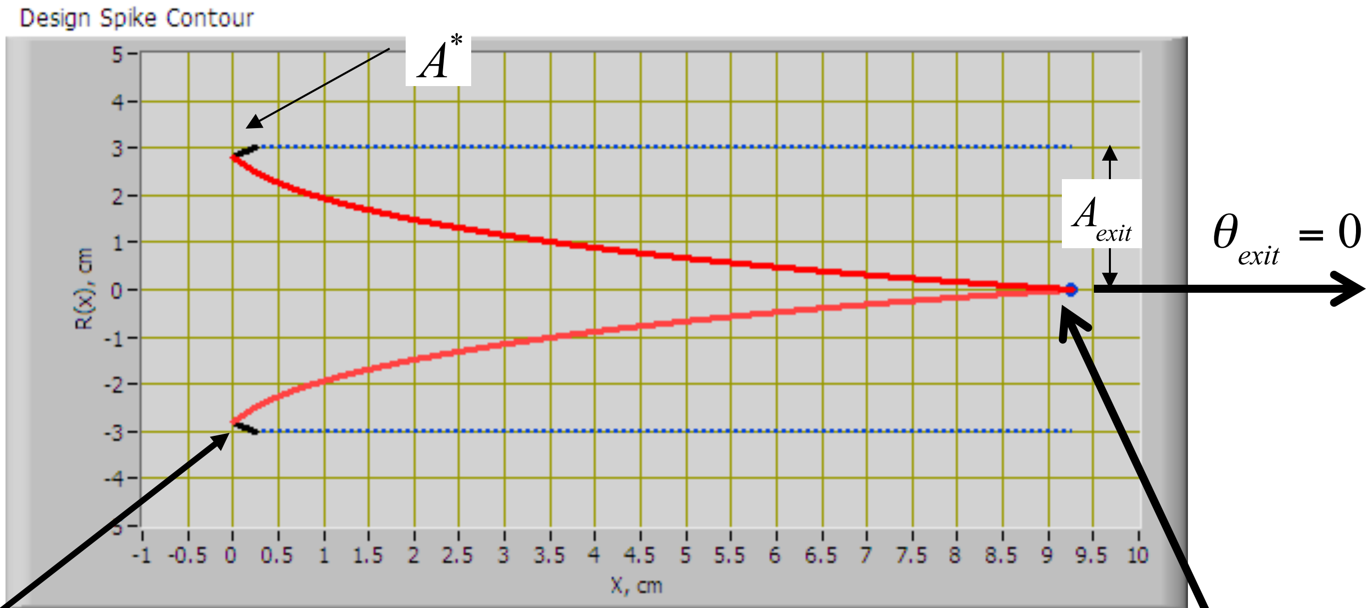
Homework *1.3, Part 2 Solution*

1) Re-derive the Conical (3-D) Aerospike Contour Design Rules (*Slide 31)* for a two dimensional (Linear) Nozzle

2-D Nozzle Contour Design, Choked throat

Apply Method of Characteristics to Aerospike Nozzle (1)



$$\nu_{throat} = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(1^2-1\right)}\right\} - \tan^{-1}\sqrt{1^2-1} = 0$$

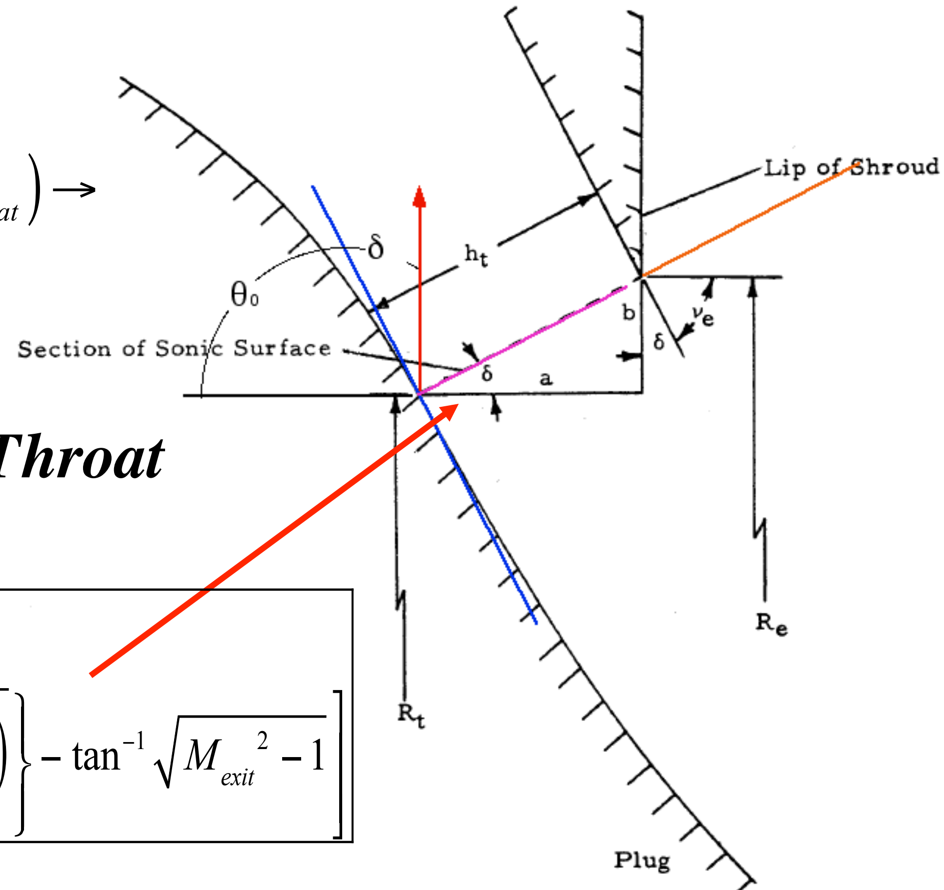
$$\frac{A_{exit}}{A^*} = \left[\frac{1}{M_{exit}}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{(\gamma-1)}{2}M_{exit}{}^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right]$$

Apply Method of Characteristics to Aerospike Nozzle (2)

$$\theta_{throat} + \nu_{throat} = \theta_x + \nu_x = \theta_{exit} + \nu_{exit}$$

$$\delta = 90^\circ - \theta_{throat} = 90^\circ - \left(\theta_{exit} + \nu_{exit} - \nu_{throat}\right) \rightarrow$$

$$\rightarrow \nu_{throat} = 0,\ \theta_{exit} = 0$$



$$\delta = 90^\circ - \left(\nu_{exit}\right) =$$

$$\rightarrow 90^\circ - \left[\left(M_{exit}\right) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_{exit}^{2}-1\right)}\right\} - \tan^{-1}\sqrt{M_{exit}^{2}-1} \right]$$

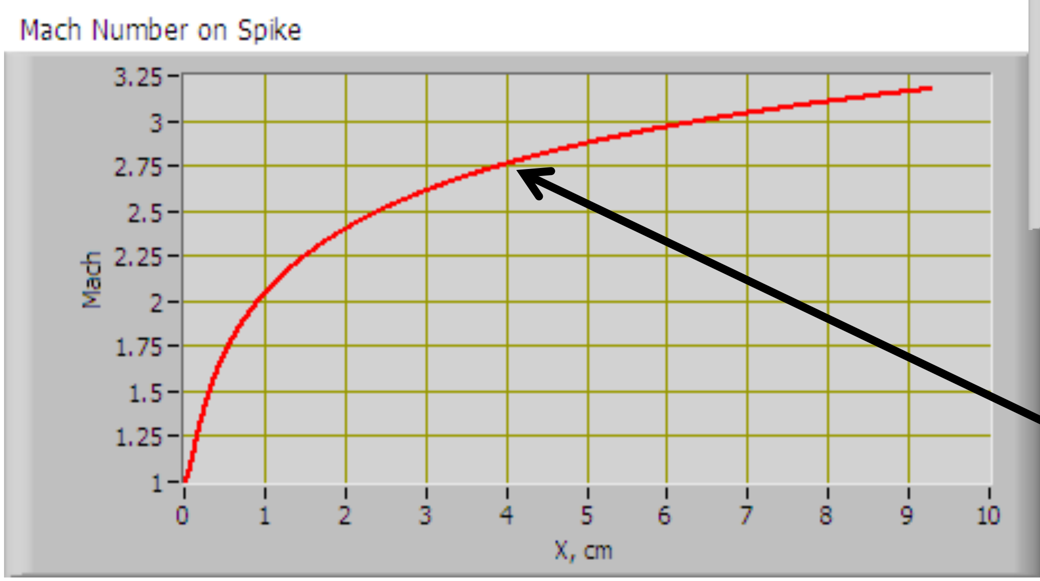
Throat Exit Angle Fixed by Spike
Design expansion ratio

Apply Method of Characteristics to Aerospike Nozzle (3)

$$\theta_x + \nu_x = \nu_{exit}$$

$$\rightarrow \boxed{\theta_x = \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_{exit}^{2}-1\right)}\right\} - \tan^{-1}\sqrt{M_{exit}^{2}-1}\right] - \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left\{\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_{x}^{2}-1\right)}\right\} - \tan^{-1}\sqrt{M_{x}^{2}-1}\right]}$$

Along spike surface

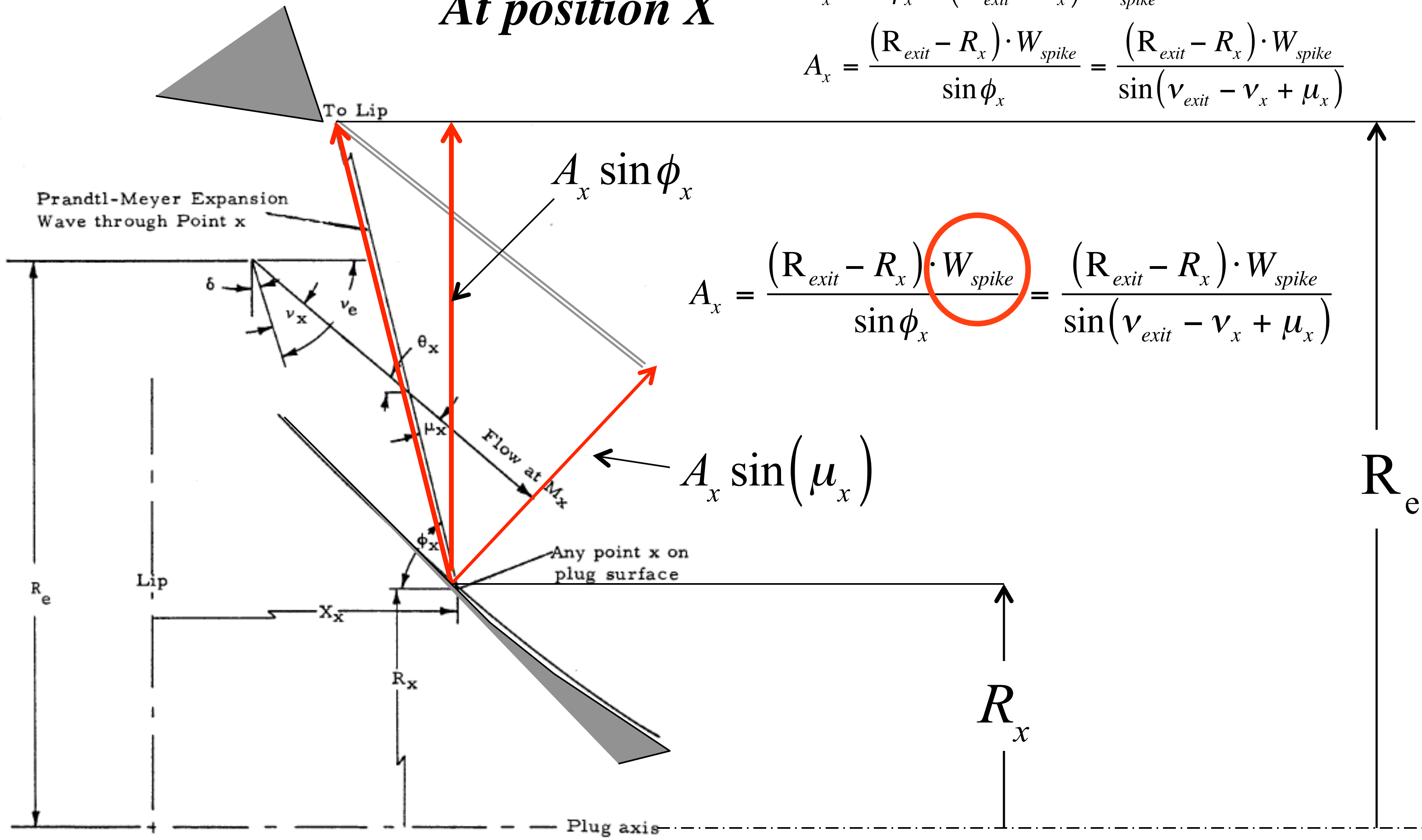


2-D
Plug Geometry At position X

$$\left.\begin{array}{l}\phi_x = \theta_x + \mu_x \\ \theta_x = \nu_{exit} - \nu_x\end{array}\right| \rightarrow \phi_x = \nu_{exit} - \nu_x + \mu_x$$

$$A_x \cdot \sin\phi_x = \left(R_{exit} - R_x\right)\cdot W_{spike} \rightarrow$$

$$A_x = \frac{\left(R_{exit} - R_x\right)\cdot W_{spike}}{\sin\phi_x} = \frac{\left(R_{exit} - R_x\right)\cdot W_{spike}}{\sin\left(\nu_{exit} - \nu_x + \mu_x\right)}$$



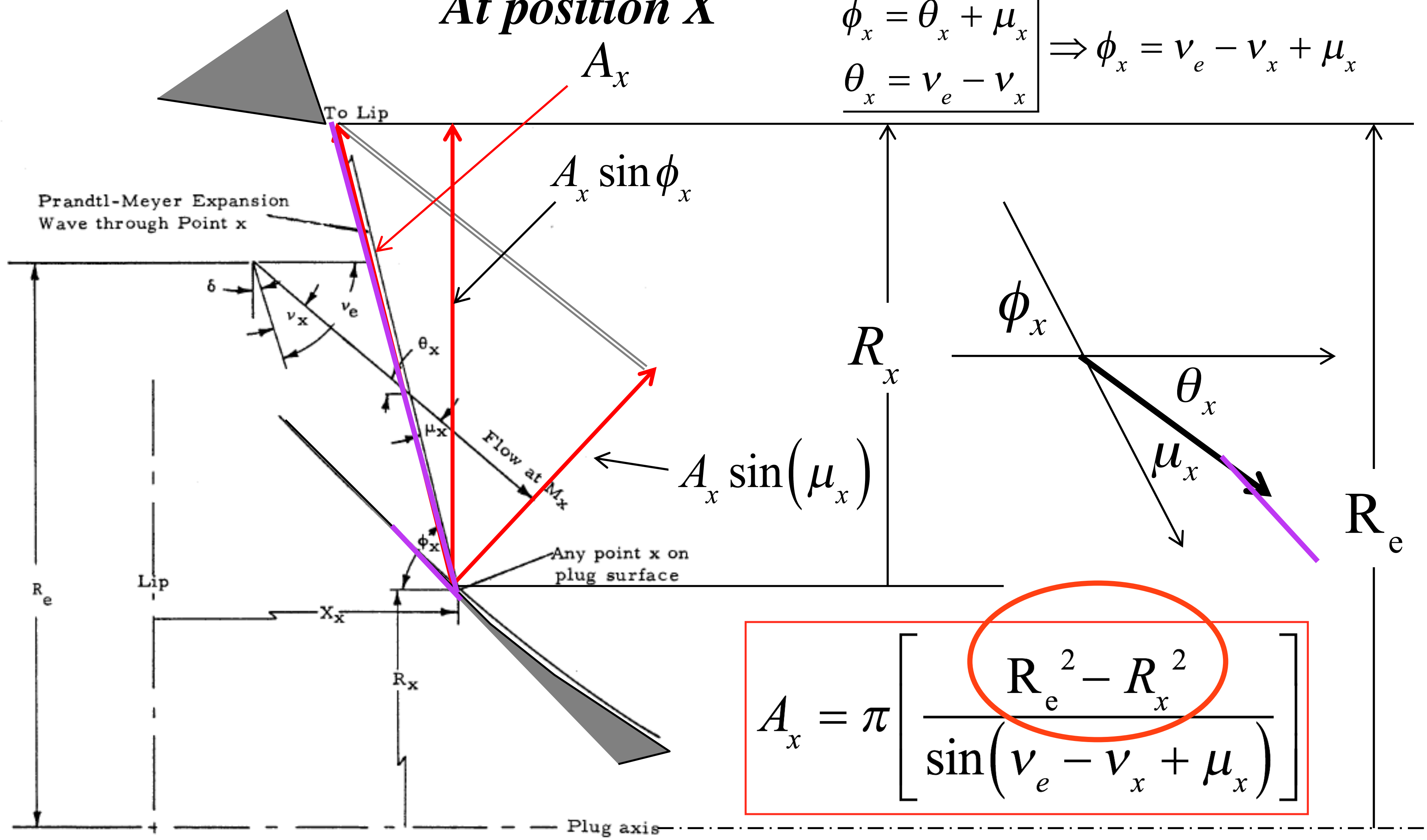
$$A_x = \frac{\left(R_{exit} - R_x\right)\cdot W_{spike}}{\sin\phi_x} = \frac{\left(R_{exit} - R_x\right)\cdot W_{spike}}{\sin\left(\nu_{exit} - \nu_x + \mu_x\right)}$$

Compare to 3-D

Plug Geometry At position X

$$A_x \sin\phi_x = \pi\left[R_e^{\,2} - R_x^{\,2}\right]$$

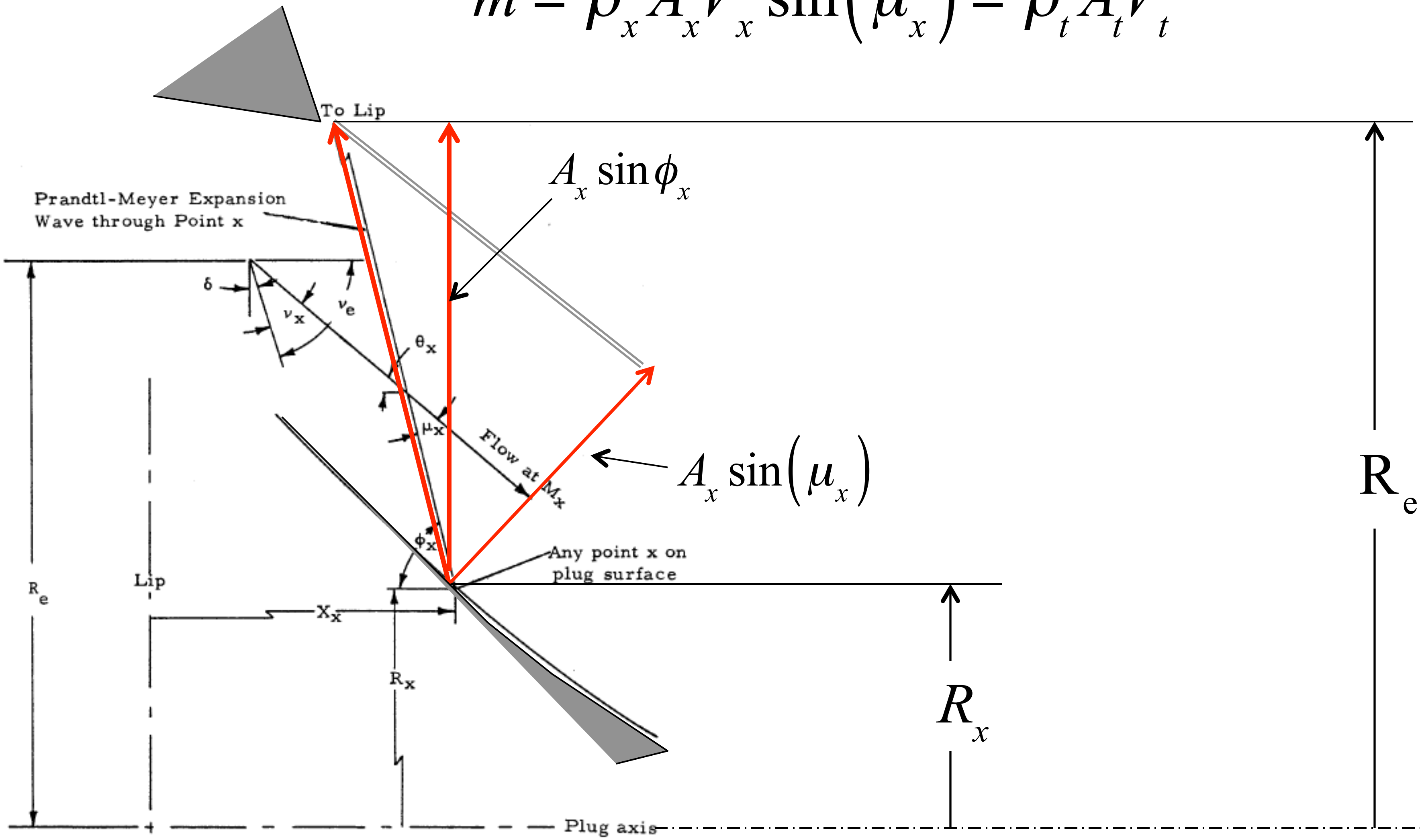
$$\left.\begin{aligned}\phi_x &= \theta_x + \mu_x \\ \theta_x &= \nu_e - \nu_x\end{aligned}\right| \Rightarrow \phi_x = \nu_e - \nu_x + \mu_x$$



$$A_x = \pi\left[\frac{R_e^{\,2} - R_x^{\,2}}{\sin\left(\nu_e - \nu_x + \mu_x\right)}\right]$$

Apply Continuity equation

$$\dot{m} = \rho_x A_x V_x \sin(\mu_x) = \rho_t A_t V_t$$



Apply Method of Characteristics to Aerospike Nozzle (5)

- Solving for A_x

$$Solve\ for\ A_x \rightarrow A_x = \frac{\rho_t \cdot A_t}{\rho_x \cdot \frac{V_x}{V_t} \cdot \sin\mu_x} = \frac{P_t \cdot A_t}{P_x \sqrt{\frac{T_t}{T_x}} \cdot \frac{\sqrt{T_t}}{V_t} \cdot \frac{V_x}{\sqrt{T_x}} \cdot \sin\mu_x} = \frac{P_t \cdot A_t}{P_x \sqrt{\frac{T_t}{T_x}} \cdot \frac{M_x}{M_t} \cdot \sin\mu_x}$$

- Divide by throat area

$$\rightarrow \frac{A_x}{A_t} = \frac{P_t}{P_x \sqrt{\frac{T_t}{T_x}} \cdot \frac{M_x}{M_t} \cdot \sin\mu_x} = \frac{\frac{P_0}{P_x} \cdot \sqrt{\frac{T_x}{T_0}}}{\frac{P_0}{P_t} \sqrt{\frac{T_t}{T_0}} \cdot \frac{M_x}{M_t} \cdot \sin\mu_x} = \frac{\left(1 + \frac{\gamma - 1}{2} \cdot M_x^{\ 2}\right)^{\frac{\gamma}{\gamma - 1}} \cdot \sqrt{\frac{1}{\left(1 + \frac{\gamma - 1}{2} \cdot M_x^{\ 2}\right)}} \frac{1}{M_x}}{\left(1 + \frac{\gamma - 1}{2} \cdot M_t^{\ 2}\right)^{\frac{\gamma}{\gamma - 1}} \cdot \sqrt{\frac{1}{\left(1 + \frac{\gamma - 1}{2} \cdot M_t^{\ 2}\right)}} \frac{1}{M_t} \cdot \sin\mu_x}$$

Apply Method of Characteristics to Aerospike Nozzle (6)

- Simplifying

$$\frac{A_x}{A_t} = \frac{\left(1+\frac{\gamma-1}{2}\cdot M_x^{\,2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\cdot\frac{1}{M_x}}{\left(1+\frac{\gamma-1}{2}\cdot M_t^{\,2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\cdot\frac{1}{M_t}\cdot\sin\mu_x} = \frac{\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\cdot M_x^{\,2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\cdot\frac{1}{M_x}}{\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\cdot M_t^{\,2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\cdot\frac{1}{M_t}\cdot\sin\mu_x}$$

$$\frac{A_t}{A^*} = \left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\cdot M_t^{\,2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\cdot\frac{1}{M_t} \rightarrow \frac{A_x}{A_t} = \frac{\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\cdot M_x^{\,2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\cdot\frac{1}{M_x}}{\frac{A_t}{A^*}\cdot\sin\mu_x}$$

$$\boxed{\frac{A_x}{A_t}\cdot\frac{A_t}{A^*} = \frac{\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\cdot M_x^{\,2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\cdot\frac{1}{M_x}}{\sin\mu_x}}$$

Apply Method of Characteristics to Aerospike Nozzle (7)

- Simplifying again

$$Simplify \rightarrow A_x = A^* \frac{\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\cdot M_x^{\ 2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \cdot \frac{1}{M_x}}{\sin\mu_x} \rightarrow \sin\mu_x = \frac{1}{M_x}$$

$$A_x = A^* \left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\cdot M_x^{\ 2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = \frac{A_{exit}}{\varepsilon}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\cdot M_x^{\ 2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$A_{exit} = R_{exit}\cdot W_{spike} \rightarrow \boxed{A_x = \frac{R_{exit}\cdot W_{spike}}{\varepsilon}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}\cdot M_x^{\ 2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

Apply Method of Characteristics to Aerospike Nozzle (8)

- Solve for R_x

$$\text{from earlier} \rightarrow A_x = \frac{\left(R_{exit} - R_x\right) \cdot W_{spike}}{\sin\left(\nu_{exit} - \nu_x + \mu_x\right)} = \frac{R_{exit} \cdot W_{spike}}{\varepsilon} \left[\left(\frac{2}{\gamma + 1} \right) \left(1 + \frac{\gamma - 1}{2} \cdot M_x^{\ 2} \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\text{Solve for } R_x \rightarrow \left(1 - \frac{R_x}{R_{exit}} \right) = \frac{\sin\left(\nu_{exit} - \nu_x + \mu_x\right)}{\varepsilon} \left[\left(\frac{2}{\gamma + 1} \right) \left(1 + \frac{\gamma - 1}{2} \cdot M_x^{\ 2} \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\boxed{\frac{R_x}{R_{exit}} = 1 - \frac{\sin\left(\nu_{exit} - \nu_x + \mu_x\right)}{\varepsilon} \left[\left(\frac{2}{\gamma + 1} \right) \left(1 + \frac{\gamma - 1}{2} \cdot M_x^{\ 2} \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

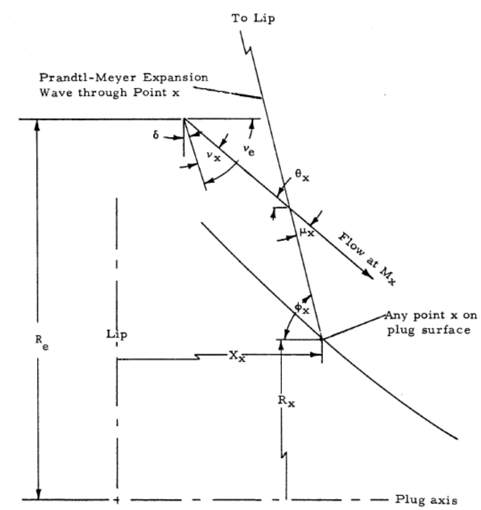
Apply Method of Characteristics to Aerospike Nozzle (9)

- and since by geometry of the surface

$$\textit{Surface geometry} \rightarrow \tan\phi_x = \frac{R_{exit} - R_x}{X_x} \rightarrow \textit{from earlier} \rightarrow \phi_x = \nu_{exit} - \nu_x + \mu_x$$

2 – D Spike Contour Lines

$$\left[\begin{array}{l} X_x = \dfrac{R_{exit} - R_x}{\tan\left(\nu_{exit} - \nu_x + \mu_x\right)} \\ R_x = R_{exit}\left(1 - \dfrac{\sin\left(\nu_{exit} - \nu_x + \mu_x\right)}{\varepsilon}\left[\left(\dfrac{2}{\gamma+1}\right)\left(1 + \dfrac{\gamma-1}{2}\cdot M_x^{\,2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right) \\ \sin\mu_x = \dfrac{1}{M_x} \qquad \tan\phi_x = \dfrac{R_{exit} - R_x}{X_x} \qquad \phi_x = \nu_{exit} - \nu_x + \mu_x \end{array}\right]$$



2-D Nozzle Algorithm

- These equations define the isentropic spike profile

Compare to 3-D Nozzle Algorithm
Aerospike Nozzle

$$\tan\phi_x = \frac{R_e - R_x}{X_x} \rightarrow \phi_x = \nu_e - \nu_x + \mu_x \rightarrow$$

$$X_x = \frac{R_{exit} - R_x}{\tan\left(\nu_e - \nu_x + \mu_x\right)}$$

$$R_x = R_{exit}\sqrt{1 - \frac{\sin\left(\nu_e - \nu_x + \mu_x\right)}{\varepsilon}\left[\left(\frac{2}{\gamma+1}\right)\cdot\left(1+\frac{\gamma-1}{2}M_x^{\ 2}\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

$$\sin\left(\mu_x\right) = \frac{1}{M_x}$$

Note "Square Root" Sign

Compare 2-D and 3-D Spike Contours

$Surface\ geometry \rightarrow \tan\phi_x = \frac{R_{exit} - R_x}{X_x} \rightarrow from\ earlier \rightarrow \phi_x = \nu_{exit} - \nu_x + \mu_x$

$2 - D\ Spike\ Contour\ Lines$

2-D Nozzle Algorithm

$$X_x = \frac{R_{exit} - R_x}{\tan\left(\nu_{exit} - \nu_x + \mu_x\right)}$$

$$R_x = R_{exit}\left(1 - \frac{\sin\left(\nu_{exit} - \nu_x + \mu_x\right)}{\varepsilon}\left[\left(\frac{2}{\gamma+1}\right)\left(1 + \frac{\gamma-1}{2}\cdot M_x^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right)$$

$$\sin\mu_x = \frac{1}{M_x} \qquad \tan\phi_x = \frac{R_{exit} - R_x}{X_x} \qquad \phi_x = \nu_{exit} - \nu_x + \mu_x$$

3-D Nozzle Algorithm

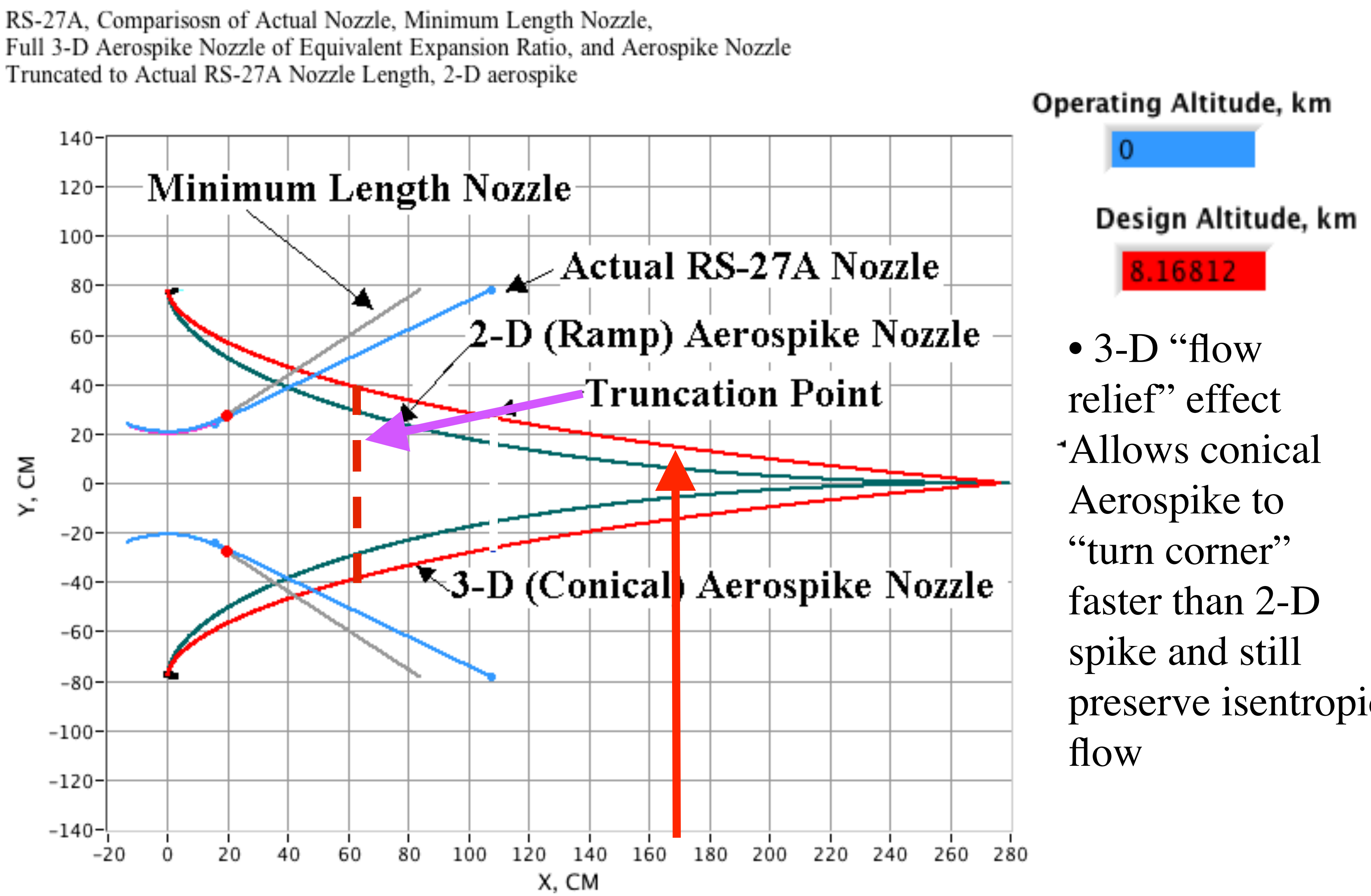
$$\tan\phi_x = \frac{\mathrm{R}_e - R_x}{X_x} \rightarrow \phi_x = \nu_e - \nu_x + \mu_x \rightarrow$$

$$X_x = \frac{\mathrm{R}_{exit} - R_x}{\tan\left(\nu_e - \nu_x + \mu_x\right)}$$

$$R_x = \mathrm{R}_{exit}\sqrt{1 - \frac{\sin\left(\nu_e - \nu_x + \mu_x\right)}{\varepsilon}\left[\left(\frac{2}{\gamma+1}\right)\cdot\left(1 + \frac{\gamma-1}{2}M_x^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

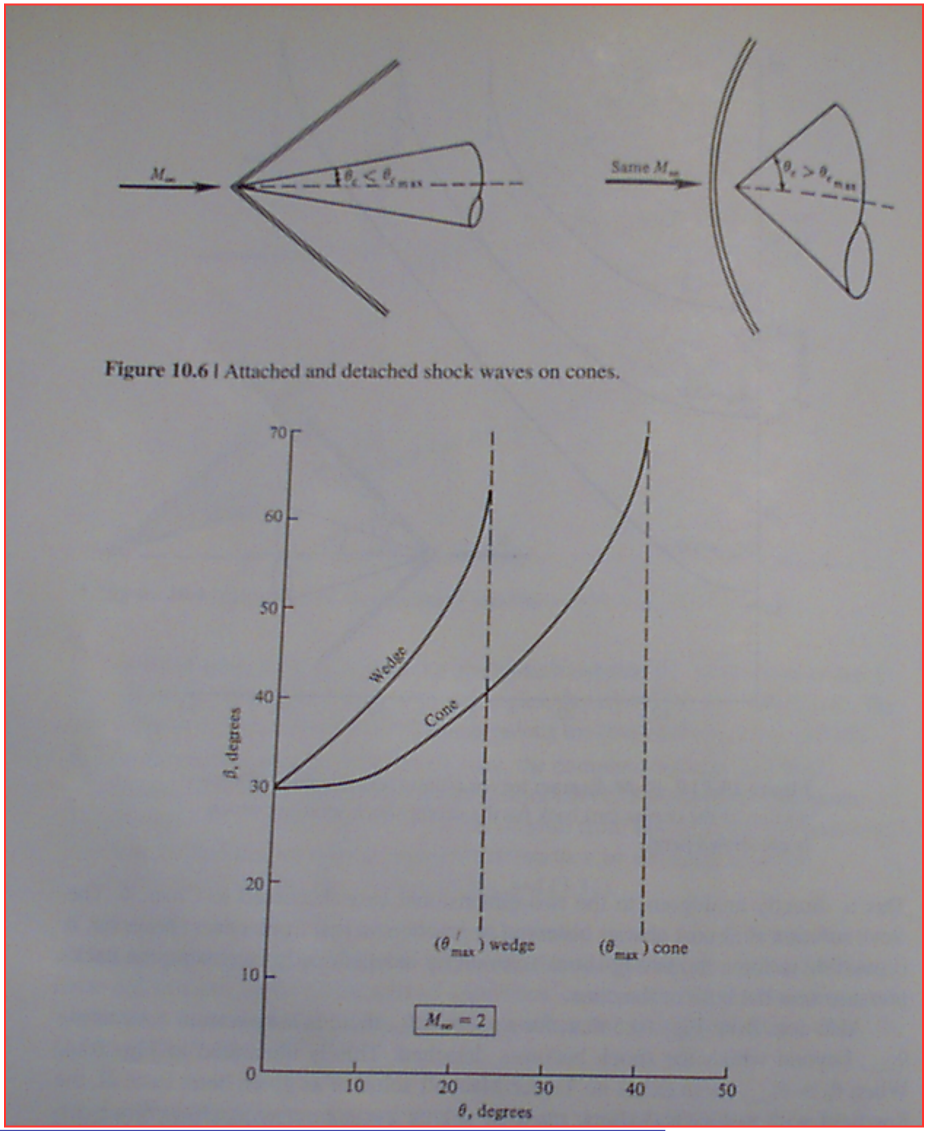
Note "Square Root" Sign

Include 2-D Spike Contour on Plot



- 3-D "flow relief" effect
- Allows conical Aerospike to "turn corner" faster than 2-D spike and still preserve isentropic flow

Physical Aspects of Cone Flow (Anderson)



- Three-dimensional "relieving" effect

- Cone shock wave is Effectively weaker Than shock wave for Corresponding wedge angle