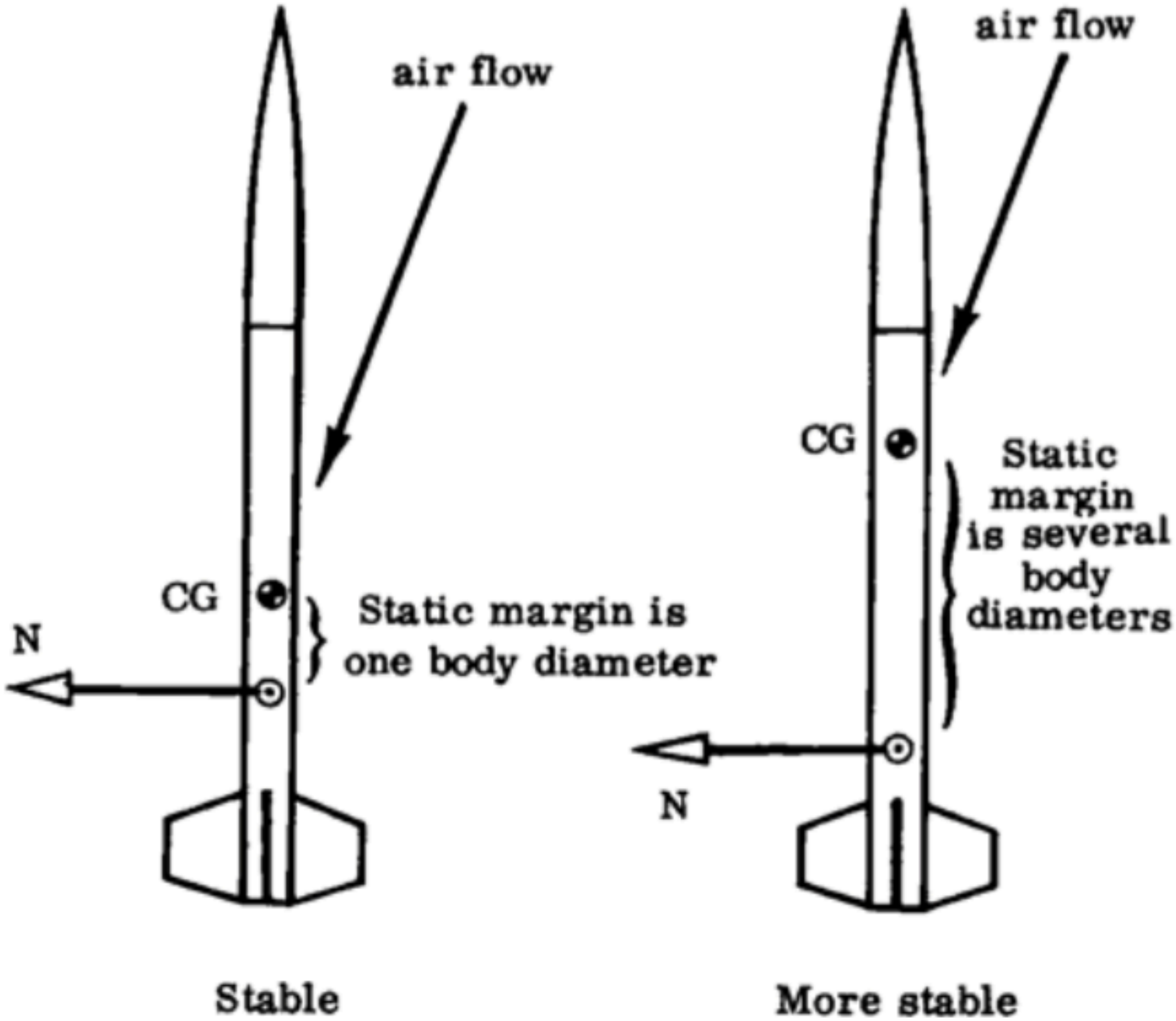


40a
Unstable configuration

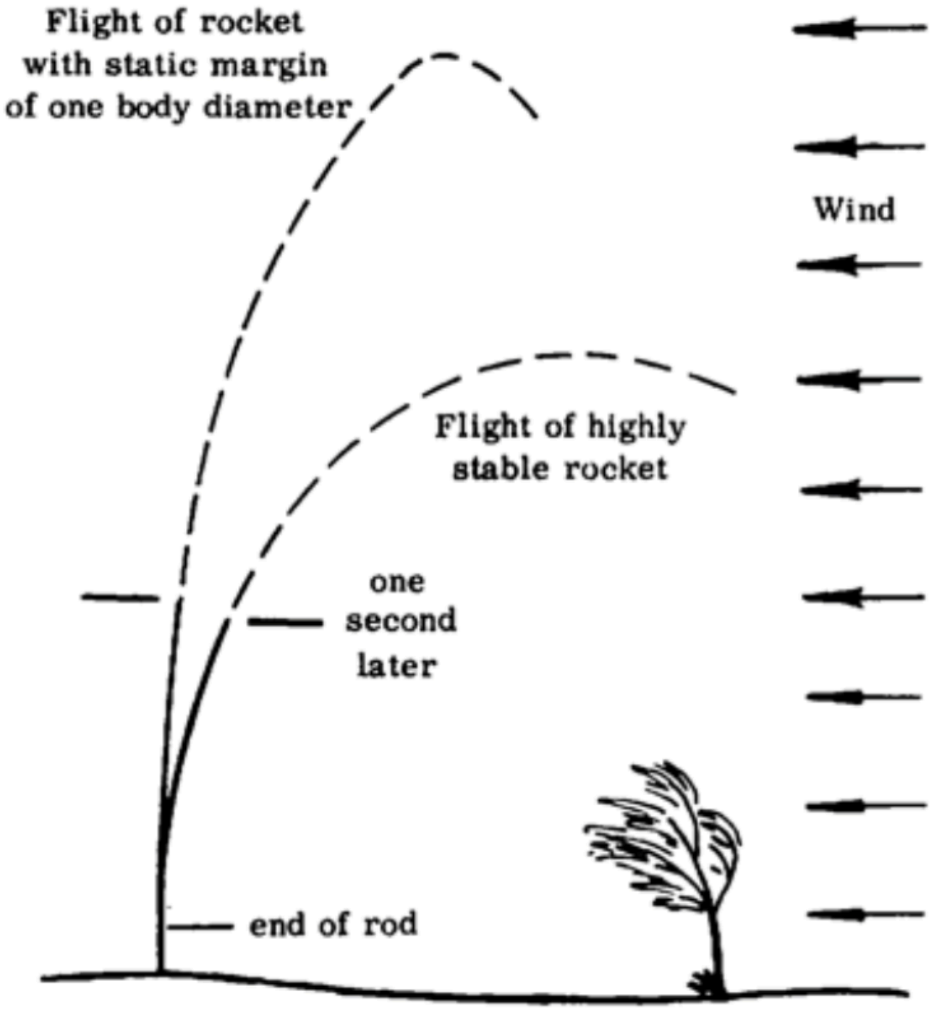
40b
Add weight to nose

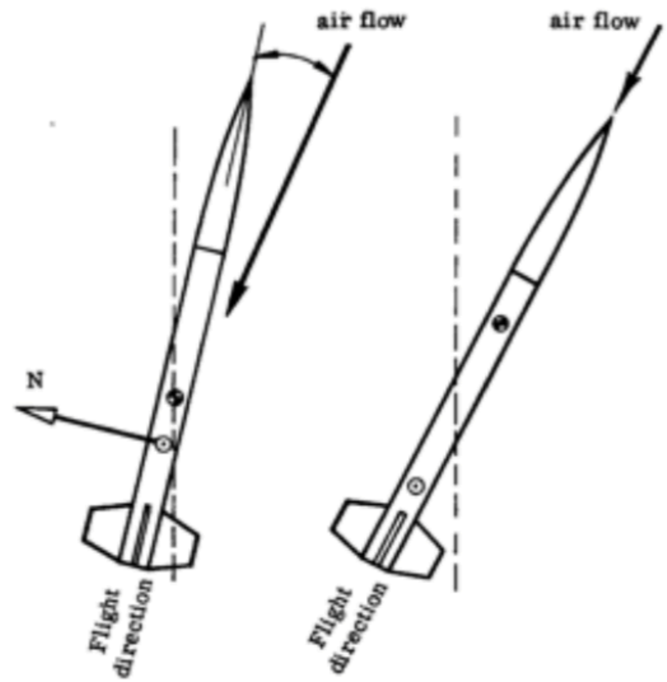
40c
Increase fin area (note that CG shifts slightly to the rear)

Static Margin = Degree of Static Stability



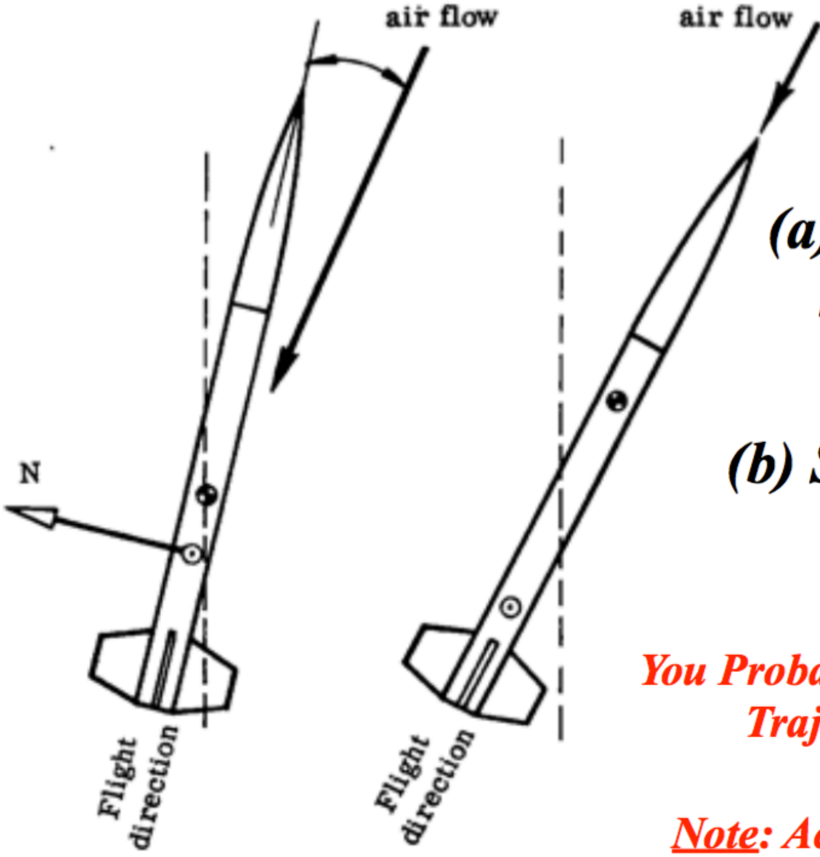
How Much Static Stability?





One Second After Launch

How Much Static Stability? (2)



(a) Modest Stability → N May Have Significant Effect on Trajectory

(b) Strong Stability → N Small (Rocket Quickly Reorients to α = 0)

You Probably Want Strong Stability to Enable Accurate Trajectory Calculation Without Modeling N

<u>Note</u>: Achieving Suitable Crosswind Flight Might Require Weaker Stability = More Difficult Simulation = More "Expensive "Rocket

Calculating the Static Margin

•Key to calculating static margin is estimate of location of longitudinal center of pressure at low angles of attack

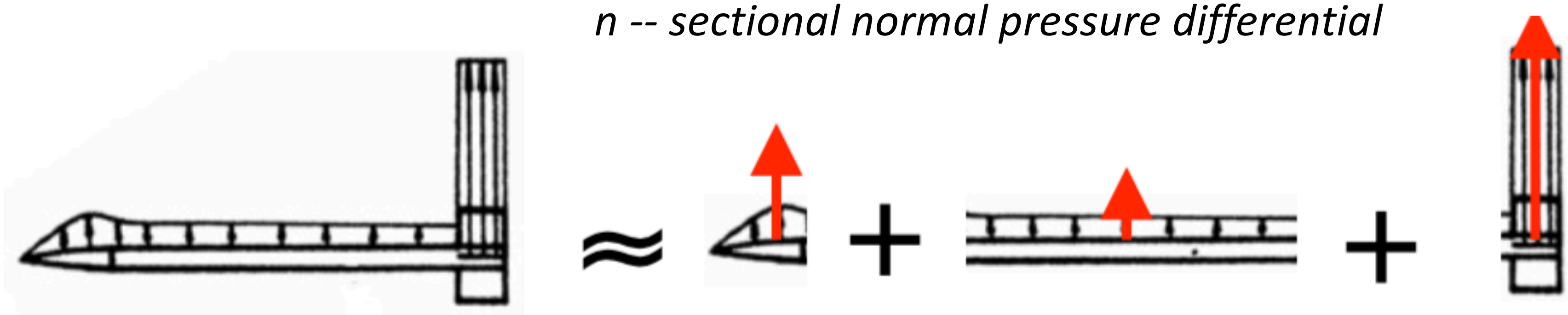
•Barrowman equations provide simple, accurate technique for Axi-symmetric rockets

•*cg* is measured as the longitudinal balance point of the rocket.

•As a rule of thumb, *Cp* distance should be aft of the *cg* by at least one rocket diameter. -- *"One Caliber stability"*.

Calculating the Static Margin (2)

N -- total normal FORCE
n -- sectional normal pressure differential



- Neglect Small Contribution of Body (Valid for Small Angles Only)
- Compute Point Force and "Center of Pressure" For Nose and Tail Distributed Loads Independently (Assume No Aerodynamic Interactions / Interference) … + transitions + boat tail

$$N = \int_{x=0}^{x=L} n(x)dx \approx \int_{NOSE} n(x)dx + 0 + \int_{TAIL} n(x)dx$$

Calculating the Static Margin (3)

$$C_N @ \frac{N}{\frac{1}{2}\rho V^2 A} \approx = \frac{N_{nose} + N_{tail} + N_{trns} + N_{boat} + \ldots}{\bar{q} \cdot A_{ref}} =$$

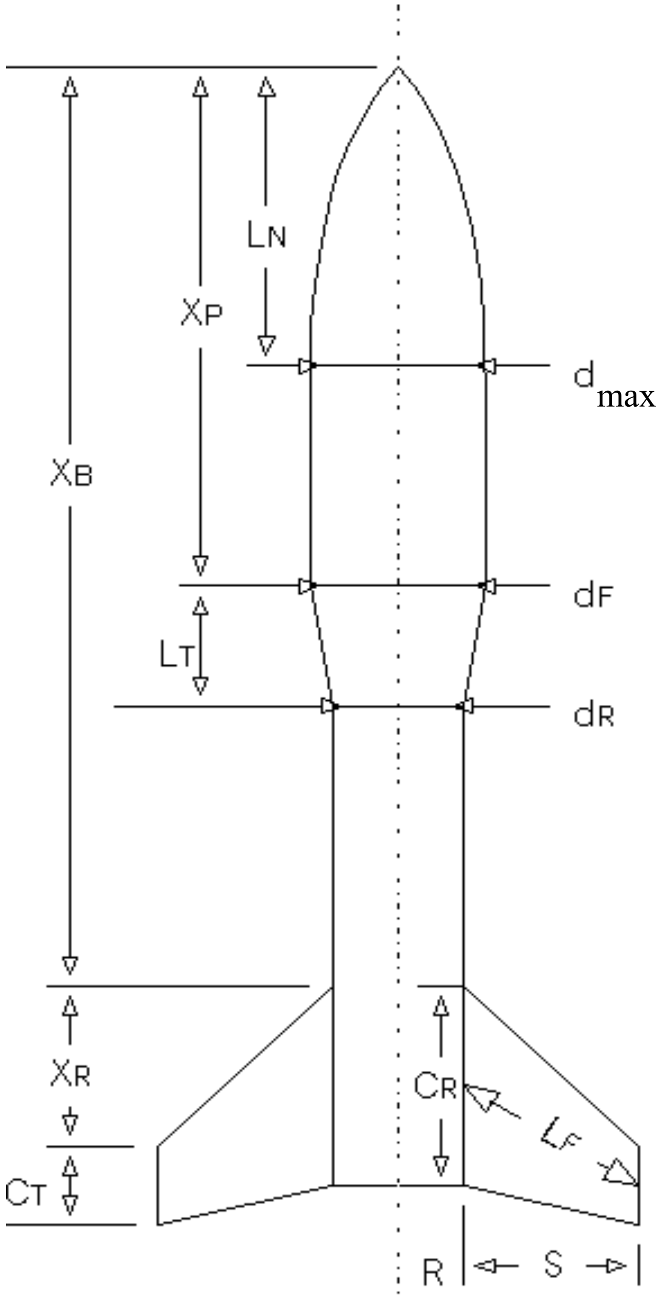
$$C_{N_{nose}} + C_{N_{tail}} + C_{N_{trns}} + C_{N_{boat}} + \ldots$$

For Small Angles of Attack, α, Force Coefficients Are Linearly Related to α

$$C_N = C_{N_\alpha}\alpha = \left(C_{N\alpha_{nose}} + C_{N\alpha_{tail}} + C_{N\alpha_{trns}} + C_{N\alpha_{boat}} + \ldots\right)\cdot\alpha$$

Simplified "Barrowman Equations" Give Formulas for Derivatives on RHS in Terms of Nose, Fin & Transition Geometry

Calculating the Static Margin (4)



Parameter Definitions

L_N = length of nose

d_{max} = *maximum body diameter*

d_F = diameter at front of transition

d_R = diameter at rear of transition

L_T = length of transition

X_P = distance from tip of nose to front of transition

C_R = fin root chord

C_T = fin tip chord

S = fin semispan

L_F = length of fin mid-chord line

R = radius of body at aft end

X_R = distance between fin root leading edge and fin tip leading edge parallel to body

X_B = distance from nose tip to fin root chord leading edge

N = number of fins

Calculating the Static Margin (3)

XN = center of pressure location for nose section

XT = center of pressure location for transitions

$(CN_\alpha)N$ = normal force derivative for nose

$(CN_\alpha)T$ = normal force derivative for transition

Nose Cone Terms

- $(C_{N_\alpha})_N = 2$
- For Cone: $X_N = 0.666L_N$
- For Ogive: $X_N = 0.466L_N$

Conical Transition Terms

$$(C_{N_\alpha})_T = 2\left[\left(\frac{d_R}{d_{max}}\right)^2 - \left(\frac{d_F}{d_{max}}\right)^2\right] \rightarrow X_T = X_P + \frac{L_T}{3}\left[1 + \frac{1 - \frac{d_F}{d_R}}{1 - \left(\frac{d_F}{d_R}\right)^2}\right]$$

Calculating the Static Margin (4)

Fin Terms

XF = center of pressure location for fin groups

$(CN_\alpha)F$ = Normal force derivative for fin group

$$(C_{N_\alpha})_F = \left[1+\frac{R}{S+R}\right]\left[\frac{4N\left(\frac{S}{d_{max}}\right)^2}{1+\sqrt{1+\left(\frac{2L_F}{C_R+C_T}\right)^2}}\right]$$

$$X_F = X_B + \frac{X_R}{3}\frac{(C_R+2C_T)}{(C_R+C_T)} + \frac{1}{6}\left[(C_R+C_T) - \frac{(C_R C_T)}{(C_R+C_T)}\right]$$

Finding the Center of Pressure

- Sum up coefficients:

$$(C_{N_\alpha})_R = (C_{N_\alpha})_N + (C_{N_\alpha})_T + (C_{N_\alpha})_F$$

- Find CP Distance from Nose Tip:

$$X_{cp} = \frac{(C_{N_\alpha})_N X_N + (C_{N_\alpha})_T X_T + (C_{N_\alpha})_F X_F}{(C_{N_\alpha})_R}$$

Static Margin (X_{sm}) =

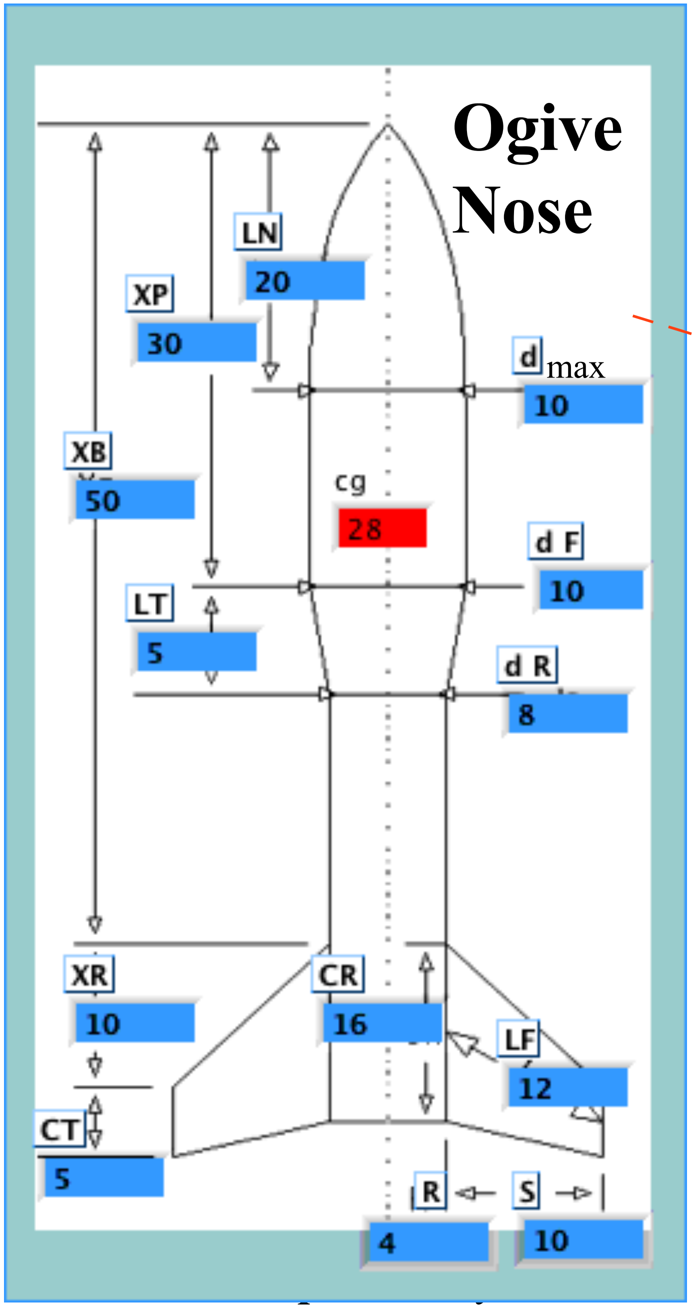
$(X_{cp} - X_{cg})/d_{max}$

$(CN_\alpha)R$ = total normal force derivative

X – measured aft from nose of vehicle

$Small\ \alpha \rightarrow C_{N_\alpha} \cong C_{L_\alpha}$

Example: Stable Static Margin Vehicle



L_N = *20 cm*

d_{max} = *10 cm*

d_F = *10 cm*

d_R = *8 cm*

L_T = *5 cm*

X_P = *30 cm*

C_R = *10 cm*

C_T = *5 cm*

S = *10 cm*

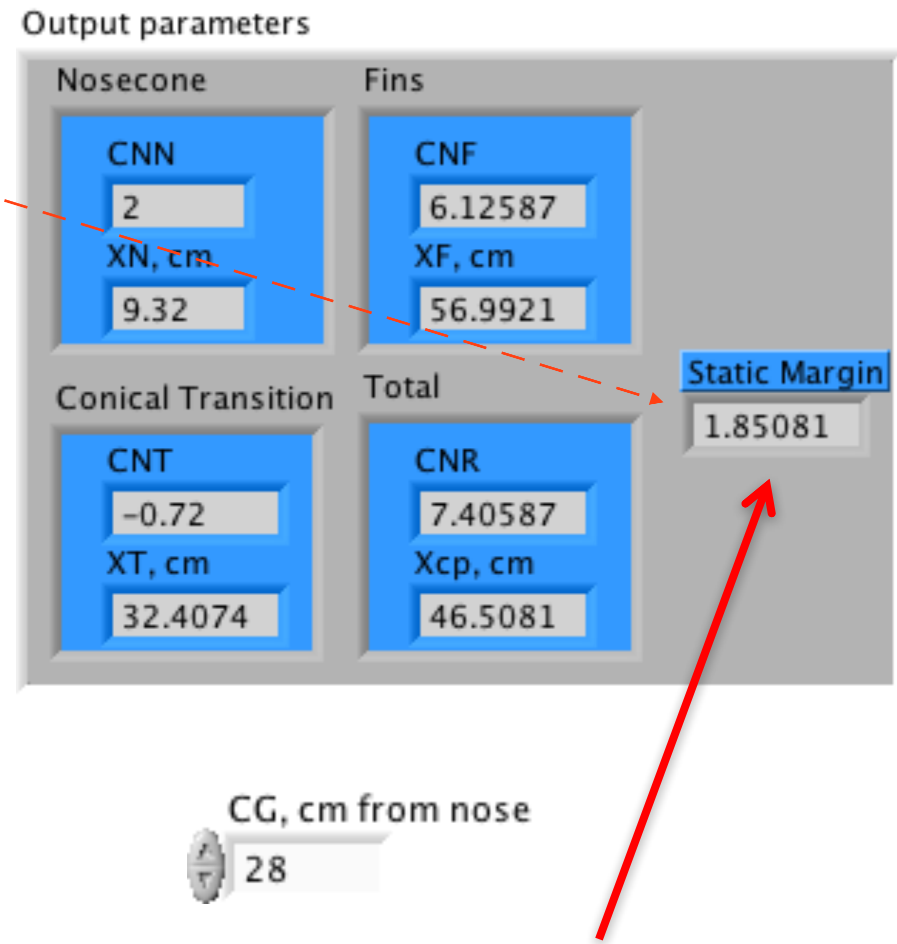
L_F = *12 cm*

R = *4 cm*

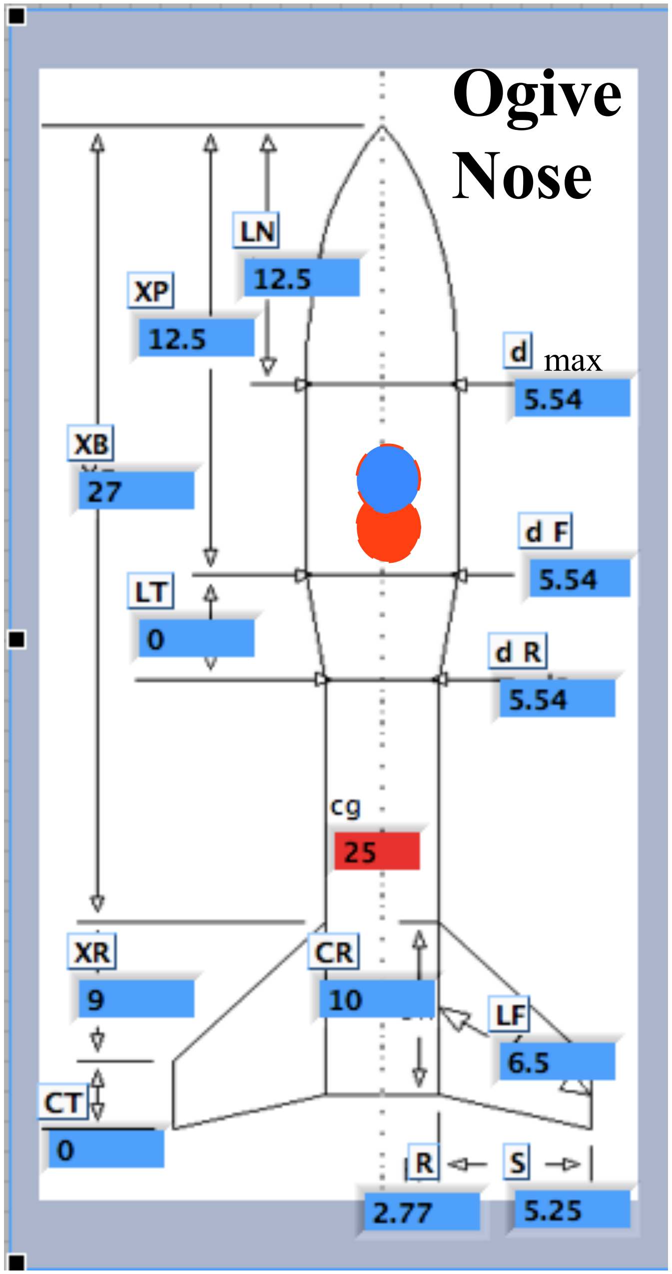
X_R = *16 cm*

X_B = *50 cm*

N = *3*



Example: Unstable Static Margin Calculation



L_N = *12.5 cm*

d_{max} = *5.54 cm*

d_F = *5.54 cm*

d_R = *5.54 cm*

L_T = *0 cm*

X_P = *12.5 cm*

C_R = *10 cm*

C_T = *0 cm*

S = *5.25 cm*

L_F = *6.5 cm*

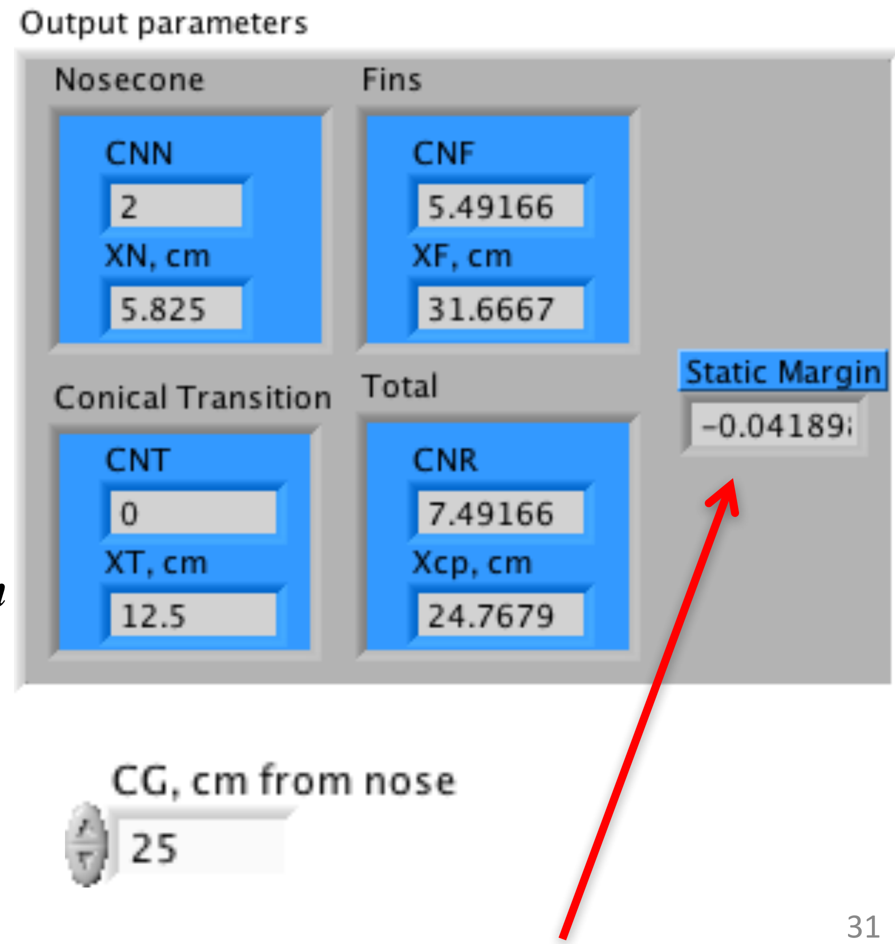
R = *2.77 cm*

X_R = *9 cm*

X_B = *27 cm*

N = *3*

Constant diameter tube (No transition section)



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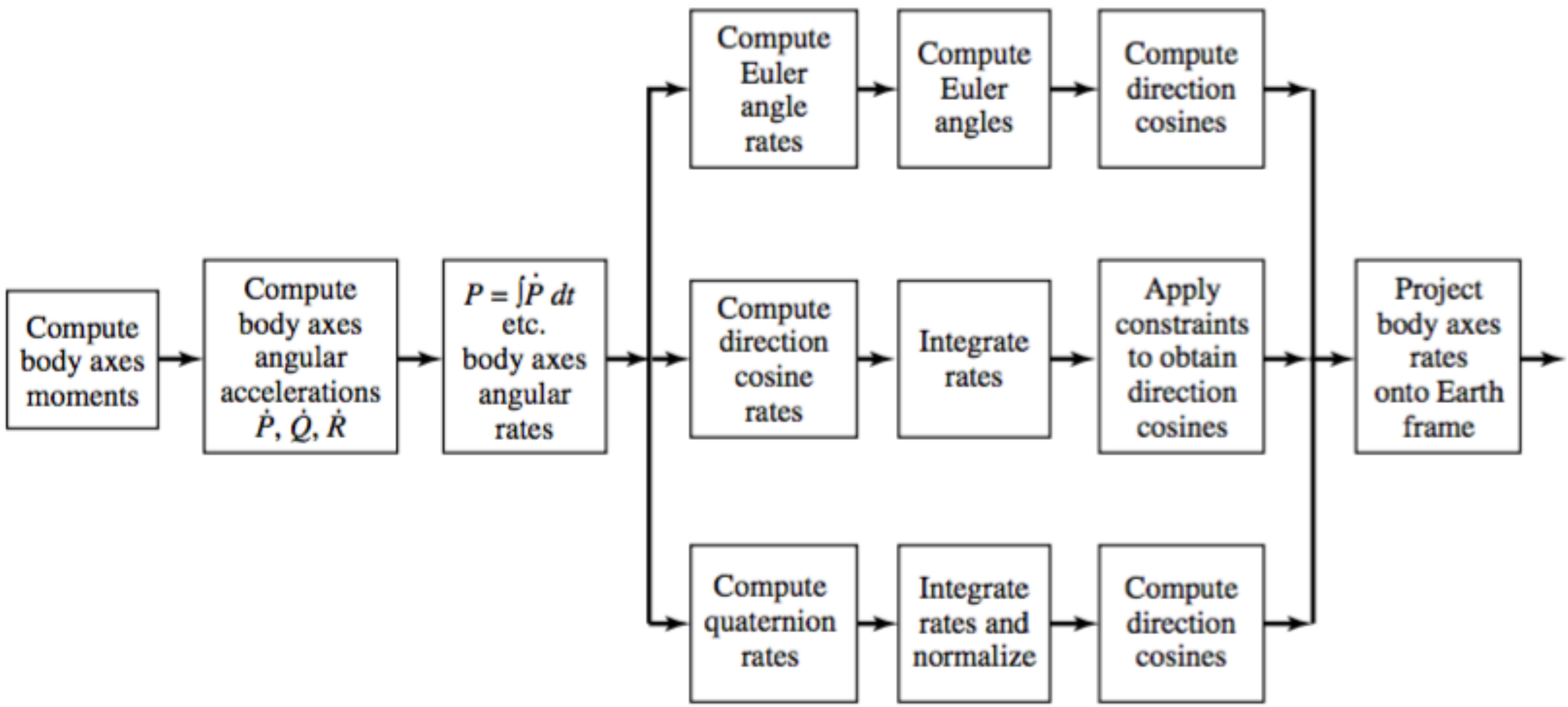
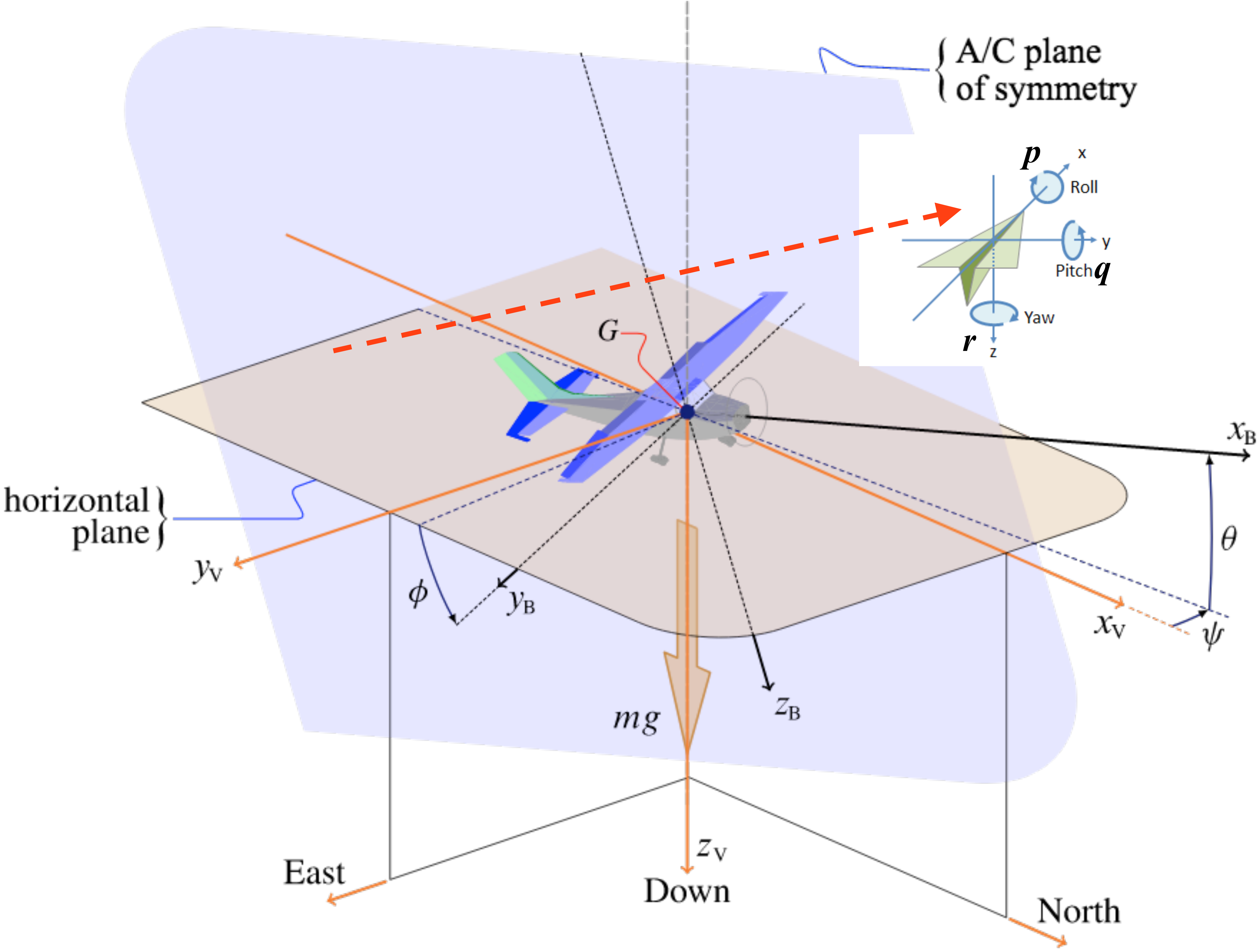
Rotational Dynamics of a Rigid Body.





A/C plane
of symmetry
p
x
Roll
y
Pitch q
Yaw
r
z
G
horizontal
plane
x_B
θ
y_V
ϕ
y_B
x_V
ψ
z_B
mg
z_V
East
Down
North

Simplified Pitch Axis Dynamics

General Rotational Dynamics

$$\begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \cdot \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{pmatrix} q \cdot r\left(I_y - I_z\right) + \left(q^2 - r^2\right) I_{yz} + p \cdot q\left(I_{xz}\right) - r \cdot p\left(I_{xy}\right) \\ r \cdot p\left(I_z - I_x\right) + \left(r^2 - p^2\right) I_{xz} + q \cdot r\left(I_{xy}\right) - p \cdot q\left(I_{yz}\right) \\ p \cdot q\left(I_x - I_y\right) + \left(p^2 - q^2\right) I_{xy} + r \cdot p\left(I_{yz}\right) - q \cdot r\left(I_{xz}\right) \end{pmatrix} + \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

Collected Equations

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \cdot \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \left(I_{yy}-I_{zz}\right)\cdot q\cdot r + I_{yz}\cdot\left(q^2-r^2\right) + I_{xz}\cdot p\cdot q - I_{xy}\cdot p\cdot r \\ \left(I_{zz}-I_{xx}\right)\cdot p\cdot r + I_{xz}\cdot\left(r^2-p^2\right) + I_{xy}\cdot q\cdot r - I_{yz}\cdot p\cdot q \\ \left(I_{xx}-I_{yy}\right)\cdot p\cdot q + I_{xy}\cdot\left(p^2-q^2\right) + I_{yz}\cdot p\cdot r - I_{xz}\cdot q\cdot r \end{bmatrix} + \begin{bmatrix} M_x \\ L_y \\ N_z \end{bmatrix}$$

Principal Axes

$$I \approx Real,symmetrix \rightarrow Find\ Axis\ Where \rightarrow \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} = U^T\cdot\Gamma\cdot U \rightarrow \Gamma = \begin{bmatrix} I'_{xx} & 0 & 0 \\ 0 & I'_{yy} & 0 \\ 0 & 0 & I'_{zz} \end{bmatrix}$$

with no inertia cross products

$$Euler's\ Equations \rightarrow I_{xy} = I_{xz} = I_{yz} = 0 \rightarrow Simplify$$

$$\begin{bmatrix} I_{xx}\cdot\dot{p} \\ I_{yy}\cdot\dot{q} \\ I_{zz}\cdot\dot{r} \end{bmatrix} = \begin{bmatrix} \left(I_{yy}-I_{zz}\right)\cdot q\cdot r \\ \left(I_{zz}-I_{xx}\right)\cdot p\cdot r \\ \left(I_{xx}-I_{yy}\right)\cdot p\cdot q \end{bmatrix} + \begin{bmatrix} M_x \\ L_y \\ N_z \end{bmatrix} \rightarrow \begin{bmatrix} \dot{p} = \left(\dfrac{I_{yy}-I_{zz}}{I_{xx}}\right)\cdot q\cdot r + \dfrac{M_x}{I_{xx}} \\ \dot{q} = \left(\dfrac{I_{zz}-I_{xx}}{I_{yy}}\right)\cdot p\cdot r + \dfrac{L_y}{I_{yy}} \\ \dot{r} = \left(\dfrac{I_{xx}-I_{yy}}{I_{zz}}\right)\cdot p\cdot q + \dfrac{N_z}{I_{zz}} \end{bmatrix}$$

Collected Euler Equations

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{I_{yy}-I_{zz}}{I_{xx}}\right)\cdot q\cdot r \\ \left(\dfrac{I_{zz}-I_{xx}}{I_{yy}}\right)\cdot p\cdot r \\ \left(\dfrac{I_{xx}-I_{yy}}{I_{zz}}\right)\cdot p\cdot q \\ p + \tan\theta\cdot\sin\phi\cdot q + \tan\theta\cdot\cos\phi\cdot r \\ \cos\phi\cdot q - \sin\phi\cdot r \\ \dfrac{\sin\phi}{\cos\theta}\cdot q + \dfrac{\cos\phi}{\cos\theta}\cdot r \end{bmatrix} + \begin{bmatrix} \dfrac{M_x}{I_{xx}} \\ \dfrac{M_y}{I_{yy}} \\ \dfrac{N_z}{I_{zz}} \\ 0 \\ 0 \\ 0 \end{bmatrix} + disturbance\ torques$$

Simplified Pitch Axis Dynamics

General Rotational Dynamics

$$\begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \cdot \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{pmatrix} q \cdot r\left(I_y - I_z\right) + \left(q^2 - r^2\right) I_{yz} + p \cdot q\left(I_{xz}\right) - r \cdot p\left(I_{xy}\right) \\ r \cdot p\left(I_z - I_x\right) + \left(r^2 - p^2\right) I_{xz} + q \cdot r\left(I_{xy}\right) - p \cdot q\left(I_{yz}\right) \\ p \cdot q\left(I_x - I_y\right) + \left(p^2 - q^2\right) I_{xy} + r \cdot p\left(I_{yz}\right) - q \cdot r\left(I_{xz}\right) \end{pmatrix} + \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

$$\rightarrow \boxed{\dot{q} = \ddot{\theta} = \frac{M_y}{I_y} + r \cdot p \frac{\left(I_z - I_x\right)}{I_y}}$$

Neglect Cross Products of Inertia

Forcing moment

Second order Disturbance torque (neglected when r, p are small)

Simplified Pitch Axis Dynamics (2)

Neglecting Disturbance torques

$$\ddot{\theta} = \dot{q} = \frac{M_y}{I_y} \rightarrow M_y \approx \text{"}pitching\ moment\text{"}$$

Consider Only Pitch Axis Dynamics for axi-symmetrix Missile

$$\text{"}pitching\ moment\ coefficient\text{"} \equiv C_m = \frac{M_y}{\bar{q} \cdot A_{ref} \cdot c_{ref}}$$

$$\bar{q} = \left(\frac{1}{2} \cdot \rho \cdot V^2\right)$$

$$A_{ref} = \frac{\pi}{4} \cdot D_{ref}^{\ 2}$$

$$\text{"}reference\ length\text{"} \rightarrow c_{ref} = D_{\max}$$

Check Units

$$\frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \sim \frac{\frac{Nt}{m^2} \cdot m^2 \cdot m}{kg \cdot m^2} \sim$$

$$\left(\frac{kg-m}{\sec^2 m^2} \cdot m^2 \cdot m\right) \cdot \frac{1}{kg \cdot m^2} \sim \frac{1}{\sec^2}$$

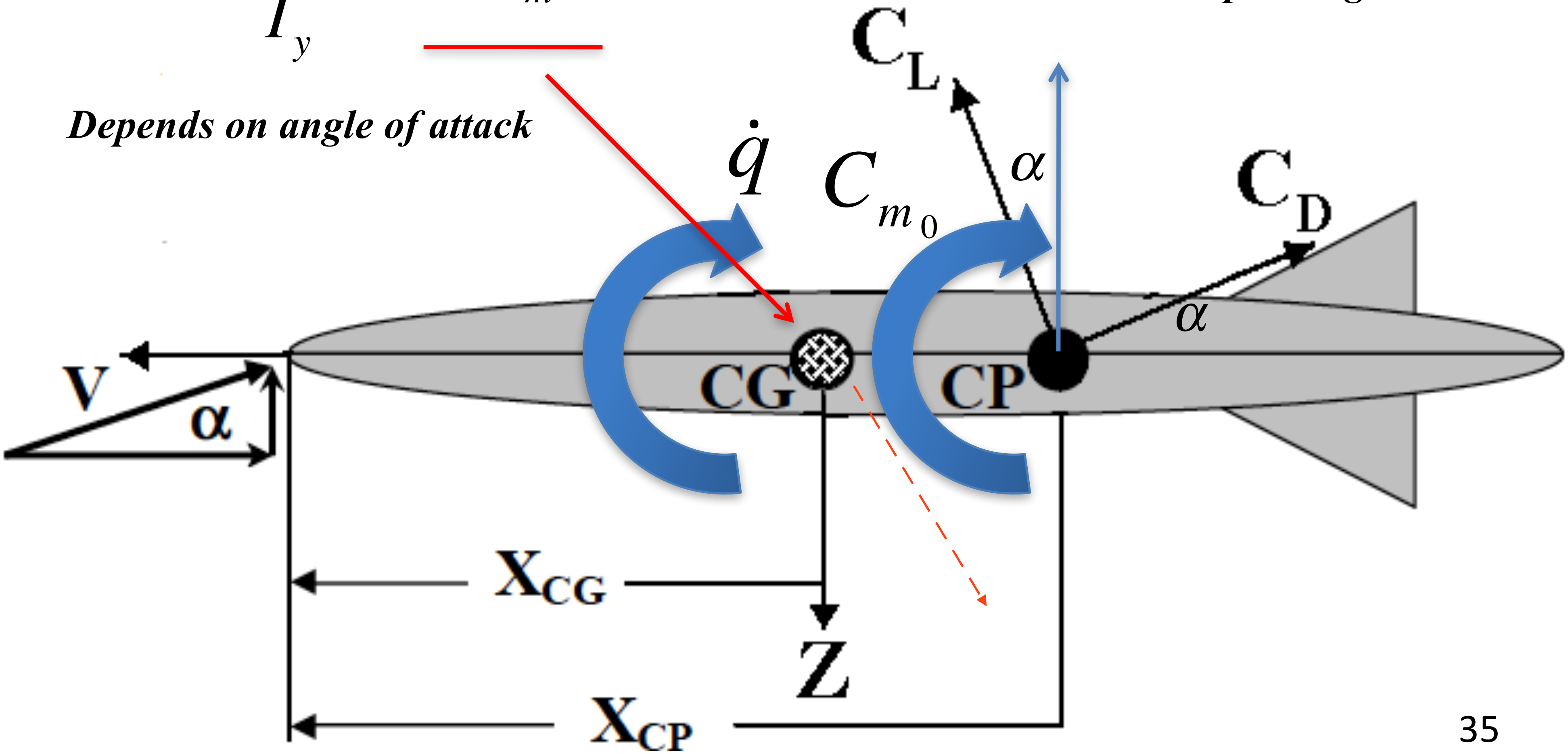
Simplified Pitch Axis Dynamics (3)

Pitching moment about cg

$$\dot{q} = \frac{\overline{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_m$$

Depends on angle of attack

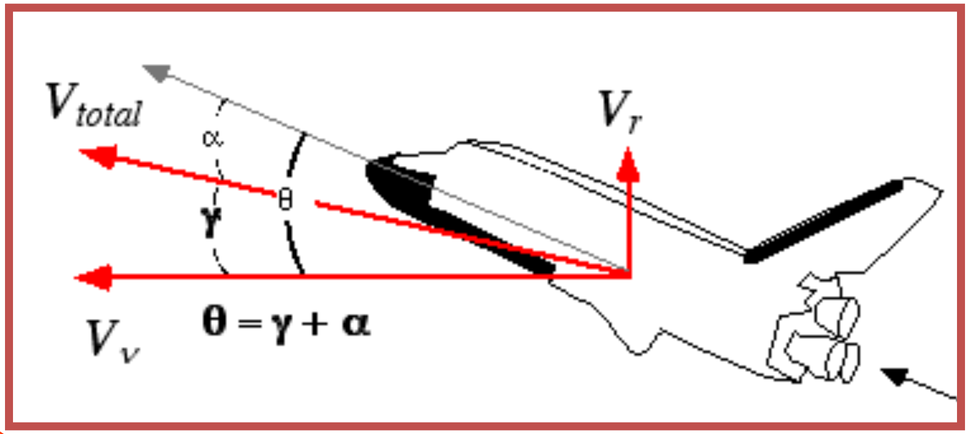
Gravity acts at cg and Cannot induce pitching moment



Simplified Pitch Axis Dynamics (4)

$$\ddot{\theta} = \dot{q} = \frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_m$$

Depends on angle of attack + Control inputs



It can also be shown that … (NASA RP-1168 pp 10-22) that for angle of attack

$$\dot{\alpha} = -\frac{\bar{q} \cdot A_{ref}}{m \cdot V} \cdot C_L + q + \frac{g_{(r)}}{V} \cos(\theta - \alpha) - \frac{F_{thrust} \sin\alpha}{m \cdot V}$$

See appendix I for derivation

Collected, Simplified Longitudinal Axis-Dynamics (5)

•a

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -\frac{\bar{q} \cdot A_{ref}}{m \cdot V} \cdot C_L + q + \frac{g_{(r)}}{V} \cos(\theta - \alpha) - \frac{F_{thrust} \sin \alpha}{m \cdot V} \\ \frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_m \\ q \end{bmatrix}$$

a

Linear Analysis of Longitudinal Axis-Dynamics

Start with

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -\frac{\bar{q} \cdot A_{ref}}{m \cdot V} \cdot C_L + q + \frac{g_{(r)}}{V} \cos(\theta - \alpha) - \frac{F_{thrust} \sin \alpha}{m \cdot V} \\ \frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_m \\ q \end{bmatrix}$$

$$\boxed{\begin{array}{l} \alpha \rightarrow small \\ C_L \approx C_{L_\alpha} \cdot \alpha \\ C_m \approx C_{m_\alpha} \cdot \alpha + C_{m_q} \cdot q \\ q = \dot{\theta} \\ \dot{q} = \ddot{\theta} \end{array}} \rightarrow \begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -\frac{\bar{q} \cdot A_{ref} \cdot C_{L\alpha} + T_{thrust}}{m \cdot V_\infty} & 1 & 0 \\ \frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_{m_\alpha} & \frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_{m_q} & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \frac{g \cdot \cos(\theta)}{V_\infty} \\ 0 \\ 0 \end{bmatrix}$$

$C_{m_\alpha} \rightarrow$ "*stability derivative*"

$C_{m_q} \rightarrow$ "*damping derivative*"

Linear Analysis (2)

Let $q = \dot{\theta}$, $\dot{q} = \ddot{\theta}$ & Reorder States …

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -\dfrac{\bar{q} \cdot A_{ref} \cdot C_{L\alpha} + T_{thrust}}{m \cdot V_\infty} & 1 & 0 \\ \dfrac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_{m_\alpha} & \dfrac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_{m_q} & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \dfrac{g \cdot \cos(\theta)}{V_\infty} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -\dfrac{\bar{q} \cdot A_{ref} \cdot C_{L\alpha} + T_{thrust}}{m \cdot V_\infty} & 0 & 1 \\ 0 & 0 & 1 \\ \dfrac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_{m_\alpha} & 0 & \dfrac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_{m_q} \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} \dfrac{g \cdot \cos(\theta)}{V_\infty} \\ 0 \\ 0 \end{bmatrix}$$

Linear Analysis (3)

Write as $\dot{X} = A \cdot X + G(t)$

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \alpha \\ \theta \\ \dot{\theta} \end{bmatrix} \rightarrow X = \begin{bmatrix} \alpha \\ \theta \\ \dot{\theta} \end{bmatrix} \rightarrow A = \begin{bmatrix} -\frac{\bar{q} \cdot A_{ref} \cdot C_{L\alpha} + T_{thrust}}{m \cdot V_\infty} & 0 & 1 \\ 0 & 0 & 1 \\ \frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_{m_\alpha} & 0 & \frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_{m_q} \end{bmatrix} \rightarrow G(t) = \begin{bmatrix} \frac{g \cdot \cos(\theta)}{V_\infty} \\ 0 \\ 0 \end{bmatrix}$$

Look at Eigenvalues of Linearized System

$$Det\left[A - \lambda \cdot I\right] = 0 \rightarrow \begin{bmatrix} -\frac{\bar{q} \cdot A_{ref} \cdot C_{L\alpha} + T_{thrust}}{m \cdot V_\infty} - \lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ \frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_{m_\alpha} & 0 & \frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_{m_q} - \lambda \end{bmatrix} = 0$$

Linear Analysis (4)

Eigen Values

$$Det\left[A-\lambda\cdot I\right]=0\rightarrow\begin{bmatrix} -\frac{\bar{q}\cdot A_{ref}\cdot C_{L\alpha}+T_{thrust}}{m\cdot V_\infty}-\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ \frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_{m_\alpha} & 0 & \frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_{m_q}-\lambda \end{bmatrix}=0$$

$$-\left(\frac{\bar{q}\cdot A_{ref}\cdot C_{L\alpha}+T_{thrust}}{m\cdot V_\infty}+\lambda\right)\cdot\left(\lambda^2-\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_{m_q}\cdot\lambda\right)+\left(\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_{m_\alpha}\right)\cdot\lambda=0$$

$\rightarrow$ *divide thru by* $-\lambda$

$$\left(\lambda+\frac{\bar{q}\cdot A_{ref}\cdot C_{L\alpha}+T_{thrust}}{m\cdot V_\infty}\right)\cdot\left(\lambda-\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_{m_q}\right)-\left(\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_{m_\alpha}\right)=0$$

Linear Analysis (4)

$$\left(\lambda+\frac{\bar{q}\cdot A_{ref}\cdot C_{L\alpha}+T_{thrust}}{m\cdot V_\infty}\right)\cdot\left(\lambda-\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_{m_q}\right)-\left(\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_{m_\alpha}\right)=0$$

Expand and Collect Terms to give Characteristic Equation for Linearized System

$$\lambda^2+\left(\frac{\bar{q}\cdot A_{ref}\cdot C_{L\alpha}+T_{thrust}}{m\cdot V_\infty}-\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_{m_q}\right)\cdot\lambda-\left(\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\right)\cdot\left[\left(\frac{\bar{q}\cdot A_{ref}\cdot C_{L\alpha}+T_{thrust}}{m\cdot V_\infty}\right)\cdot C_{m_q}+C_{m_\alpha}\right]=0$$

$$Form\ like: \lambda^2+2\cdot\varsigma\cdot\omega_n\cdot\lambda+\omega_n^2=0$$

$$\rightarrow \omega_n=\sqrt{-\left(\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\right)\cdot\left[\left(\frac{\bar{q}\cdot A_{ref}\cdot C_{L\alpha}+T_{thrust}}{m\cdot V_\infty}\right)\cdot C_{m_q}+C_{m_\alpha}\right]}$$

Natural Frequency

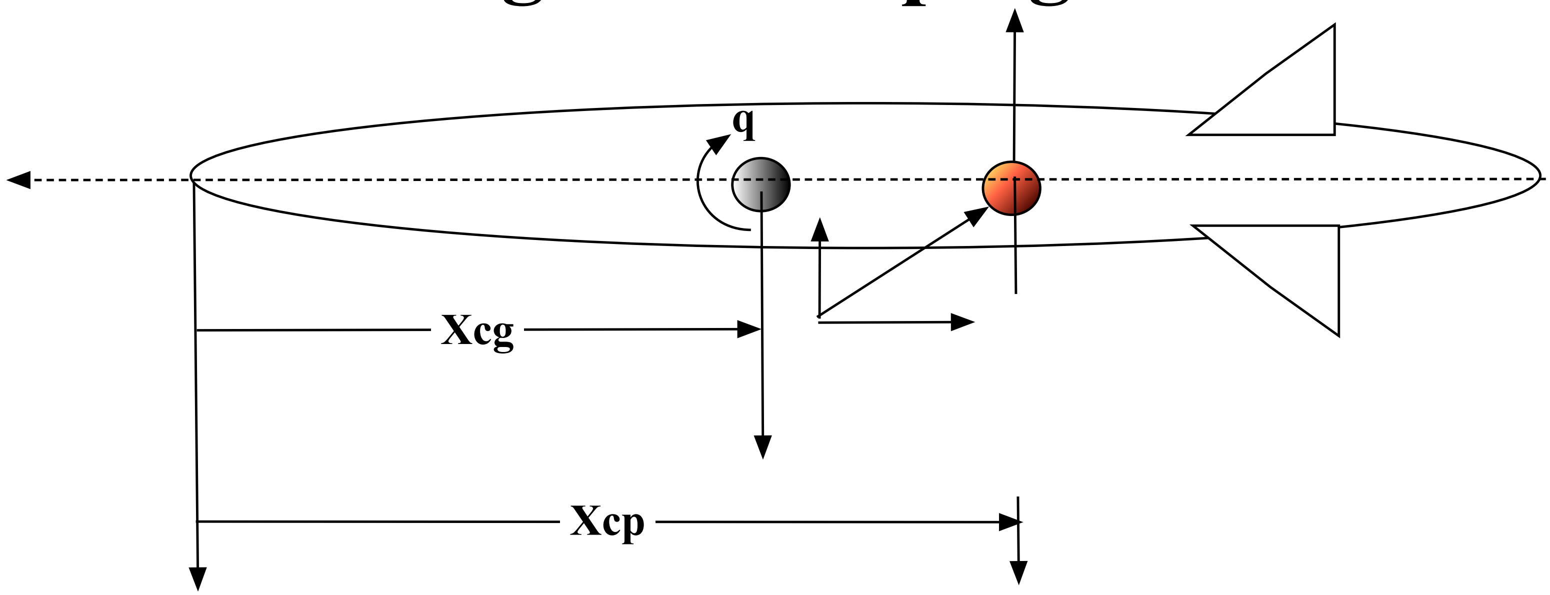
$$\frac{(2\cdot\varsigma\cdot\omega_n)^2}{2\ \omega_n^{\ 2}}\rightarrow \varsigma=\sqrt{\frac{1}{4}\frac{\left(\frac{\bar{q}\cdot A_{ref}\cdot C_{L\alpha}+T_{thrust}}{m\cdot V_\infty}-\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_{m_q}\right)^2}{-\left(\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\right)\cdot\left[\left(\frac{\bar{q}\cdot A_{ref}\cdot C_{L\alpha}+T_{thrust}}{m\cdot V_\infty}\right)\cdot C_{m_q}+C_{m_\alpha}\right]}}$$

Damping ratio

UtahState UNIVERSITY

Mechanical & Aerospace Engineering

Estimating the Damping Derivatives



$$Small\ \alpha \rightarrow C_{N_\alpha} \cong C_{L_\alpha} \rightarrow C_N = C_{L_\alpha} \cdot \left(\frac{X_{cp} - X_{cg}}{V_\infty} \right) \cdot q$$

$$C_m = -C_N \cdot \frac{\left(X_{cp} - X_{cg}\right)}{c_{ref}} = -C_{L_\alpha} \cdot \left(\frac{\left(X_{cp} - X_{cg}\right)^2}{V_\infty} \right) \cdot \frac{q}{c_{ref}} = -C_{L_\alpha} \cdot \left(\frac{\left(X_{cp} - X_{cg}\right)^2}{V_\infty c_{ref}^2} \right) \cdot q \cdot c_{ref} = -C_{L_\alpha} \cdot \left(X_{sm}^2\right) \cdot \frac{c_{ref}}{V_\infty} \cdot q$$

$$\delta C_m = \frac{\partial C_m}{\partial q} \cdot q = C_{m_q} \cdot q = -C_{L_\alpha} \cdot \left(X_{sm}^2\right) \cdot \frac{c_{ref}}{V_\infty} \cdot q \rightarrow \boxed{C_{m_q} = -C_{L_\alpha} \cdot \left(X_{sm}^2\right) \cdot \frac{c_{ref}}{V_\infty}}$$

$$\boxed{C_{m_q} = -C_{L_\alpha} \cdot \left(X_{sm}^2\right) \cdot \frac{c_{ref}}{V_\infty}} \rightarrow \textit{pitch damping opposes motion ® i.e. always negative}$$

Collected Longitudinal Equations of Motion

$$
\begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{\nu} \\ \dot{x} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{m} \end{bmatrix} =
\begin{bmatrix}
\left(\frac{V_v^2}{r}\right) + \left(\frac{\bar{q}\cdot A_{ref}}{m}\right)\cdot\left(C_L\cos\gamma - C_D\sin\gamma\right) + \left(\frac{F_{thrust}\sin\theta}{m}\right) - g_{(r)} \\
-\left(\frac{V_r\cdot V_v}{r}\right) - \left(\frac{\bar{q}\cdot A_{ref}}{m}\right)\cdot\left(C_L\sin\gamma + C_D\sin\gamma\right) + \left(\frac{F_{thrust}\cos\theta}{m}\right) \\
V_r \\
\frac{V_v}{r} \\
V_v \\
-\frac{\bar{q}\cdot A_{ref}}{m\cdot V}\cdot C_L + q + \frac{g_{(h)}}{V}\cos(\gamma) - \frac{F_{thrust}\sin\alpha}{m\cdot V} \\
\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_m \\
q \\
-\frac{F_{thrust}}{g_0\cdot I_{sp}}
\end{bmatrix}
\rightarrow
$$

$$g_{(r)} = \frac{\mu}{r^2}$$

$$\gamma = \tan^{-1}\frac{V_r}{V_v} = \theta - \alpha$$

$$\dot{x} = r\cdot\dot{\nu}$$

$$V = \sqrt{V_r^2 + V_v^2}$$

$$\bar{q} = \frac{1}{2}\cdot\rho_{(h)}\cdot V^2$$

$$h = r - R_{earth}$$

Can we simplify this "mess"?

Simplified Longitudinal Equations of Motion

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -\frac{\bar{q} \cdot A_{ref}}{m \cdot V} \cdot C_L + q + \frac{g_{(r)}}{V} \cos(\theta - \alpha) - \frac{F_{thrust} \sin\alpha}{m \cdot V} \\ \frac{\bar{q} \cdot A_{ref} \cdot c_{ref}}{I_y} \cdot C_m \\ q \end{bmatrix}$$

Augment with longitudinal Acceleration Equation (See Appendix II)

$$\dot{V}_\infty = \frac{T_{thrust} \cdot \cos\alpha}{m} - g \cdot \sin(\theta - \alpha) - \frac{\bar{q} \cdot A_{ref} \cdot C_D}{m}$$

Complete with altitude and downrange

$$\dot{h} = V_\infty \cdot \sin\gamma = V_\infty \cdot \sin(\theta - \alpha)$$

$$\dot{x} = V_\infty \cdot \cos\gamma = V_\infty \cdot \cos(\theta - \alpha)$$

Simplified Longitudinal Equations of Motion

Collected Body-Axis Equations

$$
\begin{bmatrix} \dot{V}_\infty \\ \dot{\alpha} \\ \dot{\theta} \\ \dot{q} \\ \dot{h} \\ \dot{x} \\ \dot{m} \end{bmatrix} =
\begin{bmatrix}
\frac{T_{thrust} \cdot \cos\alpha}{m} - g \cdot \sin(\theta - \alpha) - \frac{\bar{q} \cdot A_{ref} \cdot C_D}{m} \\
\dot{\alpha} = q + \frac{g \cdot \cos(\theta - \alpha)}{V_\infty} - \frac{\bar{q} \cdot A_{ref} \cdot C_L + T_{thrust} \cdot \sin\alpha}{m \cdot V_\infty} \\
q \\
\frac{\bar{q} \cdot A_{ref} \cdot c_{ref} \cdot C_m}{I_{yy}} \\
V_\infty \sin(\theta - \alpha) \\
V_\infty \cos(\theta - \alpha) \\
-\frac{T_{thrust}}{g_0 \cdot I_{sp}}
\end{bmatrix}
$$

$$C_L = C_L(\alpha) \simeq C_{N_\alpha} \cdot \alpha$$

$$C_m = C_m(\alpha, q) \approx C_{m_\alpha} \cdot \alpha + C_{mq} \cdot q =$$
$$-X_{sm} \cdot (C_{N_\alpha} + C_D) \cdot \alpha - X_{sm}{}^2 \cdot \left(C_{N_\alpha} \cdot \frac{c_{ref}}{V_\infty} \right) \cdot q$$

Model Comparison

Body Axis with Pitch Dynamics Model

One extra degree of freedom

$$\begin{bmatrix} \dot{V}_\infty \\ \dot{\alpha} \\ \dot{\theta} \\ \dot{q} \\ \dot{h} \\ \dot{x} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \frac{T_{thrust} \cdot \cos\alpha}{m} - g \cdot \sin(\theta - \alpha) - \frac{\bar{q} \cdot A_{ref} \cdot C_D}{m} \\ \dot{\alpha} = q + \frac{g \cdot \cos(\theta - \alpha)}{V_\infty} - \frac{\bar{q} \cdot A_{ref} \cdot C_L + T_{thrust} \cdot \sin\alpha}{m \cdot V_\infty} \\ q \\ \frac{\bar{q} \cdot A_{ref} \cdot c_{ref} \cdot C_m}{I_{yy}} \\ V_\infty \sin(\theta - \alpha) \\ V_\infty \cos(\theta - \alpha) \\ -\frac{T_{thrust}}{g_0 \cdot I_{sp}} \end{bmatrix}$$

Local Vertical/Local Horizontal Ballistic Model

$$\begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ v \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \frac{V_v^2}{r} - \frac{\mu}{r^2} + \left[\frac{F_{thrust}}{m} - \frac{\rho V_\infty^2}{2\beta}\right]\sin(\gamma) \\ -\frac{V_r V_v}{r} + \left[\frac{F_{thrust}}{m} - \frac{\rho V_\infty^2}{2\beta}\right]\cos(\gamma) \\ V_r \\ \frac{V_v}{r} \\ -\frac{F_{thrust}}{g_0 I_{sp}} \end{bmatrix}_{\alpha=0}$$

$$\gamma = \tan^{-1}\left[\frac{V_r}{V_v}\right]$$

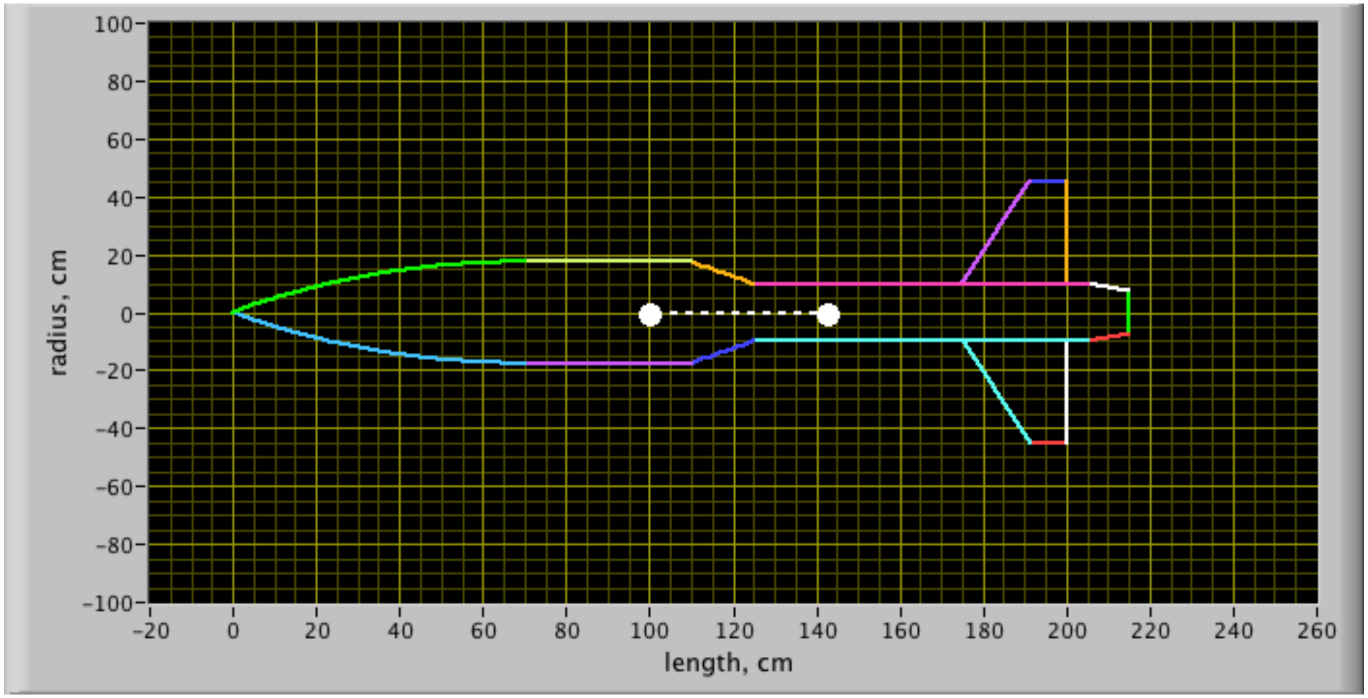
$$\beta = \frac{M}{C_D \cdot A_{ref}}$$

$$\dot{X} = f[X, F_{thrust}]$$

$$\vec{X} = \begin{bmatrix} V_r \\ V_v \\ r \\ v \\ M \end{bmatrix}$$

$$\left(\dot{\vec{X}}\right) = \begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ v \\ -\dot{m}_{motor} \end{bmatrix}$$

Example Calculation



Fin Coordinate Array

0

Root Length, cm: 25

Leading Edge Taper angle, deg: 25

Leading edge longitudunal coordinate,: 175

Width, cm: 35

Trailing Edge Taper Angle, deg: 0

Distance from Centerline, cm: 10

Rotate Rocker Roll Axis, deg.: 30

Fin Leading Edge Thickness, cm: 0.5

OF FINS: 3

Mean fin Angle of Attack, deg.: 1

ROTATE Fins? 2

Rocket Dimensions 2

Nose Cone Length, cm: 70

Axial Nose Points: 119

Body Tube Sections, Start, End Diameters, cm

0

0

35	35
35	20
20	20
0	0

Body Tube Sections, Lengths, cm

0

40

15

80

0

Body Tube Sections, # of Points

0

15

15

15

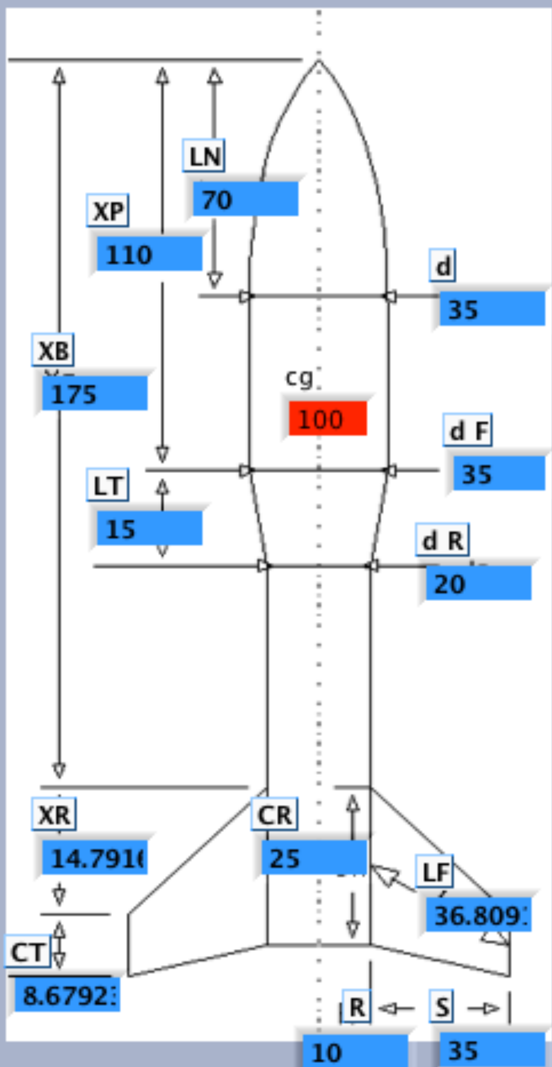
Boat Tail Length, cm: 10

Boat Tail End diameter: 15

Boat tail points: 15

Approximate cg., cm: 100

Example Calculation (2)



Parameter Definitions

L_N = length of nose

d = diameter at base of nose

d_F = diameter at front of transition

d_R = diameter at rear of transition

L_T = length of transition

X_P = distance from tip of nose to front of transition

C_R = fin root chord

C_T = fin tip chord

S = fin semispan

L_F = length of fin mid-chord line

R = radius of body at aft end

X_R = distance between fin root leading edge and fin tip leading edge parallel to body

X_B = distance from nose tip to fin root chord leading edge

N = number of fins

CG, cm from nose: 100

Noze Shape: Ogive / Conical

Parameter Definitions, cm

0

70
35
35
20
15
110
25
8.6792
35
36.809
10
14.791
175
3

Example Calculation (2)

Fin Coefficients

Barrowman …

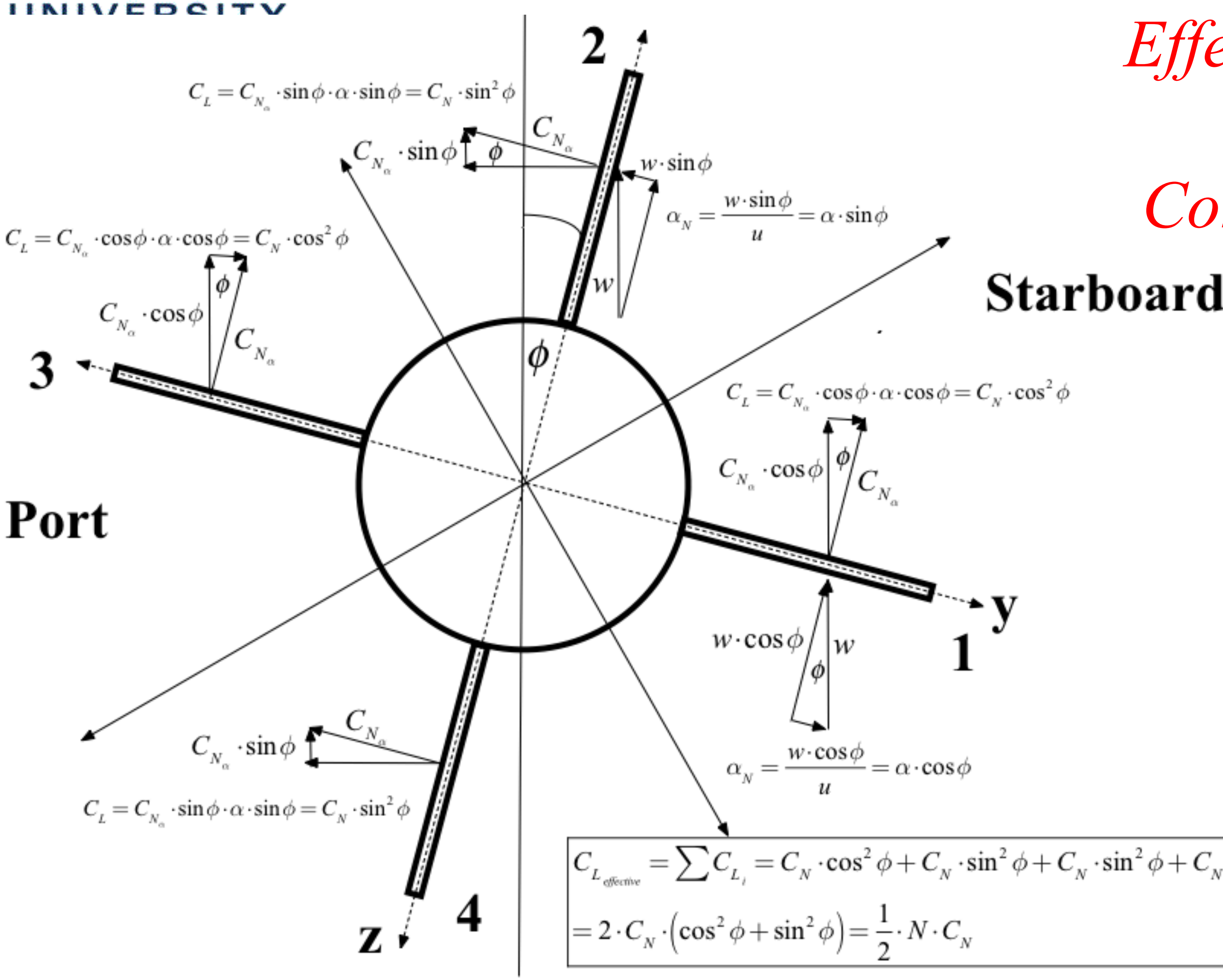
$$(C_{N_\alpha})_F = \left[1+\frac{R}{S+R}\right]\left[\frac{4N\left(\frac{S}{d_{max}}\right)^2}{1+\sqrt{1+\left(\frac{2L_F}{C_R+C_T}\right)^2}}\right] = \left(1+\frac{10}{(10+35)}\right)\left(\frac{4\left(\frac{35}{35}\right)^2}{1+\left(1+\left(\frac{2\cdot 36.8091}{25+8.67923}\right)^2\right)^{0.5}}\right)3 = 4.30898$$

Helmbold …

$$AR = \frac{LF^2}{A_{fin}} = \frac{36.8091^2}{589.387} = 2.29885$$

$$C_{L_\alpha} = \frac{2\pi\cdot A_R}{2+\sqrt{A_R^2+4}}\times\frac{N_{fins}}{2} = \frac{2\pi 2.29885}{2+(2.29885^2+4)^{0.5}}\frac{3}{2} = 4.29281$$

Effective C_L for 4 Fin Configuration



$$C_{L_{effective}} = \sum C_{L_i} = C_N \cdot \cos^2\phi + C_N \cdot \sin^2\phi + C_N \cdot \sin^2\phi + C_N \cdot \cos^2\phi$$
$$= 2 \cdot C_N \cdot \left(\cos^2\phi + \sin^2\phi\right) = \frac{1}{2} \cdot N \cdot C_N$$

4-Fin Rocket Tail End Looking Forward

Effective C_L for 4 Fin Configuration

$$w \cdot \cos\phi \rightarrow \alpha_N = \frac{w \cdot \cos\phi}{u} = \alpha \cdot \cos\phi$$

$$C_L = C_{N_\alpha} \cdot \cos\phi \cdot \alpha \cdot \cos\phi = C_N \cdot \cos^2\phi$$

$$\alpha_N = \frac{w \cdot \sin\phi}{u} = \alpha \cdot \sin\phi$$

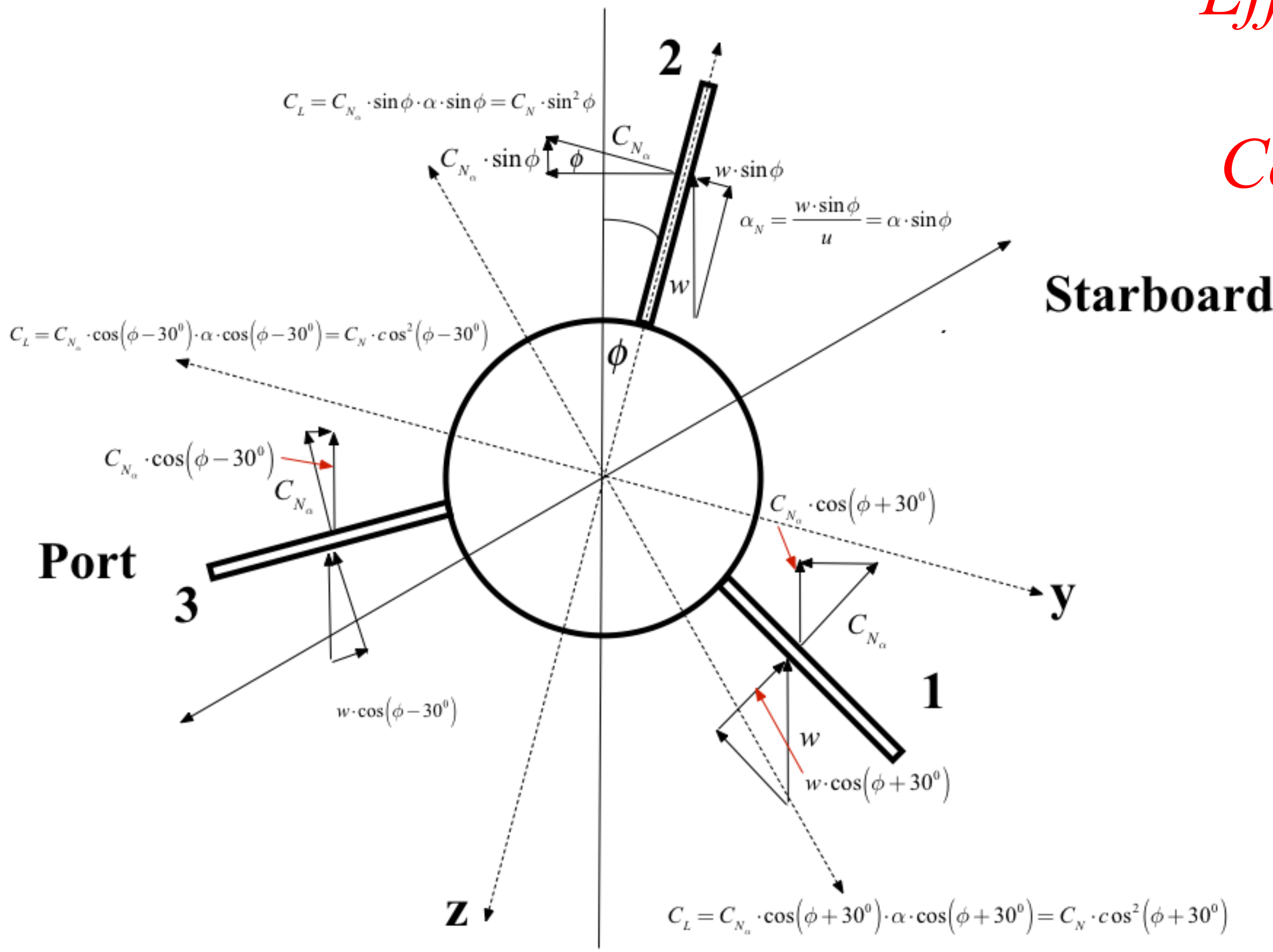
$$C_L = C_{N_\alpha} \cdot \sin\phi \cdot \alpha \cdot \sin\phi = C_N \cdot \sin^2\phi$$

$$C_{L_{effective}} = \sum C_{L_i} = C_N \cdot \cos^2\phi + C_N \cdot \sin^2\phi + C_N \cdot \sin^2\phi + C_N \cdot \cos^2\phi$$

$$= 2 \cdot C_N \cdot \left(\cos^2\phi + \sin^2\phi\right)$$

$$\rightarrow \boxed{C_{L_{effective}} = \frac{1}{2} \cdot N \cdot C_N \rightarrow N = 4}$$

Effective C_L for 3 Fin Configuration



3-Fin Rocket Tail End Looking Forward

Effective C_L for 3 Fin Configuration

$$C_L = C_{N_\alpha} \cdot \cos\left(\phi + 30^0\right) \cdot \alpha \cdot \cos\left(\phi + 30^0\right) = C_N \cdot \cos^2\left(\phi + 30^0\right)$$

$$C_L = C_{N_\alpha} \cdot \cos\left(\phi - 30^0\right) \cdot \alpha \cdot \cos\left(\phi - 30^0\right) = C_N \cdot \cos^2\left(\phi - 30^0\right)$$

$$C_{L_{effective}} = \sum C_{L_i} = C_N \cdot \cos^2\left(\phi + 30^0\right) + C_N \cdot \cos^2\left(\phi - 30^0\right) + C_N \cdot \sin^2\phi$$

$$\cos^2\left(\phi + 30^0\right) = \left(\cos\phi\cos 30^0 - \sin\phi\sin 30^0\right)^2 = \left(\frac{\sqrt{3}}{2}\cos\phi - \frac{1}{2}\sin\phi\right)^2 = \frac{3}{4}\cos^2\phi - \frac{\sqrt{3}}{2}\cos\phi\sin\phi + \frac{1}{4}\sin^2\phi$$

$$\cos^2\left(\phi - 30^0\right) = \left(\cos\phi\cos 30^0 + \sin\phi\sin 30^0\right)^2 = \left(\frac{\sqrt{3}}{2}\cos\phi + \frac{1}{2}\sin\phi\right)^2 = \frac{3}{4}\cos^2\phi + \frac{\sqrt{3}}{2}\cos\phi\sin\phi + \frac{1}{4}\sin^2\phi$$

$$\sum C_{L_i} = C_N \cdot \left(\sin^2\phi + \frac{3}{4}\cos^2\phi - \frac{\sqrt{3}}{2}\cos\phi\sin\phi + \frac{1}{4}\sin^2\phi + \frac{3}{4}\cos^2\phi + \frac{\sqrt{3}}{2}\cos\phi\sin\phi + \frac{1}{4}\sin^2\phi\right) = \frac{3}{2}C_N \cdot \left(\sin^2\phi + \cos^2\phi\right) = \frac{3}{2}C_N$$

$$\boxed{C_{L_{effective}} = \frac{N}{2}C_N \rightarrow N = 3}$$

QED!

UtahState UNIVERSITY

Mechanical & Aerospace Engineering

Example Calculation (3)

Total parameter calculation

Output parameters

Nosecone

CNN: 2

XN, cm: 32.62

Fins

CNF: 4.30898

XF, cm: 185.741

Conical Transition

CNT: -1.34694

XT, cm: 116.818

Total

CNR: 4.96204

Xcp, cm: 142.733

Static Margin: 1.22093

Moment Coefficient Data

Mean Cl-alpha, 1/radian: 4.96204

Mean Cm-alpha, 1/radian: -6.44348

Mean Cm-q, 1/radian/sec: -0.038475

DRAG DATA

CF BASED ON AWET: 0.002547

CF BASED ON MAX CROSS SECTION: 0.0529207

BASE DRAG Coeff BASED MAX CROSS SECTION AREA: 0.126062

Compressible BASE DRAG Coeff Adjusted for Boat tail: 0.07091

Nose Cone Profile Drag: 0.00935593

Adjusted Total drag Coefficient: 0.315467

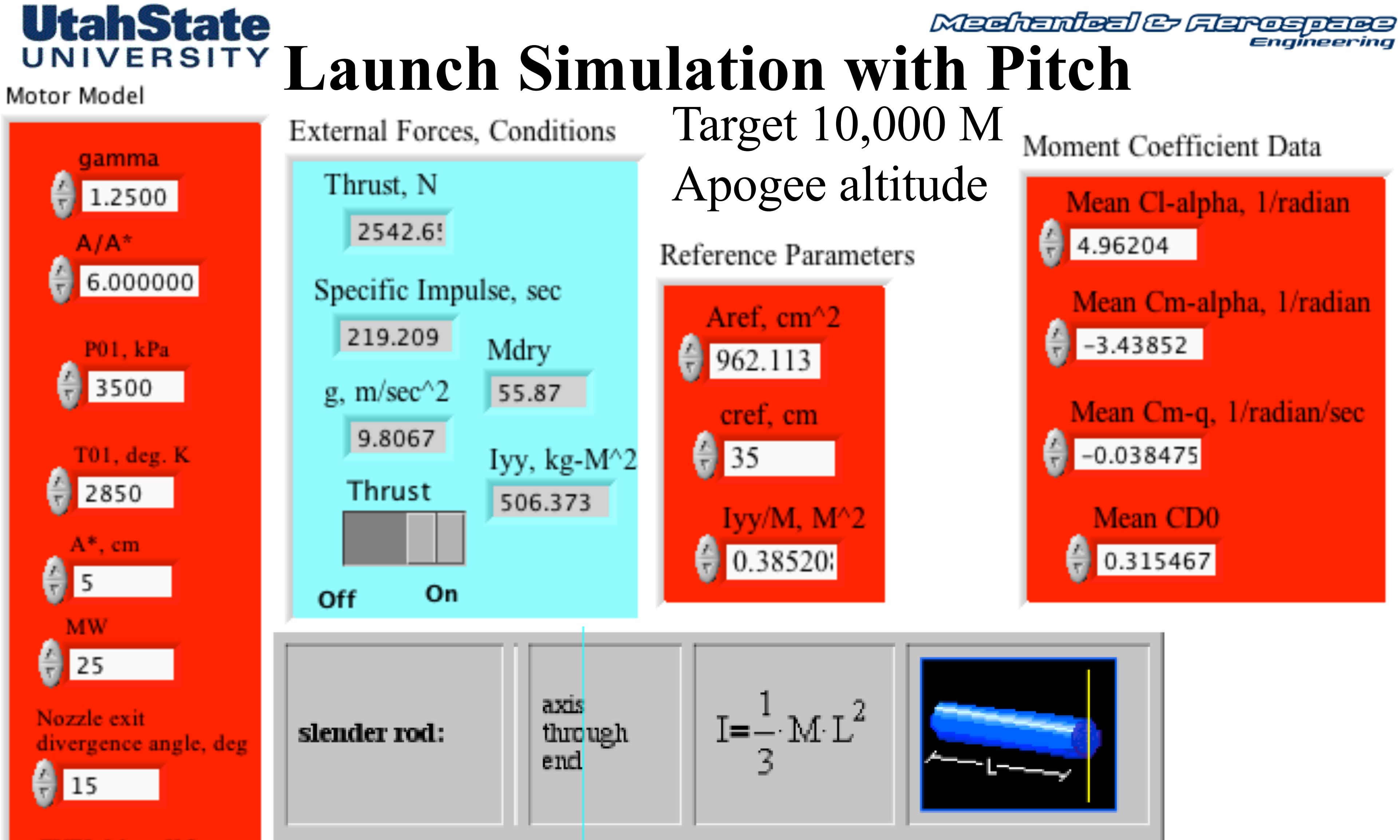
$$C_{L_\alpha} \approx C_{N_R}$$

$$C_{m_\alpha} = -X_{sm} \cdot \left(C_{L_\alpha} + C_{D_0}\right) =$$
$$-1.22093\,(4.96204 + 0.315467)$$
$$= -6.44348$$

$$\left(C_{m_q}\right)_0 = -X_{sm}^2 \cdot C_{L_\alpha} \cdot \left(\frac{c_{ref}}{V_{\infty 0}}\right) =$$
$$\frac{(-1.22093^2\,(4.96204))}{67.2868}\;\frac{35}{100}$$
$$= -0.038475$$

Launch Simulation with Pitch

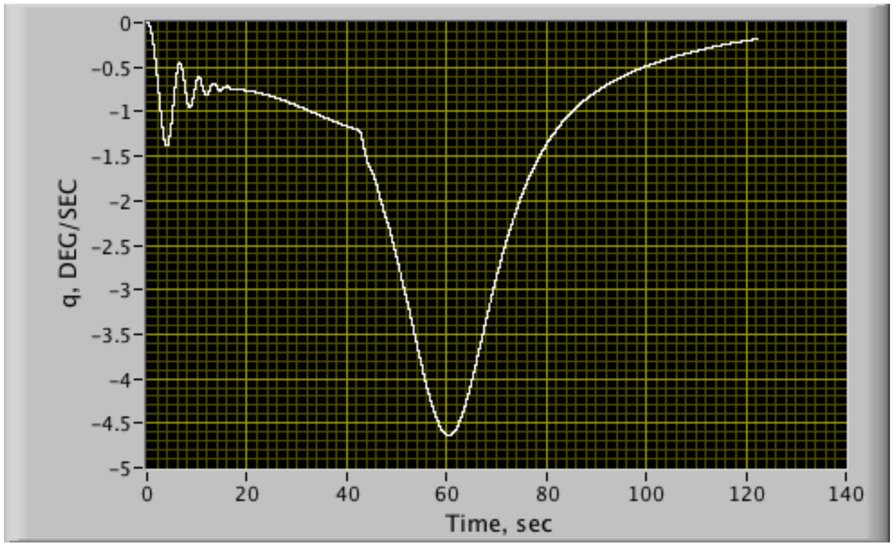
Target 10,000 M
Apogee altitude



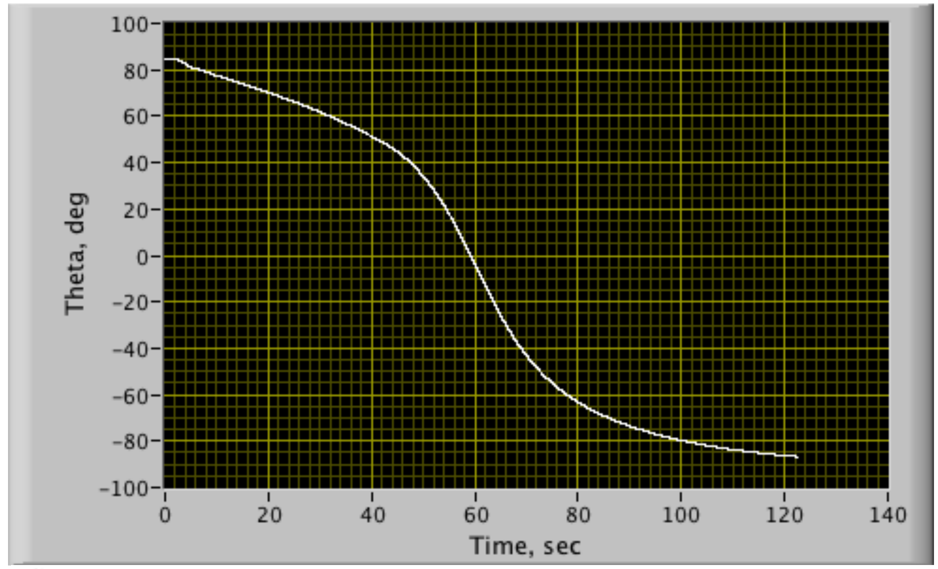
$$\left(I_{yy}\right)_0 = M_0 \cdot L^2 = 105.87 \cdot 2.15^2 = 506.373_{kg-m^2}$$

Example Calculation, Pitch Sim

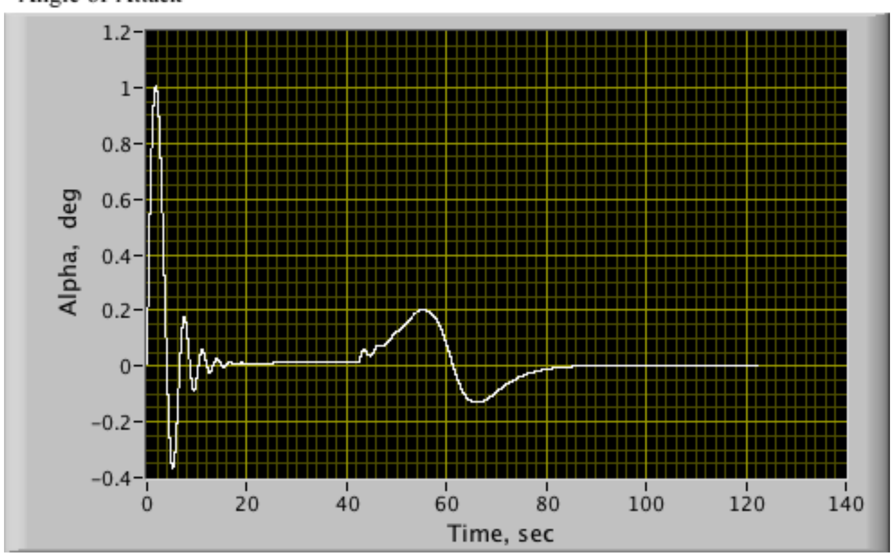
PITCH RATE



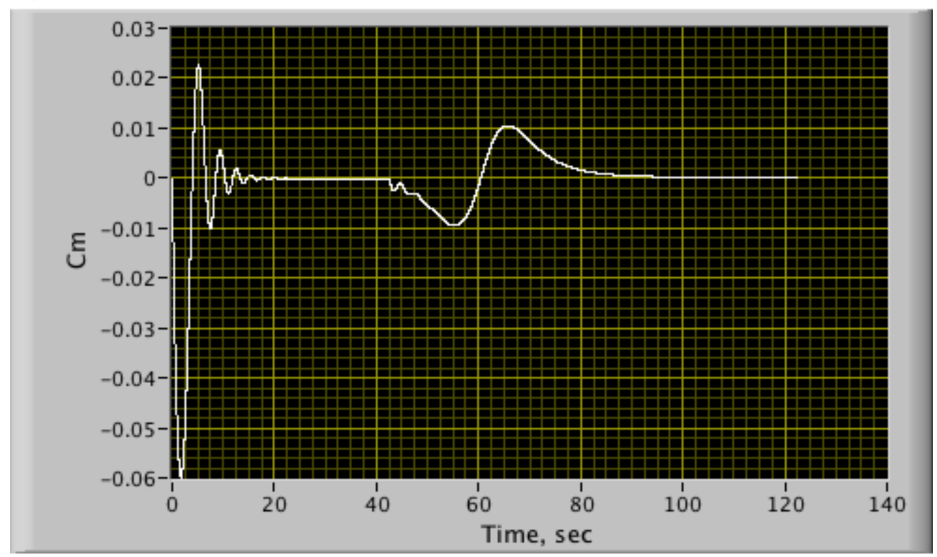
PITCH Angle



Angle of Attack

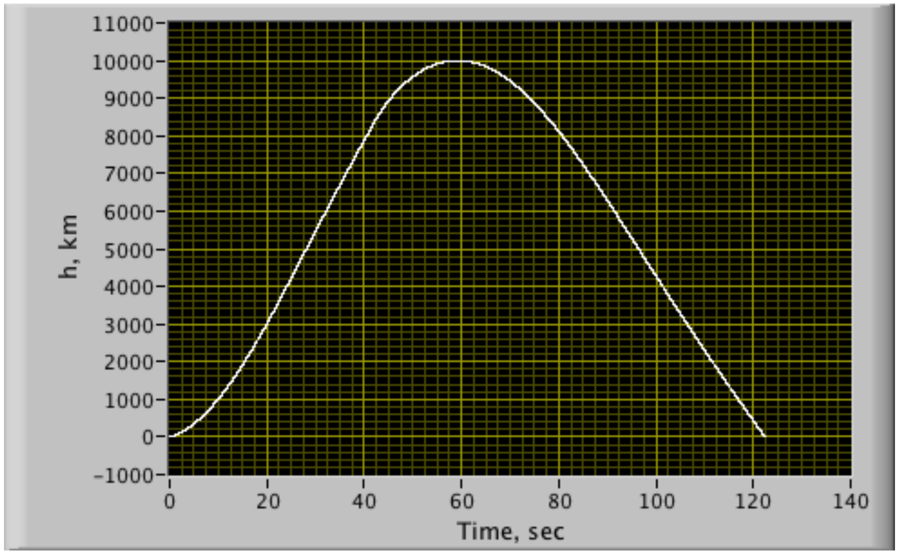


Cm

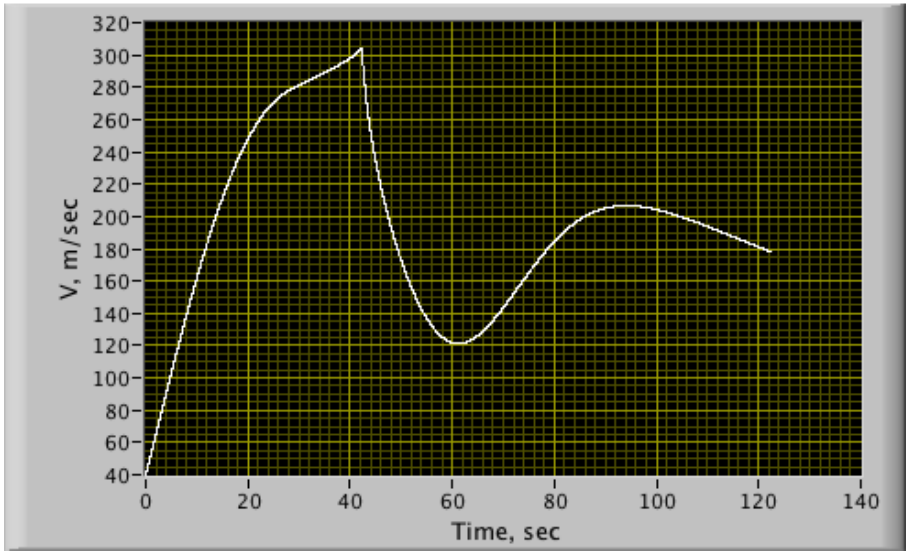


Example Calculation, Pitch Sim (2)

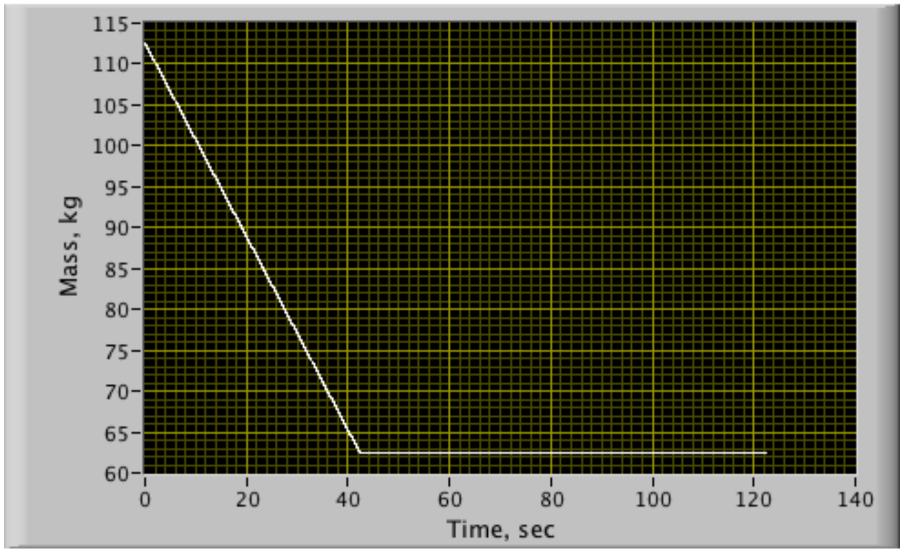
Altitude



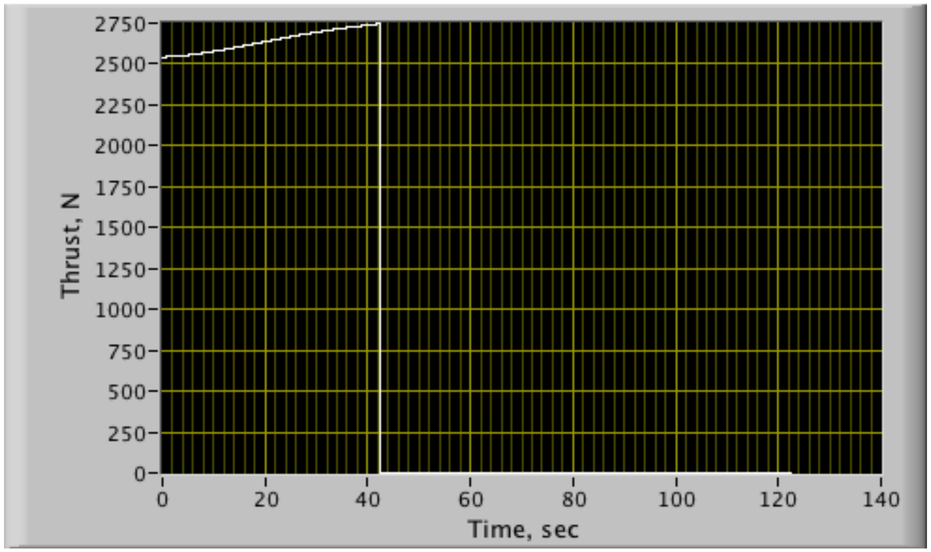
Airspeed



Mass,

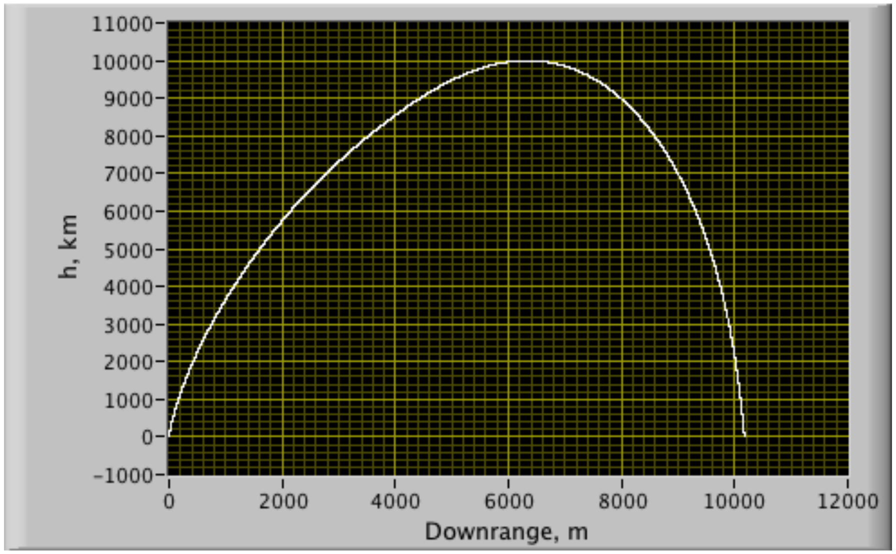


Thrust

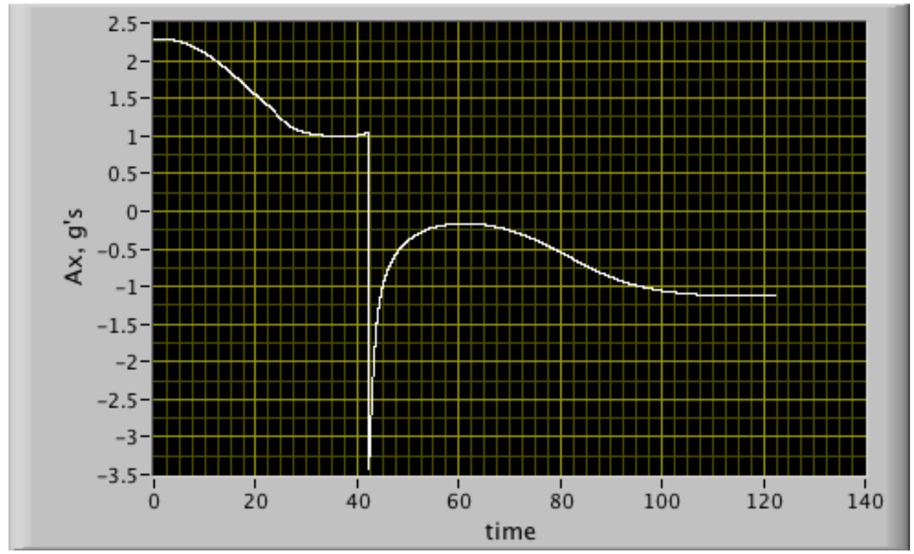


Example Calculation, Pitch Sim (3)

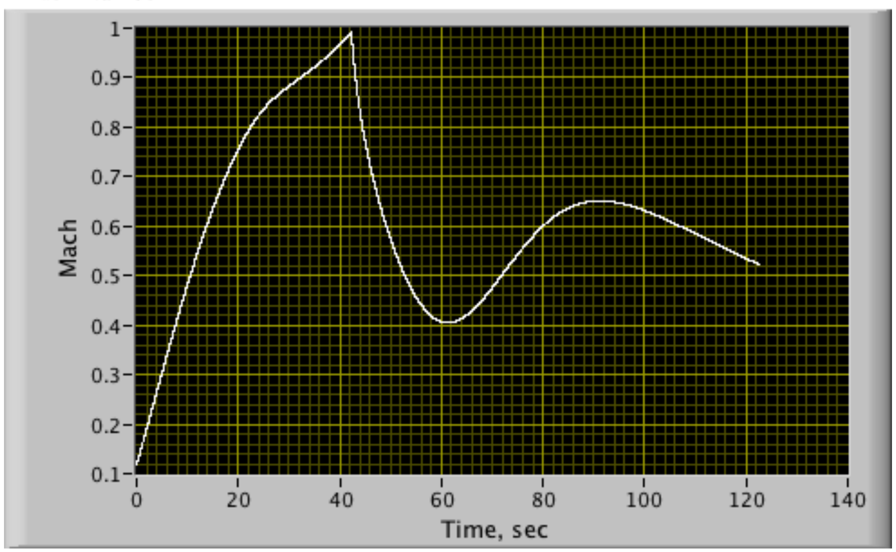
Downrange



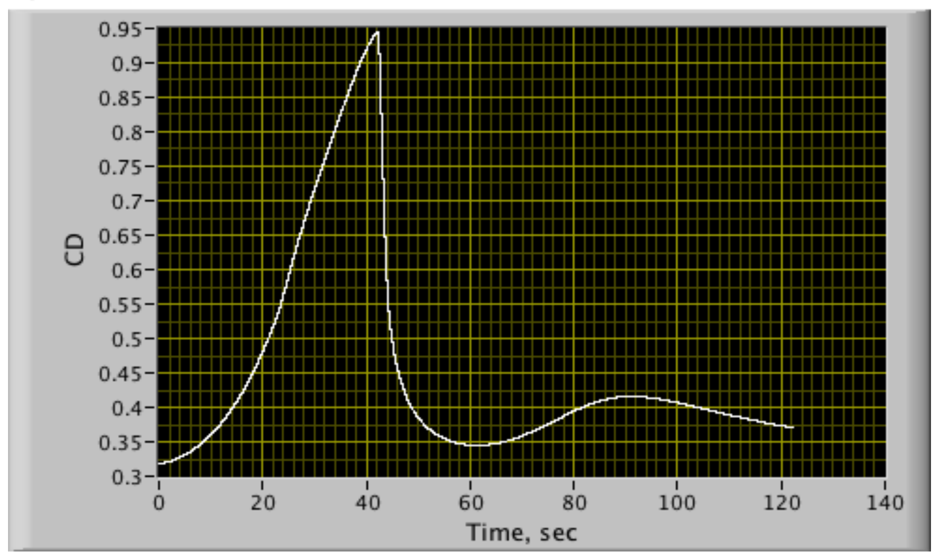
Acceleration, g's



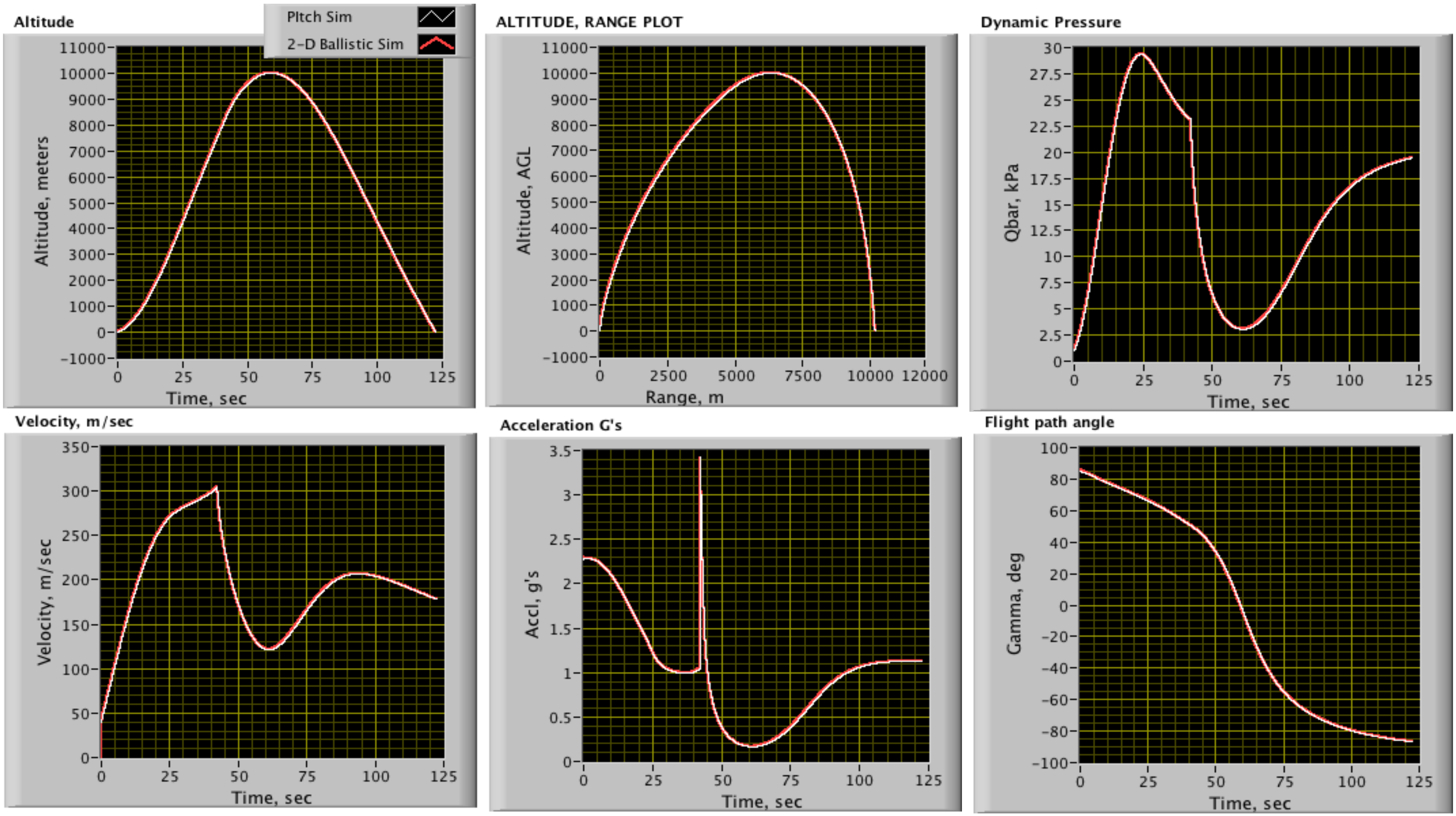
Mach Number



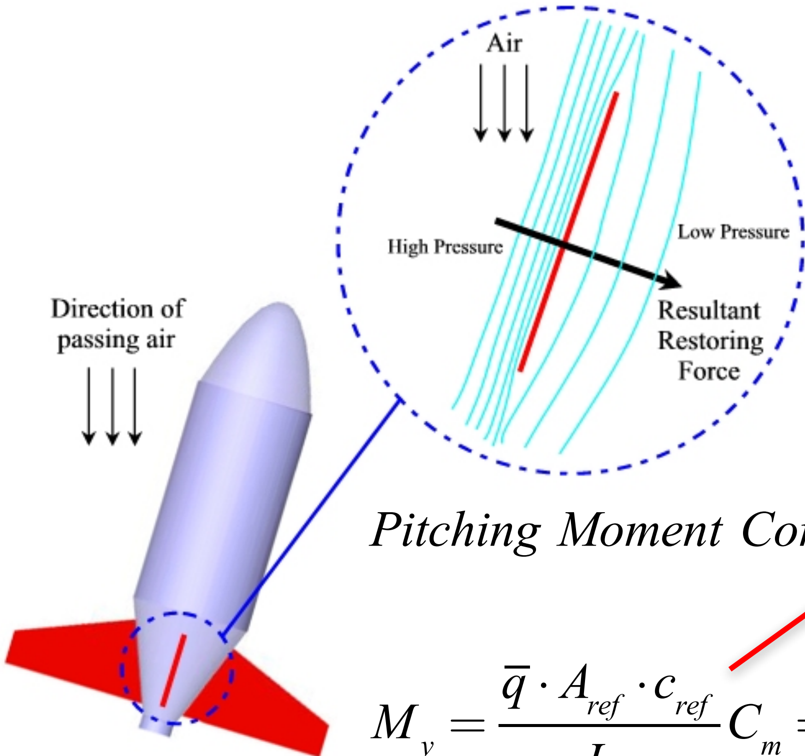
CD



Simulation Comparisons



Pitching Moment Control of Vehicle



$$\begin{bmatrix}\dot{\alpha}\\ \dot{q}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix} -\frac{\bar{q}\cdot A_{ref}}{m\cdot V}\cdot C_L + q + \frac{g_{(r)}}{V}\cos(\theta-\alpha) - \frac{F_{thrust}\sin\alpha}{m\cdot V} \\ \frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_m \\ q \end{bmatrix}$$

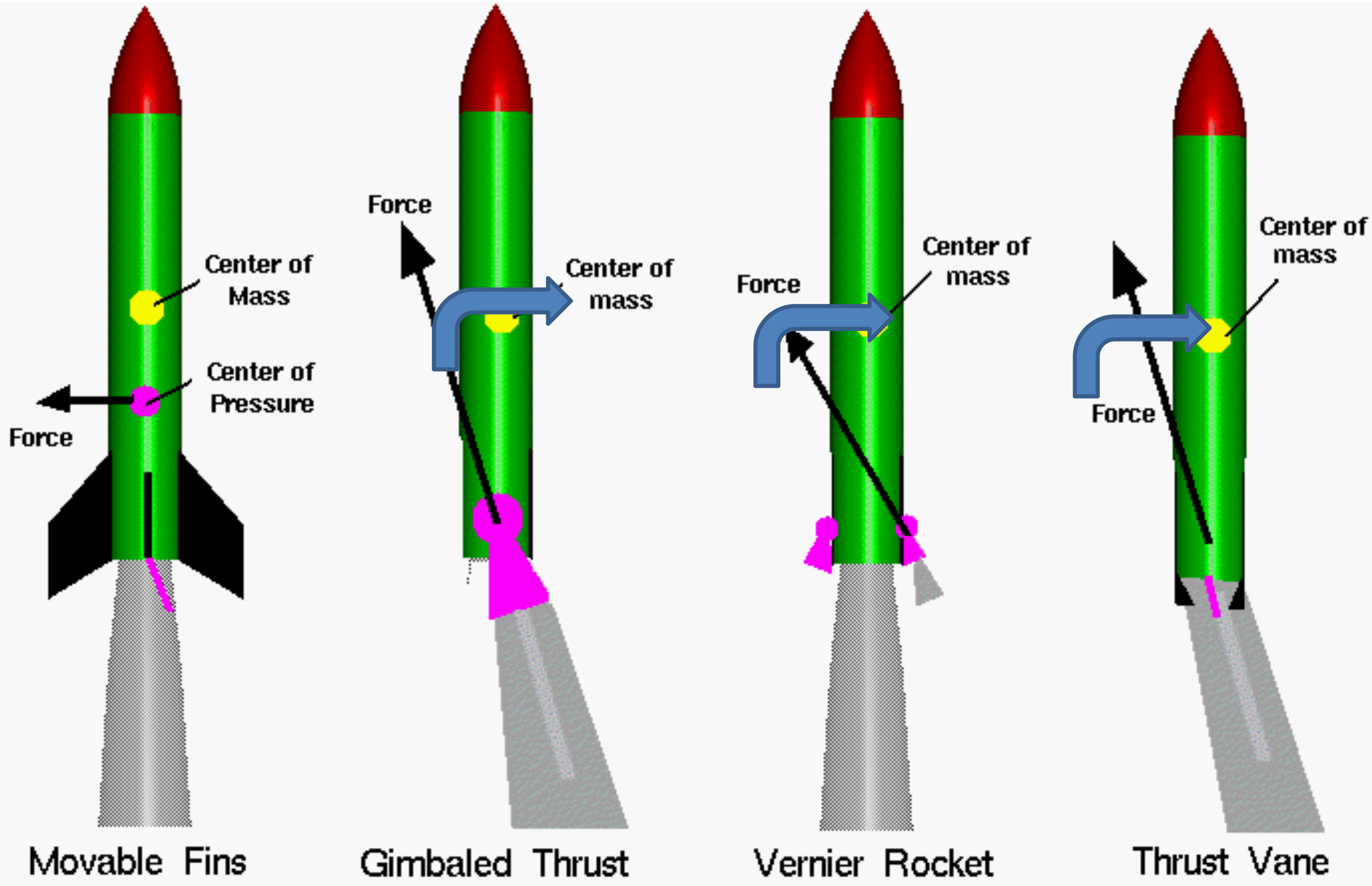
Pitching Moment Control

$$M_y = \frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_{yy}} C_m = \frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_{yy}}\left(C_{m(\alpha,q)} + C_{m_d}\cdot\delta\right)$$

Natural Response
Depends on α, q

Controlled Response
Depends on Actuation

Launch (Rocket) Controls



Decoupled Equations of Motion

$$
\begin{bmatrix} \dot{V}_r \\ \dot{V}_v \\ \dot{r} \\ \dot{\nu} \\ \dot{x} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{m} \end{bmatrix} =
\begin{bmatrix}
\left(\frac{V_v^2}{r}\right) + \left(\frac{\bar{q}\cdot A_{ref}}{m}\right)\cdot\left(C_L\cos\gamma - C_D\sin\gamma\right) + \left(\frac{F_{thrust}\sin\theta}{m}\right) - g_{(r)} \\
-\left(\frac{V_r\cdot V_v}{r}\right) - \left(\frac{\bar{q}\cdot A_{ref}}{m}\right)\cdot\left(C_L\sin\gamma + C_D\sin\gamma\right) + \left(\frac{F_{thrust}\cos\theta}{m}\right) \\
V_r \\
\frac{V_v}{r} \\
V_v \\
-\frac{\bar{q}\cdot A_{ref}}{m\cdot V}\cdot C_L + q + \frac{g_{(h)}}{V}\cos(\gamma) - \frac{F_{thrust}\sin\alpha}{m\cdot V} \\
\frac{\bar{q}\cdot A_{ref}\cdot c_{ref}}{I_y}\cdot C_m \\
q \\
-\frac{F_{thrust}}{g_0\cdot I_{sp}}
\end{bmatrix}
$$

Outer Loop

Inner Loop

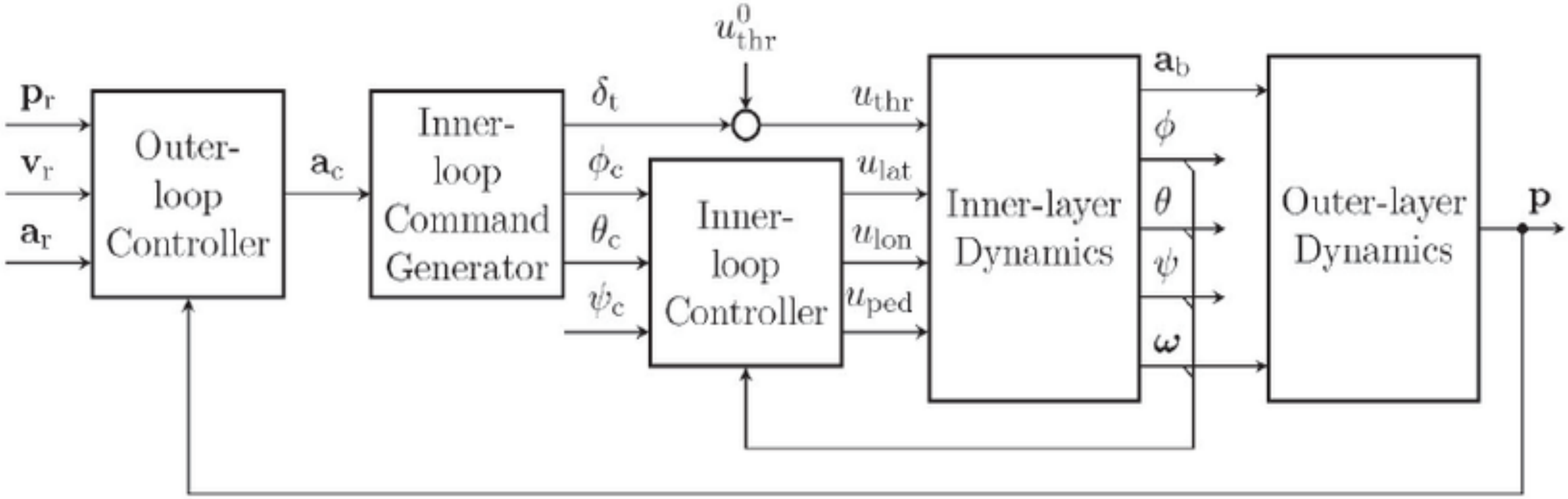
- *Very loose coupling between longitudinal aerodynamics and pitch dynamics and overall vehicle trajectory*

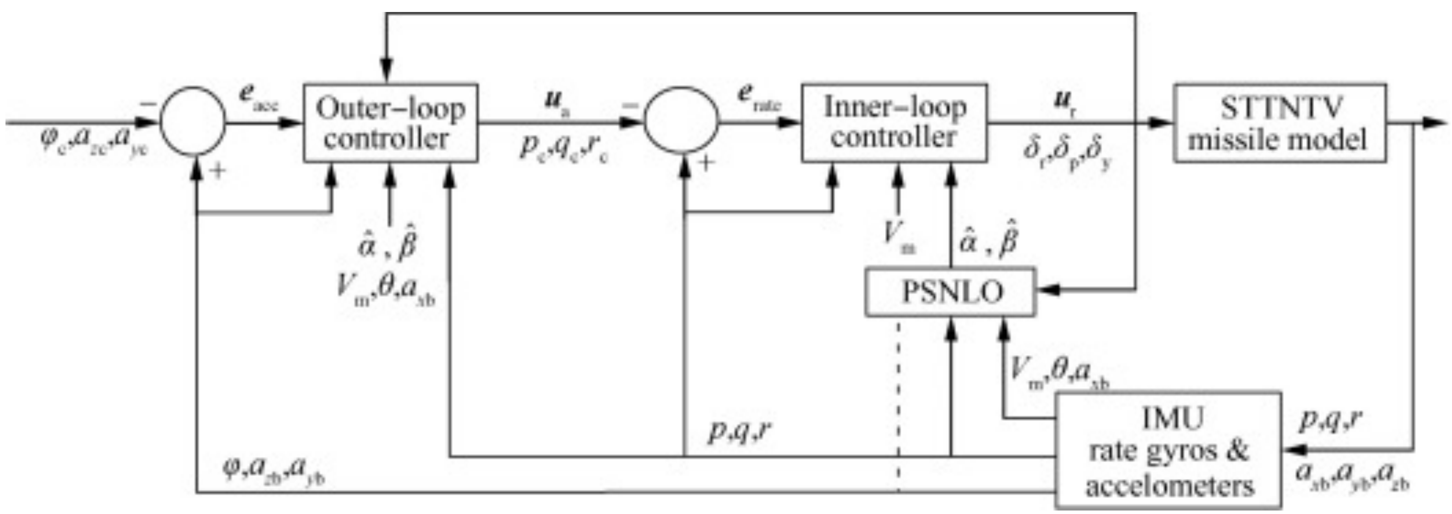
- Generally pitch dynamics and vehicle trajectory are controlled independently

- Trajectory controlled about some Prescribed *$\theta_0(t)$ (outer loop)*

- *Pitch tracking controlled about nominal $\theta_0(t)$ (inner loop)*

Example Control Strategies



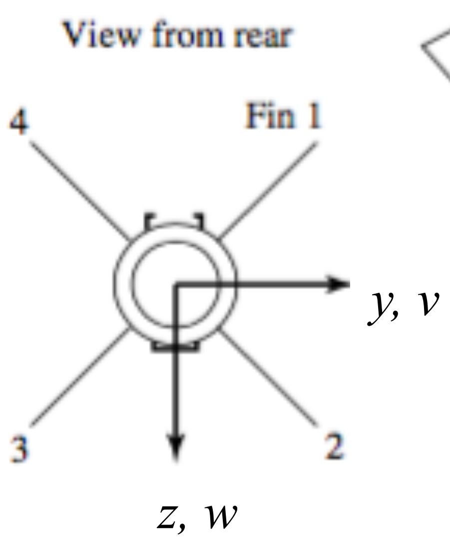


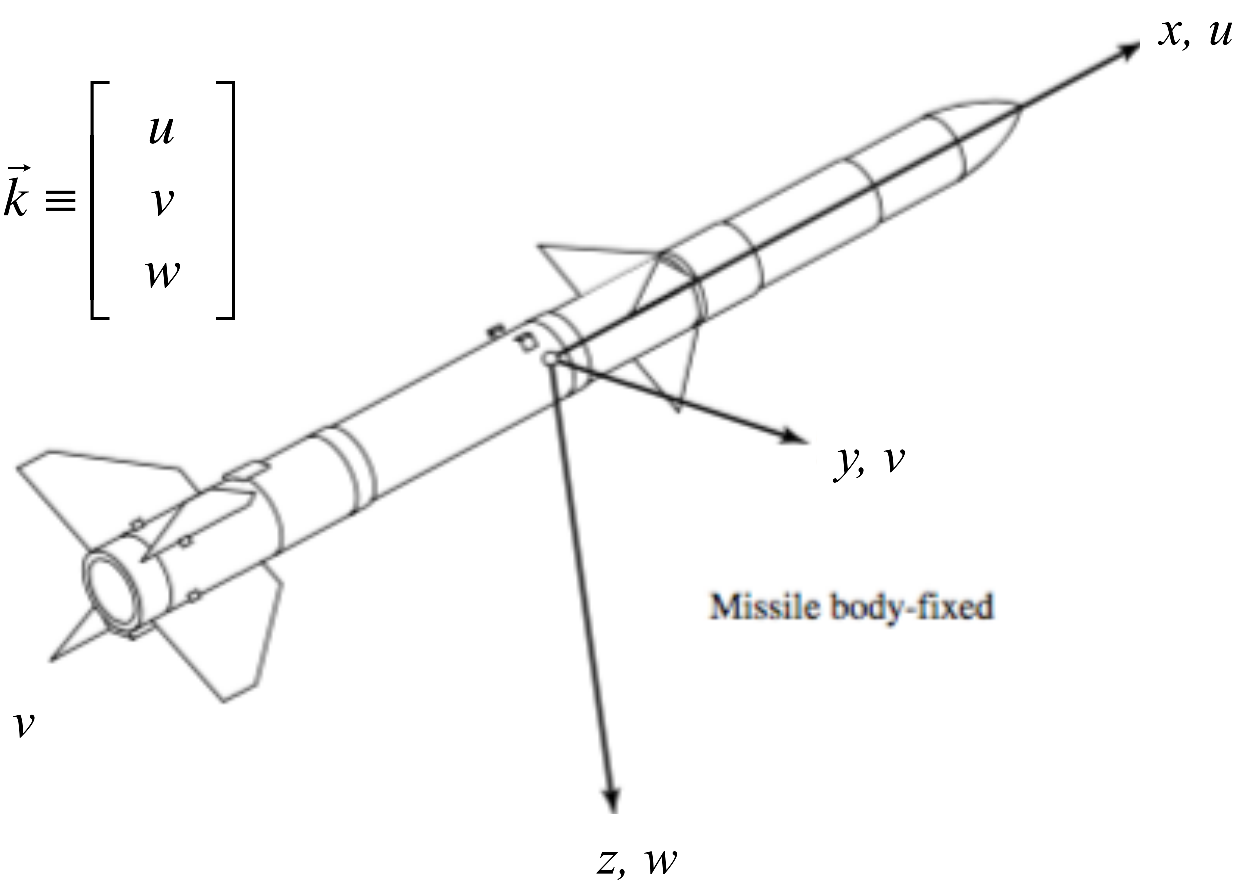

Questions??

Appendix I: Derivation of the Longitudinal “AlphaDot” Equation

• **Body Axis Coordinates, Position and Airspeed Components**

$$\vec{V}_\infty = u \cdot \vec{i} + v \cdot \vec{j} + w \cdot \vec{k} \equiv \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

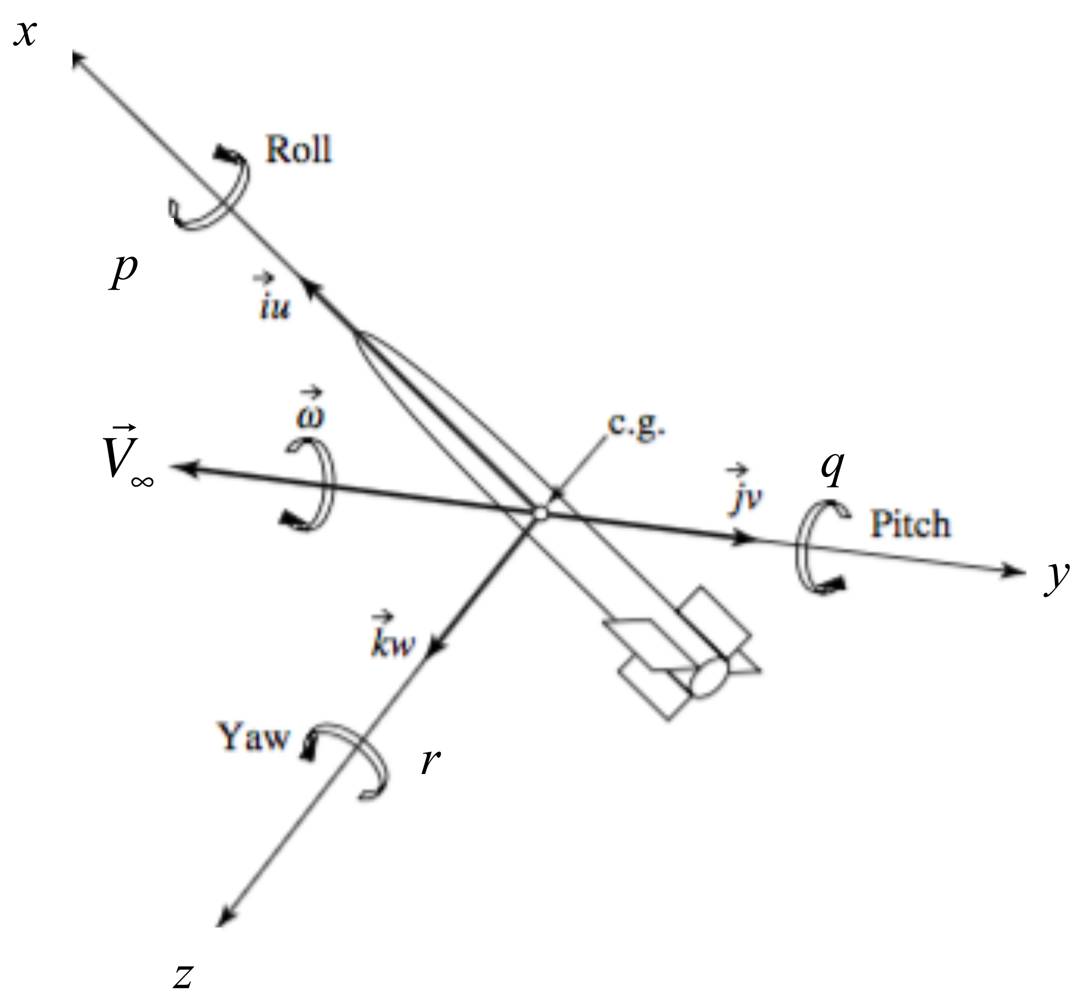




Derivation of the Longitudinal "AlphaDot" Equation (2)

- **Body Axis Coordinates, Angular Rate Components**

$$\vec{\Omega} = p \cdot \vec{i} + q \cdot \vec{j} + r \cdot \vec{k} \equiv \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$



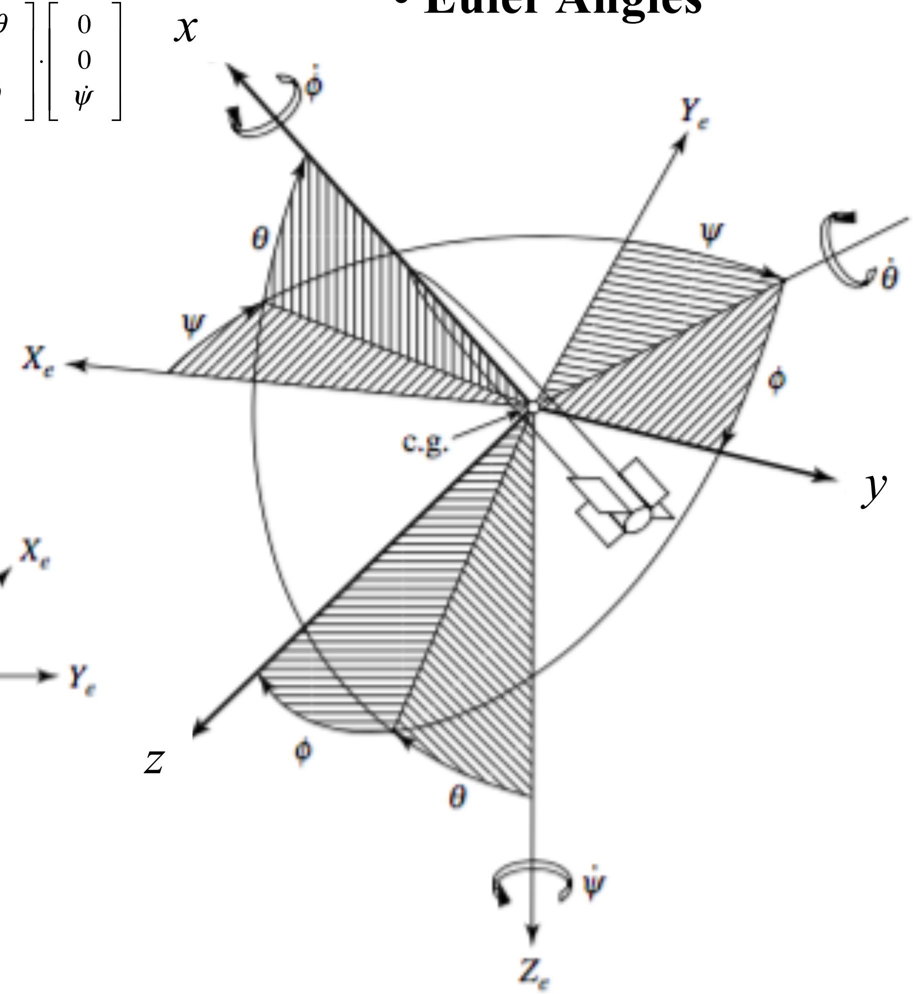
Derivation of the Longitudinal "AlphaDot" Equation (3)

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

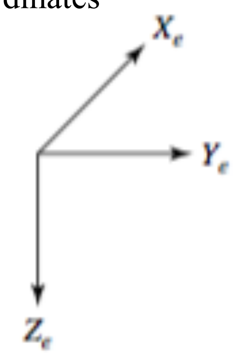
$$= \begin{bmatrix} 1 & 0 & -\sin\phi \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \cdot \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \dot{\phi} - \sin\phi \cdot \dot{\psi} \\ \cos\phi \cdot \dot{\theta} + \cos\theta\sin\phi \cdot \dot{\psi} \\ -\sin\phi \cdot \dot{\theta} + \cos\theta\cos\phi \cdot \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix} \cdot \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} p + q \cdot \tan\theta\sin\phi + r \cdot \tan\theta\cos\phi \\ q \cdot \cos\phi - r \cdot \sin\phi \\ q \cdot \sin\phi/\cos\theta + r \cdot \cos\phi/\cos\theta \end{bmatrix}$$

- **Euler Angles**



Earth-Fixed Coordinates



Derivation of the Longitudinal "AlphaDot" Equation (4)

- **From Dynamics**

$$\frac{\vec{F}}{m} = \vec{A} = \frac{d}{dt}\left(\vec{V}\right) + \vec{\Omega} \times \vec{V} \rightarrow \frac{1}{m}\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{Vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ u & v & w \end{Vmatrix}$$

$$\rightarrow \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = -\begin{Vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ u & v & w \end{Vmatrix} + \frac{1}{m}\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

- **Evaluating Cross Product**

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \frac{1}{m}\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

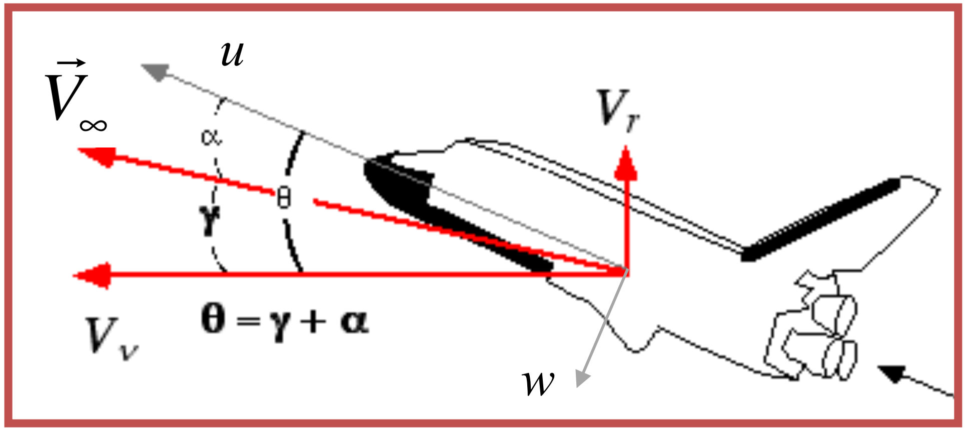
Derivation of the Longitudinal "AlphaDot" Equation (5)

- **Assume Axi-Symmetric Missile Profile, 2-D Flight Path**

$$(v,\ \beta,\ F_y = 0,\ V_\infty = \sqrt{u^2 + w^2}\ )$$

- **Translational EOM Reduce to**

$$V_\infty = \sqrt{u^2 + w^2} \rightarrow \begin{bmatrix} \dot{u} = & -q \cdot w + \dfrac{F_x}{m} \\ \dot{w} = & q \cdot u + \dfrac{F_z}{m} \end{bmatrix}$$



$\rightarrow$ *From Kinematics*

$$\tan\alpha = \frac{w}{u} \rightarrow \left(1 + \tan^2\alpha\right)\dot{\alpha} = \frac{\dot{w}}{u} - \frac{w}{u^2}\cdot\dot{u} = \frac{\dot{w}\cdot u - \dot{u}\cdot w}{u^2}$$

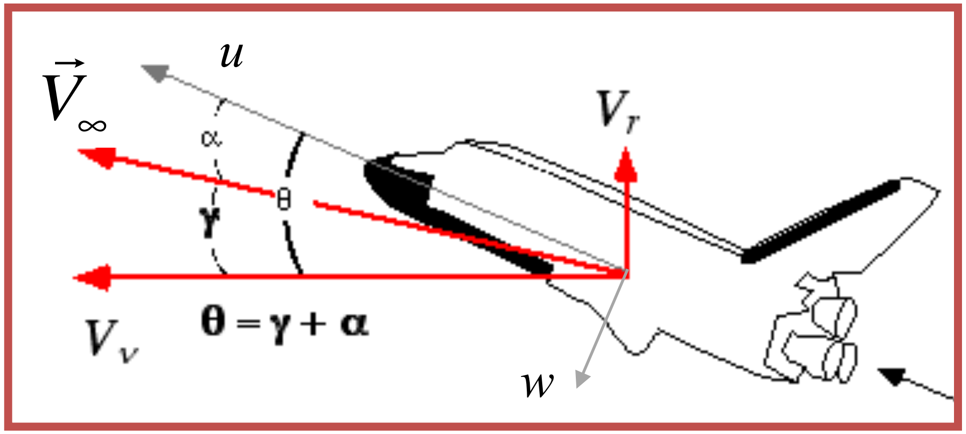
$$\left(1 + \tan^2\alpha\right) = \left(1 + \left(\frac{w}{u}\right)^2\right) = \frac{u^2 + w^2}{u^2} \rightarrow \dot{\alpha} = \frac{u^2}{u^2 + w^2}\cdot\frac{\dot{w}\cdot u - \dot{u}\cdot w}{u^2} = \frac{\dot{w}\cdot u - \dot{u}\cdot w}{u^2 + w^2}$$

$$\dot{\alpha} = \frac{\dot{w}\cdot u - \dot{u}\cdot w}{u^2 + w^2} = \frac{\dot{w}\cdot u}{u^2 + w^2} - \frac{\dot{u}\cdot w}{u^2 + w^2}$$

Derivation of the Longitudinal "AlphaDot" Equation (6)

→ *From Kinematics*

$$\dot{\alpha} = \frac{\dot{w}\cdot u - \dot{u}\cdot w}{u^2+w^2} = \frac{\dot{w}\cdot u}{u^2+w^2} - \frac{\dot{u}\cdot w}{u^2+w^2}$$



→ *From Dynamics*

$$\begin{bmatrix} \dot{u} = & -q\cdot w + \frac{F_x}{m} \\ \dot{w} = & q\cdot u + \frac{F_z}{m} \end{bmatrix}$$

→ *Substitute*

$$\dot{\alpha} = \frac{\left(q\cdot u + \frac{F_z}{m}\right)\cdot u}{V_\infty^2} - \frac{\left(-q\cdot w + \frac{F_x}{m}\right)\cdot w}{V_\infty^2} = \frac{\left(q\cdot u^2 + \frac{F_z}{m}\cdot u + q\cdot w^2 - \frac{F_x}{m}\cdot w\right)}{V_\infty^2} = \frac{\left(q\cdot\left(u^2+w^2\right) + \frac{F_z}{m}\cdot u + -\frac{F_x}{m}\cdot w\right)}{V_\infty^2}$$

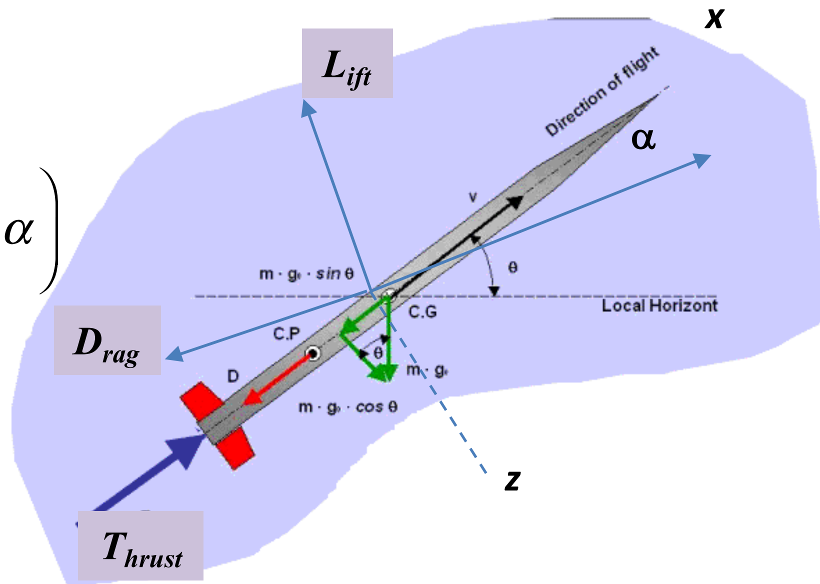
→ *Simplify*

$$\dot{\alpha} = \frac{\left(q\cdot V_\infty^2 + \frac{F_z}{m}\cdot u + -\frac{F_x}{m}\cdot w\right)}{V_\infty^2} = \frac{\left(q\cdot V_\infty^2 + \frac{F_z}{m}\cdot u + -\frac{F_x}{m}\cdot w\right)}{V_\infty^2} = \left(q + \frac{F_z}{m\cdot V_\infty}\cdot\frac{u}{V_\infty} - \frac{F_x}{m\cdot V_\infty}\cdot\frac{w}{V_\infty}\right)$$

Derivation of the Longitudinal "AlphaDot" Equation (7)

inematics

$$\rightarrow \begin{bmatrix} \dfrac{u}{V_\infty} = \cos\alpha \\ \dfrac{w}{V_\infty} = \sin\alpha \end{bmatrix} \rightarrow \dot{\alpha} = \left(q + \frac{F_z}{m \cdot V_\infty} \cdot \cos\alpha - \frac{F_x}{m \cdot V_\infty} \cdot \sin\alpha \right)$$



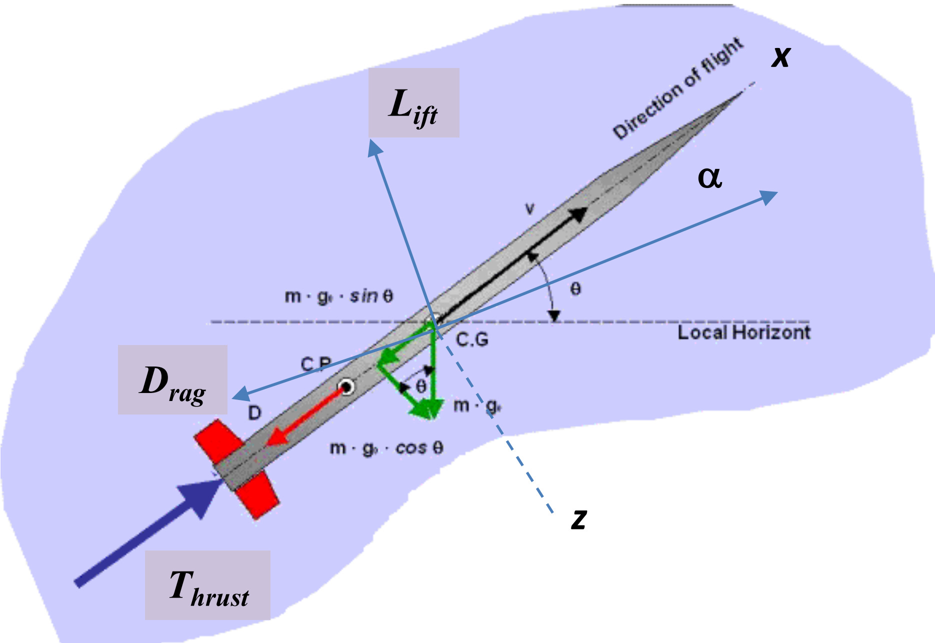
$\rightarrow$ *Resolving Forces*

$$\begin{bmatrix} F_x = T_{thrust} - m \cdot g \cdot \sin\theta - D_{rag} \cdot \cos\alpha + L_{ift} \cdot \sin\alpha \\ F_z = \quad m \cdot g \cdot \cos\theta - D_{rag} \cdot \sin\alpha - L_{ift} \cdot \cos\alpha \end{bmatrix}$$

$\rightarrow$ *Substituting*

$$\dot{\alpha} = \left(q + \frac{m \cdot g \cdot \cos\theta - D_{rag} \cdot \sin\alpha - L_{ift} \cdot \cos\alpha}{m \cdot V_\infty} \cdot \cos\alpha - \frac{T_{thrust} - m \cdot g \cdot \sin\theta - D_{rag} \cdot \cos\alpha + L_{ift} \cdot \sin\alpha}{m \cdot V_\infty} \cdot \sin\alpha \right)$$

Derivation of the Longitudinal "AlphaDot" Equation (7)



$$C_L = \frac{L_{ift}}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A_{ref}}, \quad C_D = \frac{D_{rag}}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A_{ref}}$$

$C_L = lift\ coefficient, \quad C_D = drag\ coefficient$

$\frac{1}{2}\cdot\rho\cdot V^2 = \text{"dynamic pressure"}(\bar{q})$

$A_{ref} = \text{"reference area"} -> typically\ maximum\ frontal\ area$

$\rho = \text{"local air density"} -> function\ of\ altitude$

$\rightarrow$ *Collecting*

$$\dot{\alpha} = q + \frac{m\cdot g\cdot\cos\theta\cdot\cos\alpha + m\cdot g\cdot\sin\theta\cdot\sin\alpha}{m\cdot V_\infty} + \frac{-D_{rag}\cdot\sin\alpha\cdot\cos\alpha + D_{rag}\cdot\cos\alpha\cdot\sin\alpha}{m\cdot V_\infty} - \frac{L_{ift}\cdot\cos^2\alpha + L_{ift}\cdot\cos^2\alpha}{m\cdot V_\infty} - \frac{T_{thrust}\cdot\sin\alpha}{m\cdot V_\infty}$$

$\rightarrow$ *Simplifying*

$$\dot{\alpha} = q + \frac{g\cdot\cos(\theta-\alpha)}{V_\infty} - \frac{L_{ift}}{m\cdot V_\infty} - \frac{T_{thrust}\cdot\sin\alpha}{m\cdot V_\infty} \rightarrow\rightarrow \boxed{\dot{\alpha} = q + \frac{g\cdot\cos(\theta-\alpha)}{V_\infty} - \frac{\bar{q}\cdot A_{ref}\cdot C_L + T_{thrust}\cdot\sin\alpha}{m\cdot V_\infty}}$$

Appendix II: Derivation of the Longitudinal "VDot" Equation

Kinematics

$$\dot{V}_\infty = \frac{1}{2\sqrt{u^2+w^2}}\cdot\left(2\cdot u\cdot\dot{u}+2\cdot w\cdot\dot{w}\right)=\left(\frac{u\cdot\dot{u}+w\cdot\dot{w}}{\sqrt{u^2+w^2}}\right)=$$

$$\left(\frac{u\cdot\left(-q\cdot w+\frac{F_x}{m}\right)+w\cdot\left(q\cdot u+\frac{F_z}{m}\right)}{V_\infty}\right)=\left(\frac{u}{V_\infty}\cdot\frac{F_x}{m}+\frac{w}{V_\infty}\cdot\frac{F_z}{m}\right)=\cos\alpha\cdot\frac{F_x}{m}+\sin\alpha\cdot\frac{F_z}{m}$$

$\rightarrow$ *Resolving Forces*

$$\begin{bmatrix} F_x = T_{thrust} - m\cdot g\cdot\sin\theta - D_{rag}\cdot\cos\alpha + L_{ift}\cdot\sin\alpha \\ F_z = \qquad m\cdot g\cdot\cos\theta - D_{rag}\cdot\sin\alpha - L_{ift}\cdot\cos\alpha \end{bmatrix}$$

Substituting

$$\dot{V}_\infty = \cos\alpha\cdot\frac{\left(T_{thrust}-m\cdot g\cdot\sin\theta-D_{rag}\cdot\cos\alpha+L_{ift}\cdot\sin\alpha\right)}{m}+\sin\alpha\cdot\frac{\left(m\cdot g\cdot\cos\theta-D_{rag}\cdot\sin\alpha-L_{ift}\cdot\cos\alpha\right)}{m}$$

Derivation of the Longitudinal "VDot" Equation

Simplifying

$$\dot{V}_\infty = \cos\alpha \cdot \frac{\left(T_{thrust} - m\cdot g\cdot\sin\theta - D_{rag}\cdot\cos\alpha\right)}{m} + \sin\alpha\cdot\frac{\left(m\cdot g\cdot\cos\theta - D_{rag}\cdot\sin\alpha\right)}{m} =$$

$$\dot{V}_\infty = \cos\alpha \cdot \frac{\left(T_{thrust} - m\cdot g\cdot\sin\theta - D_{rag}\cdot\cos^2\alpha\right)}{m} + \sin\alpha\cdot\frac{\left(m\cdot g\cdot\cos\theta - D_{rag}\cdot\sin^2\alpha\right)}{m} =$$

$$\frac{T_{thrust}\cdot\cos\alpha}{m} + g\cdot\left(\cos\theta\sin\alpha - \sin\theta\cos\alpha\right) - \frac{D_{rag}}{m}\left(\cos^2\alpha + \sin^2\alpha\right) =$$

$$\frac{T_{thrust}\cdot\cos\alpha}{m} + g\cdot\left(\cos\theta\sin\alpha - \sin\theta\cos\alpha\right) - \frac{\bar{q}\cdot A_{ref}\cdot C_D}{m} = -\frac{\bar{q}\cdot A_{ref}\cdot C_D}{m} - g\cdot\sin\left(\theta-\alpha\right) + \frac{T_{thrust}\cdot\cos\alpha}{m}$$

$$\boxed{\dot{V}_\infty = \frac{T_{thrust}\cdot\cos\alpha}{m} - g\cdot\sin\left(\theta-\alpha\right) - \frac{\bar{q}\cdot A_{ref}\cdot C_D}{m}}$$